APPLICATIONS OF GROUP THEORY TO ELEMENTARY PARTICLE PHYSICS

by

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PREFACE

The work presented in this thesis was carried out in the Department of Theoretical Physics, Imperial College, London between October 1963 and May 1966 under the supervision of Professor P.T. Matthews.

Except where stated in the text, the work described is original and has not been submitted in this or any other University for any other degree. The thesis is based upon two papers written by myself and three papers in collaboration with P. Rotelli whom I would like to thank for giving his permission to include the joint work in this thesis.

I am indebted to Professor P.T. Matthews, Professor A. Salam and Dr. J.M. Charap for their continued help, guidance and encouragement during the course of this work. I would also like to take this opportunity to express my gratitude for the numerous helpful discussions I have had with my colleagues at Imperial College.

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ABSTRACT

The use of non-compact groups is considered in constructing Poincare invariant S-matrix elements. This is done in the manner of Matthews and Feldman. Later, the invariance of the S-matrix is extended to broken inhomogeneous U(6,6), thus including SU(3) as well, and the corresponding fields are constructed using U(6,6) as the auxiliary group. The connection of this group with $SU(6)_W$ is shown. Finally, proton-antiproton annihilation at rest into mesons is considered in detail and the predictions are found to be in disagreement with experiment.

INTRODUCTION

Theoretical physicists, for the last few years have been striving to perceive some order in the morass of elementary particles which are continually being found by the experimentalists. SU(2) was successfully used to classify the proton and neutron states way back in the 1930's. Since then strong interactions have been assumed to be invariant under this isospin group and most of the hadrons (strongly interacting particles) have been classified as multiplets of this group. In 1959, Ikeda, Ogawa and Ohnuki[1] suggested U(3) as a possible symmetry group for the strong interactions, combining both SU(2) and the hypercharge gauge group which was well established. Ne'eman and Gell-Mann[2] then proposed the eight-fold way of SU(3) in which the $\frac{1}{2}^{+}$ baryons and 0^{-} mesons are classified as octets. Subsequently the 1^{-} mesons were fitted in an octet and the most remarkable achievement of SU(3) was the prediction of the Ω^{-3}, strangeness -3, $\frac{3}{2}^{+}$ particle which was needed for the completion of the $\frac{3}{2}^{+}$ decouplet. This particle was found early in 1964[4].

Even though all the SU(3) predictions were not successful[5], it was at least successful in bringing some order. In 1964 an old idea of Wigner[6] was applied to elementary particles. Wigner had postulated the independence of nuclear interactions under isospin and spin separately. These two groups could then be combined to form SU(4). Similarly SU(3) and the spin group were included in SU(6)[7]. This then states that strong interactions are spin and unitary spin independent. This group is clearly non-relativistic and can only be used to classify static states. It can only be applied to 'static' problems. Its chief achievement was in classifying all the baryons and baryon resonances in one multiplet, the <u>56</u>, and the pseudoscalar and vector mesons in the <u>35</u> multiplet.

There was an obvious need to make this group relativistic. Notwithstanding the general theorems of O'Raifeartaigh[8] et. al about the difficulties of combining SU(3) and the Poincare group, several attempts were made towards this end. The most notable of these were the U(6,6) theory developed by Salam, Delbourgo and Strathdee[9] and the SL(6,C) theory of Fulton and Wess, and Rühl[10]. In this thesis we shall be concerned with a particular application of U(6,6) to proton-antiproton annihilations at rest. This work was done in collaboration with P. Rotelli and the results have already been published.[11]

CHAPTER 1

USE OF NON-COMPACT GROUPS

This chapter is a review of fairly well-known material and is based to a great extent on the work of Weinberg[12] and Matthews and Feldman[13]. We shall try to illustrate the use of finite, non-unitary representations of a non-compact group, in the construction of local fields associated with particles that belong to finite unitary representations of the Poincare group.

We start by considering the procedure for constructing local fields which are associated with the single particle multiplets of the Poincare group. With these fields it is a simple matter to write down Poincare invariant S-matrix elements.

The homogeneous, proper, orthochronous Lorentz group, $L_+^\uparrow$, is the group of transformations

$$x_\mu \rightarrow \Lambda_\mu^{\ \nu} x_\nu \qquad (1.1)$$

for which $\det \Lambda = +1$

and $$\Lambda_o^o \geqslant 1 \qquad (1.2)$$

and which leave invariant the quadratic form

$$x_\mu x_\nu g^{\mu\nu} \qquad (1.3)$$

($\mu,\nu = 0,1,2,3$, with metric $(1,-1,-1,-1)$). The conditions (1.2) exclude discrete space or time reflections, so that the group elements are continuously connected with the identity. The infinitesimal transformations can be expressed in terms of six real parameters $\varepsilon_{\mu\nu}$, so that

$$\delta x_\mu = \varepsilon_{\mu\nu}\, x^\nu$$

where
$$\varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu} \tag{1.3}$$

The corresponding six infinitesimal generators $J_{\mu\nu}$, where

$$J_{\mu\nu} = -J_{\nu\mu}$$

satisfy the commutation relation

$$[J_{\mu\nu}, J_{\pi\rho}] = i\,(g_{\mu\rho}J_{\nu\pi} - g_{\mu\pi}J_{\nu\rho} + g_{\nu\pi}J_{\mu\rho} - g_{\nu\rho}J_{\mu\pi}) \tag{1.4}$$

The three operators $J_{ij}(i,j = 1,2,3)$ are interpreted physically as the angular momentum.

The basic representation of the generators $J_{\mu\nu}$ is given by the six 4x4 matrices

$$\sigma_{\mu\nu} = \frac{i}{2}\,[\gamma_\mu, \gamma_\nu] \tag{1.5}$$

where the four matrices γ_μ, whose elements we write as $(\gamma_\mu)_\alpha{}^\beta$ with $\alpha\beta = 1;\cdot 4$, are the Dirac matrices satisfying

$$\{\gamma_\mu\ \gamma_\nu\} = 2g_{\mu\nu} \tag{1.6}$$

Since γ_o is hermitian and $\gamma_i(i = 1,2,3)$ are anti-hermitian, σ_{ij} are hermitian and σ_{oi} anti-hermitian. This illustrates the non-compact character of the homogeneous Lorentz group.

We now generalise to the inhomogeneous Lorentz - or Poincare group, by allowing displacements in space - time

$$x_\mu \rightarrow x_\mu + a_\mu$$

These form a four-parameter Abelian group, with infinitesimal generators P_μ which satisfy

$$[P_\mu, P_\nu] = 0 \qquad (1.7)$$

These are interpreted physically as the total energy and momentum operators.

The Poincare group is the semi-direct product of this Abelian group with the proper homogeneous Lorentz group. It is defined by one additional commutation relation

$$[P_\lambda, J_{\mu\nu}] = i\,(g_{\lambda\mu}P_\nu - g_{\lambda\nu}P_\mu) \qquad (1.8)$$

The Poincare group is non-compact so that its unitary representations are all infinite. The physical states must be normalizable and hence form unitary representations of the group. These are specified by the eigenvalues of a complete set of commuting operators constructed from the group generators. An irreducible representation can partially be specified by giving the eigenvalues of the Casimir operator

$$\langle P^2 \rangle = m^2 \qquad (1.9)$$

and for a state in this representation we can further specify

$$\langle P_\mu \rangle = p_\mu \qquad (1.10)$$

For physical states we require

$$m^2 \geqslant 0 \quad , \; p_o \geqslant 0. \qquad (1.11)$$

We restrict our discussion to $m^2 > 0$. Physically it is clear that we have selected those states corresponding to a given mass. The manifold of states satisfying condition (1.9) and (1.11) is infinite; it is called an orbit. A particular component of this infinite-component multiplet - or point on the orbit - is given by the rest state,

$$p_\mu = (m, \underline{0})$$

Any other point on the orbit may be reached by a Lorentz transformation.

To remove the degeneracy from the state so far specified only by the orbit $p^2 = m^2$, $p_o > 0$ and the value of $\underline{p}$ (equivalent to p_μ) we must consider the "little group". By definition, the "little group" of the Poincare group is the sub-group of the homogeneous Lorentz group which leaves p_μ invariant. It may be shown that the little group is the same for all points on an orbit[14], and it is convenient to consider the special point $p_\mu = (m, \underline{0})$. It is easy to show, that in this frame, the "little group" is the rotation group with infinitesimal generators J_{ij}.

Thus, for the Poincare group, having specified the orbit $\langle p^2 \rangle = m^2$, $p_o > 0$, an irreducible representation is defined by specifying $\langle J^2 \rangle$ and a particular state in this representation by specifying J_{12}. For one particle states these reduce to the spin s, and spin component s_3. At this point, we may also specify the parity $\langle R \rangle$ where the parity operator R satisfies

$$\{R, P_i\} = 0 \quad , \quad [R, P_o] = 0$$
$$\{R, J_{oi}\} = 0 \quad , \quad [R, J_{ij}] = 0. \tag{1.12}$$

This completely determines the representations of the one particle states, as far as their space-time properties are concerned. We denote these states by

$$|m^2, s; \underline{p}, s_3\rangle \equiv |p, s\rangle , \tag{1.13}$$

where obviously m^2 and s determine a representation and $\underline{p}$ and s_3 a state in this representation.

The general Poincare transformation on the physical set of states in Hilbert space can be written as

$$U(a,\Lambda) = \exp\left[-i(P_\mu a^\mu + \tfrac{1}{2} J_{\mu\nu}\eta^{\mu\nu}\right] \tag{1.14}$$

i.e. it is parameterised by a displacement a^μ and a Lorentz transformation $\Lambda_\mu{}^\nu$. This is a unitary operator on physical states. To obtain the explicit representations of these transformations on physical states it is convenient to consider the boost operation which takes a rest frame state $|m,s\rangle$ to a moving state $|p,s\rangle$. This corresponds to a pure Lorentz transformation. Thus

$$|p,s\rangle = N \exp[-i\, \underline{\varepsilon}(p).\underline{K}]\, |m,s\rangle \tag{1.15}$$

where $K_i = J_{oi}$

and $P_\mu |p,s\rangle = p_\mu |p,s\rangle$ (1.16)

Using the general commutation relations (1.4), (1.7) and (1.8) it is easy to show that (1.16) is satisfied if

$$\cosh |\varepsilon| = \frac{p_o}{m} \quad , \quad \sinh|\varepsilon| = \frac{|\underline{p}|}{m} \tag{1.17}$$

and $\underline{\varepsilon}(p)$ is in the direction of $\underline{p}$. N is a normalising factor.

Under a pure Lorentz transformation Λ, a general one particle state transforms as

$$\begin{aligned} U(\Lambda)|p,s\rangle &= e^{-i\,\underline{\eta}.\underline{K}} |p,s\rangle \\ &\equiv e^{-i\underline{\varepsilon}'.\underline{K}} \left(e^{i\underline{\varepsilon}'.\underline{K}} \; e^{-i\underline{\eta}.\underline{K}} \; e^{-i\underline{\varepsilon}.\underline{K}} \right) |m,s\rangle \end{aligned} \tag{1.18}$$

where $\underline{\varepsilon}' = \underline{\varepsilon}(p')$ and $\underline{\varepsilon} = \underline{\varepsilon}\,(p)$.

and $p' = \Lambda^{-1}p.$ (1.19)

From the definition of $\underline{\varepsilon}$, $\underline{\eta}$ and $\underline{\varepsilon}'$ it is obvious that the effect of the three exponential factors in brackets is to induce the transformations $m \to p \to p' \to m$. Therefore, they take a rest frame state to a rest frame state and induce only a little group spin rotation of a rest frame state. We define this as the Wigner rotation

$$D(p) = e^{i\underline{\varepsilon}'.\underline{K}} \; e^{-i\underline{\eta}.\underline{K}} \; e^{-i\,\underline{\varepsilon}.\underline{K}} \tag{1.20}$$

Then equation (1.18) can be written as

$$U(\Lambda)|p,s\rangle = e^{-i\underline{\varepsilon}'.\underline{K}} |m,s'\rangle \langle m,s'|D(p)|m,s\rangle$$

$$= |p',s'\rangle \langle m,s'|D(p)|m,s\rangle \quad (1.21)$$

Under a translation a_μ

$$|p,s\rangle \rightarrow U(a)|p,s\rangle = e^{iP_\mu a^\mu}|p,s\rangle = e^{ip_\mu a^\mu}|p,s\rangle \quad (1.22)$$

These transformation properties can also be expressed in terms of Fock space creation and annihilation operators acting on a non degenerate vacuum state $|o\rangle$, such that

$$a^+(p,s)\ |o\rangle \equiv |p,s\rangle$$

These states are covariantly normalised

$$\langle p',s'|p,s\rangle\ 2\pi\ \theta\ (p_o)\delta(p^2-m^2) = (2\pi)^4\ \delta^4(p-p')\ \delta_{ss'}$$

Then by (1.21),

$$U(\Lambda)a^+(p,s)U^{-1}(\Lambda) = a^+(p',s')\langle m,s'|D(p)|m,s\rangle\ \ldots$$

and since D is unitary

$$U(\Lambda)a(p,s)\ U^{-1}(\Lambda) = \langle m,s|D^{-1}(p)|m,s'\rangle\ a(p',s') \quad (1.23)$$

and by (1.22)

$$U(a)a^+(p,s)\ U^{-1}(a) = e^{ip_\mu a^\mu}\ a^+(p,s) \quad (1.24)$$

and

$$U(a)a(p,s)\ U^{-1}(a) = e^{-ip_\mu a^\mu}\ a(p,s) \quad (1.25)$$

We have now explicitly exhibited the transformation properties of one particle states under Poincare transformations. We could use these creation and annihilation operators to construct Poincare invariant scattering elements. However, their transformation properties are complicated because the Wigner rotation D(p) does not only depend on $\eta^{\mu\nu}$, the parameter of the Lorentz transformations Λ, but also on the momentum of the state being transformed. Because of this we try to construct auxiliary operators which have simpler transformation properties under Lorentz transformations. To achieve this, we require explicit representations for the three factors appearing in the Wigner rotation D(p). Since these involve the generators, K_i, of pure Lorentz transformations we need to consider an <u>auxiliary</u> group which contains these generators. The simplest choice is the homogeneous Lorentz group and we may use any representation which contains the spin s in its decomposition. It is simplest to use the finite representations $|\alpha\rangle$ which are non-unitary. It is in this manner that the non-unitary finite representations of the Lorentz group arise in field theory. Later, we shall show how this auxiliary group can be enlarged to U(6,6). Thus

$$\langle m,s|D^{-1}(p)|m,s'\rangle$$

$$= \langle m,s|\alpha\rangle\langle\alpha|e^{i\underline{\varepsilon}.\underline{K}}|\beta\rangle\langle\beta|e^{i\underline{\eta}.\underline{K}}|\gamma\rangle\langle\gamma|e^{-i\underline{\varepsilon}'.\underline{K}}|\delta\rangle$$

$$\langle\delta|m,s'\rangle \; . \tag{1.26}$$

We now define the auxiliary operator.

$$A_\alpha(p) = \langle\alpha|e^{-i\underline{\varepsilon}.\underline{K}}|\beta\rangle\langle\beta|m,s\rangle\, a(p,s)$$

$$\equiv u_\alpha(p)^s a(p,s) \tag{1.27}$$

Then by (1.23) and (1.26)

$$U(\Lambda)A_\alpha(p)\ U^{-1}(\Lambda)$$

$$= \quad u_\alpha(p)^s\ U(\Lambda)\ a(p,s)\ U^{-1}(\Lambda)$$

(as $\langle\alpha|e^{-i\underline{\varepsilon}.\underline{K}}|m,s\rangle$ is just a number)

$$= \langle\alpha|e^{-i\underline{\varepsilon}.\underline{K}}|\beta\rangle\ \langle\beta|m,s\rangle\ \langle m,s|D^{-1}(p)|m,s'\rangle\ a(p',s')$$

$$= \langle\alpha|e^{-i\underline{\varepsilon}.\underline{K}}\ |\beta\rangle\ \langle\beta|D^{-1}(p)|\gamma\rangle\ \langle\gamma|m,s'\rangle\ a(p',s') \quad (1.28)$$

This is possible since D is an element of the rotation group, which is a subgroup of the auxiliary group, and the representation $|\alpha\rangle$ includes the spin s.

