

# Multiloop information from the QED effective lagrangian

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**Abstract.** We obtain information on the QED photon amplitudes at high orders in perturbation theory starting from known results on the QED effective Lagrangian in a constant electric field. A closed-form all-order result for the weak field limit of the imaginary part of this Lagrangian has been given years ago by Affleck, Alvarez and Manton (for scalar QED) and by Lebedev and Ritus (for spinor QED). We discuss the evidence for its correctness, and conjecture an analogous formula for the case of a self-dual field. From this extension we then obtain, using Borel analysis, the leading asymptotic growth for large  $N$  of the maximally helicity violating component of the  $l$  - loop  $N$  - photon amplitude in the low energy limit. The result leads us to conjecture that the perturbation series converges for the on-shell renormalized QED  $N$  - photon amplitudes in the quenched approximation.

## 1. Introduction: The perturbation series in quantum electrodynamics

In quantum electrodynamics, physical quantities are usually computed as a perturbative series in powers of the fine structure constant  $\alpha = \frac{e^2}{4\pi}$ :

$$F(e^2) = c_0 + c_2 e^2 + c_4 e^4 + \dots \quad (1)$$

In the early years of quantum field theory it was hoped that this type of series would turn out to be convergent in general, or at least for sufficiently small values of the coupling constant(s). However, in 1952 Dyson [1] argued on physical grounds that QED perturbation theory should be divergent. Dyson argues: “Suppose, if possible, that the series (1) converges for some positive value of  $e^2$ ; this implies that  $F(e^2)$  is an analytic function of  $e$  at  $e = 0$ . Then for sufficiently small values of  $e$ ,  $F(-e^2)$  will also be a well-behaved analytic function with a convergent power-series expansion” [1]. He then argues physically that this cannot be the case, since for  $e^2 < 0$  the QED vacuum will be unstable due to a runaway production of  $e^+e^-$  pairs which coalesce into like-charge groups. This argument does not prove divergence or convergence, but it gives

a first hint of the connection between perturbative divergence and instability in quantum field theory [2, 3, 4, 5].

To substantiate this divergence mathematically in the case of QED turns out to be exceedingly difficult [2]. Explicit all-order calculations are, of course, next to impossible, so that the natural thing to try is to prove the divergence by establishing lower bounds on the contributions of the individual Feynman diagrams representing the coefficients of the perturbation series. However this is still very hard to do for QED diagrams since here the integrands have no definite sign. On the other hand, in scalar field theories the integrands have a definite sign (in the Euclidean) so that here lower bounds can be established using inequalities such as

$$\prod_{i=1}^F \left( \frac{1}{p_i^2 + \kappa^2} \right) \geq \frac{F^F}{\left( \sum_{i=1}^F p_i^2 + F\kappa^2 \right)^F} \quad (2)$$

At about the same time that Dyson's paper appeared, Hurst [6] and Thirring [7] (see also Petermann [8]) used this lower bound strategy to show by explicit calculation that perturbation theory diverges in scalar  $\lambda\phi^3$  theory for *any* value of the coupling constant  $\lambda$ . Hurst's proof is quite straightforward; its essential steps are (i) the use of (2) in the parametric representation to find lower bounds for arbitrary diagrams (ii) a proof that the number of distinct Feynman diagrams at  $n^{\text{th}}$  loop order grows like  $(\frac{n}{2})!n!$  (iii) the fact that there are no sign cancellations between graphs. This led Hurst to conjecture that the analog statement holds true for other renormalizable quantum field theories and in particular for QED. However, QED was already well-established experimentally in 1952. To account for this fact, Hurst postulated that the QED perturbation series, while not convergent, is an asymptotic series and thus still makes sense numerically.

Today the fact that the perturbation series is asymptotic rather than convergent is not only well-established for QED, but believed to be a generic property of nontrivial quantum field theories (see [2, 3, 4, 5, 9, 10]). Convergence of the series can be expected only in trivial cases where higher-order radiative corrections are absent altogether (usually on account of some symmetry).

In the absence of convergence, the reconstruction of the exact physical quantities from their perturbation series must be attempted using summation methods. For this there are various possibilities, and we mention here only the one which has been most widely used in QFT, Borel summation [9, 10]. For a factorially divergent series

$$F(\alpha) \sim \sum_{n=0}^{\infty} c_n \alpha^{n+1} \quad (3)$$

one defines the Borel transform as

$$B(t) \equiv \sum_{n=0}^{\infty} c_n \frac{t^n}{n!} \quad (4)$$

Assuming that  $B(t)$  has no singularities on the positive real axis and does not increase too rapidly at infinity, one can also define the Borel integral

$$\tilde{F}(\alpha) \equiv \int_0^{\infty} dt e^{-\frac{t}{\alpha}} B(t) \quad (5)$$

$\tilde{F}$  is the Borel sum of the original series  $F$ .  $F$  is asymptotic to  $\tilde{F}$  by construction, although the physical quantity represented by the series  $F$  might still differ from  $\tilde{F}$  by nonperturbative terms. Even when the Borel transform has singularities it remains a useful concept, since these singularities contain detailed information on the divergence structure of the theory. In many cases they can be traced either to instantons (related to tunneling between vacua) or Euclidean bounces (related to vacuum decay) or to renormalons. Those latter typically arise in renormalizable theories and are related to large or small loop momentum behaviour. Diagrammatically, they can be analyzed in terms of “infinitely long” chains of ‘bubble’ diagrams (for a review, see [11]).

