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Optimization of electron cooling for a medium energy accumulator ring

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Abstract

A simple version of the binary collision model of electron cooling is introduced to discuss the principal issues in using electron cooling to accumulate ions at medium energies, where "medium energy" is defined by a beam velocity > 0.8c. Then a more complete treatment which gives some new general results is developed. In it longitudinal and transverse cooling rates are calculated analytically for a particle executing betatron oscillations in a storage ring. The formulas obtained are compared with numerical results. Particular attention is paid to the case in which the transverse components of the relative velocities substantially exceed the longitudinal components. The exact analytical results for finite electron temperatures are presented. The time for longitudinal cooling of a Gaussian beam is calculated as a function of the acceptable fraction of un-cooled particles; the optimum electron beam size is calculated and an optimum electron density distribution is found. The results obtained are applied to an electron cooling system for the Fermilab Recycler [1]; the cooling time, optimum parameters and tolerances are calculated.

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I. INTRODUCTION

The method of electron cooling was originally suggested by A. M. Budker [2]. It was developed and studied then both theoretically and experimentally; an ample list of the references can be found, for example, in Ref. [3]. However, some practically important problems still need to be resolved or at least need more detailed or accurate results like, for example, the problem of calculating a cooling time for a beam distribution.

Electron cooling was considered for the antiproton accumulator rings built at CERN [4] and Fermilab [5] during the 1980's, but in both cases stochastic cooling [6] was chosen because it is faster for the large emittance collected from the production target. However, the bandwidth of a stochastic cooling system limits the beam current that can be accumulated. As successive improvements in system bandwidth become more difficult and costly, electron cooling becomes increasingly attractive as a supplementary technique because its performance is practically independent of accumulated current. Its performance is, however, very sensitive to the transverse emittance of the ion beam. Fermilab is currently developing electron cooling for the 8 GeV Recycler ring, which will be commissioned in 1999 as a second-stage antiproton accumulator using stochastic cooling.

Most features of an electron cooler can be understood from a qualitative description of the scheme. The electron beam is accelerated to the same mean velocity as the ions and passed through a straight section of an ion storage ring. The electron beam is prepared with the lowest practicable momentum spread. In the coordinate frame moving at the common mean velocity, ions move randomly among electrons of much lower energy. The effect of the many Coulomb collisions of each ion is to pass the energy of random motion from the ions to the electrons. The ions may circulate for minutes or even hours through this straight section, but the electrons make a single pass and are collected at a potential close to that of the electron gun. In the co-moving frame the process looks like the exchange of heat between a hot ion gas and an electron gas that is continuously circulated at low temperature. The lab-frame energy of the electron beam is lower than the ion energy by the electron-ion mass ratio. The cooling rate is proportional to the electron current and the length of the cooling straight section. Because the Lorentz transformations that convert the cooling rate in beam-frame parameters to the laboratory frame introduce an inverse square dependence on the ion energy, a medium energy cooling system may be expected to feature high electron current and a long interaction region. However, beam accumulation may not require very fast cooling nor a very high phase space compression, so cooling system parameters other than beam energy need not differ greatly from those at existing low energy rings. In all that follows the ion is assumed to be a proton or antiproton for simplicity. Generalization for a different charge-to-mass ratio is direct but is not important for the subjects treated.

II. ELEMENTARY MODEL

A simple physical model of electron cooling can give useful quantitative results for medium energy accumulation where the beam-frame ion velocities are generally considerably greater than the electron velocities. In this case a strong magnetic field is not required in the cooling interaction region; the electron-ion collisions are simple Rutherford scattering, and the details of the electron velocity distribution are not important. It is sufficient to analyze the interaction as the non-relativistic multiple Coulomb scattering of a single ion with the electron distribution. The only collective plasma effect taken into account is limitation of the impact parameter for the collisions by the Debye screening radius. This approach has been described in both research papers and reviews, for example references [7] and [8].

If a quantity $\mathcal{L}(u,\chi)$ is some function of the center-of-mass scattering angle χ and relative velocity u in the Coulomb scattering, the average rate of change of $\mathcal{L}$ is

$$\langle\dot{\mathcal{L}}\rangle = n^* \int\!\!\int u\,\sigma(u,\chi)\,\mathcal{L}(u,\chi)\,g(\vec{v})\,\mathrm{d}u\,\mathrm{d}\Omega \quad , \tag{1}$$

where n^* is the volume density of the electrons, g is the normalized electron velocity distribution, σ is the differential Rutherford cross section for scattering by χ

$$\sigma = \left(\frac{r_e c^2}{2u^2\sin^2\chi/2}\right)^2 \quad , \tag{2}$$

$\mathrm{d}\Omega = 2\pi\,\sin\chi\,\mathrm{d}\chi$ is the differential of solid angle, r_e is the classical electron radius, and c is the velocity of light. Quantities like the density n and ion momentum p_p which will be referred to both lab and beam frames will be given the superscript $*$ for the beam frame. Other quantities like the electron velocity $\vec{v}$ and relative velocity $\vec{u}$ which always refer to the beam frame will not be superscripted. When $\mathcal{L}$ is $\Delta\vec{p}_p^{\,*}$, $\langle\dot{\mathcal{L}}\rangle$ is the average frictional or cooling force $\vec{F}^*$. The component of $\Delta\vec{p}_p^{\,*}$ along $\vec{p}_p^{\,*}$ gives the cooling, the transverse components give in second order a concomitant diffusion. The diffusion coefficients are given by a similar integral with $\mathcal{L} = \Delta p^*_{p,i}\,\Delta p^*_{p,j}$. The diffusion of the ions by the electrons is not practically significant. However, the scattering of an ion by ions — intrabeam scattering — is an important source of diffusion which is treated in a like manner in Sec. III G.

The integral is finite when χ is cut off at some $\chi_{\min} > 0$, a value determined by identifying the physical limitation on the maximum impact parameter $b_{\max}$. Usually this limit is the radius of the Debye sphere, but in particular cases the radius of the electron beam or the time the ion takes to pass through the interaction region may result in a lower maximum. In any case, the integral may be written

$$\vec{F}^* = F_\circ \Lambda \vec{I} \quad , \tag{3}$$

where

$$F_\circ = 4\pi (r_e m c^2)^2 n^*/m \quad , \tag{4}$$

$$\Lambda = \log(b_{\max}/b_{\min}) \quad , \tag{5}$$

and

$$\vec{I} = \int \frac{\vec{u}}{u^3}\, g(\vec{v})\,\mathrm{d}^3\vec{v} \quad . \tag{6}$$

$F_\circ$ is a scaling constant with dimensions force $\times$ velocity^2, the Coulomb log Λ contains the ratio of the maximum to minimum impact parameters b possible for the collisions, and the collision integral $\vec{I}$ embodies the integration over electron velocities. The constant m is the electron mass. Λ is weakly dependent on u and is generally removed from the velocity integral as shown here. The argument of the log varies about as the cube of relative velocity, but the appropriate velocity depends on whether one is considering the longitudinal or transverse component of the friction. The value generally lies in the range $\Lambda = 10 \pm 4$ with lower values for the longitudinal case and higher for the transverse.

It is helpful to observe that the velocity space integral $\vec{I}$ is the same as the coordinate space integral for the Coulomb force between a point charge at position $\vec{v}_p^*$ and an extended distribution of charges at positions $\vec{v}$. This so-called Coulomb analogy makes a trove of potential theory results and familiar concepts applicable to the analysis of the cooling process. For example, one infers immediately that

$$\vec{I} = \vec{v}_p^* / v_p^{*3} \quad (\text{for} \;\; u \gg v) \tag{7}$$

regardless of g. The maximum cooling rate occurs for $u \approx v$. For this introduction it is sufficient to assume a cold electron beam, that is to take an electron velocity equal to zero. More important than this approximation, however, is the omission of variation in ion velocity arising from betatron oscillation. Average friction should include averaging over betatron phases; this deficiency is remedied in Sec. III B.

More useful than the force $\vec{F}^*$ itself in evaluating system parameters are certain related quantities transformed to the lab frame. The beam frame is chosen with the z^* axis in the direction of the mean beam velocity $\bar{v} = \beta c$. The lab frame has the same orientation but moves with respect to the beam frame at a velocity $-\bar{v}$. A tabulation of the effects of the Lorentz transformation for this special case appears as Table I. In the preceding development no allowance has been made for the fact that cooling takes place only for ions in the cooling section; the end results must include a reduction of the force by the ratio $\eta = \ell_c / C$ of cooling section length to storage ring circumference. This packing factor for the cooling can hardly exceed a few percent. The factor η will be included when expressing quantities in the lab frame but not in beam-frame expressions.