Then

$$U(\Lambda)A_\alpha(p)\ U^{-1}(\Lambda)$$

$$= \langle\alpha|\ e^{-i\underline{\varepsilon}.\underline{K}}\ e^{i\underline{\varepsilon}.\underline{K}}\ e^{i\underline{\eta}.\underline{K}}\ e^{-i\underline{\varepsilon}'.\underline{K}}\ |\gamma\rangle\ \langle\gamma|m,s'\rangle\ \ a(p',s')$$

$$= \langle\alpha|e^{i\underline{\eta}.\underline{K}}\ |\ \beta\ \rangle\langle\beta|\ e^{-i\underline{\varepsilon}'.\underline{K}}\ |\gamma\rangle\ \langle\gamma|m,s'\rangle\ a(p',s')$$

$$(1.29)$$

Therefore, finally

$$U(\Lambda)\ A_\alpha(p)\ U^{-1}(\Lambda)$$
$$= \langle\alpha|e^{i\underline{\eta}.\underline{K}}\ \ |\beta\rangle\ A_\beta\ (p')$$

$$= S_\alpha{}^\beta \, \Lambda_\beta(p') \qquad (1.30)$$

This is just the simple transformation property we were looking for. The transformation of $A_\alpha(p)$ is now a pure index transformation parametrised by η alone and with the additional requirement that p is replaced by p'. The label α thus defines a finite non-unitary representation of the non-compact homogeneous Lorentz group. The factor $U_\alpha(p)^s$ is a generalised spinor and the relation (1.27) is the crucial link between the group theoretic analysis and operator fields. The non-unitary states $|\alpha\rangle$ have no physical significance and have been introduced to simplify the problem of constructing Poincare invariants. Their connection with the physical states $|m,s\rangle$ is the constant spinor $\langle\alpha|m,s\rangle$. To construct A_α, $|\alpha\rangle$ must contain the spin $|s\rangle$. In these non-unitary representations, $\langle\alpha|$ is the dual, not the conjugate of $|\alpha\rangle$. It is defined such that $\langle\alpha|\alpha\rangle = 1$. Thus

$$\langle\alpha|\psi\rangle \neq \langle\psi|\alpha\rangle^*$$

and
$$\langle\beta|K|\alpha\rangle = K_\beta{}^\alpha$$

is no longer a hermitian matrix. The relationship between the dual and the hermitian conjugate of $A_\alpha(p)$ has to be evaluated for each particular representation. The dual operator A^α is defined as

$$A^\alpha(p) \equiv a^+(p,s) \, \langle s,m|\beta\rangle \langle\beta|e^{i\underline{\varepsilon}.\underline{K}}|\alpha\rangle$$

$$\equiv a^+(p,s) \, u_s(p)^\alpha \qquad (1.31)$$

so that it transforms contravariantly, thus

$$U(\Lambda)\ A^{\alpha}(p)\ U^{-1}(\Lambda) = A^{\beta}(p')\langle\beta\, e^{-i\underline{\eta}\cdot\underline{K}}|\alpha\rangle$$

$$= A^{\beta}(p')\ (S^{-1})_{\beta}{}^{\alpha} \qquad (1.32)$$

so that $A^{\alpha}(p)\ A_{\alpha}(p)$ is a scalar density.

It is easy to show that under translations

$$A_{\alpha}(p) \to e^{-ip_{\mu}a^{\mu}}\ A_{\alpha}(p) \qquad (1.33)$$

$$A^{\alpha}(p) \to e^{ip_{\mu}a^{\mu}}\ A^{\alpha}(p) \qquad (1.34)$$

With these auxiliary operators it is a simple matter to construct Poincare invariants. All one has to do is to saturate indices in the product of the appropriate auxiliary operators. The factor p_{μ} transforms like a four vector when it appears as a product $p_{\mu}\ A_{\alpha}(p)$ so it can be used in constructing these scalar densities. If we take $|\alpha\rangle$ to be the Dirac $((\frac{1}{2},0)+(0,\frac{1}{2}))$ representation for spin $\frac{1}{2}$ particles we have

$$(K_i)_{\alpha}{}^{\beta} = \left(\frac{\sigma_{oi}}{2}\right)_{\alpha}{}^{\beta}$$

and four vectors and pseudo-scalars can be constructed from

$$\psi^{\alpha}(p)\ (\gamma_{\mu})_{\alpha}{}^{\beta}\ \psi_{\beta}(q)\ ,\ \psi^{\alpha}(p)(\gamma_5)_{\alpha}{}^{\beta}\ \psi_{\beta}(q)$$

respectively. $\psi^{\alpha},\psi_{\beta}$ are now the Dirac fields. The most general

Poincare invariant involving such operators is then the form

$$T = \int \psi^{\alpha}(p_1)\ldots f(\gamma_\mu p^\mu, \gamma_5)_{\alpha\ldots}^{\beta\ldots}\psi_\beta(p_2)$$

$$\delta^4(p_1+\ldots-p_2\ldots)\; d^4p_1\ldots d^4p_2$$

Also for the Dirac field $A^{\alpha}(p) \equiv \psi^{\alpha}(p) = (\psi^{+}(p)\gamma_o)^{\alpha}$.

Notice that we have been able to construct Poincare invariant S-operators without introducing the concepts of antiparticles or any of the general properties associated with them such as crossing symmetry and CTP invariance. Weinberg[12], Matthews and Feldman[13] have shown how these arise from the important notion of local fields. They also show how any equation of motion, apart from the Klein-Gordan equation, is a consequence of using a representation $|\alpha\rangle$ which runs over more values than the number of spin components, 2s+1. The equations of motion are obtained by restricting α to the s values, for example, by requiring a definite parity. We illustrate this in the next chapter where we enlarge the auxiliary group to U(2,2) and then later to U(6,6). Here, the restrictions lead naturally to the Bargmann-Wigner[15] equations.

The baryon number is introduced by taking a direct product of the gauge group U(1) with the auxiliary group. For the basic 4x4 Dirac algebra the infinitesimal generator of U(1) is the unit matrix. With this choice, since A_α and A^α transform covariantly and contravariantly respectively under these simple phase transformations, they will have opposite baryon number. They represent quark and anti-quark states respectively.

CHAPTER 2

U(2,2) AS THE AUXILIARY GROUP

In chapter 1 we discussed the rise of finite representations of an auxiliary group in constructing Poincare invariants. This group has to contain the Lorentz group and the representations used must contain the spin s in their decomposition. We now consider the use of larger auxiliary groups, in particular U(2,2).

As before, we define the auxiliary operator

$$A_{\alpha}(p) = \langle\alpha| e^{-i\underline{\varepsilon}.\underline{K}} |\beta\rangle \langle\beta|m,s\rangle \quad a(p,s) \tag{2.1}$$

but now the $|\alpha\rangle$ are representations of the group U(2,2) $|\alpha\rangle$ will also transform like some reducible representation of the Lorentz group. U(2,2) is the group of transformations whose infinitesimal generators are $F^r (r = 0,1,2....15)$ where in the basic representation (4x4)

$$F^r = \tfrac{1}{2}\, \Gamma^r$$

$$\Gamma^r = \gamma_{\mu}\ ,\ \gamma_5\ , i\ \gamma_5\gamma_{\mu}\quad \sigma_{\mu\nu}\ , 1 \tag{2.2}$$

Define $[F_r, F_s] = i\ f_{rs}{}^q F_q$.

This group is closely related to U(4) but the essential difference is that only eight $(1, \gamma_o, i\gamma_5\gamma_i, \sigma_{ij})$ of the sixteen matrices (2.2) are hermitian, the other eight being antihermitian. Here

$$\gamma_5 = \gamma_o\gamma_1\ \gamma_2\gamma_3 \tag{2.3}$$

and these matrices satisfy the relation

$$\gamma_o(\Gamma_r)^+ \gamma_o = \Gamma_r \quad \text{(+ denotes hermitian conjugate).} \tag{2.4}$$

which ensures that the relationship between hermitian conjugate operators and dual for U(2,2) is the same as for the homogeneous Lorentz group i.e. for the basic 4x4 representation (which contains the Dirac representation of the homogeneous Lorentz group) $(A_\alpha(p))^+$ transforms like $A^\beta(p)\ (\gamma_o)_\beta{}^\alpha$.

$$A^\alpha(p) = (A^+(p)\gamma_o)^{\alpha} \tag{2.5}$$

Thus under pure Lorentz transformations

$$U(\Lambda)A_\alpha(p)\ U^{-1}(\Lambda) = \langle\alpha|e^{i\underline{\eta}\cdot\underline{K}}|\beta\rangle A_\beta(p')$$

$$= S_\alpha{}^\beta A_\beta(p') \tag{2.6}$$

which is exactly the same as equation (1.30). However we now have the further possibility of performing U(2,2) transformations on the auxiliary operators.

$$A_\alpha(p) \rightarrow \langle\alpha|e^{i\varepsilon_r F^r}|\beta\rangle A_\beta(p)$$

$$\equiv T_\alpha{}^\beta\ A_\beta(p). \tag{2.7}$$

Again the dual operator $A^\alpha(p)$ is

$$A^\alpha(p) = a^+(p,s)\langle m,s|\beta\rangle\langle\beta|e^{i\underline{\varepsilon}\cdot\underline{K}}|\alpha\rangle \tag{2.8}$$

and under U(2,2) transformations

$$A^{\alpha}(p) \rightarrow \quad A^{\beta}(p)(T^{-1})_{\beta}{}^{\alpha}. \tag{2.9}$$

We now proceed to the construction of local fields from the auxiliary operators. For the case of spin $\frac{1}{2}$ particles, the lowest U(2,2) representation which contains spin $\frac{1}{2}$ is the 4-dimensional basic representation. In the reduction to the Lorentz group this representation reduces to the $(\frac{1}{2},0) + (0,\frac{1}{2})$ representation. This has been discussed in detail by Matthews and Feldman and they have also studied auxiliary operators of the form $A_{\alpha}{}^{\beta}(p)$ $(\alpha,\beta = 0, \ldots .3)$. We shall here construct fields made out of auxiliary operators $A_{\alpha\beta\gamma}(p)$ without particular symmetrization of the indices so that it transforms like a reducible representation of U(2,2). Such operators will contain particles of spin $\frac{1}{2}$ and $\frac{3}{2}$.

ii) <u>Auxiliary operator of rank 3</u>

Let us define the 64-component object

$$A_{\alpha\beta\gamma}(p) \quad = \quad (U(p)^{s})_{\alpha\beta\gamma} \quad a(p.s) \tag{2.10}$$

where

$$(U(p)^{s})_{\alpha\beta\gamma} = \langle\alpha\beta\gamma|e^{-i\underline{\varepsilon}.\underline{K}}|\alpha'\beta'\gamma'\rangle \langle\alpha'\beta'\gamma'|m,s\rangle \tag{2.11}$$

$$= \frac{1}{6}\left\{\left(e^{-i\varepsilon_i\sigma_{oi}/2}\right)_{\alpha}^{\alpha'}\left(e^{-i\varepsilon_i\sigma_{oi}/2}\right)_{\beta}^{\beta'}\left(e^{-i\varepsilon_i\sigma_{oi}/2}\right)_{\gamma}^{\gamma'} + \text{permutations of } \alpha'\beta'\gamma'\right\} \langle\alpha'\beta'\gamma'|m,s\rangle \qquad (2.12)$$

since in the basic representation

$$\langle\alpha|e^{-i\underline{\varepsilon}.\underline{K}}|\beta\rangle = (e^{-i\varepsilon_i\sigma_{oi}/2})_{\alpha}^{\beta} \qquad (2.13)$$

We can now define a field

$$\psi_{\alpha\beta\gamma}(x) = \int (A_{\alpha\beta\gamma}(p)\, e^{-ip.x} + \tilde{B}_{\alpha\beta\gamma}(p)\, e^{ip.x})\, 2\pi\theta(p_o) \times \delta(p^2-m^2)\,\frac{d^4p}{(2\pi)^4} \qquad (2.14)$$

where, to allow for later developments we have introduced a second particle of mass m and spin s with auxiliary field $\tilde{B}_{\alpha\beta\gamma}(p)$,

$$\tilde{B}_{\alpha\beta\gamma}(p) = \langle\alpha\beta\gamma|e^{-i\underline{\varepsilon}.\underline{K}}|\alpha'\beta'\gamma'\rangle\langle\alpha'\beta'\gamma'|B|m,\bar{s}\rangle\, b^+(p,\bar{s})$$

$$\equiv \tilde{v}_{\alpha\beta\gamma}(p)^{\bar{s}}\, b^+(p,\bar{s}) \qquad (2.15)$$

The operator b^+ creates the antiparticle of the particle destroyed by a. This combination of annihilation and creation operators for the definition of $\psi_{\alpha\beta\gamma}(x)$ is possible because

Weinberg has demonstrated that the auxiliary operator associated with a creation operator can be made to have the same transformation properties as that associated with an annihilation operator. For any pure rotation D there exists a matrix B, such that

$$< m,s|D|m,s' > \;=\; <m,s'|B^{-1}D^{-1}B|m,s>$$

or in other words there is a matrix B which relates a representation D of the rotation group with its dual representation

$$D' = -D^T$$

through the relation

$$D \;=\; BD'B^{-1}$$

i.e. these two representations D and D' are equivalent. Therefore, we can re-write (1.22) as

$$U(\Lambda)a^+(p,s)U^{-1}(\Lambda) \;=\; <m,s\;|B^{-1}D^{-1}B|m,s'>a^+(p',s') \qquad (2.16)$$

and hence we can introduce an alternative auxiliary operator

$$\tilde{A}_\alpha(p) \;=\; <\alpha|e^{-i\underline{\varepsilon}\cdot\underline{K}}|\beta><\beta|B|m,s>\;a^+(p,s) \qquad (2.17)$$

where since B is a matrix in spin space $<\beta|B|m,s>$ is to be interpreted as

$$<\beta|m,s'><m,s'|B|m,s> \quad .$$

Under a pure Lorentz transformation

$$U(\Lambda)\ \tilde{A}_\alpha(p)\ U^{-1}(\Lambda) = \langle\alpha| e^{i\underline{\eta}\cdot\underline{K}} |\beta\rangle \tilde{A}_\beta(p') \qquad (2.18)$$

which is the same as (1.30). But, under translations

$$\tilde{A}_\alpha(p) \rightarrow e^{ip_\mu a^\mu}\ \tilde{A}_\alpha(p)$$

which is opposite to (1.33). We now see that our definition (2.14) is valid. We combine a and b^+, so that $\psi(x)$ behaves simply under gauge transformations.

The variable $\bar{s}$ in (2.15) runs over the same range of values as s, but the matrix $\langle\alpha\beta\gamma|B|m,\bar{s}\rangle$ may be different from $\langle\alpha\beta\gamma|m,s\rangle$. We should now like to demonstrate how the causality requirement is linked to equations of motion for the field and to the parities of the anti-particles. By causality we mean that the fields satisfy local commutation relations

$$[\psi(x),\psi^+(y)]_\pm = 0, \quad (x-y)^2 < 0 \qquad (2.19)$$

where the commutator (anti-commutator) refers to Bose (Fermi) fields.

Assuming that the particle operators a(p,s), b(p,$\bar{s}$) etc are Fermi operators we have $\{\psi_{\alpha\beta\gamma}(x),\psi^+_{\pi\rho\sigma}(y)\}$

$$= \int\Big[u_{\alpha\beta\gamma}(p)^s (u_{\pi\rho\sigma}(p)^s)^*\ e^{-ip\cdot(x-y)}$$

$$+ \tilde{v}_{\alpha\beta\gamma}(p)^{\bar{s}} \ (\tilde{v}_{\pi\rho\sigma}(p)^{\bar{s}})^* \ e^{ip.(x-y)} \Big] \ 2\pi\theta(p_o)\delta(p^2-m^2) \ \frac{d^4p}{(2\pi)^4} \qquad (2.20)$$

This is obtained by using the relations

$$\{a(p,s),a^+(q,s')\} \ 2\pi\theta(p_o)\delta(p^2-m^2) = (2\pi)^4\delta^4(p-q)\delta_{ss'} \qquad (2.21)$$

and

$$\{a,b\} = 0 \quad \text{etc.}$$

which are consistent with the normalisation of the states.