Despite of the many insights which have been gained along these lines, a point which remains poorly understood is the influence of gauge cancellations on the divergence structure of a gauge theory. Generally, in gauge theory individual diagrams do not give gauge invariant results; gauge invariance is recovered only after summing over certain classes of diagrams. In QED a textbook example is provided by one-loop photon-photon scattering, where a gauge invariant (as well as UV finite) result is obtained only after performing a sum over the six inequivalent orderings of the external momenta for the basic diagram in fig. 1.

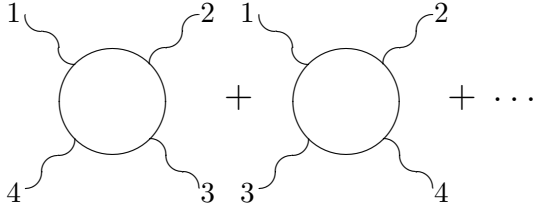


Fig. 1: Sum of one loop photon scattering diagrams.

This recovery of gauge invariance generally implies cancellations in the sum over gauge related diagrams, and leads one to expect that the coefficients of the perturbation series for such amplitudes should come out smaller in magnitude than predicted by a naive combinatorial analysis. However, it is presently not known what implications this has for the large order behaviour of amplitudes in either QED or other gauge theories. In 1977 Cvitanovic [12] performed a detailed diagrammatic analysis of the effect of gauge cancellations in a previous calculation of the sixth order contribution to the electron magnetic moment [13]. Based on this example, he suggested that in QED their effect is sufficiently strong to modify estimates based on the basic factorial growth in the number of Feynman diagrams; more realistic estimates might be obtained by counting gauge invariant classes of diagrams instead. The asymptotic growth of the number of these classes is, however, less than factorial. For the case at hand, the electron  $g-2$ , he conjectured that its perturbation series even *converges* in the ‘quenched’ approximation, i.e. for the contribution represented by diagrams not involving electron loops.

In the present paper, we present an analogous conjecture for the  $N$  - photon amplitudes, in scalar and spinor QED. Here the ‘quenched’ approximation corresponds to taking only the diagrams involving just one electron loop, which is also the  $O(N_f)$  part of the amplitude in QED with  $N_f$  flavors. Our conjecture is that this part of the amplitude converges in perturbation theory when renormalized on-shell. We use known results on the QED effective Lagrangian to obtain information on the large order behaviour of this amplitude for the special case of ‘all +’ polarizations, in the limit of low photon energies and large photon number  $N$ .

## 2. The Euler-Heisenberg Lagrangian and the $N$ - photon amplitudes: general case

Let us start with some basic facts about the QED effective Lagrangian in a constant external field  $F_{\mu\nu}$ . At one loop in spinor QED, this is the well-known Euler-Heisenberg (‘EH’) Lagrangian,

obtained 1936 by Heisenberg and Euler [14] in form of the following integral:

$$\mathcal{L}_{\text{spin}}^{(1)}(F) = -\frac{1}{8\pi^2} \int_0^\infty \frac{dT}{T^3} e^{-m^2 T} \left[ \frac{(eaT)(ebT)}{\tanh(eaT) \tan(ebT)} - \frac{1}{3}(a^2 - b^2)T^2 - 1 \right] \quad (6)$$

Here  $T$  is the proper-time of the loop fermion,  $m$  its mass, and  $a, b$  are the two Maxwell invariants, related to  $\mathbf{E}, \mathbf{B}$  by  $a^2 - b^2 = B^2 - E^2$ ,  $ab = \mathbf{E} \cdot \mathbf{B}$ . A similar representation exists for scalar QED [15, 16]:

$$\mathcal{L}_{\text{scal}}^{(1)}(F) = \frac{1}{16\pi^2} \int_0^\infty \frac{dT}{T^3} e^{-m^2 T} \left[ \frac{(eaT)(ebT)}{\sinh(eaT) \sin(ebT)} + \frac{1}{6}(a^2 - b^2)T^2 - 1 \right] \quad (7)$$

By standard QFT (see, e.g., [17]) these effective Lagrangians contain the information on the one-loop photon S-matrix in the limit where all photon energies can be neglected compared to the electron mass,  $\omega_i \ll m$ . Thus diagrammatically  $\mathcal{L}_{\text{spin}}^{(1)}(F)$  is equivalent to the sum of the Feynman graphs in fig. 2 at zero momentum.