Because force is $\mathrm{d}p/\mathrm{d}t$, the force on a particle divided by its momentum is the instantaneous fractional rate of change of momentum, *i. e.*, the cooling rate. For the large v_p case,

$$\vec{F}^* = F_\circ \Lambda \vec{v}_p^* / v_p^{*3} \tag{8}$$

so, using the non-relativistic expression for the momentum, the rate is

$$\alpha^* = F_\circ \Lambda / (m_p v_p^{*3}) \quad . \tag{9}$$

In this case one can obtain the time dependence of the velocity by direct integration of the force equation:

$$v_p^{*3} = v_{p\circ}^{*3} - \frac{3 F_\circ \Lambda}{m_p} t^* \quad . \tag{10}$$

For the t^* value

$$t^*_{\rm stop} = m_p v^{*3}_{p\circ}/(3F_\circ\Lambda) \tag{11}$$

the ion velocity would be zero if the v^{-2} force law carried through to small v^*_p. Even though it does not signify the time for complete cooling, $t_{\rm stop}$ is useful because it has the full scaling properties for the cooling time and because it approximates the time required to achieve the cooling needed for useful accumulation, *viz.*, the time to lower ion velocity spread to about that of the electrons. Notice that $t^*_{\rm stop} = (\alpha^*_\circ)^{-1}/3$, where the subscript $\circ$ denotes the rate evaluated at $v^*_{p\circ}$. The lab-frame expression is

$$t_{\rm stop} = \gamma m_p v^{*3}_{p\circ}/(3F_\circ\Lambda\eta) \quad . \tag{12}$$

It is not especially complicated to express v^*_p in terms of lab-frame variables, but a convenient approximation is to replace $\vec{v}^*_p$ with $v^*_{p\perp}$ which typically constitutes the greater part of it and is expressible in the particularly simple form

$$v^*_{p\perp} = c\varepsilon_\perp/r_b \quad , \tag{13}$$

where $\varepsilon_\perp$ is the invariant emittance containing all of the beam of interest and r_b is the radius of the ion beam in the cooling section. The approximated $t_{\rm stop}$

$$t_{\rm stop} = \gamma m_p c^3\varepsilon^3_\perp/(3F_\circ\Lambda\eta r^3_b) \tag{14}$$

has the same practical usefulness as the original expression. The constant $F_\circ$ incorporates the beam-frame electron density n^* which introduces another power of γ when expressed in terms of lab-frame density; see Table I. Writing all of this out explicitly one has

$$t_{\rm stop} = \frac{\gamma^2 a^2\beta e\varepsilon^3_\perp}{12\pi^3 r_p r_e\Lambda\eta I_e r^3_b} \quad , \tag{15}$$

where quantities not previously defined are the electron beam radius a, electron charge $e > 0$, the classical radius of the proton r_p, and the electron beam current I_e.

The maximum rate of longitudinal drag, *i. e.*, the rate at which the ion beam as a whole can be accelerated or decelerated by sweeping the electron beam energy, is

$$R_D = \left.\frac{{\rm d}E}{{\rm d}t}\right|_{\rm max} = F_\parallel|_{\rm max}\bar{v} \quad . \tag{16}$$

Table I shows that $F^*_\parallel \equiv F_\parallel$; however, the cooling fraction η must be introduced. Thus,

$$R_D = \eta F^*_\parallel|_{\rm max}\beta c = \eta F_\circ\Lambda I_{\parallel\rm max}\beta c \quad . \tag{17}$$

When the ion velocity is much less than the electron velocity spread and the longitudinal components are comparable, $I_\parallel$ has its maximum value of approximately $\Delta^{-2}_\perp$, where $\Delta_\perp$ is the width of the electron transverse velocity distribution. The drag rate for a low-emittance ion beam is thus

$$R_D = F_\circ\Lambda\eta\beta c/\Delta^2_\perp \quad . \tag{18}$$

The cooling rate α is a somewhat different quantity. It is obtained by transforming the beam-frame rate α^*, Eq. 9, to the laboratory frame and correcting for the fraction η of the circumference occupied by the cooling section:

$$\alpha = \eta\alpha^*/\gamma = \frac{\eta F_\circ \Lambda I_\parallel}{\gamma m_p v_p} \quad . \tag{19}$$

This quantity gives the rate at which the momentum spread is reduced and thus the effectiveness of the momentum stacking used to accumulate multiple injections. As for $t_{\rm stop}$, there is also the factor of γ^{-1} contained in $F_\circ$.

The roughest approximation for $\vec{I}$ is just $\vec{v}_p/v^{*3}$, which gives just the rate $\alpha = (3t_{\rm stop})^{-1}$. Various approximations can be developed for $\vec{I}$ to serve in different parameter regimes; the results of Sec. III take the important step of averaging over the betatron oscillations of the ions. The cooling rate can be stated as $r = 2\Delta E\alpha$ [eV/s] where ΔE is the energy half-width of the ion beam for direct comparison to the drag rate.

A. Estimating electron cooling system parameters

In Table 2 are listed the system and beam parameters representing a *nominal* design for an electron cooling system for an 8 GeV antiproton storage ring at Fermilab called the Recycler. Some of these parameters like beam energy and ring circumference are well established and are not subject to optimization. Other quantities like input emittances are not so precisely known but are also taken as fixed. Some technical parameters like voltage regulation are set at values representative of the current state of the art. The optimization with which this paper is concerned applies primarily to the properties of the electron beam.

The particular choice of electron beam current and cooling section length is somewhat arbitrary. The length has been taken considerably longer than what has been used for low energy coolers but less than a quarter of the available straight section in the Recycler. The 500 mA electron beam current is a reasonable value on the basis of some preliminary experience. [9] It should be understood that, except at a rather detailed level, the cooling rate scales only with the product of the two. Thus, the choice, for example, of higher current and proportionally shorter cooling section would not have a first order effect on the model analysis. However, technically the choice could be crucial. Probably higher performance can be obtained more easily by higher beam current than by a longer cooling interaction region.

1. Cooling time

The cooling time ($t_{\rm stop}$) is given in Table III. This table also records that for the Recycler

$$\vec{v}^*_{p\perp} > \vec{v}^*_{p\parallel} \gtrsim \Delta_\perp \gg \Delta_\parallel \quad , \tag{20}$$

but only in the last inequality is there a full order of magnitude difference. Therefore, the formula for $t_{\rm stop}$ is approximate on account of both the first two relations. The third power dependence of $t_{\rm stop}$ on ion velocity makes it clear that transverse emittance is a critical consideration.

2. Longitudinal drag and longitudinal cooling rate

The purpose of the electron cooling in the stacking mode is to reduce the momentum spread of an injected batch plus the stack back to the initial stack momentum spread before the next batch comes along. In this condition stacking is in equilibrium with cooling and can in principle proceed for many batches. The cooling rate for Recycler parameters (Table II) is given in Table III.

Another mode in which stacking could be carried out is by sweeping the electron beam energy through the batch momentum, using the beam as an accelerator or decelerator in much the same way as rf stacking is carried out but without disruption of the stack. The drag rate R_D given in Table III gives the maximum rate at which the beam energy can be changed. The sweep rate in the table appears to make this mode advantageous for getting higher stacking rate, but part of the stack is not being cooled during the electron beam energy change. Furthermore, the tendency for the beam to develop a high momentum density just at the electron β risks intrabeam scattering strong enough to scatter some ions back into the range from which they had been removed. The sweeping mode is not considered further, but it may be an interesting alternate approach for a system with lower electron current.

B. Sensitivity of performance to the choice of parameters

The performance of a model case has been evaluated with a particular choice of system parameters. To optimize the parameter choice, a knowledge of the sensitivity of the performance to changes of the major system parameters is needed.

The specification on accelerating voltage regulation as ± 200 V was based on what has been obtained in current practice. However, until the longitudinal velocity error, either slow or fast, becomes comparable to the transverse velocity spread, the effect on performance is small. Because

$$v_{\parallel} = c\Delta U/(\beta U) \quad , \tag{21}$$

an energy drift or jitter of $\mathcal{O}(10^{-4})$ or 500 V would be acceptable.

The transverse temperature in the cooling section can in principle be lower than the cathode temperature because the beam radius is larger than the cathode radius, but there are various likely sources of transverse velocity growth including non-uniform emission, nonlinear fields near the cathode, redistribution of space charge free energy, transport aberrations, *etc.* The specified value is such that the transverse electron velocity is comparable to the longitudinal ion velocity in the beam frame. For large $\Delta_\perp$,

$$\alpha \propto \Delta_\perp^{-2} \propto T_\perp^{-1} \quad , \tag{22}$$

but as long as $\Delta_\perp$ is a factor of two or so smaller than $c\beta_{p,\perp}$, the loss in longitudinal cooling is small. Thus, from the standpoint of stacking time, the temperature could be several times greater. To preserve transverse cooling capability, $\Delta_\perp$ should be less than $5 \cdot 10^{-4}$ allowing $T_\perp$ up to $\lesssim 2000$ K. Therefore, even though electron beam temperature is a sensitive parameter, there is margin for non-ideal electron transport in this case.