To show that the causality condition is satisfied we need to evaluate the spin sums

$$U_{\alpha\beta\gamma}(p)^s(U_{\pi\rho\sigma}(p)^s)^*$$

$$= \langle\alpha\beta\gamma|e^{-i\underline{\varepsilon}.\underline{K}}|\alpha'\beta'\gamma'\rangle \langle\alpha'\beta'\gamma'|m,s\rangle \langle m,s|\pi'\rho'\sigma'\rangle \langle\pi'\rho'\sigma'|e^{-i\underline{\varepsilon}.\underline{K}}|\pi\rho\sigma\rangle \qquad (2.22)$$

and

$$\tilde{v}_{\alpha\beta\gamma}(p)^{\bar{s}} \ (\tilde{v}_{\pi\rho\sigma}(p)^{\bar{s}})^*$$

$$= \langle\alpha\beta\gamma|e^{-i\underline{\varepsilon}.\underline{K}}|\alpha'\beta'\gamma'\rangle \langle\alpha'\beta'\gamma'|B|m,\bar{s}\rangle \quad \times$$

$$\langle m,\bar{s}|B^{-1}|\pi'\rho'\sigma'\rangle \langle\pi'\rho'\sigma'|e^{-i\underline{\varepsilon}.\underline{K}}|\pi\rho\sigma\rangle \qquad (2.23)$$

These spin sums depend on what we choose for the operators

$$0_{\alpha'\beta'\gamma'}{}^{\pi'\rho'\sigma'} = \langle\alpha'\beta'\gamma'|m,s\rangle\langle m,s|\pi'\rho'\sigma'\rangle \qquad (2.24)$$

and

$$\tilde{0}_{\alpha'\beta'\gamma'}{}^{\pi'\rho'\sigma'} = \langle\alpha'\beta'\gamma'|B|m,\bar{s}\rangle\langle m,\bar{s}|B^{-1}|\pi'\rho'\sigma'\rangle \qquad (2.25)$$

We must now decide how many particles are present in the representations $|m,s\rangle$ and $|m,\bar{s}\rangle$. If s runs over the same number of values as α then we would have

$$\langle\alpha|m,s\rangle = \delta_\alpha^{\ s} .$$

We can then generalise from this basic quark representation to find that

$$0_{\alpha'\beta'\gamma'}{}^{\pi'\rho'\sigma'} = \tilde{0}_{\alpha'\beta'\gamma'}{}^{\pi'\rho'\sigma'}$$

$$= \tfrac{1}{6}\,(\delta_{\alpha'}^{\ \pi'}\delta_{\beta'}^{\ \rho'}\delta_{\gamma'}^{\ \sigma'} + \text{all other combinations}) \qquad (2.26)$$

if s and $\bar{s}$ run over as many values as $\alpha\beta\gamma$ (64). We are assuming no symmetrization of the indices $\alpha\beta\gamma$.

Thus, (2.20) reduces to

$$\{\psi_{\alpha\beta\gamma}(x)\ ,\ \psi^{+}_{\pi\rho\sigma}(y)\}$$

$$= f_{\alpha\beta\gamma}{}^{\pi\rho\sigma}(\partial)\,\Delta(x-y) \tag{2.27}$$

$$\text{where } \Delta(x-y) = \int \left[e^{-ip(x-y)} - e^{+ip.(x-y)} \right] 2\pi\theta(p_o)\delta(p^2-m^2)\,\frac{d^4p}{(2\pi)^4} \tag{2.28}$$

where $f(\partial)$ is some function of the derivative operator $\partial = \frac{\partial}{\partial x_\mu}$.
This is realised by observing that in the basic representation

$$\langle\alpha| e^{-i\underline{\varepsilon}.\underline{K}} |\beta\rangle = (e^{-i\varepsilon_i\sigma_{oi}/2})_\alpha{}^\beta \tag{2.29}$$

and then from (1.23) we find that

$$(e^{-i\varepsilon_i\sigma_{oi}})_\alpha{}^\beta = (\tfrac{\not{p}}{m}\gamma_o)_\alpha{}^\beta \tag{2.30}$$

Generalising this to the representation $|\alpha\beta\gamma\rangle$ we find that (2.27) is true. The expression (2.27) then is known to vanish for $(x-y)^2 < 0$ and hence the causality condition (2.19) is satisfied. We see that we have been able to find causal fields without having the necessity of any equation of motion apart from the Klein-Gordan equation. We get equations of motion by putting restrictions on the spinor $\langle\alpha\beta\gamma|m,s\rangle$ which appears in (2.22).

In the basic representation we see that γ_o fulfills the role of the parity operator as it satisfies the relations (1.12). In the representation $|\alpha\beta\gamma\rangle$ the parity operator is

$$\langle\alpha\beta\gamma|R|\pi\rho\sigma\rangle = \frac{1}{6}\left((\gamma_o)^{\pi}_{\alpha}(\gamma_o)_{\beta}^{\rho}(\gamma_o)_{\gamma}^{\sigma} + \text{all combinations}\right) \tag{2.31}$$

We start by specifying the parity of the state $|m,s\rangle$ by requiring that

$$\frac{1}{6}\left\{(\gamma_o)^{\pi}_{\alpha}(\gamma_o)^{\rho}_{\beta}(\gamma_o)^{\sigma}_{\gamma} + \ldots\right\}\langle\pi\rho\sigma|m,s\rangle = +\langle\alpha\beta\gamma|m,s\rangle \tag{2.32}$$

and, therefore,

$$\langle\alpha\beta\gamma|m,s\rangle = \frac{1}{12}\left\{(\delta_{\alpha}^{\ \pi}\delta_{\beta}^{\ \rho}\delta_{\gamma}^{\ \sigma} + (\gamma_o)_{\alpha}^{\ \pi}(\gamma_o)_{\beta}^{\ \rho}(\gamma_o)_{\gamma}^{\ \sigma}) + \ldots\right\}$$
$$\langle\pi\rho\sigma|m,s\rangle \tag{2.33}$$

Thus, the operators 0 become

$$0_{\alpha'\beta'\gamma'}^{\ \ \ \ \ \pi'\rho'\sigma'} = \frac{1}{6}\left\{\frac{1}{2}\left(\delta_{\alpha'}^{\ \pi'}\delta_{\beta'}^{\ \rho'}\delta_{\gamma'}^{\ \sigma'} + (\gamma_o)_{\alpha'}^{\ \pi'}(\gamma_o)_{\beta'}^{\ \rho'}(\gamma_o)_{\gamma'}^{\ \sigma'}\right)\right.$$
$$\left. + \text{permutations of } \pi',\rho',\sigma'\right\} \tag{2.34}$$

This leads to

$$U_{\alpha\beta\gamma}(p)^{s}(U_{\pi\rho\sigma}(p)^{s})^{*}$$
$$= \frac{1}{12}\left\{\left(\left(\frac{\not p}{m}\right)_{\alpha}^{\ \pi}\left(\frac{\not p}{m}\right)_{\beta}^{\ \rho}\left(\frac{\not p}{m}\right)_{\gamma}^{\ \sigma} + (\gamma_o)_{\alpha}^{\ \pi}(\gamma_o)_{\beta}^{\ \rho}(\gamma_o)_{\gamma}^{\ \sigma}\right)\right.$$
$$\left. + \text{permutations}\right\} \tag{2.35}$$

by using (2.30) and the fact that γ_o anti-commutes with σ_{oi}.

To proceed further we must restrict parity of the state $|m,\bar{s}\rangle$. If we take this to be the same as $|m,s\rangle$ i.e.

$$\frac{1}{6}\left\{(\gamma_o)^{\pi}_{\alpha}(\gamma_o)^{\rho}_{\beta}(\gamma_o)^{\sigma}_{\gamma} + \ldots\right\}\langle\pi\rho\sigma|B|m,\bar{s}\rangle = \langle\alpha\beta\gamma|B|m,\bar{s}\rangle \quad (2.36)$$

and proceeding as before we find

$$\tilde{v}_{\alpha\beta\gamma}(p)^{\bar{s}}\;(\tilde{v}_{\pi\rho\sigma}(p)^{\bar{s}})^{*}$$

$$= \frac{1}{12}\left\{\left(\frac{\not p}{m}\right)^{\pi}_{\alpha}\left(\frac{\not p}{m}\right)^{\rho}_{\beta}\left(\frac{\not p}{m}\right)^{\sigma}_{\gamma} + (\gamma_o)^{\pi}_{\alpha}\,(\gamma_o)^{\rho}_{\beta}\,(\gamma_o)^{\sigma}_{\gamma} + \text{permutations}\right\} \quad (2.37)$$

and the condition (2.27) is not satisfied. However, if we take

$$\frac{1}{6}\left\{(\gamma_o)^{\pi}_{\alpha}\,(\gamma_o)^{\rho}_{\beta}\,(\gamma_o)^{\sigma}_{\gamma} + \ldots\right\}\langle\pi\rho\sigma|B|m,s\rangle = -\langle\alpha\beta\gamma|B|m,\bar{s}\rangle \quad (2.38)$$

then

$$\tilde{v}\tilde{v}^{*} = \frac{1}{12}\left\{\left(\frac{\not p}{m}\right)^{\pi}_{\alpha}\left(\frac{\not p}{m}\right)^{\rho}_{\beta}\left(\frac{\not p}{m}\right)^{\sigma}_{\gamma} - (\gamma_o)^{\pi}_{\alpha}(\gamma_o)^{\rho}_{\beta}(\gamma_o)^{\sigma}_{\gamma} + \text{permutations}\right\} \quad (2.39)$$

and then the causality condition is satisfied. In this representation we have then found that, to satisfy the causality condition, the antiparticles have opposite parity to particles. This is the conventional theory for fermions.

We can now boost the rest condition (2.32) by multiplying (2.12) on the left by

$$\frac{1}{6}\Big\{(e^{-i\varepsilon_i\sigma_{oi}\gamma_o})_\pi^{\ \alpha}\,(e^{-i\varepsilon_i\sigma_{oi}\gamma_o})_\rho^{\ \beta}\,(e^{-i\varepsilon_i\sigma_{oi}\gamma_o})_\sigma^{\ \gamma} + \dots\Big\}$$

and then use (2.32) and (2.30) to obtain the equation

$$(\not{p})_{\alpha\beta\gamma}^{\ \ \ \pi\rho\sigma}\; U_{\pi\rho\sigma}(p)^s = m^3\, U_{\alpha\beta\gamma}(p)^s \qquad (2.40)$$

where

$$(\not{p})_{\alpha\beta\gamma}^{\ \ \ \pi\rho\sigma} = \frac{1}{6}\Big\{(\not{p})_\alpha^{\ \pi}\,(\not{p})_\beta^{\ \rho}\,(\not{p})_\gamma^{\ \sigma} + \dots\Big\}$$

Similarly,

$$(\not{p})_{\alpha\beta\gamma}^{\ \ \ \pi\rho\sigma}\; \tilde{v}_{\pi\rho\sigma}(p)^s = -m^3\, \tilde{v}_{\alpha\beta\gamma}(p)^s \qquad (2.41)$$

and hence

$$(i\gamma_\mu\partial_\mu)_{\alpha\beta\gamma}^{\ \ \ \pi\rho\sigma}\; \psi_{\pi\rho\sigma}(x) = m^3\, \psi_{\alpha\beta\gamma}(x) \qquad (2.42)$$

that is, the field operator in configuration space satisfies an equation of motion. We have found that to satisfy causality we have had to introduce antiparticles and to assume anti-commutation relations for the field $\psi_{\alpha\beta\gamma}(x)$. We have therefore obtained Fermi statistics for this field which we shall show is an half-integer spin field. We have also found that if we specify the parity of the particles we are forced to choose the opposite parity for the anti-particles to be able to satisfy the causality condition. Also, the equation of motion (2.43) is only a boost of the parity condition (2.32).

We can put further restrictions on the spinors $\langle\alpha\beta\gamma|m,s\rangle$ and we show below how these restrictions lead to the Bargmann-Wigner equations[15]. We specify the 'quark' parity (i.e. parity of the basic representation). Consistent with (2.32) the state $|m,s\rangle$ has a positive quark parity

$$(\gamma_o)_{\alpha}^{\ \alpha'}\langle\alpha'\beta\gamma|m,s\rangle = \langle\alpha\beta\gamma|m,s\rangle \tag{2.43}$$

Therefore

$$\langle\alpha\beta\gamma|m,s\rangle = \tfrac{1}{2}\,(1+\gamma_o)_{\alpha}^{\ \alpha'}\langle\alpha'\beta\gamma|m,s\rangle \tag{2.44}$$

and

$$0_{\alpha'\beta'\gamma'}^{\ \ \pi'\rho'\sigma'} = \frac{1}{48}\left\{(1+\gamma_o)_{\alpha'}^{\ \pi'}(1+\gamma_o)_{\beta'}^{\ \rho'}(1+\gamma_o)_{\gamma'}^{\ \sigma'} + \ldots\ldots\right\} \tag{2.45}$$

giving

$$(uu^*)_{\alpha\beta\gamma}^{\ \pi\rho\sigma} = \frac{1}{48m^3}\left\{((\not p+m)\gamma_o)_{\alpha}^{\ \pi}\,((\not p+m)\gamma_o)_{\beta}^{\ \rho}((\not p+m)\gamma_o)_{\gamma}^{\ \sigma} + \ldots\right\} \tag{2.46}$$

Consistent with (2.38) we have to now choose

$$(\gamma_o)_{\alpha}^{\ \alpha'}\langle\alpha'\beta\gamma|B|m,\bar{s}\rangle = -\langle\alpha\beta\gamma|B|m,\bar{s}\rangle \tag{2.47}$$

that is, the parity of the antiquark has to be opposite to the parity of the quark. This leads to

$$(\tilde{v}\tilde{v}^*)_{\alpha\beta\gamma}^{\;\;\pi\rho\sigma} = \frac{1}{48m^3}\Big\{((\not{p}-m)\gamma_o)_\alpha^{\;\pi}((\not{p}-m)\gamma_o)_\beta^{\;\rho}((\not{p}-m)\gamma_o)_\gamma^{\;\sigma} + \ldots\Big\} \quad . \qquad (2.48)$$

Substituting (2.46) and (2.48) in (2.20) we are led to the equation

$$\Big\{\psi_{\alpha\beta\gamma}(x), \psi^+_{\pi\rho\sigma}(y)\Big\}$$

$$= \frac{1}{48m^3}\Big\{((i\not{\partial}+m)\gamma_o)_\alpha^{\;\pi}((i\not{\partial}+m)\gamma_o)_\beta^{\;\rho}((i\not{\partial}+m)\gamma_o)_\gamma^{\;\sigma} + \ldots\Big\}\,\Delta(x-y) \qquad (2.49)$$

which is in the causal form (2.27).

We can now boost the condition (2.43), as before, but this time we are led to the equation

$$(\not{p}-m)_\alpha^{\;\alpha'}\, U_{\alpha\beta\gamma}(p)^s = 0 \qquad (2.50)$$

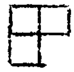
and on boosting (2.47) we get

$$(\not{p}+m)_\alpha^{\;\alpha'}\, \tilde{v}_{\alpha'\beta\gamma}(p)^s = 0 \qquad (2.51)$$

and hence

$$(i\,\not{\partial}-m)_\alpha^{\;\alpha'}\, \psi_{\alpha\beta\gamma}(x) = 0 \qquad (2.52)$$

which is the well-known Bargmann-Wigner equation[15] in configuration space. We see here that the Bargmann-Wigner equations are essentially the specifications of the quark and antiquark parities.

The representation $|\alpha\beta\gamma\rangle$ is reducible. It contains the irreducible representations which are specified by the Young tableaux, ,  and  i.e. fully symmetric, mixed symmetry and fully antisymmetric.

The restriction of the quark parity positive and anti-quark parity negative leads to $A_{\alpha\beta\gamma}$ describing + ve parity particles and $A_\beta^{\ \alpha}$ describing negative parity particles.

The quark parity boost for $A_\beta^{\ \alpha}(p)$ gives the Bargmann - Wigner[13] equations

$$(\not{p}-m)_\alpha^{\ \alpha'}\, A_{\alpha'}^{\ \beta}(p) = A_\alpha^{\ \beta'}(p)\,(\not{p}+m)_{\beta'}^{\ \beta} = 0. \qquad (2.53)$$

and in momentum space

$$(i\not{\partial}-m)_\alpha^{\ \alpha'}\, \phi_{\alpha'}^{\ \beta}(x) = \phi_\alpha^{\ \beta'}(x)\,(i\not{\partial}+m)_{\beta'}^{\ \beta} = 0. \qquad (2.54)$$

CHAPTER 3

REDUCTION OF U(2,2) to $\mathcal{L}_4$.

If we are to use the group U(2,2) as the auxiliary group we need to know the Lorentz group content of any U(2,2) representation. In the fundamental (quark) representation of U(2,2) an infinitesimal transformation of the 4-component spinor $\Lambda_\alpha (\alpha = 1,2,3,4)$ is given by

$$\delta\Lambda_\alpha = \frac{i}{2}\,\varepsilon_r(\Gamma_r)_\alpha{}^\beta\,\Lambda_\beta$$

$$= \frac{i}{2}\,(\varepsilon+\varepsilon_\mu\gamma_\mu+\tfrac{1}{2}\varepsilon_{\mu\nu}\sigma_{\mu\nu}+\varepsilon_5\gamma_5+i\varepsilon_{\mu 5}\gamma_\mu\gamma_5)_\alpha{}^\beta\Lambda_\beta. \qquad (3.1)$$

with real parameters ε_r. With

$$\Lambda^\beta = (\Lambda_\alpha)^+\,(\gamma_o)_\alpha{}^\beta \qquad (3.2)$$

this definition of the U(2,2) group has the property of leaving $\Lambda^\alpha\Lambda_\alpha$ invariant since

$$\delta\,\Lambda^\alpha = -\,i\,\frac{\varepsilon_r}{2}\,\Lambda^\beta(\Gamma_r)_\beta{}^\alpha \qquad (3.3)$$

because of (2.4)

All finite-dimensional, non-unitary representations can be obtained by constructing multi-spinors which transform as direct products of quarks and antiquarks, namely

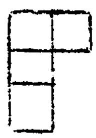
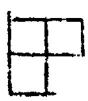
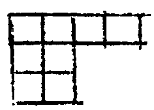
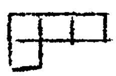
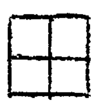
$$A_{\alpha\beta\ldots\ldots}^{\gamma\delta\ldots\ldots} \rightarrow S_\alpha^{\alpha'}\,S_\beta^{\beta'}\ldots(S^{-1})_{\gamma'}^{\gamma}\,(S^{-1})_{\delta'}^{\delta}\,A_{\alpha'\beta'\ldots\ldots}^{\gamma'\delta'\ldots\ldots}$$

(3.4)

where

$$S = \exp\,(i\varepsilon_r F_r) \tag{3.5}$$

The irreducible representations of SU(2,2) correspond to traceless tensors of well-defined symmetry characters. We give below a list of some low dimensional representations[16] where we have introduced brackets [ ] and { } to denote antisymmetry and symmetry in the enclosed indices.