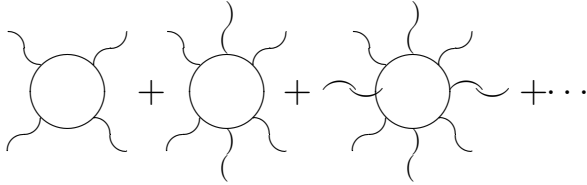


Fig. 2: Diagrams equivalent to the EH Lagrangian.

To obtain the one-loop  $N$  - photon amplitude from the Lagrangian, introduce for each photon leg the field strength tensor

$$F_i^{\mu\nu} = k_i^\mu \varepsilon_i^\nu - k_i^\nu \varepsilon_i^\mu \quad (8)$$

with  $\varepsilon_i, k_i$  the photon polarization and momentum. Define

$$F_{\text{total}} = \sum_{i=1}^N F_i \quad (9)$$

Then

$$\Gamma^{(1)}[k_1, \varepsilon_1; \dots; k_N, \varepsilon_N] = \mathcal{L}^{(1)}(iF_{\text{total}}) |_{F_1 \dots F_N} \quad (10)$$

On the right hand side it is understood that, after expanding out  $\mathcal{L}^{(1)}(iF_{\text{total}})$  to the  $N$ th order in the field, only those terms are retained which involve each  $F_1, \dots, F_N$  linearly. Using this expansion together with a convenient choice of polarizations, one can obtain a closed-form expression for the low energy limit of the one-loop on-shell photon amplitudes, valid for arbitrary  $N$  and all polarization components [18, 19]. We will be concerned here only with the ‘all +’ or ‘maximally helicity violating’ component of the amplitude, defined by choosing the polarization

vectors for all photons as positive helicity eigenstates (the ‘all +’ and ‘all −’ components are related by a parity transformation). This component becomes particularly simple:

$$\begin{aligned}\Gamma_{\text{spin}}^{(1)}[k_1, \varepsilon_1^+; \dots; k_N, \varepsilon_N^+] &= -2 \frac{(2e)^N}{(4\pi)^2 m^{2N-4}} c_{\text{spin}}^{(1)}\left(\frac{N}{2}\right) \chi_N \\ c_{\text{spin}}^{(1)}(n) &= -\frac{\mathcal{B}_{2n}}{2n(2n-2)}\end{aligned}\tag{11}$$

Here the  $\mathcal{B}_k$  are the Bernoulli numbers, and  $\chi_N$  is a kinematical factor which in spinor helicity notation (see, e.g., [20]) can be written as

$$\chi_N = \frac{\left(\frac{N}{2}\right)!}{2^{\frac{N}{2}}} \left\{ [12]^2 [34]^2 \dots [(N-1)N]^2 + \text{permutations} \right\}\tag{12}$$

( $[ij] = \langle k_i^+ | k_j^- \rangle$ ). Its precise form is not essential for our purpose. Here and in the following it should be understood that we deal with the low energy limits of the amplitudes only.

### 3. Two loop: the self-dual case

The two-loop correction  $\mathcal{L}^{(2)}(F)$  to the Euler–Heisenberg Lagrangian, involving an additional photon exchange in the loop, has been considered by various authors starting with V.I. Ritus in 1975 [21, 22, 23, 24]. However, for a general constant field  $F_{\mu\nu}$  no representation is known for  $\mathcal{L}^{(2)}(F)$  which would be explicit enough to derive closed all- $N$  formulas for the two-loop photon amplitudes. Things simplify dramatically, though, if one specializes the field to a (euclidean) self dual (‘SD’) field, obeying

$$F_{\mu\nu} = \tilde{F}_{\mu\nu} \equiv \frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}\tag{13}$$

The square of  $F$  becomes then proportional to the identity matrix,  $F^2 = -f^2 \mathbb{1}$ , and we can use the number  $f$  to parametrize  $F$ .  $f$  can be either real or imaginary. The real  $f$  case corresponds to a field which in Minkowski space has a real magnetic component and an imaginary electric component. In Euclidean space (with  $x_4 = it$ ) the field strength matrix can be Lorentz transformed to the form

$$F = \begin{pmatrix} 0 & f & 0 & 0 \\ -f & 0 & 0 & 0 \\ 0 & 0 & 0 & f \\ 0 & 0 & -f & 0 \end{pmatrix}\tag{14}$$

Thus it describes a “doubling up” of the magnetic case, and indeed the EH Lagrangian for such a field turns out to have the same qualitative properties as the EH Lagrangian for a purely magnetic field. The self-dual case with a real  $f$  will therefore be called ‘magnetic’ in the following. Similarly, for a self dual field with a purely imaginary  $f$  the properties of the Lagrangian are similar to the electric EH Lagrangian and we call this case ‘electric’.