Besides random thermal velocity, there are other ways in which the electron beam can have greater transverse velocity in the beam frame. These include beam-beam alignment, stray magnetic field, and space charge field. Allowing transverse velocity from beam alignment to be no more than $\Delta_\perp/2$ gives an alignment requirement of $\Delta\vartheta < \beta_\perp/2$. This is 100 μrad for nominal parameters but could be relaxed to 250 μrad or so with a relaxed requirement for $T_\perp$. The results obtained in Sec. III are just a bit more restrictive than these estimates.

The simple model illustrates that the nominal parameters are reasonable for 8 GeV $\bar{\mathrm{p}}$ accumulation. Generally they are rather comparable to those for existing electron cooling systems. The clear exception and most obvious technical challenge to medium energy cooling is obtaining reliable electron beam at 4.3 MeV, several hundred mA transported over tens of meters. More than two megawatts of beam power are required but not unprecedented electron emittance or voltage regulation nor beam alignment at the few-microradian level. From each of these last matters incremental quantitative gains can be made over time, but reliable high voltage dc and efficient beam transport are paramount. The next section takes a more quantitative approach to several of the issues raised above but is consistent with these qualitative conclusions.

III. DETAILED ANALYSIS

Each ion is cooled at the rate particular to its own relative velocity, but practically it is important to know how long one must wait until all or almost all ions are cooled. In fact, the answer depends on the definition of "almost all". Electron cooling works in such a way that the higher the ion's velocity relative to the electron beam, the less efficient is its cooling. However long the ions are cooled, a portion of them still have practically their initial velocities. Thus, the time required for the beam cooling depends on the acceptable fraction of insufficiently cooled ions; it increases without bound as this fraction goes to zero. These un-cooled tails of the ion distribution will be referred to as a loss fraction. The calculation of beam cooling time is the first goal of this section, but it is necessary to derive accurate formulas for cooling rates of ions executing betatron oscillations, the subject of the Sec. III A below. Beam loss as function of cooling time is calculated analytically in Sec. III E for the longitudinal cooling. From this an optimum electron beam radius is found, which minimizes the required time of cooling for a given loss. In the Sec. III F, the shape of the electron beam profile optimizing the longitudinal cooling is found for a cold electron beam. In Secs. IV and IV A the results obtained are applied to cooling for the Fermilab Recycler [11,12]. Many symbols are redefined in this section; usage is independent of and somewhat inconsistent with Sec. II.

A. Single particle rates

The friction force acting on the ion in the beam frame is given by Eqs. 3 – 6. It causes a slow change of the actions for the ions. The transverse actions $J_{x,y}$ and phases $\psi_{x,y}$ are defined to correspond to normalized emittances:

$$x = \sqrt{2J_x\beta_f/(\gamma\beta)}\cos\psi_x\ , \quad x' = -\sqrt{2J_x/(\beta_f\gamma\beta)}\sin\psi_x \tag{23}$$

and similarly for the y direction. Here $\beta_f = \beta_x = \beta_y$ is the Courant-Snyder beta-function. With $\beta'_f = 0$ in the cooler, only the transformation Eq. 23 is needed. The actions $J_{x,y}$ are defined to have the rms normalized emittances $\epsilon_{x,y}$ as their average values:

$$\langle J_{x,y}\rangle = \epsilon_{x,y}\ \ . \tag{24}$$

Ion velocities in the beam frame are given by the Lorentz transformation; in units where $c = 1$ they may be written

$$u_x = \gamma\beta x' = v_x\cos\psi_x\ ;\ \ u_z = \beta\Delta p/p = v_z\ ;\ \ u^2 = u_x^2 + u_y^2 + u_z^2\ \ . \tag{25}$$

When the longitudinal motion of the cooled ions is free, the role of the longitudinal action is played by the relative momentum spread $\Delta p/p$. The cooling rates can be determined as logarithmic time derivatives of the actions

$$\tau_x^{-1} = -\frac{1}{J_x}\frac{dJ_x}{dt} \quad \tau_z^{-1} = -\frac{1}{\Delta p/p}\frac{d\Delta p/p}{dt} \tag{26}$$

averaged over the betatron oscillations. For cold electron beam they can be expressed as

$$\begin{aligned} \tau_x^{-1} &= -\frac{2}{\gamma v_x^2}\langle F_x u_x\rangle = \frac{4(I_e/e)r_e r_p\eta}{\beta\gamma^2 a^2}\left\langle \frac{L_C}{u^3}\frac{2u_x^2}{v_x^2}\right\rangle \\ \tau_z^{-1} &= -\frac{1}{\gamma}\langle F_z\rangle = \frac{4(I_e/e)r_e r_p\eta}{\beta\gamma^2 a^2}\left\langle\frac{L_C}{u^3}\right\rangle\ , \end{aligned} \tag{27}$$

where a is the electron beam radius, I_e is the electron current, e is the electron charge, r_e, r_p are the classical electron and ion radii and, η is the fraction of the orbit occupied by the cooler. The angle brackets stand for the betatron averaging:

$$\langle ...\rangle = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} ... \frac{d\psi_x d\psi_y}{(2\pi)^2}.$$

B. Longitudinal rate for cold electrons

In this subsection the electron beam is treated as cold; that is, the electron velocities in the beam frame are small compared to the corresponding velocities of the ions.

The longitudinal rate in Eq. 26 is proportional to the average inverse cube of the ion velocity:

$$\left\langle\frac{1}{u^3}\right\rangle = \frac{1}{\pi^2}\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2}\frac{d\psi_x d\psi_y}{(v_x^2\sin^2\psi_x + v_y^2\sin^2\psi_y + v_z^2)^{3/2}}\ \ . \tag{28}$$

This integral can be calculated analytically for both of the limiting cases $v_x \gg v_z$ and $v_x \ll v_z$. In the first case, the integral over phases converges at $|\psi_x| \le v_z/v_x \ll 1$ and can be calculated by the substitution $\sin\psi = \psi$ and an expansion of the integrations on the whole real axis:

$$\left\langle \frac{1}{u^3} \right\rangle = \frac{1}{\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\psi_x d\psi_y}{(v_x^2 \psi_x^2 + v_y^2 \psi_y^2 + v_z^2)^{3/2}} = \frac{2}{\pi v_x v_y v_z} \quad . \tag{29}$$

In the opposite case $v_x \ll v_z$, the result is obvious: $\langle 1/u^3 \rangle = 1/v_z^3$. The transition between these alternatives is smooth; a simple way to join them is

$$\left\langle \frac{1}{u^3} \right\rangle = \frac{1}{v_z \sqrt{(\pi v_x^2/2 + v_z^2)(\pi v_y^2/2 + v_z^2)}} \quad . \tag{30}$$

This gives for the longitudinal rate

$$\tau_z^{-1} = \frac{2}{\pi} \frac{(I_e/e) r_e r_p \eta L_{\|}}{\gamma^2 \beta^2 J_e \sqrt{\tilde{J}_x \tilde{J}_y} (\Delta p/p)} \tag{31}$$

with

$$\tilde{J}_x = J_x(1 + (2/\pi)(v_z^2/v_x^2)) = J_x + (2/\pi)\beta\beta_f(\Delta p/p)^2/\gamma \quad . \tag{32}$$

Here $L_{\|}$ is the Coulomb logarithm calculated with the longitudinal velocity as its argument, and J_e is the invariant corresponding to the electron beam radius:

$$a = \sqrt{2 J_e \beta_f/(\gamma\beta)} \quad . \tag{33}$$

The Coulomb logarithm is a function of the relative velocity between the ion and the electron beam; its argument normally scales as velocity cubed. If the longitudinal velocity is small compared to the transverse velocity amplitudes, the ion is mainly cooled when its transverse velocity is smaller than or comparable to the longitudinal one. Therefore, the logarithm has to be calculated with the longitudinal velocity as its argument in this case. In the opposite situation, the relative velocity is equal to the longitudinal one; hence, the logarithm again has to be calculated for the longitudinal velocity. Thus it can be concluded that, in any case, the longitudinal cooling rate must be evaluated with the Coulomb logarithm taken with the longitudinal velocity as its argument. [10] The integral in Eq. 28 was calculated numerically for $v_z = 0.1$ and $0.03 \le \sqrt{v_x^2 + v_y^2} \le 1$ with $0.03 \le \arctan(v_y/v_x) \le \pi/2 - 0.03$ and compared to the analytical approximation Eq. 30; the results are presented in Figs. 1 and 2. The agreement between the analytical and numerical calculations allows use of the analytical expressions Eqs. 30 and 31 for any relation between longitudinal and transverse velocities of the ions.