Dimensionality		Young Tableaux
ψ_α	$\underline{4}$	
$\phi_{[\alpha\beta]}$	$\underline{6}$	
$\phi_{\{\alpha\beta\}}$	$\underline{10}$	
$\phi_\alpha{}^\beta$	$\underline{15}$	
$\psi_{\{\alpha\beta\gamma\}}$	$\underline{20}$	
$\psi_{[\alpha\beta]\gamma}$	$\underline{20}'$	
$\psi_{[\alpha\beta\gamma]}$	$\underline{4}^*$	
$\phi^{\{\alpha\beta\}}_{\{\gamma\delta\}}$	$\underline{84}$	
$\phi^{[\alpha\beta]}_{\{\gamma\delta\}}$	$\underline{45}$	
$\phi^{[\alpha\beta]}_{[\gamma\delta]}$	$\underline{20}''$	

We find the spin content of these representations by going down to the little group of the homogeneous Lorentz group. We first descend from U(2,2) to the Lorentz group by noting that the irreducible representations defined above are now in general reducible representations of the Lorentz group. We can now think of the indices α, β ... as Lorentz indices and the Young tableaux are still maintained. In terms of the transformations (3.1), we reduce from U(2,2) to $\mathcal{L}_4$ by putting

$$\varepsilon_5 = \varepsilon_\mu = \varepsilon_{\mu 5} = 0 \qquad (3.6)$$

We can now introduce a lowering (charge-conjugation) matrix $C_{\alpha\beta}$[16] within the Dirac algebra, such that

$$A_\alpha^{(C)}(p) \equiv C_{\alpha\beta} \; A^\beta(p) \qquad (3.7)$$

transforms like $A_\alpha(p)$ under the Poincare group. This requires that

$$C_{\alpha\beta}\, A^\beta \to C_{\alpha\beta}\, A^\gamma\, (S^{-1})_\gamma{}^\beta$$

$$= S_\alpha{}^\beta\, C_{\beta\gamma}\, A^\gamma \qquad (3.8)$$

where S is a Lorentz transformation. Thus

$$C_{\alpha\beta}\, (S^{-1})_\gamma{}^\beta = S_\alpha{}^\beta\, C_{\beta\gamma} \qquad (3.9)$$

In terms of the infinitesimal generators, this becomes

$$(C^{-1})^{\alpha\beta}\, (\sigma_{\mu\nu})_\beta{}^\gamma C_{\gamma\delta} = -(\sigma_{\mu\nu})_\delta{}^\alpha \qquad (3.10)$$

The matrix C has the properties

$$(C^{-1})^{\alpha\beta} C_{\beta\gamma} = \delta_{\gamma}^{\ \alpha} \tag{3.11}$$

$$C_{\alpha\beta} = - C_{\beta\alpha} \tag{3.12}$$

and

$$(C^{-1})^{\alpha\beta}(\gamma_\mu)_\beta{}^\gamma C_{\gamma\delta} = -(\gamma_\mu)_\delta{}^\alpha \tag{3.13}$$

from which it can be shown that $A_\alpha^{(c)}(p)$ has the same parity transformation properties as $A_\alpha(p)$, that is

$$(\not{p}-m)_\alpha{}^\beta A_\beta^{(c)}(p) = A_\alpha^{(c)}(p) \tag{3.14}$$

We relate the representation $A^\alpha(p)$ with $\bar{A}^\alpha(p)$ by noting that the anti-quark parity condition

$$< m,\bar{s}|\beta > (\gamma_o)_\beta{}^\alpha = - < m,\bar{s}|\alpha > \tag{3.15}$$

can be boosted to the equation

$$A^\alpha(p)\ (\not{p}+m)_\alpha{}^\beta = 0 \tag{3.16}$$

and the Dirac equation

$$(\not{p}-m)\ A_\alpha(p) = 0 \tag{3.17}$$

by Hermitian conjugation gives

$$\bar{A}^\alpha(p)\ (\not{p}-m)_\alpha{}^\beta = 0 \tag{3.18}$$

and therefore

$$\bar{A}^\alpha(-p) = (A_\alpha(p))^+ \gamma_o \equiv A^\alpha(p) \tag{3.19}$$

We then see that the bahaviour under translation of $\Lambda_\alpha^{(c)}(p)$ is the same as $\Lambda_\alpha(p)$, since $A(p)$ and $A^*(-p)$ behave similarly under translation. Of course, $\Lambda_\alpha^{(c)}(p)$ does not have the same transformation properties under the baryon gauge group.

We now get back to the problem of finding the spin content of any representation of U(2,2) which we have reduced to $\mathcal{L}_4$. All the upper indices can now be lowered by means of $C_{\alpha\beta}$

$$\phi^{\alpha \,\dots\, \beta}_{\gamma \,\dots\, \delta\varepsilon} = C_{\varepsilon\theta}\, \phi^{\alpha \,\dots\, \beta\theta}_{\gamma \,\dots\, \delta} \tag{3.21}$$

The resultant n-component spinor can then be reduced according to the standard theory of Lie groups by specifying the symmetry properties of the n indices by means of an n-rank Young tableaux. Such tableaux refer to four-component spinor labels, and thus may have up to four boxes in any column. The rest condition

$$(\gamma_o)^{\alpha}_{\alpha}\ \langle\alpha'\beta\gamma\dots|m,s\rangle = \langle\alpha\beta\gamma\dots|m,s\rangle \tag{3.22}$$

for each index, reduce the spinor labels effectively to two valued labels. The same Young tableaux, now interpreted as referring to these two-component spinors determine the representations of the little group $SU(2) \approx 0_3$, physically interpreted as the particle spin.

The spinor $\Lambda_{\{\alpha\beta\gamma\}}$ then becomes a 4-component object describing a spin $\frac{3}{2}$ particle. $\Lambda_{[\alpha\beta]\gamma}$ reduces from 20-component to a 2-component object and hence spin $\frac{1}{2}$. $\Lambda_{[\alpha\beta\gamma]}$ vanishes identically when reduced to SU(2). These states all have quark number three. For describing mesons we form the quark number zero representations $A_\beta^{\ \alpha}$, $A_\alpha^\alpha = 0$, from which one constructs

$$A_{\alpha\beta} = C_{\alpha\gamma} A_\beta^{\ \gamma} \tag{3.23}$$

Out of these we form the ten and five component objects $A_{\{\alpha\beta\}}$ and $A_{[\alpha\beta]}$. After reduction we find that these contain the spin 1 and spin 0 representations of the little group, respectively.

Since we shall be using these representations when we enlarge the auxiliary group to U(6,6), to include SU(3), we now explicitly exhibit their Lorentz structure[9].

(ii) Auxiliary operator of rank 2.

Write

$$A_\alpha^{\ \beta} = [\phi + \gamma_5\phi_5 + i\gamma_\mu\gamma_5\phi_{\mu 5} + \gamma_\mu\phi_\mu + \tfrac{1}{2}\sigma_{\mu\nu}\phi_{\mu\nu}]_\alpha^{\ \beta} \tag{3.24}$$

We have the two Bargmann-Wigner equations (boosts of the quark-antiquark parities),

$$(\not{p} - m)_\gamma^{\ \alpha} A_\alpha^{\ \beta}(p) = A_\alpha^{\ \beta}(p)\,(\not{p}+m)_\beta^{\ \gamma} = 0.$$

Using this on (3.24) we find

$$\left.\begin{aligned} &\phi = 0 \\ &p_\mu \phi_5 = i m \phi_{\mu 5}, \quad p_\mu \phi_{\mu 5} = -im\phi_5 \\ &p_\mu\phi_\nu - p_\nu\phi_\mu = im\phi_{\mu\nu}, p_\nu \phi_{\nu\mu} = -im\phi_\mu \end{aligned}\right\} \tag{3.25}$$

Thus $(\phi_5, \phi_{\mu 5})$ together form a 5-component object describing a pseudo-scalar particle [17] and $(\phi_\mu, \phi_{\mu\nu})$ a 10-component object describing a vector particle as was expected. We can use (3.25) then to write (3.24) as

$$A_\alpha^{\ \beta} = \frac{1}{m} \{(\not{p}+m)\gamma_5\phi_5 + (\not{p}+m)\gamma_\mu\phi_\mu\} \qquad (3.26)$$

using $p_\mu\phi_\mu = 0$ which can be derived from (3.25).

(iii) <u>Fully symmetric auxiliary operator of rank 3.</u>

To write out explicitly the $\mathcal{L}_4$ symmetry of any operator we use the fact that the 16 Dirac matrices $(\Gamma_r C)_{\alpha\beta}$ fall into two distinct classes; the matrices $(\gamma_\mu C)_{\alpha\beta}$ and $(\sigma_{\mu\nu}C)_{\alpha\beta}$ are symmetric, and $C_{\alpha\beta}$, $(\gamma_5 C)_{\alpha\beta}$, $(i\gamma_\mu\gamma_5 C)_{\alpha\beta}$ are antisymmetric.

Consider symmetry in α,β in the fully symmetric operator $A_{\alpha\beta\gamma}$, to write

$$A_{\alpha\beta\gamma} = (\gamma_\mu C)_{\alpha\beta}\, \psi_{\mu,\gamma} + \tfrac{1}{2}\, (\sigma_{\mu\nu}C)_{\alpha\beta}\, \psi_{\mu\nu,\gamma} \qquad (3.27)$$

To find full symmetry, we see that $A_{\alpha\beta\gamma}$ must be annihilated by the anti-symmetric tensors $(C^{-1})^{\beta\gamma}$, $(C^{-1}\gamma_5)^{\beta\gamma}$ and $(iC^{-1}\gamma_\mu\gamma_5)^{\beta\gamma}$. This gives three conditions

$$(\gamma_\mu)_\alpha^{\ \gamma}\, \psi_{\mu,\gamma} + \tfrac{1}{2}\, (\sigma_{\mu\nu})_\alpha^{\ \gamma}\psi_{\mu\nu,\gamma} = 0$$

$$(\gamma_\mu\gamma_5)_\alpha^{\ \gamma}\psi_{\mu,\gamma} + \tfrac{1}{2}\, (\sigma_{\mu\nu}\gamma_5)_\alpha^{\ \gamma}\psi_{\mu\nu,\gamma} = 0 \qquad (3.28)$$

$$(i\ \gamma_\mu\gamma_\lambda\gamma_5)_\alpha{}^\gamma\ \psi_{\mu,\gamma} + \tfrac{1}{2}\ (i\sigma_{\mu\nu}\ \gamma_\lambda\gamma_5)_\alpha{}^\gamma\ \psi_{\mu\nu,\gamma} = 0 \tag{3.28}$$

The first two equations give

$$\left.\begin{aligned} (\gamma_\mu)_\alpha{}^\gamma\psi_{\mu,\gamma} &= 0 \\ \text{and} \qquad (\sigma_{\mu\nu})_\alpha{}^\gamma\psi_{\mu\nu,\gamma} &= 0 \end{aligned}\right\} \tag{3.29}$$

The last equation of (3.28) gives, using (3.29),

$$(\gamma_\mu)_\alpha{}^\beta\psi_{\mu\lambda,\beta} + i\psi_{\lambda,\alpha} = 0\ . \tag{3.30}$$

As a result of (3.29) the 40-component object of (3.27) is now reduced to the expected 20-independent components.

We now use the Bargmann-Wigner equation

$$(\not{p}-m)_\alpha{}^{\alpha'}\ A_{\alpha'\beta\gamma} = 0 \tag{3.31}$$

and substitute (3.27) for $A_{\alpha\beta\gamma}$ and then contract with $(C^{-1}\gamma_\mu)^{\beta\gamma}$, $(C^{-1}\gamma_\mu)^{\alpha\beta}$ and $(C^{-1}\sigma_{\mu\nu})^{\alpha\beta}$ to find

$$(\not{p}-m)_\alpha{}^\beta\ \psi_{\mu,\beta} = 0 \tag{3.32}$$

$$p_\nu\psi_{\nu\mu,\alpha} + im\ \psi_{\mu,\alpha} = 0,\ p_\mu\psi_{\nu,\alpha} - p_\nu\ \psi_{\mu,\alpha} = im\ \psi_{\mu\nu,\alpha} \tag{3.33}$$

These equations along with (3.29) are equivalent to the Rarita-Schwinger[18] formalism for a particle of spin $\frac{3}{2}{}^+$.

Finally, using (3.33) we can write (3.27) as

$$\Lambda_{\alpha\beta\gamma}(p) = \frac{1}{m} [(\not{p}+m)\gamma_\mu C]_{\alpha\beta} \quad \psi_{\mu,\gamma}(p) \tag{3.34}$$

with conditions (3.29) and (3.30).

(iv). <u>Mixed auxiliary operator of rank 3.</u>$\Lambda_{[\alpha\beta]\gamma}$

$\Lambda_{[\alpha\beta]\gamma}$ has 24 components. We get the 20-independent components by stating that the fully antisymmetric part vanishes, by

$$\Lambda_{[\alpha\beta]\gamma}+\Lambda_{[\beta\gamma]\alpha} + \Lambda_{[\gamma\alpha]\beta} = 0 \tag{3.35}$$

This can then be written as

$$\Lambda_{[\alpha\beta]\gamma} = (\gamma_5 C)_{\alpha\beta} \; \psi_\gamma + i(\gamma_\mu\gamma_5 C)_{\alpha\beta} \; \psi_{\mu,\gamma} + C_{\alpha\beta} K_\gamma \tag{3.36}$$

Now, we use the equations

$$(\not{p}-m)_\alpha^{\;\alpha'} \Lambda_{[\alpha'\beta]\gamma} = 0 \tag{3.37}$$

$$(\not{p}-m)_\alpha^{\;\gamma'} \Lambda_{[\alpha\beta]\gamma'} = 0 \tag{3.38}$$

(3.38) gives

$$(\not{p}-m)_\gamma^{\;\gamma'} \psi_\gamma = (\not{p}-m)_\gamma^{\;\gamma'} \psi_{\mu,\gamma'} = (\not{p}-m)_\gamma^{\;\gamma'} K_{\gamma'} = 0 \tag{3.39}$$

and (3.37) gives

$$\left.\begin{aligned} K = 0, \quad & p_\mu \psi_{\nu,\alpha} - p_\nu \psi_{\mu,\alpha} = 0 \\ p_\mu \psi_\alpha = im\psi_{\mu,\alpha} \quad & , \; p_\mu \; \psi_{\mu,\alpha} = -im\psi_\alpha \end{aligned}\right\} \qquad (3.40)$$

This system clearly describes a particle of spin $\frac{1}{2}$ as was expected. (3.40) now allows us to write

$$\Lambda_{[\alpha\beta]\gamma} = \frac{1}{m} [(\not{p}+m)\gamma_5 C]_{\alpha\beta} \; \psi_\gamma(p) \quad . \qquad (3.41)$$

(v) We have uptil now considered in detail the use of U(2,2) as the auxiliary group. The problem of including the internal symmetry group is fairly easy. The rest states are now defined as $|m,s,\mu,I^2;s_3,Y,I_3\;>$ where μ defines the SU(3) representation, I is the total isospin, Y and I_3 are the hypercharge and the third component of isospin respectively.