In this self dual case all integrals turn out to be elementary, leading to the following simple closed-form expression for the two-loop effective Lagrangian [18],

$$\mathcal{L}_{\text{spin}}^{(2)}(\kappa) = -2\alpha \frac{m^4}{(4\pi)^3} \frac{1}{\kappa^2} \left[ 3\xi^2(\kappa) - \xi'(\kappa) \right] \quad (15)$$

Here  $\kappa \equiv \frac{m^2}{2e\sqrt{f^2}}$  and

$$\xi(x) \equiv -x \left( \psi(x) - \ln(x) + \frac{1}{2x} \right) \quad (16)$$

where  $\psi(x) = \Gamma'(x)/\Gamma(x)$  is the digamma function. Since self dual fields are helicity eigenstates [25] the effective action for such fields carries precisely the information on the ‘all +’ (or ‘all -’) photon amplitudes. Thus at the two-loop level we are still able to write down a closed-form all- $N$  expression for this particular polarization choice:

$$\begin{aligned} \Gamma_{\text{spin}}^{(2)}[k_1, \varepsilon_1^+; \dots; k_N, \varepsilon_N^+] &= -2\alpha\pi \frac{(2e)^N}{(4\pi)^2 m^{2N-4}} c_{\text{spin}}^{(2)}\left(\frac{N}{2}\right) \chi_N \\ c_{\text{spin}}^{(2)}(n) &= \frac{1}{(2\pi)^2} \left\{ \frac{2n-3}{2n-2} \mathcal{B}_{2n-2} + 3 \sum_{k=1}^{n-1} \frac{\mathcal{B}_{2k}}{2k} \frac{\mathcal{B}_{2n-2k}}{(2n-2k)} \right\} \end{aligned} \quad (17)$$

(the kinematic factor  $\chi_N$  is the same as in (11)).

#### 4. Higher loop orders: $\text{Im } \mathcal{L}$

Beyond two loops, it gets rather hard to obtain information on the EH Lagrangian by a direct calculation. Nevertheless, the work of Ritus and Lebedev quoted above [21, 22] leads to an all-order prediction for the *imaginary part* of this Lagrangian. As is well-known, the EH Lagrangian will have an imaginary part whenever the field is not purely magnetic. Physically it represents the possibility of vacuum pair creation by the electric field component. At one loop for the case of a purely electric field, Schwinger [16] found for it the following representation,

$$\text{Im } \mathcal{L}_{\text{spin}}^{(1)}(E) = \frac{m^4}{8\pi^3} \beta^2 \sum_{k=1}^{\infty} \frac{1}{k^2} \exp \left[ -\frac{\pi k}{\beta} \right] \quad (18)$$

with  $\beta = \frac{eE}{m^2}$ . Here the term with index  $k$  describes the coherent production of  $k$  pairs by the field. This representation is manifestly nonperturbative in the field and coupling, making it clear that the imaginary part cannot be seen unless one sums up the whole set of diagrams in fig. 2.

At the two-loop level, the imaginary part of  $\mathcal{L}(E)$  was first investigated by Lebedev and Ritus [22] who found the following generalization of Schwinger’s formula (18):

$$\text{Im}\mathcal{L}_{\text{spin}}^{(2)}(E) = \frac{m^4}{8\pi^3}\beta^2 \sum_{k=1}^{\infty} \alpha\pi K_k(\beta) \exp\left[-\frac{\pi k}{\beta}\right] \quad (19)$$

Here  $\alpha = \frac{e^2}{4\pi}$  is the fine-structure constant. The exponential factors are the same as in (18), however at two loop the  $k$ th exponential comes with a prefactor  $K_k(\beta)$  which is itself a function of the field. These prefactor functions are not known explicitly, but [22] were able to show that they have small  $\beta$  expansions of the following form:

$$K_k(\beta) = -\frac{c_k}{\sqrt{\beta}} + 1 + \mathcal{O}(\sqrt{\beta})$$

$$c_1 = 0, \quad c_k = \frac{1}{2\sqrt{k}} \sum_{l=1}^{k-1} \frac{1}{\sqrt{l(k-l)}}, \quad k \geq 2 \quad (20)$$

In the following we will concentrate on the weak field limit  $\beta \ll 1$  of  $\text{Im}\mathcal{L}$ . In this limit the  $k$  - series in (18), (19) are dominated by the  $k = 1$  term, and from (20)  $\lim_{\beta \rightarrow 0} K_1(\beta) = 1$ . Thus one has

$$\text{Im}\mathcal{L}_{\text{spin}}^{(1)}(E) + \text{Im}\mathcal{L}_{\text{spin}}^{(2)}(E) \stackrel{\beta \rightarrow 0}{\sim} \frac{m^4\beta^2}{8\pi^3} (1 + \alpha\pi) e^{-\frac{\pi}{\beta}} \quad (21)$$