C. Longitudinal rate for flattened distributions

When the longitudinal velocity is small, $v_z^2 \ll v_{x,y}^2$, the expression Eq. 31 for the longitudinal rate is simplified:

$$\tau_z^{-1} = \frac{2}{\pi} \frac{(I_e/e) r_e r_p \eta L_{\|}}{\gamma^2 \beta^2 J_e \sqrt{J_x J_y} (\Delta p/p)} \quad . \tag{34}$$

The dependence $\tau_z^{-1} \propto (\Delta p/p)^{-1}$ means that the longitudinal cooling force actually does not depend on the momentum offset. It also means that the locally determined cooling time τ_z (Eq. 26) is equal to the total cooling time. In fact, the Eq. 34 result comes mainly from those phases of the ion betatron oscillation where it is nearly stopped, $\psi_{x,y} \leq v_z/v_{x,y}$. That is why the electron beam has to be wide enough to cover the amplitudes of the ions. The cooling time as a function of the electron beam size has a singularity at the ion's maximum offset, when $a = \sqrt{x_m^2 + y_m^2}$, with $x_m = \sqrt{2J_x\beta_f/(\gamma\beta)}$. For $a > \sqrt{x_m^2 + y_m^2}$, the cooling time grows with the radius $\propto a^2$ because of the electron density decrease. It grows much faster when the radius goes down below the singularity, $a < \sqrt{x_m^2 + y_m^2}$. This dependence is shown in Fig. 3 for a case of the equal betatron amplitudes, $v_x = v_y$ at $v_z = 0.1v_x$ and $v_z = 0.25v_x$. It can be concluded that the ions stopped outside the electron beam are practically lost to the cooling process.

To this point, the electron velocities in the beam frame have been considered negligible. When both ion and electron longitudinal velocities are small compared to their respective transverse velocities, the cooling rate can be calculated analytically for arbitrary relation between the transverse velocities. This case is called that of flattened distributions. Assuming $f(\vec{w}, \vec{r}_\perp)$ to be a normalized electron distribution over the 3D velocities $\vec{w} = (w_x, w_y, w_z)$ and the transverse (2D) coordinates $\vec{r}_\perp = (x, y)$, the longitudinal rate (Eq. 27) can be presented as follows:

$$\tau_z^{-1} = -\frac{1}{\gamma}\langle F_z \rangle = \frac{4\pi(I_e/e)r_e r_p \eta L_C}{\beta\gamma^2}\left\langle \int \frac{d\vec{w} f(\vec{w}, \vec{r}_\perp)(u_z - w_z)}{|\vec{u} - \vec{w}|^3 u_z} \right\rangle \ , \tag{35}$$

with

$$\int d^3w d^2r_\perp f(\vec{w}, \vec{r}_\perp) = 1 \ . \tag{36}$$

For the flattened distribution, $|u_{x,y} - w_{x,y}| \gg |u_z - w_z|$, the integral over the electron transverse velocities $\vec{w}_\perp$ is dominated by the vicinity of the ion transverse velocity $\vec{w}_\perp = \vec{u}_\perp$, and it can be performed analytically:

$$\tau_z^{-1} = \frac{8\pi^2(I_e/e)r_e r_p \eta L_C}{\beta\gamma^2}\left\langle \int dw_z f(u_x, u_y, w_z)\mathrm{sign}(u_z - w_z)/u_z \right\rangle \ . \tag{37}$$

Assuming the distribution to be factorized, $f(\vec{w}, \vec{r}_\perp) = f_\perp(\vec{w}_\perp, \vec{r}_\perp) f_z(w_z)$, the longitudinal integral $\int dw_z f_z(w_z)\mathrm{sign}(u_z - w_z)/u_z$ can be performed separately for the limiting cases of one or another longitudinal velocity dominating:

$$\int dw_z f_z(w_z)\mathrm{sign}(u_z - w_z)/u_z = \begin{cases} 1/|u_z| & \text{if } |u_z| \gg \Delta w_z \\ f_z(0) & \text{if } |u_z| \ll \Delta w_z \end{cases} \ , \tag{38}$$

where Δw_z is the width of the electron longitudinal velocity distribution. It is convenient here to define this width as $\Delta w_z = 1/f_z(0)$. Then, the two limits in the Eq. 38 can be joined by an approximate formula:

$$\int dw_z f_z(w_z)\mathrm{sign}(u_z - w_z)/u_z = 1/\tilde{u}_z \ , \quad \tilde{u}_z = \sqrt{u_z^2 + \Delta w_z^2} \ . \tag{39}$$

For a Gaussian distribution with the rms velocity $\hat{w}_z$, this effective width is $\Delta w_z = \sqrt{2\pi}\hat{w}_z$. Substituting Eq. 39 into Eq. 37,

$$\tau_z^{-1} = \frac{8\pi^2 (I_e/e) r_e r_p \eta L_C}{\beta\gamma^2 \tilde{u}_z} \left\langle f_\perp(\vec{u}_\perp, \vec{r}_\perp) \right\rangle \quad . \tag{40}$$

Eq. 40 expresses the cooling rate in terms of the electron distribution function taken at the ion trajectory and averaged $\langle ... \rangle$ over betatron oscillations.

For certain cases, the betatron averaging in Eq. 40 can be performed analytically. In particular, it can be done for a constant distribution over the transverse coordinates $\vec{r}_\perp$ within the radius a, and Gaussian distribution over the transverse velocities $\vec{w}_\perp$ with the rms $\hat{w}_\perp$, *i. e.*

$$f_\perp(\vec{w}_\perp, \vec{r}_\perp) = \frac{\exp(-\vec{w}_\perp^2/(2\hat{w}_\perp^2))}{2\pi^2 \hat{w}_\perp^2 a^2} \quad . \tag{41}$$

Using an integral representation of the modified Bessel function

$$\mathrm{I}_0(\kappa) = \langle \exp(\kappa \cos\psi) \rangle \quad , \tag{42}$$

the result for the rate can be expressed as

$$\tau_z^{-1} = \frac{2(I_e/e) r_e r_p \eta L_C}{\pi\beta\gamma^2 v_x v_y \tilde{u}_z a^2} \sqrt{\mathrm{A}(v_x^2/(4\hat{w}_\perp^2))\mathrm{A}(v_y^2/(4\hat{w}_\perp^2))} \quad , \tag{43}$$

where a special function $\mathrm{A}(x) = 2\pi\ x\ \exp(-2x)\mathrm{I}_0^2(x)$ has been introduced. This function is plotted in Fig. 4. The asymptotic value $\mathrm{A}(\infty) = 1$ corresponds to zero-temperature electron beam, described by Eq. 34. It is interesting that the rate (Eq. 43) increases with the electron transverse temperature up to $\hat{w}_\perp^2 = v_{x,y}^2/3$, where the rate has a temperature maximum. For higher temperature, the rate drops as one over the temperature. At the optimum, the rate is 40% higher than the zero-temperature limit.

D. Transverse rate for the cold electron beam

The emphasis throughout has been on longitudinal cooling, which is crucial for accumulation in longitudinal phase space. A single-particle transverse rate formula analogous to the longitudinal rate formula Eq. 31 is useful for predicting the concurrent transverse cooling. A general analytical expression for the transverse integral $\langle (1/u^3)(2u_x^2/v_x^2) \rangle$ in Eq. 27 probably does not exist. It can be evaluated, however, for various limiting cases where one of the velocities is much higher than the others. Then, one or another formula can be tried to join the limits and compared to the exact numerical results. Without going into details, an approximate formula of such kind is

$$\left\langle (1/u^3)(2u_x^2/v_x^2) \right\rangle = \frac{1}{\tilde{v}_x^3} \begin{cases} 1 + \ln(\tilde{v}_x/\tilde{v}_y) & \text{if } \tilde{v}_x > \tilde{v}_y \\ \tilde{v}_x/\tilde{v}_y & \text{otherwise} \end{cases} \quad , \tag{44}$$

with $\tilde{v}_{x,y} = \sqrt{v_{x,y}^2 + v_z^2}$. A comparison of the numerical and the analytical calculations is shown in Figs. 5 and 6.

E. Beam cooling time

Up to this subsection, the cooling rates of single particles were of interest. The values calculated above are functions of particle actions. The ions have a certain distribution over the actions, and a rather high percentage of the ions should be cooled. So, the beam cooling time needs to be calculated when the single-particle rates are already known. Then the parameters of the cooler can be optimized to have a minimum for the beam cooling time within the given constraints. Also tolerances can be found. The problem is extensive; the consideration is limited here to the case of longitudinal cooling with small longitudinal velocity $v_z^2 \ll v_{x,y}^2$, a typical case for relativistic beams.

Ions cool with different rates. In a given time t, only a certain part of the beam can be cooled. The ions with low rates $\tau^{-1} < t^{-1}$ are not cooled sufficiently. The longer the cooling, the less is the loss. Thus, beam cooling time is a function of acceptable loss. This time can be calculated by first getting the expression for the loss fraction for a given time and then inverting it to give the time for a desired loss fraction.

One must specify the ion distribution as well as the properties of the electron beam. Assume a Gaussian distribution for the ions and a constant density within the circle $x^2+y^2 \leq a^2$ for the electron beam. Small loss is of the interest,giving a small parameter for the problem and allowing almost all calculations to be made analytically.