If we want to have Poincare and SU(3) invariance for our theory, it follows that the smallest auxiliary group we need is $\mathcal{L}_4 \otimes$ SU(3) and the auxiliary operator in the fundamental representation would be

$$\Lambda_{\alpha,p} \qquad \alpha = 1,2,3,4, \quad p = 1,2,3.$$

The next step is to extend $\mathcal{L}_4$ to U(2,2) to give the auxiliary group U(2,2) $\otimes$ SU(3) where again the fundamental auxiliary operator is

$$\Lambda_{\alpha,p} \qquad \alpha = 1,2,3,4, \quad p = 1,2,3$$

but now the transformation on α are of the full U(2,2) group rather than its subgroup $\mathcal{L}_4$. The theory of Salam et. al. is obtained by embedding U(2,2) $\otimes$ SU(3) into the group U(6,6) which has the generators

$$F_a = \sqrt{6}\Gamma_r T_i \qquad \begin{array}{l} r = 0, \ldots\ldots 15 \\ i = 0, \ldots\ldots 8 \\ a = 1 \ \ldots\ldots 144 \end{array} \tag{3.42}$$

where Γ_r are as defined by (2.2) and T_i are the SU(3) generators (See appendix). The basic (quark) auxiliary operator is now (12-dimensional)

$$A_\Lambda = A_{\alpha p} \ (\Lambda = 1, \ldots\ldots\ldots\ldots 12) \text{ which has the general } U(6,6), \cdots$$

transformation

$$A \rightarrow SA$$

with

$$S = \exp\left[\tfrac{1}{2}\, i \sum_a F_a \varepsilon^a\right] \tag{3.43}$$

and real parameters ε^a.

We now proceed exactly as we did for U(2,2) and construct the dual (anti-quark) representation and we can then go on to show the causality condition is satisfied as before. Here, the parity operator in the basic representation is

$$< A|R|B> = <\alpha, p|R|\beta, q> = (\gamma_o)_\alpha{}^\beta \ \ \delta_p{}^q \ . \tag{3.44}$$

We arrive at the well known Bargmann Wigner equations when we specify the parity of the quark as positive and the anti-quark as negative. The higher representations are constructed as before. The equations of motion for an arbitrary representation are

$$(\not{p}-m)_{\alpha}^{\ \alpha'} \Lambda_{\alpha' p, \beta q, \ldots\ldots}^{\delta r, \varepsilon s \ldots\ldots} = \Lambda_{\alpha p, \beta q \ldots\ldots}^{\delta' r, \varepsilon s, \ldots\ldots} (\not{p}+m)_{\delta'}^{\ \delta} = 0 \tag{3,45}$$

Salam[9] et. al. found that some of the well known meson and meson resonances and baryons and baryon resonances are combined very neatly in only two low-dimensionsal representations of U(6,6).

(α) <u>2nd rank Auxiliary operator Φ_B^A</u> <u>(traceless)</u>.

This is the <u>143</u> dimensional regular representation. Its $U(2,2) \otimes SU(3)$ structure can be determined by considering it to be made of a product of a quark and an anti-quark representation. Thus

$$\underline{12} \otimes \underline{12}^* = \underline{143} \oplus \underline{1} \, .$$

The <u>12</u> decomposes to (4,3) where the first number refers to U(2,2) and the second to SU(3). We then find that the structure of <u>143</u> is

$$\underline{143} = (15,8) + (15,1) + (1,8).$$

We have seen that 15 reduces to 10+5, on stepping down to $\mathcal{L}_4$, which then describe 1^- and 0^- particles. <u>143</u> then describes a nonet of vector and pseudoscalar particles. The U(2,2) singlet vanishes identically on reduction to $\mathcal{L}_4$.

Thus, Φ_B^A (p) can be written as

$$\Phi_B^A (p) = \frac{1}{m} \left\{ (\not{p}+m)\ {}_5\phi_5^i(p) + (\not{p}+m)\gamma_\mu \phi_\mu^i(p) \right\}_\beta^\alpha (T^i)_q^p \tag{3.46}$$

(β) The baryons are constructed from three quark states as we assign the quark the baryon number $B = \frac{1}{3}$. The structure of the fully symmetric third rank representation is

$$\underline{364} = (20,10) + (20',8) + (4,1)$$

where 20 and 20' are the fully symmetric and mixed symmetry third rank tensors of U(2,2). We have shown that these describe $\frac{3}{2}^+$ and $\frac{1}{2}^+$ particle respectively. The fully anti-symmetric (4) third rank representation of U(2,2) vanishes identically on reduction to $\mathcal{L}_4$. Thus the $\underline{364}$ describes an octet of $\frac{1}{2}^+$ particles and a decuplet of $\frac{3}{2}^+$ particles. We can assign the eight baryons and the well-known ten baryon resonances of $\frac{3}{2}^+$ to this representation with remarkable neatness.

Finally, $B_{\{ABC\}}$ can be written as

$$B_{\{ABC\}}(p) = B_{\alpha\beta\gamma,pqr}(p)$$

$$= \frac{1}{m} [(\not{p}+m)\gamma_\mu C]_{\alpha\beta} D_{\mu,\gamma,pqr}(p) +$$

$$+ \frac{1}{2\sqrt{6}} (\varepsilon_{pqs} N_{[\alpha\beta]\gamma;r}^{\ \ s}(p) + \varepsilon_{qrs} N_{[\beta\gamma]\alpha,p}^{\ \ s}(p) + \varepsilon_{rps} N_{[\gamma\alpha],q}^{\ \ s}(p)$$

(3.47)

where $N_{[\alpha\beta]\gamma,\ r}^{\ \ \ \ s}(p) = \frac{1}{m} [(\not{p}+m)\gamma_5 C]_{\alpha\beta} \ N_{\gamma,\ r}^{\ \ s}(p)$

$$\text{and} (\not{p}-m)_\alpha^{\ \alpha'} \ N_{\alpha',\ r}^{\ \ s}(p) = 0. \qquad (3.48)$$

$$\text{and } (\not{p}-m)_\alpha^{\ \alpha'} D_{\mu,\alpha',pqr}(p) = 0, \ (\gamma_\mu)_\alpha^{\ \alpha'} D_{\mu,\alpha',pqr}(p) = 0$$

(3.49)

and $D_{\mu,\gamma,pqr}(p)$ is fully symmetric in pqr.

We have now constructed all the U(6,6) tensors which we need for our calculations.

CHAPTER 4

INHOMOGENEOUS U(2,2), INHOMOGENEOUS U(6,6) and $SU(6)_W$.

We have now constructed causal field operators which transform like finite, irreducible non-unitary representations of a non-compact auxiliary group. This allows us, in a very natural manner, to combine different spin and SU(3) multiplets in one field. We now consider the use of these fields.

Since SU(3) only introduces non-essential complications, we shall restrict the discussion to the enlargement of the space-time symmetry. The most natural requirement for theories constructed from fields like $\psi_{\{\alpha\beta\gamma\}}$ and $\phi_{\alpha}^{\cdot\beta}$ is that the effective interaction Lagrangian, or S-matrix elements, are to be invariant under the index transformation of U(2,2), namely (2.7) and (2.9).

$$\Lambda_\alpha(p) \quad \rightarrow T_\alpha^{\ \beta}\Lambda_\beta(p)$$

and

$$\Lambda^\alpha(p) \quad \rightarrow \Lambda^\beta(p)(T^{-1})_\beta^{\ \alpha}$$

Such invariants are very easily formed by taking trace products like

$$A^\alpha(p) \quad \Lambda_\beta(q) \ .$$

This restriction to index U(2,2) invariances, leads to a restricted subclass of Poincare invariants. This leads to terms like

$$\Lambda^{\alpha}(p_1)\ (\gamma_{\mu})_{\alpha}^{\ \beta}\Lambda_{\beta}\ (p_2)q_{\mu}$$

which is Poincare invariant, being excluded. Also, terms like

$$\Lambda^{\alpha}(\gamma_{\mu})_{\alpha}^{\ \beta}\Lambda_{\beta}\ B^{\delta}(\gamma^{\mu})_{\delta}^{\ \varepsilon}B_{\varepsilon}$$

which are $\mathcal{L}_4$ invariant are excluded by index U(2,2) invariance which requires the complete combination

$$\Lambda^{\alpha}(\Gamma^{r})_{\alpha}^{\ \beta}\Lambda_{\beta}\ B^{\delta}\ (\Gamma_{r})_{\delta}^{\ \varepsilon}B_{\varepsilon} \quad .$$

This procedure therefore excludes the well-known derivative couplings which arise naturally in a Poincare invariant theory.

However, even if the interaction Lagrangian is restricted in this way, as a prescription it is not possible to construct a consistent theory which leads in general to S-matrix elements which show this index U(2,2) invariance. By a theory we mean the specification of an interaction Lagrangian and free-particle propagators from which S-matrix elements can be calculated to all orders in the coupling strength. Even if the calculation is not possible, it should be possible at least to determine the invariance properties of the S-matrix elements, from the given information. Any theory based on the U(2,2) local fields leads to S-matrix elements which involve not only the interaction Lagrangian but also propagators and summation over spins such as (2.35) and (2.46). These introduce factors of the form $p_{\mu}\gamma^{\mu}$ or $\gamma_{\mu}\ \ldots\gamma^{\mu}$ which destroy the generalised invariance, which has been built into the interaction, and give rise to an S-matrix element which is invariant only under the Poincare group. If the auxiliary group is extended to include SU(3), as in SL(6,C) or U(6,6), the remaining symmetry is just SU(3) $\otimes$ P. We can also see this in a different light by considering the unitarity condition

$$\text{Im } T = T_{\rho} T^{+}$$

The factor ρ also involves sums over spins and even if T and T^{+} are assumed to be U(2,2) invariant, this factor introduces expressions into ImT which contradict this assumption.

We have seen that index invariance leads to a subclass of Poincare invariants. Any interaction which is index U(2,2) invariant is automatically Poincare invariant but not vice-versa. This is because we have specified invariance under index transformations rather than true transformations. We can have true transfomations by specifying a larger space-time than the normal four dimensions. In fact, we require invariance of our theory under the inhomogeneous U(2,2), IU(2,2), rather than the Poincare group.

(ii) <u>Inhomogeneous U(2,2).</u>

We extend the Poincare group to inhomogeneous U(2,2)[19] by forming the semi-direct product of U(2,2) with the 16-parameter Abelian group, the infinitesimal generators P_r of which satisfy the commutation relations

$$[P_r, P_s] = 0$$

$$[F_r, P_s] = i f_{rs}{}^q P_q$$

where $f_{rs}{}^q$ are the structure constants of the U(2,2) algebra (2.2). This last relation states that P_r transforms as a 16- vector under U(2,2). The operations generated by the P_r relate to 'translations' in a generalized 16-dimensional space, and include the space-time displacements of the Poincare group as a sub-group. In other words, the P_r are generalised energy-momentum operators and include the physical energy-momentum operators P_μ. We can combine these with Γ_r to form a U(2,2) spinor operator

$$\hat{p} \equiv \Gamma^r P_r$$

The particle operators a(p,s) now depend on 16p's and the 's' labels the representations of the little group of U(2,2) which we now show to be U(2) ⊗ U(2)[19,20]. As before the little group is defined to leave P_r invariant. We make contact with the physical world by restricting the P_r to p_μ i.e. to the physical four momenta. As mentioned in Chapter 1 the little group is the same for all points on an orbit, and it is convenient to consider the special point $p = (m, \underline{o})$. Thus the little group can be determined by considering those transformations which leave invariant the form

$$m\, a^{+}(o,s)\; a(o,s) = m(\Lambda_\alpha(o))^{+}\langle\alpha|m,s\rangle\,\langle m,s|\beta\rangle\Lambda_\beta(o)$$

$$= m(\Lambda_\alpha(o))^{+} 0_\alpha{}^{\beta}\Lambda_\beta(o)$$

which is just the rest frame projection of the free particle energy-momentum operator. The auxiliary group which we use is U(2,2).

So we need those elements Γ of the **generators of** U(2,2) which commute with $0_\alpha{}^\beta$

$$(\Gamma^{+}\; 0\Gamma)_\alpha{}^\beta = 0_\alpha{}^\beta$$

and, since we are assuming antiparticles **we** must also have

$$\Gamma^{+}\,\tilde{0}\,\Gamma = \tilde{0}.$$

The matrix $0_\alpha{}^\beta$ always contains the unit matrix, which implies that

$$\Gamma = \Gamma^{+}$$

If the restrictions on $\langle\alpha|m,s\rangle$ are always of the parity type considered in Chapter 2, $0_\beta{}^\alpha$ only involves γ_o and then the little group generators are

$$1, \gamma_o, \sigma_i, \gamma_5\gamma_i \sim \gamma_o\sigma_i$$

which generate U(2) $\otimes$ U(2). We get this result also when we use the auxiliary fields $\Lambda_{\alpha\beta\gamma}$ and $\Lambda_\beta{}^\alpha$ as the 0 operators only involve γ_o as long as the only restrictions on $\langle\alpha\beta\gamma|m,s\rangle$ and $\langle\beta^\alpha|m,s\rangle$ are of the parity type.

Our auxiliary operators are now of the form

$$\Lambda_\alpha(p) = \langle\alpha|e^{-i\varepsilon_r G^r}|\beta\rangle\langle\beta|m,s\rangle\, a(p,s)$$

where G^r are the eight generators $\frac{1}{2}\,(\gamma_i,\sigma_{oi},\gamma_5,\ i\gamma_o\gamma_5)$[19], to be contrasted with (2.1) where only the $\underline{K}$ part of G^r are included. Here, the p runs over the sixteen values of the regular representation.

Proceeding as before we find under IU(2,2) transformations

$$\Lambda_\alpha(p) \rightarrow \langle\alpha|e^{i\eta_r F^r}|\beta\rangle\,\Lambda_\beta(p')$$

where $\delta p_r = \frac{i}{2}\,\varepsilon^r[F_r,p_s]$. This transformation is now a 'true' transformation under U(2,2). We can construct higher fields, as before, by considering the product of quark and antiquark fields. The spin content of these fields is the same as when U(2,2) was considered the auxiliary group for Poincare invariance. But we have acquired an additional freedom in constructing invariants in as much that we can write down quantities like

$$\Lambda^\beta(p)\ (\Gamma^r)^\alpha_\beta\,\Lambda_\alpha(p')\ q_r$$

and $\Lambda^\beta(p_1)\ (\Gamma^r)_\beta{}^\alpha\Lambda_\alpha(p_2)\quad B^\delta(p_3)\ (\Gamma_r)_\delta{}^\varepsilon B_\varepsilon(p_4)$

apart from index invariant quantities like

$$\Lambda^{\alpha}(p_1)\ \Lambda_{\beta}(p_2)\ .$$

As before, we have physics when we restrict the 16-dimensional space-time to the physical four dimensional world.

We have seen that if our theory is IU(2,2) invariant then for one particle states the rest symmetry is $U(2) \otimes U(2)$ i.e. the little group. We now show the hierarchy of symmetries as we proceed from one particle state to many particle states[19,21]:

(iii) For collinear processes like vertex functions and annihilation at rest, specialising to the z-direction we find that there are only two independent moment $p_o\gamma_o$ and $p_3\gamma_3$. Therefore, to see what invariance is left we look for those generators of U(2,2) which commute with γ_o and γ_3. These are the 3 hermitian generators

$$\tfrac{1}{2}\, i\gamma_1\gamma_5\ ,\ \tfrac{1}{2}\, i\gamma_2\gamma_5 \text{ and } \tfrac{1}{2}\, \sigma_{12}\ .$$

It is easy to show that these are the generators for an SU(2) group and this subgroup of U(2,2) is usually written as $SU(2)_W$. These generators can alternatively be written as

$$\tfrac{1}{2}\gamma_o\sigma_1,\ \tfrac{1}{2}\gamma_o\sigma_2,\ \tfrac{1}{2}\sigma_3$$

where $\underline{\sigma} = (\sigma_{23},\sigma_{31},\sigma_{12})$.

We have here introduced a modified conserved spin which can be used to classify not only the states of the particles at rest, but also states of finite momentum in the z-direction. These operators commute with Lorentz transformations in the z-direction. Thus the w-spin classification for a particle state with a finite momentum in the z-direction is the same as that for the corresponding state at rest. We can now classify all particles in $SU(2)_W$ or $SU(6)_W$ when we include the internal symmetry group SU(3). So whenever we are considering processes for which all momenta are in a single direction in some Lorentz frame, we can equivalently do either $SU(6)_W$ Clebsch-Gordan tricks or inhomogeneous U(6,6).

Proceeding a stage further if we consider co-planar processes, e.g., general two particle scattering amplitudes, we find that the residual symmetry is then just $U(1) \otimes U(1)$ which generalises to $U(3) \otimes U(3)$ on inclusion of internal symmetry. For any more complicated situation only U(3) survives.