Now [22] find that the  $\alpha\pi$  term in (21) has a natural interpretation in the pair creation picture. Naively, to turn real a virtual electron-positron must separate out along the field direction a distance  $r_{\parallel}$  such that the energy gained from the field makes up for the rest masses. This is

$$r_{\parallel} = \frac{2m}{eE} \quad (22)$$

However, this does not take into account the Coulomb attraction; a pair getting “born” at a separation of  $r_{\parallel}$  comes with a negative binding energy of  $-\alpha/r_{\parallel}$ . Lebedev and Ritus argue that, since this energy reduces the amount of energy which has to be taken out of the field, it is equivalent to a lowering of the rest mass. Thus  $m$ , the physical (on-shell renormalized) vacuum mass, should be replaced by an effective field-dependent mass  $m_*(E)$ ,

$$m_*(E) = m - \frac{\alpha}{2} \frac{eE}{m} \quad (23)$$

with corrections expected to be of higher order in  $\beta$  (see [26, 27] for a detailed discussion). Then replacing  $m$  by  $m_*(E)$  in the one-loop  $k = 1$  exponential and expanding in  $\alpha$  one finds that

$$\exp\left[-\pi \frac{m_*^2(E)}{eE}\right] = \left[1 + \alpha\pi + \mathcal{O}(\alpha^2)\right] \exp\left[-\pi \frac{m^2}{eE}\right] \quad (24)$$

Thus if this interpretation is correct then the  $1 + \alpha\pi$  in the two-loop formula (21) is the truncation of an exponential series, and the missing higher order terms must show up in the higher loop corrections to  $\text{Im}\mathcal{L}_{\text{spin}}(E)$ . Moreover, in the weak field limit these should be the *only* missing terms, leading to a remarkably simple all-order prediction:

$$\sum_{l=1}^{\infty} \text{Im}\mathcal{L}_{\text{spin}}^{(l)}(E) \stackrel{\beta \rightarrow 0}{\sim} \frac{m^4 \beta^2}{8\pi^3} \exp\left[-\frac{\pi}{\beta} + \alpha\pi\right] \quad (25)$$

In the weak field limit the above considerations are spin-independent [27] so that the same formula (25) with just an additional factor of 1/2 on the right hand side applies to scalar QED. For this case, moreover, there exists a completely different and more direct derivation of the same formula due to Affleck et al. [28]. (These derivations were independent of one another: apparently Affleck et al. were not aware of Ritus's work, and Ritus and Lebedev were not aware of Affleck et al.'s work.) It uses the following 'worldline path integral' representation of the quenched effective action in scalar QED [29],

$$\Gamma_{\text{scal}}^{(\text{quenched})}(A) = \int_0^\infty \frac{dT}{T} e^{-m^2 T} \int \mathcal{D}x(\tau) e^{-S[x(\tau)]} \quad (26)$$

Here the path integral  $\mathcal{D}x(\tau)$  runs over the space of all embeddings  $x(\tau)$  of the particle trajectory into (Euclidean) spacetime, with periodic boundary conditions in proper-time  $x(T) = x(0)$ . The path integral action  $S[x(\tau)]$  has three parts,

$$S = S_0 + S_e + S_i \quad (27)$$

$$\begin{aligned} S_0 &= \int_0^T d\tau \frac{\dot{x}^2}{4} \\ S_e &= ie \int_0^T \dot{x}^\mu A_\mu(x(\tau)) \\ S_i &= -\frac{e^2}{8\pi^2} \int_0^T d\tau_1 \int_0^T d\tau_2 \frac{\dot{x}(\tau_1) \cdot \dot{x}(\tau_2)}{(x(\tau_1) - x(\tau_2))^2} \end{aligned} \quad (28)$$

$S_e$  describes the interaction of the scalar with the external field,  $S_i$  the exchange of internal photons in the scalar loop. In [28] this representation was applied to the constant electric field case, and  $\text{Im}\mathcal{L}_{\text{scal}}(E)$  was calculated using a stationary path approximation both for the  $T$  integral and the path integral  $\int \mathcal{D}x(\tau)$ . The stationary trajectory turns out to be a circle with a field dependent radius. Evaluation of the worldline action  $S[x(\tau)]$  on this trajectory yielded exactly the exponent in (25), and the second variation determinant gave precisely the same prefactor. Affleck et al. then argued that the stationary point approximation becomes exact in the weak field limit up to renormalization effects.