Because of the small longitudinal velocity, ions are effectively cooled longitudinally only when they are almost stopped transversely, that is when they have maximum offset from the axis. That is why ions which are stopped transversely beyond the electron beam boundary have rates a factor of $\simeq v_z/v_{x,y} \ll 1$ lower than those stopped within the electron beam. This condition for effective cooling can be expressed as

$$x_m^2 + y_m^2 \leq a^2 \quad , \tag{45}$$

with x_m, y_m standing for the amplitudes of the betatron oscillations in the cooler. Therefore, the loss consists of two parts. The first one, Eq. 45, includes the ions inside the electron beam with velocities too high to be cooled in a given time. The second part includes the ions outside, *i. e.*, ones satisfying the condition opposite to Eq. 45.

It is convenient to go to dimensionless variables. The transverse actions $J_{x,y}, J_e$ will be expressed in the units of the rms normalized emittance $\epsilon = \epsilon_x = \epsilon_y$; the longitudinal velocity v_z can be expressed in terms of its rms value $v_{zrms} = \beta(\Delta p/p)_{rms}$. The radius of the electron beam a or the corresponding action J_e is to be optimized; therefore, the time unit t_0 should be independent of it. It can be taken as the inverse of the rate in Eq. 34 with $J_x = J_y = J_e = \epsilon$:

$$t_0^{-1} = \frac{2}{\pi} \frac{(I_e/e) r_e r_p \eta L_\parallel}{\gamma^2 \beta^2 \epsilon^2 (\Delta p/p)_{rms}} \quad . \tag{46}$$

In these dimensionless variables the ion's cooling time is

$$\tau = v_z \sqrt{J_x J_y} J_e \quad . \tag{47}$$

At a fixed time t, the ions with $v_z > \bar{v} = t/(J_e \sqrt{J_x J_y})$ are not sufficiently cooled. The corresponding portion of the loss fraction Δ_{in} is calculated by means of an integration over the ion distribution:

$$\Delta_{in} = \sqrt{2/\pi} \iint_{J_x+J_y<J_e} dJ_x dJ_y \exp(-J_x - J_y) \int_{\bar{v}}^{\infty} dv_z \exp(-v_z^2/2) \ . \tag{48}$$

For small loss $\Delta_{in} \ll 1$, the longitudinal integral comes mainly from the vicinity of its lower limit $v_z = \bar{v}$, and the resulting transverse integrals can be evaluated by the saddle-point method. Then

$$\Delta_{in} = 2\sqrt{\pi J_*/3} \exp(-3J_*) \ . \tag{49}$$

The value

$$J_* = 2^{-1/3}(t/J_e)^{2/3} \ . \tag{50}$$

is the saddle-point of the integral over the transverse actions in Eq. 48; the loss is primarily in its vicinity, where $J_x = J_y = J_*$ and

$$v_z = v_* = t/(J_e J_*) \ . \tag{51}$$

The second part of the loss arises from the ions stopped outside the electron beam:

$$\Delta_{out} = \iint_{J_x+J_y>J_e} dJ_x dJ_y \exp(-J_x - J_y) \ . \tag{52}$$

This integral can be evaluated exactly; keeping the same accuracy as for Eq. 49,

$$\Delta_{out} = J_e \exp(-J_e) \ . \tag{53}$$

The total loss is

$$\Delta_t = 2\sqrt{\pi J_*/3} \exp(-3J_*) + J_e \exp(-J_e) \ . \tag{54}$$

It is a function of both the electron radius and the time of cooling via Eq. 50 and $J_e = a^2/x_m^2$. Note that x_m is the betatron amplitude associated with the rms emittance ϵ, so $x_m^2 = 2x_{rms}^2$. Assuming the loss to be fixed at a certain tolerable level, the cooling time can be found as a function of the electron beam radius. The radius corresponding to the minimum cooling time is an optimum. The plots for the cooling time versus the electron beam radius are presented in Fig. 7, for loss fractions of 5%, 10%, and 15%. The electron beam radius is in units of the ion beam rms amplitude $x_m = \sqrt{2\epsilon\beta_f/(\beta\gamma)}$; the time is in units of the parameter t_0 introduced in Eq. 46. The optimum electron beam radius lies within the rather narrow interval $2.1 < a/x_m < 2.4$ for any loss percentage between 5% and 15%.

F. Optimum electron density distribution

In the previous subsection the electron beam was assumed to have constant density within the circle $r \leq a$; the radius a was then optimized. However, the electron density could vary both in the radial $r = \sqrt{x^2 + y^2}$ and poloidal $\phi = \arccos(x/r)$ directions, and an optimum distribution may exist with the shortest cooling time for the given electron current and loss fraction.

This problem can be resolved in two steps. First, the optimum density distribution $n(\vec{r}_\perp)$ can be found within the circle $r \leq a$; second, the radius a can be optimized. Using the same units as in Sec. III E, the cooling time for an ion with the stopping point $\vec{r}_\perp$ can be expressed as

$$\tau = v_z\sqrt{J_xJ_y}J_e/n(\vec{r}_\perp), \quad \text{with} \tag{55}$$

$$\int_{r<a} d^2rn(\vec{r}_\perp) = \pi a^2 = 2\pi J_e \ . \tag{56}$$

The Eq. 56 normalization is needed to establish the identity between the units here and in Sec. III E. It follows that for the fixed cooling time t, the ions with

$$v_z > \bar{v} = n(\vec{r}_\perp)t/(J_e\sqrt{J_xJ_y}) \tag{57}$$

are effectively lost for cooling. With $g(J_x, J_y, v)$ to denote the ion distribution over the transverse actions J_x, J_y and the longitudinal velocity v, normalized as

$$\int_0^\infty \int_0^\infty dJ_xdJ_y \int_0^\infty dv_zg(J_x, J_y, v) = 1 \ , \tag{58}$$

the inside part of the loss fraction Δ_{in} is expressed

$$\Delta_{in} = \iint_{J_x+J_y<J_e} dJ_xdJ_y \int_{\bar{v}}^\infty dv_zg(J_x, J_y, v) \ . \tag{59}$$

To find the optimum distribution within the fixed radius a, the variational principle can be used: at the optimum, a small redistribution $\delta n(\vec{r}_\perp)$ of the density does not change the Eq. 59 loss to first order:

$$\iint_{r<a} d^2r\delta n(\vec{r}_\perp) = 0 \ . \tag{60}$$

The variation of the density results in a change of $\bar{v}$ (Eq. 57), the lower limit of the loss integral Eq. 59, causing a variation of the integral itself:

$$\delta\Delta_{in} = -\iint_{J_x+J_y<J_e} dJ_xdJ_yg(J_x, J_y, \bar{v})\delta n(x, y)t/\sqrt{J_x, J_y} \ . \tag{61}$$

For the integration over the actions, the density is taken at the stopping point of the ion, $x = \sqrt{2J_x}$, $y = \sqrt{2J_y}$. The loss variation follows by converting from the integration over actions to integration over the coordinates. The loss variation is zero at the optimum:

$$\delta\Delta_{in} \propto \iint_{r<a} dxdyg(J_x, J_y, \bar{v})\delta n(x, y) = 0 \ . \tag{62}$$

The last condition has to be satisfied by any density redistribution restricted only by the Eq. 60 requirement of electron current conservation. It can be fulfilled only when the ion distribution is constant at the cooling boundary:

$$g(J_x, J_y, \bar{v}) = \text{const} \quad . \tag{63}$$

This equation gives the condition for the optimum distribution of the electron density. For a given ion distribution g, it imposes a restriction on the maximum longitudinal velocity $\bar{v}$, and so on the electron density $n(x,y) = \bar{v} J_e \sqrt{J_x J_y}/t$.

For the Gaussian distribution

$$g(J_x, J_y, v) = \sqrt{2/\pi}\exp(-J_x - J_y - v^2/2), \tag{64}$$

the optimum condition Eq. 63 results in

$$J_x + J_y + \bar{v}^2/2 = J_m, \tag{65}$$

where J_m is a constant. The electron density can be obtained from this equation, while the constant J_m is found from the Eq. 56 normalization. The electron density associated with the maximum velocity $\bar{v}$ (Eq. 57) is found from

$$\begin{aligned} n(\vec{r}_\perp) =& \ n_r(r)\sin\phi\cos\phi \\ n_r(r) =& \begin{cases} J J_e\sqrt{2(J_m - J)}/t & \text{if } J \le J_e \\ 0 & \text{otherwise} \end{cases} \quad . \\ \int_0^{J_e} n_r dJ =& \ \pi J_e, \quad J = r^2/2 \end{aligned} \tag{66}$$

The profile of the optimum radial distribution n_r is presented in Fig. 8.