(iv) Extension to Inhomogeneous U(6,6)

As in the case of IU(2,2) we take the semi-direct product of U(6,6) with a 143-dimensional abelian group T_{143}. We have thus introduced 143 energy-momentum operators P_a which transform like the regular representation of SU(6,6). i.e.

$$[P_a, P_b] = 0 \qquad a,b,c = 1.......143$$

$$[J_a, P_b] = if_{ab}{}^{c}P_c$$

where the structure constants $f_{ab}{}^{c}$ of U(6,6) are defined by (appendix)

$$[J_a, J_b] = if_{ab}{}^{c}J_c.$$

As a reminder, in the fundamental representation the J's are given by

$$F_a = \sqrt{6}\,(\Gamma_r T^i)_B^A \qquad \begin{matrix} A= 1.....12 \\ B= 1.....12 \end{matrix}$$

We can write the P_a's in an equivalent, alternative representation as

$$\tilde{P}_B{}^A = (P_a F_a)_B{}^A$$

and similarly

$$J_B{}^A = (J_a F_a)_B{}^A .$$

We also have the relation

$$\tilde{P}_A{}^A = 0 , \qquad J_A{}^A = 0$$

We use U(6,6) as the auxiliary group in constructing local fields for IU(6,6) and we find also in this case that the little group is U(6) ⊗ U(6) if the restrictions on the constant spinors $\langle A|m,s\rangle$ are of the usual γ_o-parity variety. So we can set up the theory as outlined earlier but in this case we have a wider range of invariants including the so-called irregular couplings. We make contact with physics by restricting the P_a to the physical four energy-momentum operators P_μ. We therefore have essentially the same theory as U(6,6) but the difference lies in the fact that IU(6,6), plus restrictions to real space-time, leads to the possibility of the irregular or derivative couplings in the interaction Lagrangian which is less restrictive than pure U(6,6) index invariance. As we have shown in the previous section this leads to $SU(6)_W$ invariance for collinear processes.

The practical procedure for applying inhomogeneous U(6,6) is to construct all possible invariants from the particle fields $\Lambda_A(p)$ and the different momentum operators appearing in the process and then specialise by taking the physical four-momenta.

CHAPTER 5

SOME CALCULATIONS IN IU(6,6)

(i) Before we go on to evaluate certain scattering processes we look at charge conjugation more explicitly. We assume that strong interactions are invariant under charge conjugation or particle-antiparticle interchange. When we are working with the Lorentz group and the Dirac representation we denote the auxiliary field by $\psi_\alpha(p)$ instead of $\Lambda_\alpha(p)$. In Chapter 3 we showed that it was possible to introduce a lowering (charge-conjugation) matrix $C_{\alpha\beta}$ within the Dirac algebra, such that

$$\psi_\alpha^{(c)}(p) \equiv C_{\alpha\beta}\psi^\beta(p) \equiv C_{\alpha\beta}\,\bar{\psi}^\beta(-p)$$

transforms like $\psi_\alpha(p)$ under the Poincare group. However, it is well known that within the group U(2,2) it is not possible to introduce a lowering matrix $C_{\alpha\beta}$ such that

$$\Lambda_\alpha^{(c)} \equiv C_{\alpha\beta}\bar{\Lambda}^\beta$$

transforms like Λ_α under this group[19]. This is a consequence of the fact that the transformation properties of the C matrix are not indicated by the two lower indices, i.e.

$$C^{-1}\Gamma_r\, C \neq -\,\Gamma_r^{\,T} \qquad\qquad r = 0\ldots\ldots\ldots 15$$

However since we are finally only concerned with the Lorentz group we define the charge conjugate field in U(2,2) by

$$\Lambda_\alpha^{(c)}(p) \equiv C_{\alpha\beta}\Lambda^\beta(p) \equiv C_{\alpha\beta}\bar{\Lambda}^\beta(-p) \qquad\qquad (5.1)$$

and $$\bar{\Lambda}^{\alpha}(p) \rightarrow \bar{\Lambda}^{\alpha}(p)^{c} = -\Lambda_{\beta}(-p)(C^{-1})^{\beta\alpha} \quad . \qquad (5.2)$$

Consistent with the Dirac (Bargmann-Wigner) equations, the baryons and mesons can be considered in momentum space to be the following combination of quark fields

$$\Lambda_{\alpha\beta\gamma}(p) = \Lambda_{\alpha}(p)\, \Lambda_{\beta}(p)\, \Lambda_{\gamma}(p)$$

and $$\Lambda_{\beta}{}^{\alpha}(p) = \bar{\Lambda}^{\alpha}(-p)\, \Lambda_{\beta}(p)$$

with the appropriate symmetrization implied. Then the charge conjugation properties follow immediately

$$\Lambda_{\alpha\beta\gamma}(p) \rightarrow C_{\alpha\delta}\; C_{\beta\varepsilon}\; C_{\gamma\lambda}\; \bar{\Lambda}^{\delta\varepsilon\lambda}\,(-p)$$

and $$\Lambda_{\beta}{}^{\alpha}(q) \rightarrow (C^{-1})^{\alpha\delta}\, \Lambda_{\delta}{}^{\varepsilon}(q)\; C_{\varepsilon\beta} \qquad (5.3)$$

This choice of meson transformation is necessary to give the conventional C-parity of the π^{0} and ρ^{0} mesons.

This definition of charge conjugation would seem to indicate that a quantity like

$$(\bar{\Lambda}^{\alpha})^{c}\; (\Lambda_{\beta})^{c}\; (\Lambda_{\alpha}{}^{\beta})^{c} \qquad (c = \text{charge-conjugate})$$

is not a U(2,2) invariant. But if we substitute

$$(\Lambda_{\alpha})^{c} = C_{\alpha\beta}\bar{\Lambda}^{\beta}$$

$$(\bar{\Lambda}^{\alpha})^{c} = (C^{-1})^{\alpha\beta}\Lambda_{\beta}$$

$$(\Lambda_{\alpha}{}^{\beta})^{c} = C_{\alpha\delta}\; \Lambda_{\varepsilon}{}^{\delta}\; (C^{-1})^{\beta\varepsilon}$$

we get $\bar{\Lambda}^{\alpha}\Lambda_{\beta}\Lambda_{\alpha}{}^{\beta}$ which is a U(2,2) invariant. However, a quantity like

$\Lambda^{\alpha}C_{\alpha\beta}\Lambda^{\beta}$ is not U(2,2) invariant, but is Lorentz invariant.

We construct a charge-conjugation operator in U(6,6) by generalising C such that under charge conjugation

$$\bar{\psi}^{A}(p) \rightarrow (\bar{\psi}^{A}(p))^{c} = (C^{-1})^{AB}\ \psi_{B}(-p)$$

$$\psi_{A}(p) \rightarrow (\psi_{A}(p))^{c} = C_{AB}\ \bar{\psi}^{B}(-p) \qquad (5.4)$$

(For a fuller discussion see P. Rotelli's thesis).
where our generalised C's are antisymmetric in their U(6,6) indices A,B.

The properties of these C's are

$$C_{AB}\ (C^{-1})^{BD} = \delta_{A}{}^{D}$$

$$(C^{-1})^{AB}\ (\gamma_{\mu}\lambda^{i})_{B}{}^{C}(C)_{CD} = -\ (\gamma_{\mu}\lambda_{i}{}^{T})_{D}{}^{A} \qquad (5.5)$$

for $\mu = 0,\ldots 3$ and $i = 0\ldots.8$.
(Under charge conjugation the SU(3) representation is changed into its adjoint representation). The charge conjugation properties of the baryon and meson fields are now

$$\Psi_{ABC}(p) \rightarrow C_{AD}C_{BE}C_{CF}\,\Psi^{DEF}(-p)$$

and $\Phi_B{}^A(q) \rightarrow (C^{-1})^{AD}\,\bar{\Phi}_D{}^E(q)\,C_{EB}$. (5.6)

We are now ready to test any IU(2,2) invariant to see whether it is charge-conjugation invariant. We have to be careful to symmetrize and antisymmetrize any Bose and Fermi fields in the amplitudes before applying charge conjugation.

(ii) <u>Three meson vertex MMM.</u>

The reduction of $\underline{143} \otimes \underline{143}$ is

$$\underline{143} \times \underline{143} = 1 \oplus \underline{143}_D \oplus \underline{143}_F \oplus \underline{4212} \oplus \underline{5005} \oplus \underline{5005}^* \oplus \underline{5940}$$

So in general we would expect to get two 'regular' couplings

$$g[\Phi_A{}^B(p_1)\,\Phi_B^C(p_2) \pm \Phi_A^B(p_2)\Phi_B^C(p_1)]\,\Phi_C^A\,(p_3) \qquad (5.7)$$

with

$$p_1 + p_2 + p_3 = 0 \qquad (5.8)$$

and all the mesons being on the mass shell. We now apply the charge conjugation operator developed in the first section of this chapter. Consider the first term

$$\Phi_A^B\,(p_1)\,\Phi_B^C\,(p_2)\,\Phi_C^A(p_3).$$

Under charge conjugation this transforms to

$$(C^{-1})^{BE}\Phi_E{}^F(p_1)\ C_{FA}(C^{-1})^{CG}\Phi_G{}^H(p_2)C_{HB}(C^{-1})^{AI}\Phi_I{}^J(p_3)C_{JC}$$

$$= \delta^E_H\ \delta^I_F\ \delta^G_J\ \Phi_E{}^F(p_1)\ \Phi_G{}^H(p_2)\Phi_I{}^J(p_3)$$

$$= \Phi^E_G(p_2)\Phi^F_E(p_1)\Phi^G_F(p_3)\ .$$

So we see that under charge-conjugation the first term is transformed into the second term and vice-versa. Therefore, the positive sign gives a C conserving amplitude while the negative sign gives a C violating amplitude. Thus, since we assume that strong interactions are C-invariant, we get a unique amplitude

$$g[\Phi_A{}^B(p_1)\Phi_B{}^C(p_2)\ +\Phi_A{}^B(p_2)\Phi_B{}^C(p_1)]\Phi_C{}^A(p_3). \qquad (5.9)$$

What about 'irregular' amplitudes? For this we are entitled to use the momentum operators $(\not{p}_1)^A_B$, $(\not{p}_2)^A_B$, $(\not{p}_3)^A_B$ defined as

$$(\not{p}_1)_B{}^A = p_{1\mu}\ (\gamma_\mu)_\beta{}^\alpha\delta_s{}^r .$$

Remembering the condition (5.8) and the fact that all the mesons are on the mass shell i.e.

$$(\not{p}_1)^A_B\ \Phi_A{}^C(p_1)\ \ =\ \ m\ \Phi_B{}^C(p_1)\ \text{etc.},$$

we find that the only irreducible invariants are of the form

$$g'\Phi_B{}^A(p_1)\Phi_A{}^B(p_2)\ (\not{p}_1-\not{p}_2)^D_E\ \Phi_D{}^E(p_3)\ \text{and}\ 1,2,3 \qquad (5.10)$$

permutations. However, these all violate C-parity as is easily seen by applying the C operator (4.10) then transforms to

$$g'(C^{-1})^{AF}\Phi_F{}^G(p_1)\ C_{GB}(C^{-1})^{BH}\Phi_H{}^I(p_2)\ C_{IA}$$

$$(\not p_1 - \not p_2)_E{}^D\ (C^{-1})^{EJ}\Phi_J{}^K(p_3)\ C_{KD}$$

$$= -g'\ \Phi_B{}^A(p_1)\Phi_A{}^B(p_2)\ (\not p_1 - \not p_2)_E{}^D\Phi_D{}^E(p_3)$$

using (5.5). We see also explicitly that this amplitude is C-violating by observing that it gives rise to a $\emptyset^o\pi^o\pi^o$ coupling which is obviously C-violating. ($\emptyset^o$ is the vector singlet).

We now evaluate the unique coupling (5.10). Dropping indices this can be written as

$$g\ \mathrm{Tr}\ \{(\Phi(p_1)\Phi(p_2)\ +\ \Phi(p_2)\Phi(p_1))\Phi(p_3)\ \}$$

Now we substitute for the Φ's from (3.46) to give

$$\frac{g}{\mu^3}\ \mathrm{Tr}\Big\{\Big[\{(\not p_1+\mu)\gamma_5\emptyset_5{}^i(p_1)+(\not p_1+\mu)\gamma_\mu\emptyset^i_\mu\ (p_1)\}$$

$$\{(\not p_2+\mu)\gamma_5\emptyset^j_5(p_2)+(\not p_2+\mu)\gamma_\nu\emptyset^j_\nu\ (p_2)\}\ T^iT^j$$

$$+\{(\not p_2+\mu)\gamma_5\emptyset^j_5(p_2)+(\not p_2+\mu)\gamma_\nu\emptyset^j_\nu(p_2)\}\{(\not p_1+\mu)\gamma_5\emptyset^i_5(p_1)$$

$$+ (\not{p}_1+\mu)\gamma_\mu \phi^i_\mu(p_1)\} \, T^j T^i][\{(\not{p}_3+\mu)\gamma_5 \phi_5{}^k(p_3)$$

$$+ (\not{p}_3+\mu)\gamma_\kappa \phi^k_\kappa(p_3)\}T^k] \Big\}$$

This was first evaluated by Delbourgo et.al and Sakita and Wali[9]. This gives rise to the F-type coupling of VVV, VPP and the D-type coupling of VVP as expected from charge conjugation[22]. There are very few new predictions. One is the absence of the $\varphi\to\rho\pi$ mode[9,23] because $g_{\varphi\rho\pi} = 0$ if we make the identification of the physical φ and ω.

$$\omega_\mu = \frac{\sqrt{2}}{\sqrt{3}}\,\phi_\mu{}^o + \frac{1}{\sqrt{3}}\,\phi^8_\mu$$

$$\varphi_\mu = \frac{1}{\sqrt{3}}\,\phi_\mu{}^o - \sqrt{\frac{2}{3}}\,\phi_\mu{}^8. \qquad (5.11)$$

Another prediction is the ratio $g_{\rho\omega\pi/\rho\pi\pi} = 2/\mu$ first stated by Sakita and Wali. With $\mu \approx 700$ MeV this compares favourably with the ratio $g_{\rho\omega\pi}/g_{\rho\pi\pi} \approx 2.4/\mu$ obtained by Gell-Mann, Sharp and Wagner[24] for ω decay.

(iii) <u>Proton-antiproton annihilation at rest into two mesons.</u>

We shall as before take equal mass for the two mesons irrespective of whether they are both the same spin. We denote the mass of the proton and antiproton by m and the mass of the mesons by μ. The various momenta are

$$
\begin{aligned}
p &= (m,o) = \text{momentum of proton} \\
& = \text{momentum of antiproton} \\
q_1 &= (\varepsilon,\underline{q}_1) = \text{momentum of meson} \\
q_2 &= (\varepsilon,-\underline{q}_1) = \text{momentum of other meson.}
\end{aligned}
\tag{5.12}
$$

We are in the laboratory frame where the baryons are really at rest. We have the relation

$$
q_1 + q_2 = 2p \quad \text{i.e.} \quad \varepsilon = m. \tag{5.13}
$$

We define the independent momenta q (p being the other) as

$$
q = q_1 - q_2 = (0,2\underline{q}_1) = (0,\underline{q}) \tag{5.14}
$$

First the regular amplitudes. Using the reductions

$$
\underline{143} \otimes \underline{143} = 1 \oplus \underline{143}_F \oplus \underline{143}_D \oplus \underline{5940} \oplus \underline{4212} \oplus \underline{5005} \oplus \underline{5005}^*
$$

and $\underline{364} \otimes \underline{364}^* = 1 \oplus \underline{143} \oplus \underline{5940} \oplus \underline{126412}$

we see that there are four U(6,6) invariant amplitudes denoted by $\underline{1}$, $\underline{143}_D$, $\underline{143}_F$ and $\underline{5940}$. Using the U(6,6) baryon and meson tensors constructed previously, these couplings can be written, respectively, as :

$$\bar{\Psi}^{ABC}(-p)\ \ \Psi_{ABC}(p)\ \ \Phi_E{}^D(q_1)\ \ \Phi_D{}^E(q_2)$$

$$\bar{\Psi}^{ABC}(-p)\ \ \Psi_{ABD}(p)\left\{\Phi_C{}^E(q_1)\ \ \Phi_E{}^D(q_2)\ \pm\ \Phi_E{}^D(q_1)\ \ \Phi_C{}^E(q_2)\right\}$$

$$\bar{\Psi}^{ABC}(-p)\ \Psi_{ADE}(p)\ \ \Phi_B{}^D(q_1)\ \ \Phi_C{}^E(q_2) \tag{5.15}$$

where the $\pm$ signs indicate the $\underline{143}$ D and F couplings respectively. We use $\bar{\Psi}(-p)$ as this indicates an incoming anti-particle with momentum p.