Formulas analogous to (18), (19) exist also for the self dual case [30]. In the 'magnetic' case (real  $f$ ) the EH Lagrangian is real, while in the 'electric' case (imaginary  $f$ ) it has an imaginary part. At one and two loops, this imaginary part has the following Schwinger type expansion,



$$\begin{aligned}
\text{Im}\mathcal{L}_{\text{spin}}^{(1)}(i\kappa) &= -\frac{m^4}{(4\pi)^2} \frac{1}{\kappa} \sum_{k=1}^{\infty} \left( \frac{1}{k} + \frac{1}{2\pi\kappa k^2} \right) e^{-2\pi k\kappa} \\
\text{Im}\mathcal{L}_{\text{spin}}^{(2)}(i\kappa) &= -\alpha\pi \frac{m^4}{(4\pi)^2} \frac{1}{\kappa} \sum_{k=1}^{\infty} K_k(\kappa) e^{-2\pi k\kappa}
\end{aligned} \tag{29}$$

Here the functions  $K_k(\kappa)$  are the SD analogues of the Lebedev–Ritus functions  $K_k(\beta)$ . In contrast to the  $K_k(\beta)$ , the  $K_k(\kappa)$  are known explicitly as power series in  $1/\kappa$  [31]:

$$K_k(\kappa) = k - \frac{1}{2\pi\kappa} - \frac{3\kappa}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^m \mathcal{B}_{2m}}{2m\kappa^{2m}} \tag{30}$$

In the weak field limit  $f \rightarrow 0$  or  $\kappa \rightarrow \infty$  one finds the same relation between the one and two loop contributions to  $\text{Im}\mathcal{L}$  as in (21),

$$\text{Im}\mathcal{L}_{\text{spin}}^{(2)}(i\kappa) \stackrel{f \rightarrow 0}{\sim} \alpha\pi \text{Im}\mathcal{L}_{\text{spin}}^{(1)}(i\kappa) \tag{31}$$

Although neither the Lebedev–Ritus arguments nor the approach of Affleck et al. carry over to the self dual case in an obvious way, we take this as evidence that the simple exponentiation (25) applies also to the self dual case. Thus we conjecture that

$$\sum_{l=1}^{\infty} \text{Im}\mathcal{L}_{\text{spin}}^{(l)}(i\kappa) \stackrel{f \rightarrow 0}{\sim} -\frac{m^4}{(4\pi)^2} \frac{1}{\kappa} \exp[-2\pi\kappa + \alpha\pi] \tag{32}$$

## 5. $\text{Re}\mathcal{L} \leftrightarrow \text{Im}\mathcal{L}$ via Borel dispersion relations

We would now like to convert the result (25) for  $\text{Im}\mathcal{L}$  into information about  $\text{Re}\mathcal{L}$  and the photon amplitudes. The appropriate dispersion relation involves the same Borel technique which we described in the introduction for the loop expansion, now applied to the weak field expansion. Let us first demonstrate it at the one loop level [32]. In the purely magnetic case, it is easy to obtain from (6) the following closed formula for the coefficients of the weak field expansion of  $\mathcal{L}_{\text{spin}}^{(1)}$ ,

$$\mathcal{L}_{\text{spin}}^{(1)}(B) = \frac{2m^4}{\pi^2} \sum_{n=0}^{\infty} a_n^{(1)} g^{n+2} \tag{33}$$

where  $g = \left(\frac{eB}{m^2}\right)^2$  and

$$a_n^{(1)} = -\frac{2^{2n} \mathcal{B}_{2n+4}}{(2n+4)(2n+3)(2n+2)} \tag{34}$$

Using properties of the Bernoulli numbers one can then show that asymptotically the  $a_n^{(1)}$  behave as

$$a_n^{(1)} \stackrel{n \rightarrow \infty}{\sim} \frac{(-1)^n \Gamma(2n+2)}{8\pi^4 \pi^{2n}} \left(1 + \frac{1}{2^{2n+4}} + \frac{1}{3^{2n+4}} + \dots\right) \quad (35)$$

If one replaces  $a_n^{(1)}$  with just the leading term in this expansion, the series (33) turns into an alternating series which is divergent but Borel summable, i.e. the Borel integral (5) is well-defined. The weak field expansion of  $\mathcal{L}_{\text{spin}}^{(1)}$  for the purely electric case differs from (33) only by an additional factor  $(-1)^n$ :

$$\mathcal{L}_{\text{spin}}^{(1)}(E) = \frac{2m^4}{\pi^2} \sum_{n=0}^{\infty} (-1)^n a_n^{(1)} g^{n+2} \quad (36)$$

where now  $g = \left(\frac{eE}{m^2}\right)^2$ . In this case the same leading approximation produces a non-alternating series for which the Borel integral diverges. Nevertheless, the imaginary part of this integral is well-defined through a dispersion relation (using the discontinuity across the cut along the negative  $g$  axis) and yields just the  $k=1$  term in the Schwinger expansion of  $\text{Im}\mathcal{L}_{\text{spin}}^{(1)}$ , eq. (18). Applying the same procedure to the whole asymptotic expansion (35) term by term one reproduces the complete Schwinger expansion.