It follows that for a given constant $J_m \ge J_e$, the beam cooling time is determined by the equation

$$t = \int_0^{J_e} J\sqrt{2(J_m - J)}dJ/\pi \quad . \tag{67}$$

Then, the inside part of the loss, Eq. 59, can be expressed as

$$\Delta_{in} = \sqrt{\frac{2}{\pi}}\exp(-J_m)\int_0^{J_e}\frac{dJJ}{\sqrt{2(J_m - J)}} = \sqrt{2\pi}\exp(-J_m)\frac{\partial t}{\partial J_m} \quad . \tag{68}$$

The total loss consists of Δ_{in} and the outside part from Eq. 53:

$$\Delta_t = \sqrt{2\pi}\exp(-J_m)\frac{\partial t}{\partial J_m} + J_e\exp(-J_e) \quad . \tag{69}$$

Now it can be demonstrated that the constant J_m introduced in Eq. 65 is equal to J_e at the optimum. First, minimization of the total loss for the given time t requires $J_m = J_e + \mathcal{O}(1)$ so that the cooling time is

$$t = \frac{4\sqrt{2}}{15\pi}J_m^{5/2}. \tag{70}$$

Then, if $J_e < J_m$, the total loss (69) can be diminished by increasing J_e up to its maximum allowable level $J_e = J_m$, where they reach their minimum while the time t of Eq. 70 is not changed. Thus, the final expressions for the cooling time and the total loss as functions of the electron beam radius $a = J_e^2/2$ are

$$\begin{array}{ll} \Delta_t = & \frac{4}{3\pi} J_e^{3/2} \exp(-J_e) + J_e \exp(-J_e) \\ t = & \frac{4\sqrt{2}}{15\pi} J_e^{5/2} \end{array} \quad , \tag{71}$$

giving the dependence of the cooling time on the loss fraction. This dependence, together with the corresponding dependence of the optimum electron beam radius, is shown in Fig. 9. It can be concluded that the optimization of the electron beam density gives rather modest gain beyond the case of constant density with optimized radius. At the accepted loss of 5 %, the beam cooling time for these two cases differs by a factor of 1.6.

G. Longitudinal heating by intrabeam scattering

Intrabeam Coulomb scattering (IBS) drives the ion distribution toward thermal equilibrium. Consequently, the relatively hot degrees of freedom, usually transverse for relativistic beams, are cooled, and the cold one, longitudinal, is heated. General formulas for intrabeam scattering [15,16] are rather complicated and are generally used by means of computer programs. Here a handy formula for the IBS longitudinal diffusion is derived for the special case of a storage ring below transition with rather smooth lattice. The following approximations are used:

1. The longitudinal temperature of ions is much lower than the transverse. Typically their ratio is $\mathcal{O}(10^{-1})$.
2. The width of the ion beam attributable to $\Delta p/p$ can be neglected.
3. The smooth approximation to betatron oscillation is used.

In a collision a ion gets a longitudinal velocity kick; in the beam frame

$$\delta v_z = 2r_p \cos\theta/(\rho u) \ , \quad \overline{\delta v_z} = 0 \ , \quad \overline{(\delta v_z)^2} = 2r_p^2/(\rho^2 u^2) \ , \tag{72}$$

where r_p is the classical proton radius, θ is the angle between the longitudinal axis and the plane of the relative motion, ρ is the impact parameter, u is the relative velocity, an the bar $\overline{(...)}$ stands for the averaging over θ. All of the expressions assume c=1. From here the IBS diffusion coefficient $D = d(\Delta p/p)^2/dt$ in the laboratory frame is found by integrating over the impact parameter and over the velocities of the scattered ions $\vec{v}_2$. This result has to be averaged over the betatron phases ψ_x, ψ_y of the ion considered:

$$\begin{array}{l} D = \dfrac{4\pi r_p^2 L_{\bar{p}}}{\gamma\beta^2} \displaystyle\int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} \frac{d\psi_x d\psi_y}{\pi^2} \int \frac{d^2 v_2 f_\perp(\vec{r}, \vec{v}_2)}{|\vec{v} - \vec{v}_2|} \\ \vec{v} = (v_x \sin\psi_x, v_y \sin\psi_y) \ ; \quad \vec{r} = -(v_x \cos\psi_x, v_y \cos\psi_y)/(\gamma\omega_b) \end{array} \tag{73}$$

with $L_{\bar{p}}$ as the IBS Coulomb logarithm and ω_b as the betatron frequency for both degrees of freedom. The relativistic factor γ appears here because of the time transformation in going to the laboratory frame. The transverse distribution of the ions is assumed to be Gaussian,

$$f_\perp(\vec{r}, \vec{v}_2) = n_p \exp(-r^2/(2a_p^2)) \exp(-v_2^2/(2v_\perp^2))/(2\pi\gamma v_\perp^2) \quad , \tag{74}$$

where $n_p = (I_p/e)/(2\pi\beta a_p^2)$ is the beam density at the axis in the laboratory frame and I_p is the peak ion current. In this case, the diffusion coefficient can be written as

$$D = \frac{4\pi n_p r_p^2 L_{\bar{p}} \mathcal{I}}{\gamma^2 \beta^2 v_\perp} \quad . \tag{75}$$

Here a dimensionless transverse factor was introduced:

$$\mathcal{I} = \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} \frac{d\psi_x d\psi_y}{\pi^2} \exp(-r^2/(2a_p^2)) \int \frac{d^2 v_2 \exp(-v_2^2/(2v_\perp^2))}{2\pi v_\perp |\vec{v} - \vec{v}_2|} \tag{76}$$

with $a_p = v_\perp/(\gamma\omega_b)$. This factor can be easily calculated for the limiting cases $v \gg v_\perp$ or $v \ll v_\perp$; then the results can be joined by a suitable analytical expression. The formula

$$\mathcal{I} = \frac{2}{\pi} \frac{v_\perp^3}{\sqrt{(v_x^2 + v_y^2 + 2v_\perp^2/\pi)(v_x^2 + 2v_\perp^2/\pi)(v_y^2 + 2v_\perp^2/\pi)}} \tag{77}$$

gives an exact result for the limiting cases; its inaccuracy in the intermediate area, $v \simeq v_\perp$, is not worse than 20%.

The longitudinal distribution of the ions is determined by the competition between the diffusion and the cooling force, rewritten below for convenience:

$$\frac{d(\Delta p/p)}{dt} = -F \; , \quad (\Delta p/p) > 0 \tag{78}$$

with

$$F = \frac{8(I_e/e) r_e r_p \eta L_\parallel}{\pi \beta^2 \gamma^2 a^2 v_x v_y} \quad . \tag{79}$$

The evolution of the longitudinal distribution function $f_\parallel$ is described by the Fokker-Planck equation:

$$\frac{\partial f_\parallel}{\partial t} + \frac{\partial}{\partial w}\left(-F f_\parallel - \frac{D}{2}\frac{\partial f_\parallel}{\partial w}\right) = 0 \quad . \tag{80}$$

Here the symbol $w = \Delta p/p$ is introduced to simplify the notation. Taking into account that neither the cooling force F nor the diffusion depend on the momentum w, a stationary solution of this equation is

$$f_\parallel^0(w) = \exp(-w/\bar{w})/\bar{w}, \quad \bar{w} = D/(2F) \quad . \tag{81}$$

The lower limit for the longitudinal 95% emittance (phase space area) $\mathcal{A}_{\min}$ is then

$$\mathcal{A}_{\min} = 6\bar{w}C = 3DC/F \ \ , \tag{82}$$

where C is the storage ring circumference. Substituting the expressions for the diffusion coefficient (Eq. 75) and the force (Eq. 79) permits rewriting this limit as

$$\mathcal{A}_{\min} = \frac{3\pi v_\perp p C I_p}{4 I_e \eta} \frac{r_p}{r_e} \frac{L_{\bar{p}}}{L_\parallel} \frac{a^2}{a_p^2} \mathcal{J} \ , \quad \mathcal{J} = \frac{\mathcal{I} v_x v_y}{v_\perp^2} \ \ . \tag{83}$$

The maximum value of the transverse velocity factor $\mathcal{J}$ is reached at $v_x = v_y = \sqrt{(1+\sqrt{5})/\pi} v_\perp \approx v_\perp$ with $\mathcal{J}_{\max} = 0.24$.

IV. ELECTRON COOLING FOR THE FERMILAB RECYCLER

Electron cooling is planned for longitudinal phase space stacking of antiprotons in the Fermilab Recycler storage ring. [1,11,12] The considerations of Sec. III give the optimum choice for the electron beam radius and a resulting beam cooling time.