All these four-particle vertices are found to vanish because

$$\bar{\Psi}^{ABC}(p)\ \Psi_{ADE}(p) = 0 \qquad \text{when } p = (m,0).$$

i.e. whenever there is at least one direct summation the vertices vanish for the nucleon part. Picking up just the spin $\frac{1}{2}$ octet part of the $\underline{364}$ tensor, we have

$$\bar{\Psi}^{ABC}(-p)\ \Psi_{ADE}\ (p)$$

$$= \frac{1}{24}\left\{\varepsilon^{pqs}\ \bar{N}^{[\alpha\beta]\gamma}(-p)^{,r}{}_{s} + \varepsilon^{qrs}\ \bar{N}^{[\beta\gamma]\alpha}(-p)^{,p}{}_{s} + \varepsilon^{rps}\ \bar{N}^{[\gamma\alpha]\beta}(-p)^{,q}{}_{s}\right\}$$

$$\times\ \left\{\varepsilon_{ptv}\ N_{[\alpha\delta]\varepsilon,u}(p)^{v} + \varepsilon_{tuv}\ N_{[\delta\varepsilon]\alpha,p}(p)^{v} + \varepsilon_{upv}\ N_{[\varepsilon\alpha]\delta,t}(p)^{v}\right\}$$

This function vanishes only because of the $\mathcal{L}_4$ parts.

There are essentially only three different $\mathcal{L}_4$ traces.

$\bar{N}^{[\alpha\beta]\gamma}(-p)\, N_{[\alpha\delta]\varepsilon}(p)$, $\bar{N}^{[\alpha\beta]\gamma}(-p)\, N_{[\delta\varepsilon]\alpha}(p)$, and $\bar{N}^{[\beta\gamma]\alpha}(-p)\, N(p)_{[\delta\varepsilon]\alpha}$

$$\bar{N}^{[\alpha\beta]\gamma}(-p)\, N_{[\alpha\delta]\varepsilon}(p) = \frac{1}{m^2}\, \bar{N}^{\gamma}(-p)[C^{-1}\gamma_5(-\not{p}+m)]^{\alpha\beta}[(\not{p}+m)\gamma_5 C]_{\alpha\delta} N_{\varepsilon}(p)$$

$$= -\frac{1}{m^2}\, \bar{N}^{\gamma}(-p)\, [C^{-1}\gamma_5(-\not{p}+m)]^{\beta\alpha}[(\not{p}+m)\gamma_5 C]_{\alpha\delta}\, N_{\varepsilon}(p)$$

$$= -\frac{1}{m^2}\, \bar{N}^{\gamma}(-p)\left\{ C^{-1}\gamma_5\, (-p^2+m^2)\gamma_5 C \right\}^{\beta}{}_{\delta}\, N_{\varepsilon}(p)$$

$$= 0$$

as $p^2 = m^2$.

$$\bar{N}^{[\alpha\beta]\gamma}(-p)\, N_{[\delta\varepsilon]\alpha}(p) = \frac{1}{m^2}\, \bar{N}^{\gamma}(-p)\, [C^{-1}\gamma_5(-\not{p}+m)]^{\alpha\beta}$$

$$\times\, [(\not{p}+m)\gamma_5 C]_{\delta\varepsilon} N_{\alpha}(p)$$

$$= \frac{1}{m^2}\, \bar{N}^{\gamma}(-p)[(\not{p}+m)\gamma_5 C]_{\delta\varepsilon}\, [C^{-1}\gamma_5(\not{p}-m)]^{\beta\alpha} N_{\alpha}(p)$$

$$= \ 0$$

as $(\not{p}-m)_{\beta}{}^{\alpha}\, N_{\alpha}(p) \ = \ 0$

$$\bar{N}^{[\beta\gamma]\alpha}(-p)\, N_{[\delta\varepsilon]\alpha}(p) \ = \ \frac{-1}{m^2}\, \bar{N}^{\alpha}(-p)[C^{-1}\gamma_5(-\not{p}+m)]^{\beta\gamma}[(\not{p}+m)\gamma_5 C]_{\delta\varepsilon}\, N_{\alpha}(p)\ .$$

This term is zero because $\bar{u}(-p)\, u(p) = 0$.

We have shown, therefore, that all the regular amplitudes vanish. It has been argued that this is a reasonably good result as the two meson annihilation modes are considerably damped as is evidenced by the rates, apart from the mode $p\bar{p} \to \rho\pi$ which accounts for about 4°/o of all meson annihilation

<u>Rates for two-body annihilations of antiprotons at rest.</u>

C. Baltay et.al. Phys.Rev.Lett. 15,532(1966)

Channel	Rate
$\pi^+\pi^-$	$(3.2) \pm 0.3) \times 10^{-3}$
K^+K^-	$(1.1 \pm 0.1) \times 10^{-3}$
$K^+K^-/\pi^+\pi^-$	0.33 ± 0.023
$K_1^oK_1^o \pm K_2^oK_2^o$	$(0.88 {+1.1 \atop -0.9}) \times 10^{-5}$
$K_1^o\ K_2^o$	$(0.61 \pm 0.09) \times 10^{-3}$
$\pi^\pm\ \rho^\mp$	$(2.9 \pm 0.4) \times 10^{-2}$
$\pi^o\ \rho^o$	$(1.4 \pm 0.2) \times 10^{-2}$

-continued -

Channel	Rate
$\rho^o\eta^o$	$(2.2 \pm 1.7) \times 10^{-3}$
$\rho^o\omega^o$	$(7.0 \pm 3.0) \times 10^{-3}$
$\rho^o\rho^o$	$(3.8 \pm 3.0) \times 10^{-3}$
K^oK^{o*}	$(1.2 \pm 0.2) \times 10^{-3}$
$K^{\pm}K^{\mp*}$	$(0.92 \pm 0.16) \times 10^{-3}$
$K^{\pm*}K^{\mp*}$	$(1.3 \pm 0.5) \times 10^{-3}$
$K^{o*}\bar{K}^{o*}$	$(2.9 \pm 0.5) \times 10^{-3}$

What about irregular couplings? In general, the number of irregular couplings is very large for scattering as there are a great number of ways of inserting momenta. But in the special case of annihilation at rest the number of non-vanishing irreducible four-point functions is greatly reduced. This is because there are only two independent momenta p and q to be inserted and also the further "rest condition" that no direct summation over baryon indices is allowed. We insert p and q in all possible ways. Whenever we have $\not{p}$ acting on either Ψ or $\bar{\Psi}$, the vertex is reducible because of the Bargmann-Wigner equations. Similarly, we see that we cannot insert either p or q between the Φ's or between a Φ and Ψ or $\bar{\Psi}$.

Finally, we are left with the following vertices

$$f_1\ \bar{\Psi}^{ABC}(-p)\ (\not{q})_A{}^D(\not{q})_B{}^E(\not{q})_C{}^F\ \Psi_{DEF}(p)\ \Phi_H{}^G(q_1)\ \Phi_G{}^H(q_2)$$

$$f_{D,F}\ \bar{\Psi}^{ABC}(-p)\ (\not{q})_A{}^D(\not{q})_B{}^E\ \Psi_{DEF}(p)\ \{\Phi_C{}^G(q_1)\Phi_G{}^F(q_2)\pm\Phi_G{}^F(q_1)\Phi_C{}^G(q_2)\}$$

$$f_2\ \bar{\Psi}^{ABC}(-p)(\not{q})_A{}^D\ \Psi_{DEF}(p)\ \Phi_B{}^E(q_1)\ \Phi_C{}^F(q_2)$$

$$f_3\ \bar{\Psi}^{ABC}(-p)\ (\not{q})_A{}^D(\not{q})_B{}^E\ \Psi_{DEF}(p)\ \Phi_C{}^F(q_1)\ (\not{q})_H{}^G\ \Phi_G{}^H(q_2)$$

$$f_4\ \bar{\Psi}^{ABC}(-p)\ (\not{q})_A{}^D(\not{q})_B{}^E\ \Psi_{DEF}(p)\ \Phi_C{}^F(q_2)\ (\not{q})_H{}^G\ \Phi_G{}^H(q_2). \quad (5.16)$$

where the f's are invariant functions of p and q.

Before we evaluate any of these expressions we have to test which of these are charge conjugation invariant as we assume the invariance of strong interactions under C. For this we use the generalised C-matrix constructed in the first section of this chapter. We have to symmetrize and anti-symmetrize the Bose and Fermi operators, respectively. Consider the general function

$$\left[\bar{\Psi}^{ABC}(p_1)\ (\not{q})_A{}^D\ \Psi_{DEF}(p_2) - \Psi_{DEF}(p_2)\ (\not{q})_A{}^D\ \bar{\Psi}^{ABC}(p_1)\ \right]$$
$$\times\left[\ \Phi_B{}^E(q_1)\ \Phi_C{}^F(q_2)\ \right].\quad F(s,u)$$

where s and u are the usual Mandelstam invariants. The symmetrization of the Bose operators M is implied.

Under charge conjugation the first term transforms to

$$(C^{-1})^{AA'}(C^{-1})^{BB'}(C^{-1})^{CC'}\ \Psi_{A'B'C'}(-p_1)(\not{d})_A{}^D\ C_{DD'}C_{EE'}C_{FF'}\bar{\Psi}^{D'E'F'}(-p_2)$$

$$(C^{-1})^{EG}\Phi_G{}^H(q_1)\ C_{HB}\ (C^{-1})^{FI}\,\Phi_I{}^J(q_2)\ C_{JC}$$

$$=\ +\ \Psi_{A'B'C'}(-p_1)\ \ (\not{d})_{D'}{}^{A'}\ \bar{\Psi}^{D'E'F'}(-p_2)\ \ \Phi_{E'}{}^{B'}(q_1)\,\Phi_{F'}{}^{C'}(q_2)$$

$$=\ +\ \Psi_{ABC}(-p_1)\ \ (\not{d})_D{}^A\ \bar{\Psi}^{DEF}(-p_2)\,\Phi_E{}^B(q_1)\ \Phi_F{}^C(q_2)$$

using (5.5). Doing this to the other term as well we find that the amplitude

$$\bar{\Psi}^{ABC}(p_1)\ \ (\not{d})_A{}^D\Psi_{DEF}(p_2)\ \Phi_B{}^E(q_1)\ \ \Phi_C{}^F(q_2)\ \ F(s,u)$$

transforms to

$$-\ \bar{\Psi}^{ABC}(-p_2)\ \ (\not{d})_A{}^D\ \Psi_{DEF}(-p_1)\ \ \Phi_B{}^E(q_1)\ \ \Phi_C{}^F(q_2)\ \ F(s,u)$$

where now proper statistical symmetrization is implied. To get back to the original amplitude we have to perform the replacement

$$p_1\ \rightarrow\ -p_2\qquad ,\qquad q_i\ \rightarrow\ q_i$$

(These must be chosen in such a manner that the equation representing conservation of momentum, $p_1-p_2 = q_1-q_2$, is left invariant). This replacement gives us

$$- \bar{\Psi}^{ABC}(p_1)\ (\not{q})_A{}^D\ \Psi_{DEF}(p_2)\ \Phi_B{}^E(q_1)\ \Phi_C{}^F(q_2)F(u,s).$$

Therefore the amplitude could be made C-invariant if we could find a form factor such that

$$F(s,u) = -F(u,s).$$

This is possible in general scattering cases but for the special case of $p\bar{p}$ annihilation at rest into mesons no such form factor can be found.[25] Therefore, the corresponding amplitude for $p\bar{p}$ annihilations at rest is C-violating.

We find that the only C-conserving amplitude is

$$g\ \bar{\Psi}^{ABC}(-p)(\not{q})_A{}^D(\not{q})_B{}^E\Psi_{DEF}(p)\{\Phi_C{}^G(q_1)\Phi_G{}^F(q_2) + \Phi_G{}^F(q_1)\ \Phi_C{}^G(q_2)\ \} \qquad (5.17)$$

The $\Phi\Phi$ part is the $\underline{143}_D$ in the reduction of $\underline{143} \times \underline{143}$. Winternitz, Makarov[26] et. al. have considered the same process but have used a generalized Pauli principle i.e. the total amplitude must be symmetrical with respect to the interchange of the two meson functions (including the unitary parts). It follows that all amplitudes, antisymmetrical with respect to the meson interchange, must be multiplied by antisymmetrical functions of the kinematical invariants, s, t and u. They consider these coefficients as functions of s and $\nu = t - u$. Under meson interchange, they have for the antisymmetric coefficients

$$f(s,\nu) = -f(s,-\nu)$$
$$f(s,o) = 0.$$

However, at rest $\nu = t-u = 2p.q = 0$ and all antisymmetric amplitudes vanish. This however gives them the additional invariant

$$f(s,v)\ \bar{\Psi}^{ABC}(-p)(\not{q})_A{}^D(\not{q})_B{}^E\Psi_{DEF}(p)\ (\Phi_C{}^F(q_1)(\not{q})_H{}^G\Phi_G{}^H(q_2)$$
$$- \Phi_C{}^F(q_2)\ (\not{q})_H{}^G\ \Phi_G{}^H(q_1))$$

which we exclude because of C invariance.

We now proceed to evaluate the remaining amplitude. We consider the baryon part first.

$$\bar{\Psi}^{ABC}(-p)(\not{q})_A{}^D\ (\not{q})_B{}^E\ \Psi_{DEF}(p)$$

$$= \bar{\Psi}^{pqr,\alpha\beta\gamma}(-p)\ \ (\not{q})_\alpha{}^\delta\ \ \delta_p{}^t(\not{q})_\beta{}^\varepsilon\ \delta_q{}^u\ \Psi_{tus,\delta\varepsilon\lambda}(p)$$

$$= \bar{\Psi}^{pqr,\alpha\beta\gamma}(-p)\ \ (\not{q})_\alpha{}^\delta(\not{q})_\beta{}^\varepsilon\ \ \Psi_{pqs,\delta\varepsilon\lambda}(p)$$

$$= \frac{1}{24}\left(\varepsilon^{pqt}\ \bar{N}^{[\alpha\beta]\gamma,r}{}_t + \varepsilon^{qrt}\ \bar{N}^{[\beta\gamma]\alpha,p}{}_t + \varepsilon^{rpt}\ \bar{N}^{[\gamma\alpha]\beta,q}{}_t\right)$$
$$\left(\varepsilon_{pqu}N_{[\delta\varepsilon]\lambda,s}{}^u + \varepsilon_{qsu}\ N_{[\varepsilon\lambda]\delta,p}{}^u + \varepsilon_{spu}\ N_{[\lambda\delta]\varepsilon,q}{}^u\right)$$
$$\times\ (\not{q})_\alpha{}^\delta\ \ (\not{q})_\beta{}^\varepsilon$$

(We drop the momentum arguments of $\bar{N}$ and N for convenience).

$$= \frac{1}{24}\Big(2\bar{N}^{[\alpha\beta]\gamma,r}{}_{t}\, N_{[\delta\varepsilon]\lambda,s}{}^{t} - \bar{N}^{[\alpha\beta]\gamma,r}{}_{u}\, N_{[\varepsilon\lambda]\delta,s}{}^{u}$$

$$- \bar{N}^{[\alpha\beta]\gamma,r}{}_{u}\, N_{[\lambda\delta]\varepsilon,s}{}^{u} - \bar{N}^{[\beta\gamma]\alpha,r}{}_{u}\, N_{[\delta\varepsilon]\lambda,s}{}^{u}$$

$$+ \delta^{r}_{s}\, \bar{N}^{[\beta\gamma]\alpha,p}{}_{u}\, N_{[\varepsilon\lambda]\delta,p}{}^{u} - \bar{N}^{[\beta\gamma]\alpha,p}{}_{s}\, N_{[\varepsilon\lambda]\delta,p}{}^{r}$$

$$+ \varepsilon^{qrt}\varepsilon_{spu}\, \bar{N}^{[\beta\gamma]\alpha,p}{}_{t}\, N_{[\lambda\delta]\varepsilon,q}{}^{u} + \varepsilon^{rpt}\varepsilon_{qsu}\, \bar{N}^{[\gamma\alpha]\beta,q}{}_{t}\, N_{[\varepsilon\lambda]\delta,p}{}^{u}$$

$$- \bar{N}^{[\gamma\alpha]\beta,r}{}_{u}\, N_{[\delta\varepsilon]\lambda,s}{}^{u} + \delta^{r}_{s}\, \bar{N}^{[\gamma\alpha]\beta,q}{}_{u}\, N_{[\lambda\delta]\varepsilon,q}{}^{u}$$

$$- \bar{N}^{[\gamma\alpha]\beta,q}{}_{s}\, N_{[\lambda\delta]\varepsilon,q}{}^{r} \Big) \times (\not{q})_{\alpha}{}^{\delta}\, (\not{q})_{\beta}{}^{\varepsilon} \qquad (5.18)$$