At the two loop level, no closed formula is known for the corresponding coefficients  $a_n^{(2)}$ . Still, in [33] their leading and subleading asymptotic growth could be determined by calculating the first fourteen coefficients and using them for a numerical fit. This yielded

$$a_n^{(2)} \stackrel{n \rightarrow \infty}{\sim} \alpha \frac{(-1)^n \Gamma(2n+2)}{8\pi^3 \pi^{2n}} \left(1 - \frac{0.44}{\sqrt{n}} + \dots\right) \quad (37)$$

The leading term differs only by a factor  $\alpha\pi$  from the corresponding one loop term, and thus when used in the Borel dispersion relation gives the  $\alpha\pi$  term in (21). From the subleading term one obtains the second term in the small  $\beta$  expansion (20) for  $K_1$ ,

$$K_1(\beta) = 1 - (0.44) \sqrt{\frac{2}{\pi}} \beta + \mathcal{O}(\beta) \quad (38)$$

Thus the Borel technique allows one to map the asymptotic large  $n$  expansion of the weak field expansion coefficients  $a_n$  of the magnetic Lagrangian to the weak field expansion of the imaginary part of the electric Lagrangian.

In the self dual case, where closed formulas are available for the real and imaginary parts of the EH Lagrangian also at two loops, this correspondence can be checked in even more detail. In [30] we calculated, up to exponentially suppressed terms, the large  $n$  expansion of the two loop weak field expansion coefficients  $c_{\text{scal}}^{(2)}(n)$  for scalar QED (the  $c_{\text{scal}}^{(2)}(n)$  differ from the  $c_{\text{spin}}^{(2)}(n)$  given in (17) only in the coefficient of the second term in braces, which in the scalar QED case is  $3/2$  instead of  $3$ ). Using a Borel dispersion relation as above then yielded precisely the small  $f$  expansion of (the scalar QED equivalent of)  $K_1(\kappa)$ .

## 6. Large $N$ behaviour of the $l$ -loop $N$ - photon amplitudes

Assuming that this Borel dispersion relation remains valid at higher loop orders, we can apply it in reverse to the exponentiation formula (32) and obtain all-loop information on the coefficients of the weak field expansion:

$$\lim_{n \rightarrow \infty} \frac{c_{\text{spin}}^{(l)}(n)}{c_{\text{spin}}^{(1)}(n)} = \frac{1}{(l-1)!} (\alpha\pi)^{l-1} \quad (39)$$

By (10) this translates into a statement on the ‘all +’ amplitudes in the limit of large photon number  $N$ ,

$$\lim_{N \rightarrow \infty} \frac{\Gamma_{\text{spin}}^{(l)}[k_1, \varepsilon_1^+; \dots; k_N, \varepsilon_N^+]}{\Gamma_{\text{spin}}^{(1)}[k_1, \varepsilon_1^+; \dots; k_N, \varepsilon_N^+]} = \frac{1}{(l-1)!} (\alpha\pi)^{l-1} \quad (40)$$

Summing this relation over  $l$  we formally get an exponentiation formula for the complete amplitude in the large  $N$  limit:

$$\lim_{N \rightarrow \infty} \frac{\Gamma_{\text{spin}}^{(\text{total})}[k_1, \varepsilon_1^+; \dots; k_N, \varepsilon_N^+]}{\Gamma_{\text{spin}}^{(1)}[k_1, \varepsilon_1^+; \dots; k_N, \varepsilon_N^+]} = e^{\alpha\pi} \quad (41)$$

Assuming sufficient uniformity in  $l$  of the convergence of the ratio (40) for  $N \rightarrow \infty$ , one could now conclude that the amplitude must be analytic in  $\alpha$  for some sufficiently large  $N$ . But analyticity of the complete amplitude is excluded by renormalons and other arguments. Therefore uniformity must fail, and it is easy to see how this comes about diagrammatically. Fig. 3 shows the Feynman diagrams for the  $N$  photon amplitude for the first four loop orders (the external photons are not displayed, as well as diagrams differing from those depicted only by moving the end points of photon lines along the fermion loops). At  $l$  loops, it involves diagrams with up to  $l-1$  fermion loops.