Certain results can be derived for misalignment tolerances. Assume that the electron beam acquires a coherent angle θ_e because of misalignments. The resulting electron transverse velocity in the beam frame is $u_{e\perp} = \gamma\beta\theta_e$. The effect of this additional stray velocity is a reduction of the cooling rates for the small velocity ions, $v_{x,y,z} \le u_{e\perp}$ given by Eq. 27 with $\langle (1/u^3) \rangle = 1/u_{e\perp}^3$. The reduction is acceptable so long as the cooling time for these ions is smaller then the beam cooling time considered in Sec. III E. This condition can be expressed in terms of the rms betatron amplitudes $v_{x,y} = \sqrt{2\beta\gamma\epsilon/(\beta_f)}$, longitudinal velocity $v_z = \beta(\Delta p/p)_{rms}$, and the dimensionless J_e proportional to the electron beam emittance:

$$u_{e\perp} \le (\pi v_x^2 v_z t/(2J_e))^{1/3} \ , \quad \theta_e = u_{e\perp}/(\gamma\beta) \ \ , \tag{84}$$

where t is the (optimized) dimensionless beam cooling time. This formula gives the acceptable level for the stray electron velocity $u_{e\perp}$. For a given emittance, it scales as $\beta_f^{-1/3}$.

The sources of stray electron velocity $u_{e\perp}$ depend on details of cooler design not considered here. However, some sources can be discussed without going into details of electron cooler construction like, for example, stray transverse magnetic field $B_\perp$ on the electron trajectory. To cancel these fields, correctors can be installed with a certain periodicity l_s, which gives a minimum wave number for the stray field space harmonics $k = 2\pi/l_s$. The amplitude of the forced electron oscillations is

$$u_{e\perp} = (B_\perp/e) r_e l_s/(2\pi) \ \ , \tag{85}$$

where l_s is much less than the pitch of the helical trajectory in the design longitudinal field. Eq. 84 establishes the tolerable level for the stray field $B_\perp$.

The parameters obtained for the electron cooler, calculated for a certain electron current and cooling length, are given in Table IV, where some general parameters of the Recycler and the p̄ beam are also included for convenience. For longitudinal cooling, the optimum value for the beta-function in the cooling straight section is the length of the interaction region. For higher values of the beta-function, the cooling rate is the same, but all the sizes increase as $\beta_f^{1/2}$ and the tolerances decrease as $\beta_f^{-1/3}$.

The multiple scattering result of Sec. III G has been used to calculate the equilibrium emittance for the p̄ beam; the value $\mathcal{A}_{eq} = 7$ eVs is small compared to the > 50 eVs stack emittances of interest, so its neglect is valid for the Recycler example that has been used.

A. Cooling regions using thin lenses

The electron beam can be focused in the cooling section either by means of a constant solenoidal magnetic field or with thin lenses for compensation of the space charge repulsion as in Ref. [13]. The tolerances for the latter case need specific consideration.

Assuming the lenses to be separated by the distances l_l, the electron beam envelope acquires an additional divergence θ_{sc} between lenses:

$$\theta_{sc} = \frac{\lambda r_e l_l}{\beta^3 \gamma^3 a} \ , \tag{86}$$

with λ standing for the electron linear density. For the parameters listed in Tab. IV A and $l_l = 1$ m, $\theta_{sc} = 1.5\mu$rad, corresponding to the lens focusing distance $f_l = a/(2\theta_{sc}) = 7$ km. Hence, the space charge effect is so weak at these parameters, that even one lens for the total cooler length would be enough to control the divergence. However, it is noted in Ref. [14] that the electron beam focused by thin lenses is unstable because of wall image charges. The instability is convective with a growth length

$$G = \sqrt{2\lambda r_e/\gamma}/(\beta b) \ , \tag{87}$$

where b is the aperture radius. For the parameters discussed, $G^{-1} = 6$ m which is comparable with the total cooling length. In this case, the tolerable beam offset x_{in} and angle θ_{in} at the entrance of the cooler are determined by the instability growth factor on the cooler length l_c:

$$x_{in} \le \theta_e G^{-1} \exp(-G l_c) \ , \quad \theta_{in} \le \theta_e \exp(-G l_c) \ , \tag{88}$$

which gives $x_{in} \le 10\mu$m and $\theta_{in} \le 2\mu$rad in the case under consideration. Offset of the lenses is not a critical parameter because of the long focal length. It is more important to control the lens angle α. This angle introduces an electron beam angle $\theta_l = \alpha\sqrt{d_l/(2f_l)}$, where d_l is a thickness of the lens. Assuming the lens angles to be independent random values with a rms range α_{rms}, the average angle acquired by the electron beam over the instability length G^{-1} can be found:

$$\theta_l = \alpha_{rms}\sqrt{\frac{d_l}{2G f_l l_l}} \ . \tag{89}$$

θ_l does not actually depend on the lens separation l_l, because $f_l \propto l_l^{-1}$. To be tolerable, this angle has to be small at the end of the cooler, which gives

$$\alpha_{rms} \le \theta_e \sqrt{\frac{2G f_l l_l}{d_l}} \exp(-G l_c + 1) \ . \tag{90}$$

According to the previous calculations, $\theta_e = 4 \cdot 10^{-5}$; for lens thickness $d_l = 10$ cm, $\alpha_{rms} \le 4 \cdot 10^{-3}$ is required. The numbers discussed in this subsection are summarized in Table V.

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FIGURES

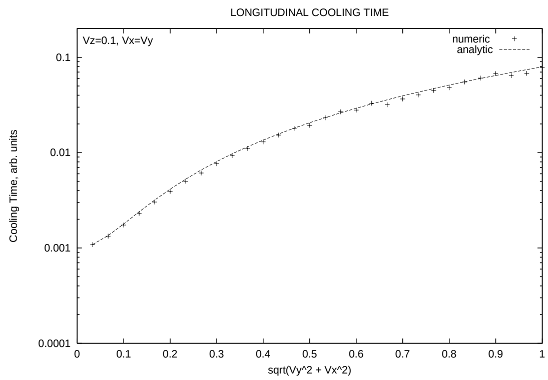

FIG. 1. Longitudinal cooling time as a function of the total transverse amplitude $v_\perp = \sqrt{v_x^2 + v_y^2}$, for $v_x = v_y$, $v_z = 0.1$, arbitrary units.

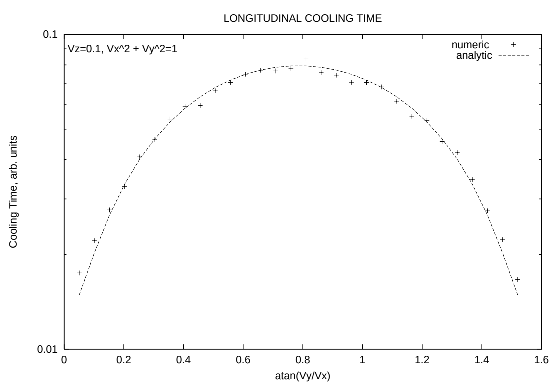

FIG. 2. Longitudinal cooling time as a function of a distribution of the transverse energy among the two degrees of freedom with $v_x^2 + v_y^2 = 1.$, $v_z = 0.1$, arbitrary units.

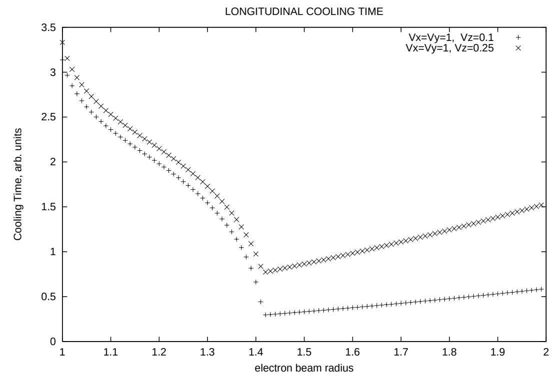

FIG. 3. Longitudinal cooling time as a function of the electron beam radius. The betatron amplitudes assumed to be equal, $v_x = v_y$. Two sets of the plotted points correspond to different longitudinal velocities, $v_z = 0.1 v_x$ and $v_z = 0.25 v_x$. The time is in arbitrary units and the radius is in units of the amplitude $x_m = v_x \beta_f / (\beta\gamma)$.

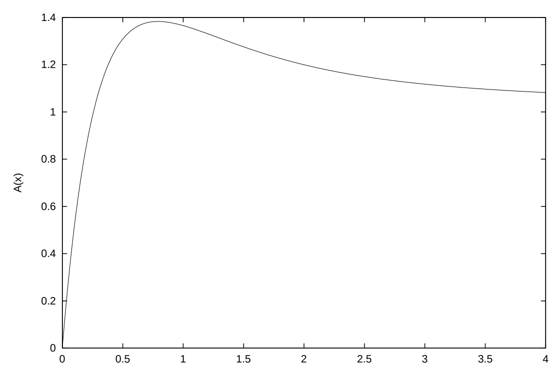

FIG. 4. Function $\mathrm{A}(x) = 2\pi\ x\ \exp(-2x)\ \mathrm{I}_0^2(x)$

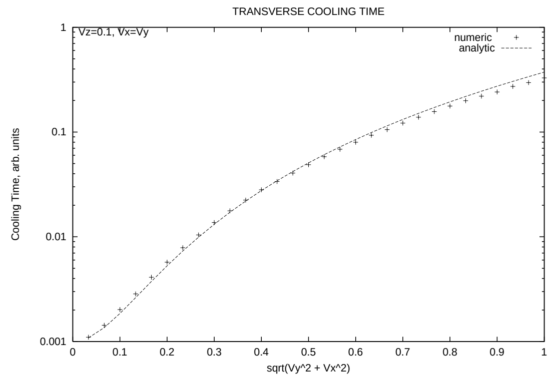

FIG. 5. Transverse (x) cooling time as a function of the total transverse amplitude $v_\perp = \sqrt{v_x^2 + v_y^2}$, for $v_x = v_y$, $v_z = 0.1$, arbitrary units.