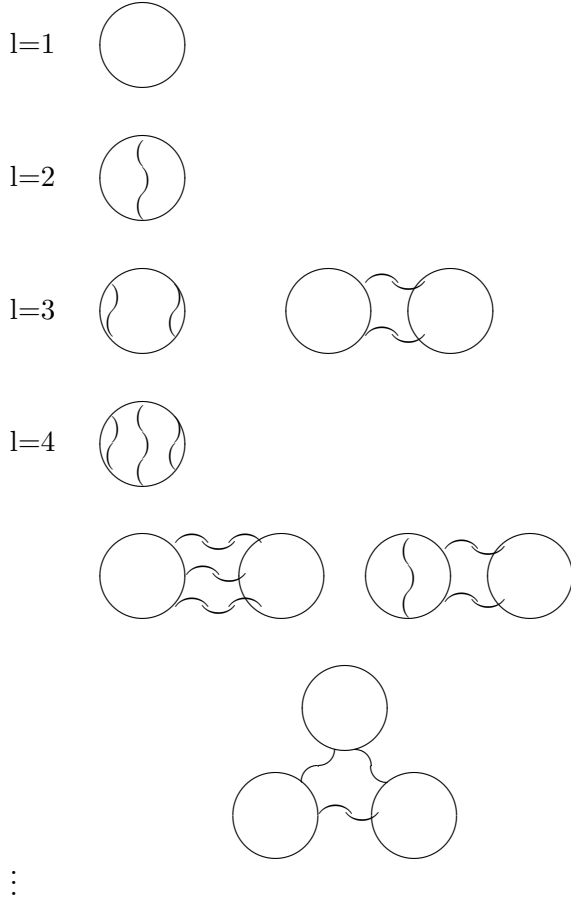


Fig. 3: Feynman diagrams up to four loop order.

From the Affleck et al. approach to the derivation of the exponentiation formula (25) we know that the weak field limit of  $\text{Im}\mathcal{L}(E)$  comes entirely from the ‘quenched’ part of the amplitude. Thus at fixed loop order  $l$  the total contribution of all non-quenched diagrams must be subleading in  $N$  compared to the quenched ones. Since the number of non-quenched classes of diagrams grows with increasing loop number, clearly one would expect the convergence in  $N$  to slow down with increasing  $l$ .

If, on the other hand, one stays inside the quenched class of diagrams from the beginning, there is no obvious mathematical reason to expect such a slowing down. Thus we believe that, for the quenched amplitude, the perturbation series indeed converges for sufficiently large  $N$ , and in fact for all  $N \geq 4$ ; since it would be very surprising to find a qualitative change in the convergence properties of these amplitudes at some definite value of  $N$  (although such a phenomenon cannot be excluded altogether, of course). Similarly, considering the essentially topological character of gauge cancellations one would not assume the restriction to zero momenta or the choice of polarizations to be essential. Thus we are led to the following generalization of Cvitanovic’s conjecture: *the perturbation series converges for all on-shell renormalized QED amplitudes at leading order in  $N_f$ .*

It must be emphasized that on-shell renormalization is essential in all of the above. The finiteness of the limits in (39) and (40) requires the weak field expansion coefficients  $c_{\text{spin}}^{(l)}(n)$

to have the same leading asymptotic growth for all  $l$ , namely  $\Gamma(2n - 2)$  in the magnetic case and  $\Gamma(2n - 1)$  in the ‘magnetic’ self dual case. At the two loop level, one can show by explicit calculation (numerically for the magnetic case [33], analytically for the ‘magnetic’ self dual one [30]) that the use of any renormalized electron mass other than the on-shell mass leads to an asymptotic growth faster than at one loop. In fact, a simple recursion argument shows that, if mass renormalization is done generically, then the  $l$  loop coefficients will grow asymptotically like  $\Gamma(2n + l - 3)$  in the magnetic and like  $\Gamma(2n + l - 2)$  in the ‘magnetic’ self dual case.

## 7. Conclusions

To summarize, we have used existing results on the imaginary part of the QED Euler-Heisenberg Lagrangian together with Borel analysis techniques and a number of explicit two loop checks to argue for convergence of the perturbation series for the physically renormalized QED photon amplitudes, to leading order in  $N_f$ . To the best of our knowledge, this statement is not at variance with known arguments against convergence for the full amplitudes, since those generally rely on the presence of an infinite number of virtual particles. In fact, it presently remains an open possibility that convergence might hold to any finite order in  $N_f$ .

It would be clearly desirable to further corroborate our conjecture by higher loop calculations. While an explicit three-loop calculation for the magnetic or electric EH Lagrangian seems presently out of the question, for the self dual case this calculation might be technically feasible [34]. Since the three loop amplitude has already a non-quenched part, we can make for it the double prediction that (i) the weak field limit of the quenched contribution to  $\text{Im}\mathcal{L}_{\text{spin}}^{(3)}(i\kappa)$  should display the  $(\alpha\pi)^2/2$  correction implicit in (32) and (ii) the non-quenched contribution should be suppressed in this limit. Moreover, one would like to see the ratio  $c^{(3)}(n)/c^{(1)}(n)$  to converge not slower than  $c^{(2)}(n)/c^{(1)}(n)$  if only the quenched contribution is used in calculating  $c^{(3)}(n)$ .

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