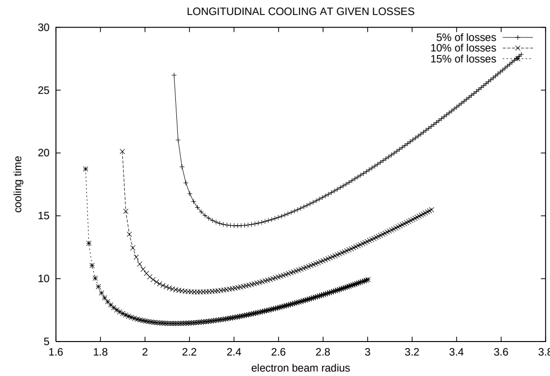

FIG. 7. Time of cooling required for the given loss as a function of the electron beam radius. The time is in units of the parameter t_0 (Eq. 46); the electron beam radius is in units of the rms amplitude $x_m = \sqrt{2\epsilon\beta_f/(\beta\gamma)}$.

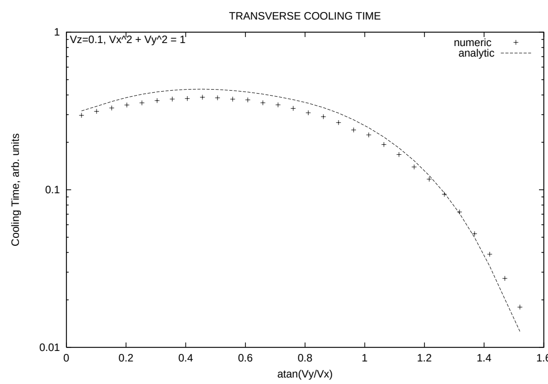

FIG. 6. Transverse (x) cooling time as a function of a distribution of the transverse energy among the two degrees of freedom with $v_x^2 + v_y^2 = 1.$, $v_z = 0.1$, arbitrary units.

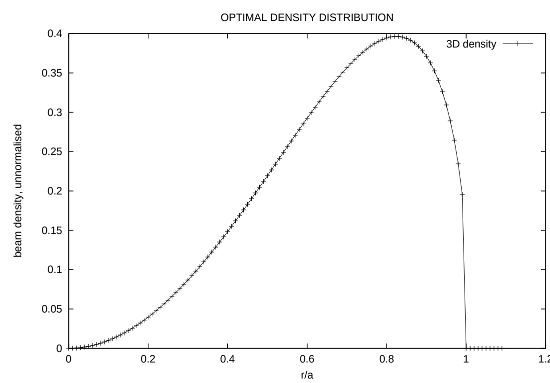

FIG. 8. Profile of the optimum radial distribution of the electron beam, unnormalized.

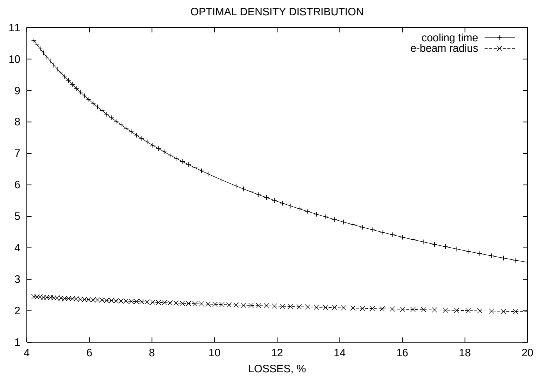

FIG. 9. Optimum radius of the electron beam and the cooling time required for given loss fraction, time in units of the parameter t_0 (Eq.46) and electron beam radius in units of the rms betatron amplitude of the ion beam $x_m = \sqrt{2\epsilon\beta_f/(\beta\gamma)}$.

TABLES

TABLE I. Transformation of variables between laboratory and beam frames

	Lab Frame	Beam Frame
velocity	$v_\perp$	$v_\perp^*/\gamma$
	$v_\parallel$	$v_\parallel^*/\gamma^{-2} + \beta c$
coordinates	$\mathrm{d}r_\perp$	$\mathrm{d}r_\perp^*$
	$\mathrm{d}r_\parallel$	$\mathrm{d}r_\parallel^*/\gamma$
	$\mathrm{d}t$	$\gamma \mathrm{d}t^*$
density	$n = \mathrm{d}N/\mathrm{d}x\mathrm{d}y\mathrm{d}z$	γn^*
momentum	$p_\perp$	$p_\perp^*$
	$p_\parallel$	$\gamma p_\parallel^*$
force	$F_\perp$	$F_\perp^*/\gamma$
	$F_\parallel$	$F_\parallel^*$
diffusion	$D_\perp$	$D_\perp^*/\gamma$
	$D_\parallel$	$\gamma D_\parallel^*$

TABLE II. Beam and system parameters for the Recycler cooling system

E_p	$\bar{\mathrm{p}}$ energy	8.94	GeV
$\varepsilon_\perp$	$\bar{\mathrm{p}}$ $\varepsilon_\perp$ (6σ, normalized)	10	π mm mrad
ΔE_p	$\bar{\mathrm{p}}$ energy spread	± 3	MeV
N_p	number of $\bar{\mathrm{p}}$ (total)	$5 \cdot 10^{12}$	
	number of $\bar{\mathrm{p}}$ (per injection)	$2 \cdot 10^{11}$	
a	electron beam radius	0.0060	m
r_c	cathode radius	0.0025	m
ε_e	elect. trans. emit. rms norm.	1	μ m
I_e	electron beam current	500	mA
ΔU	electron energy stability	± 200	eV
ℓ_c	length of cooling section	20	m
C	ring circumference	3319	m
$\bar{\beta}$	$\bar{\mathrm{p}}$ Courant-Snyder β_x and β_y	20	m
	Recycler injection frequency	2	h^{-1}

TABLE III. Model parameters

<u>input</u>				
E_e	electron energy	4.87		MeV
β	Lorentz beta of beam frame	0.994		
γ	Lorentz gamma of beam frame	9.526		
$\Delta\beta^*_{e,\parallel}$	long. el. vel. spread, beam frame	4.13	$\cdot 10^{-5}$	
$\Delta\beta^*_{e,\perp}$	trans. el. vel. spread, beam frame	3.08	$\cdot 10^{-4}$	
x	max. $\bar{\mathrm{p}}$ beam radius	5.6	$\cdot 10^{-3}$	m
$\beta^*_{p,\perp}$	trans. $\bar{\mathrm{p}}$ vel., beam frame	1.78	$\cdot 10^{-3}$	
$\beta^*_{p,\parallel}$	long. $\bar{\mathrm{p}}$ vel. in beam frame	3.38	$\cdot 10^{-4}$	
<u>output</u>				
t_{stop}	cooling time	22.8		min
r	longitudinal cooling rate	1.46	$\cdot 10^{-3}$	MeV/s
R_D	longitudinal drag	2.82	$\cdot 10^{-3}$	MeV/s

TABLE IV. Electron cooling for the Fermilab Recycler — Parameters depending on the Courant-Snyder β-function for the p̄'s are give separately for values β_f of 20 m and 200 m.

Parameter	Symbol	$\beta_f = 200$m	$\beta_f = 20$m	Unit
Circumference	C	3319.4		m
p̄ momentum	p	8.9		Gev/c
p̄ normalized rms emittance	ϵ	1.6		mm·mrad
Cooler length	l_c	20		m
Electron current	I_e	0.5		A
Losses	Δ_t	5		%
Long. beam cooling rate	r	4.49		keV/s
Electron beam radius	a	1.9	0.57	cm
Electron angle	θ_e	≤ 40	≤ 80	μrad
Maximum electron temperature	$m\hat{w}^2$	≤ 0.1	≤ 1	eV
Optimum electron temperature	$m\hat{w}^2$	0.03	0.3	eV
Electron momentum spread	$(\Delta p/p)_e$	$\leq 1 \cdot 10^{-4}$		
Corrector-corrector interval	l_s	1		m
Stray magnetic field	$B_\perp$	≤ 40	≤ 80	mG

TABLE V. Thin lens optics

Parameter	Symbol	$\beta_f = 200$m	$\beta_f = 20$m	Unit
Minimal number of lenses	N_l	1	3	
Instability growth length	G^{-1}	6		m
Beam entrance offset	x_{in}	≤ 10	≤ 20	μm
Beam entrance angle	θ_{in}	≤ 2	≤ 4	μrad
Lens rms angle	α_{rms}	≤ 4		mrad