Using the following products of ε's

$$\varepsilon^{pqt}\varepsilon_{pqu} = 2\,\delta^{t}_{u}$$

$$\varepsilon^{pqt}\varepsilon_{prs} = \delta^{q}_{r}\text{-}\delta^{t}_{s} - \delta^{q}_{s}\,\delta^{t}_{r} .$$

and the fact that $N^{p}_{p} = 0$ since SU(3) octet. Next consider the two terms

$$\Big(\varepsilon^{qrt}\ \varepsilon_{spu}\ \bar{N}^{[\beta\gamma]\alpha,p}{}_{t}\ N^{[\beta\gamma]\alpha,p}{}_{t}\ N_{[\lambda\delta]\varepsilon,q}{}^{u}$$

$$+\ \varepsilon^{rpt}\ \varepsilon_{qsu}\ \bar{N}^{[\gamma\alpha]\beta,q}{}_{t} N_{[\varepsilon\lambda]\delta,p}{}^{u}\Big)\ (\not{d})_{\alpha}{}^{\delta}(\not{d})_{\beta}^{\varepsilon}$$

$$=\varepsilon^{qrt}\ \varepsilon_{spu}\ \{\bar{N}^{[\beta\gamma]\alpha,p}{}_{t}\ (\not{d})_{\alpha}^{\delta}\ (\not{d})_{\beta}^{\varepsilon}\ N_{[\lambda\delta]\varepsilon,q}{}^{u}$$

$$+\ \bar{N}^{[\gamma\alpha]\beta,p}{}_{t}\ (\not{d})_{\alpha}^{\delta}\ (\not{d})_{\beta}{}^{\varepsilon}\ N_{[\varepsilon\lambda]\delta,\ q}{}^{u}\ \Big)$$

since p,q are dummy indices and can be relabelled.

$$=\ 2\varepsilon^{qrt}\ \varepsilon_{spu}\ \bar{N}^{[\beta\gamma]\alpha,p}{}_{t}(\not{d})_{\alpha}{}^{\delta}\ (\not{d})_{\beta}{}^{\varepsilon}\ \ N_{[\lambda\delta]\varepsilon,q}{}^{u}$$

because the position of α, β and δ, ε can be changed simultaneously.

$$=2(\not{d})_{\alpha}{}^{\delta}(\not{d})_{\beta}{}^{\varepsilon}\Big\{\ \bar{N}^{[\beta\gamma]\alpha,r}{}_{u}\ N_{[\lambda\delta]\varepsilon,s}{}^{u}\ -\ \bar{N}^{[\beta\gamma]\alpha,p}{}_{u}$$

$$N_{[\lambda\delta]\varepsilon,p}{}^{u}\ \delta^{r}_{s}\ +\ \bar{N}^{[\beta\gamma]\alpha,p}{}_{s}\ N_{[\lambda\delta]\varepsilon,p}{}^{r}\ \Big\} \tag{5.19}$$

using

$$\begin{aligned}\varepsilon^{qrt}\varepsilon_{spu}\ &=\ \delta_{s}{}^{q}(\delta^{r}_{p}\ \delta^{t}_{u}\ -\ \delta^{r}_{u}\ \ \delta^{t}_{p}\ )\\ &-\ \delta_{p}{}^{q}(\delta^{r}_{s}\ \delta^{t}_{u}\ -\ \delta^{r}_{u}\ \delta^{t}_{s})\\ &+\ \delta_{u}{}^{q}(\delta^{r}_{s}\ \delta^{t}_{p}\ -\ \delta^{r}_{p}\ \delta^{t}_{s})\ .\end{aligned}$$

Substituting (5.19) in (5.18) we get

$$\Psi^{pqr,\alpha\beta\gamma}(\phi)_\alpha{}^\delta(\phi)_\beta{}^\varepsilon \quad \Psi_{pqs,\delta\varepsilon\lambda}$$

$$= \frac{1}{24}\Big\{2\bar{N}^{[\alpha\beta]\gamma,r}{}_t \; N_{[\delta\varepsilon]\lambda,s}{}^t - \bar{N}^{[\alpha\beta]\gamma,r}{}_u \; N_{[\varepsilon\lambda]\delta,s}{}^u$$

$$- \bar{N}^{[\alpha\beta]\gamma,r}{}_u \, N_{[\lambda\delta]\varepsilon,s}{}^u - \bar{N}^{[\beta\gamma]\alpha,r}{}_u \, N_{[\delta\varepsilon]\lambda,s}{}^u$$

$$+ \delta^r_s \;\bar{N}^{[\beta\gamma]\alpha,p}{}_u \, N_{[\varepsilon\lambda]\delta,p}{}^u - \bar{N}^{[\beta\gamma]\alpha,p}{}_s \; N_{[\varepsilon\lambda]\delta,p}{}^r$$

$$+ 2\bar{N}^{[\beta\gamma]\alpha,r}{}_u \, N_{[\lambda\delta]\varepsilon,s}{}^u + 2\,\bar{N}^{[\beta\gamma]\alpha,p}{}_s \, N_{[\lambda\delta]\varepsilon,p}{}^r$$

$$- 2\bar{N}^{[\beta\gamma]\alpha,p}{}_u \; N_{[\lambda\delta]\varepsilon,p}{}^u \,\delta^r_s - \bar{N}^{[\gamma\alpha]\beta,r}{}_u \, N_{[\delta\varepsilon]\lambda,s}{}^u$$

$$+ \delta^r_s \;\bar{N}^{[\gamma\alpha]\beta,q}{}_u \; N_{[\lambda\delta]\varepsilon,q}{}^u - \bar{N}^{[\gamma\alpha]\beta,q}{}_s \, N_{[\lambda\delta]\varepsilon,q}{}^r\Big\}(\phi)_\alpha{}^\delta(\phi)_\beta{}^\varepsilon$$

$$= \frac{1}{12}\Big\{\;2\;\bar{N}^{[\alpha\beta]\gamma,r}{}_u \, N_{[\delta\varepsilon]\lambda,s}{}^u + \bar{N}^{[\beta\gamma]\alpha,r}{}_u \; N_{[\lambda\delta]\varepsilon,s}{}^u$$

$$+ \bar{N}^{[\beta\gamma]\alpha,p}{}_s \, N_{[\lambda\delta]\varepsilon,p}{}^r - \bar{N}^{[\beta\gamma]\alpha,p}{}_s \; N_{[\varepsilon\lambda]\delta,p}{}^r$$

$$+ \delta^r_s \, \bar{N}^{[\beta\gamma]\alpha,p}{}_u \, N^u_{[\varepsilon\lambda]\delta,p} - \delta^r_s \, \bar{N}^{[\beta\gamma]\alpha,p}{}_u \, N^u_{[\lambda\delta]\varepsilon,p} \Big\}$$

$$\times \, (\not{q})_\alpha{}^\delta \, (\not{q})_\beta{}^\varepsilon \qquad (5.20)$$

using symmetry properties of $N_{[\alpha\beta]\,\gamma}$ etc.

Now we calculate the $\Phi\Phi$ part.

$$\Phi_C{}^G(q_1)\,\Phi_G{}^F(q_2) + \Phi_G{}^F(q_1)\,\Phi_C{}^G(q_2)$$

$$= \frac{1}{\mu^2}\Big\{(\not{q}_1+\mu)\gamma_5\phi_5^i(q_1) + (\not{q}_1+\mu)\gamma_\mu\phi_\mu^i(q_1)\Big\}_\gamma{}^\kappa (T^i)_r{}^t$$

$$\times \Big\{(\not{q}_2+\mu)\gamma_5\phi_5^j(q_2) + (\not{q}_2+\mu)\gamma_\nu\phi_\nu^j(q_2)\Big\}_\kappa{}^\lambda (T^j)_t{}^s$$

$$+ \frac{1}{\mu^2}\Big\{(\not{q}_2+\mu)\gamma_5\,\phi_5^j(q_2) + (\not{q}_2+\mu)\gamma_\nu\,\phi_\nu^j\,(q_2)\Big\}_\gamma{}^\kappa (T^j)_r{}^t$$

$$\times \Big\{(\not{q}_1+\mu)\,\gamma_5\,\phi_5^i(q_1) + (\not{q}_1+\mu)\gamma_\mu\phi_\mu^i(q_1)\Big\}_\kappa{}^\lambda (T^i)_t{}^s$$

To simplify, for the PP contribution,

$$(\not{q}_1+\mu)\gamma_5\,(\not{q}_2+\mu)\gamma_5 = +\,(\not{q}_1+\mu)\,(\not{q}_2-\mu)$$

$$= (\not{p} + \tfrac{\not{q}}{2} + \mu)\,(\not{p} - \tfrac{\not{q}}{2} - \mu) \quad \text{using (4.13) and (4.14)}$$

$$= (m^2 - \mu^2 - \frac{q^2}{4} - \mu\not{q} + \frac{\not{q}\not{p}}{2} - \frac{\not{p}\not{q}}{2})$$

$$= \quad 2(m^2-\mu^2) - \mu \not{q} + \not{q}\not{p} \quad \text{as } -\frac{q^2}{4} = + q_1^2 = m^2 - \mu^2$$

$$= \quad \not{q}(\not{p}-\mu) \qquad \text{effectively as } 2(m^2-\mu^2) \text{ gives no contribution.}$$

Similarly

$$(q_2+\mu)\ \gamma_5\ (\not{q}_1+\mu)\gamma_5 \ = (\not{q}_2+\mu)\ (\not{q}_1 - \mu)$$

$$= \quad \not{q}\ (\mu - \not{p}) \qquad \text{effectively.}$$

So for PP contribution we get

$$\frac{1}{\mu^2}\ \not{q}\ (\not{p}-\mu)\phi_5^i(q_1)\ \phi_5^j\ (q_2)\ [T^iT^j - T^jT^i]$$

$$= \quad \frac{1}{\mu^2}\ [\not{q}(\not{p}-\mu)]_\gamma^{\ \lambda}\ \phi_5^i(q_1)\ \phi_5^j\ (q_2)\ if^{ijk}(T^k)_r^{\ s} \tag{5.21}$$

For VP, we make life easier by picking the Pseudoscalar particle to have momentum q_1 and the vector particle momentum q_2. For this case we get

$$\frac{1}{\mu^2}\left\{(\not{q}_1+\mu)\gamma_5(\not{q}_2+\mu)\ \gamma_\nu\right\}\ \phi_5^i\ (q_1)\ \phi_\nu^j(q_2)\tfrac{1}{2}(d^{ijk}+if^{ijk})\ T^k$$

$$+\frac{1}{\mu^2}\left\{(\not{q}_2+\mu)\gamma_\nu(\not{q}_1+\mu)\gamma_5\right\}\ \phi_\nu^j(q_2)\ \phi_5^i(q_1)\ \tfrac{1}{2}\ (d^{jik}+ if^{jik})T^k$$

$$= \Bigg[\ \frac{1}{2\mu^2}\left\{(\not{q}_1+\mu)\gamma_5(\not{q}_2\ +\mu)\gamma_\nu + (\not{q}_2+\mu)\gamma_\nu(\not{q}_1+\mu)\gamma_5\right\}\ d^{ijk}T^k$$

$$+ \quad \frac{1}{2\mu^2}\left\{(\not{q}_1+\mu)\gamma_5(\not{q}_2+\mu)\,\gamma_\nu - (\not{q}_2+\mu)\gamma_\nu(\not{q}_1+\mu)\gamma_5\right\} if^{ijk}T^k \Bigg]$$

$$\times\ \phi_5^i(q_1)\ \phi_\nu^j(q_2)$$

$$= \frac{1}{2\mu^2}\Bigg[\Big\{R,S\Big\}\, d^{ijk}T^k \;+\; [R,S]\ if^{ijk}T^k\Bigg]_{\gamma,r}^{\lambda,s} \phi_5{}^i(q_1)\ \phi_\nu^j(q_2)$$

(5.22)

where $R = (\not{q}_1+\mu)\ \gamma_5$

$S = (\not{q}_2+\mu)\ \gamma_\nu$

For VV we get

$$\frac{1}{2\mu^2}\Big[\{P,S\}\ d^{ijk}T^k + [P,S]\ if^{ijk}T^k\Big]_{\gamma,r}^{\lambda,s}\phi_\mu^i(q_1)\ \phi_\nu^j(q_2)$$

(5.23)

where $P = (\not{q}_1+\mu)\ \gamma_\mu$

$S = (\not{q}_2+\mu)\ \gamma_\nu.$

Now we need to combine (5.20) with (5.21) for PP, with (5.22) for VP and with (5.23) for VV.

(a) <u>Annihilation into two pseudoscalars.</u>

Since we have found an F-combination of the mesons the k-index in (5.21) cannot be zero and hence the $\delta_s{}^r$ terms in (5.20) vanish. We need to evaluate the products

$$\bar{N}^{[\alpha\beta]\gamma}\,[\not{q}(\not{p}-\mu)]_{\gamma}^{\ \lambda}\,N_{[\delta\varepsilon]\lambda}\,(\not{q})_{\alpha}^{\ \delta}(\not{q})_{\beta}^{\ \varepsilon}\ ,\ \bar{N}^{[\beta\gamma]\alpha}\,[\not{q}(\not{p}-\mu)]_{\gamma}^{\ \lambda}\,N_{[\lambda\delta]\varepsilon}\,(\not{q})_{\alpha}^{\ \delta}\,(\not{q})_{\beta}^{\ \varepsilon}$$

and $\bar{N}^{[\beta\gamma]\alpha}\,[\not{q}(\not{p}-\mu)]_{\gamma}^{\ \lambda}\ N_{[\varepsilon\lambda]\delta}\cdot(\not{q})_{\beta}^{\ \varepsilon}(\not{q})_{\alpha}^{\ \delta}$.

We evaluate $\bar{N}^{[\beta\gamma]\alpha}\,[\not{q}(\not{p}-\mu)]_{\gamma}^{\ \lambda}\,N_{[\lambda\delta]\varepsilon}$ as an example.

$$\bar{N}^{[\beta\gamma]\alpha}(-p)\ [\not{q}(\not{p}-\mu)]_{\gamma}^{\ \lambda}\,N_{[\lambda\delta]\varepsilon}(p)\ (\not{q})_{\alpha}^{\ \delta}\ (\not{q})_{\beta}^{\ \varepsilon}$$

$$= \frac{1}{m^2}\ \bar{N}(-p)\ \not{q}\ (\not{p}+m)\gamma_5 C\ (\not{p}-\mu)^T\ \not{q}^T\ C^{-1}\gamma_5(-\not{p}+m)\ \not{q}\ N(p)$$

dropping indices

$$= -\frac{1}{m^2}\ \bar{N}\ \not{q}(\not{p}+m)\ (\not{p}-\mu)\not{q}\ (-\not{p}+m)\not{q}N$$

using $\qquad C\gamma_{\mu}^{\ T}C = -\gamma_{\mu}$

$\therefore = -4\ q^2(m-\mu)\ \bar{N}\not{q}N$. Here and subsequently we evaluate these products by commuting so that p is made to act on N and $\bar{N}$ and noting that $p.q = 0$.

$$\bar{N}^{[\alpha\beta]\gamma}(-p)\ [\not{q}(\not{p}-\mu)]_{\gamma}^{\ \lambda}\,N_{[\delta\varepsilon]\lambda}(p)\ (\not{q})_{\alpha}^{\ \delta}(\not{q})_{\beta}^{\ \varepsilon}$$

$$= 8q^2(m-\mu)\ \bar{N}\not{q}N$$

$$\bar{N}^{[\beta\gamma]\alpha}(-p)\ [\not{q}(\not{p}-\mu)]_{\gamma}^{\ \lambda}\ N_{[\varepsilon\lambda]\delta}(p)\quad (\not{q})_{\alpha}^{\ \delta}\ (\not{q})_{\beta}^{\ \varepsilon}$$

$$= \quad 8q^2(m-\mu)\quad \bar{N}\not{q}N$$

So finally putting in SU(3) parts as well the PP contribution reduces to

$$\frac{q^2(m-\mu)}{\mu^2} \qquad (\bar{N}\not{q}N)_F^{\ k}\ if^{ijk}\phi_5^{\ i}(q_1)\ \phi_5^{\ j}(q_2) \qquad (5.24)$$

where

$$(\bar{N}N)_F^{\ k} \;=\; \bar{N}_u^{\ r}(T^k)_r^{\ s}\ N_s^{\ u} \;-\; \bar{N}_s^{\ u}\ N_u^{\ r}(T^k)_r^{\ s}$$

There are two states of the proton-antiproton system at rest, ${}^3S_1$ and ${}^1S_o$. The parity and charge parity of these states are

	P	C
${}^1S_o$	$-$	$+$
${}^3S_1$	$-$	$-$

Thus the triplet and singlet states are definite charge parity states. According to Gell-Mann[27] every octet that goes into itself under charge conjugation has a characteristic number $c = \pm 1$, which is the charge-conjugation quantum number C of its 1,3,4, 6 and 8 components; The charge - conjugation quantum number C of the 2,5 and 7 components is $-c$. The pseudoscalar and vector octets are just such self conjugate objects. The above rule states that the charge-conjugation properties of any self-conjugate octet are determined by the charge-conjugation quantum number of the $I_3=0$, $Y=0$ member of the octet. The normal convention is

$$C|\pi^0 > \;=\; + |\pi^0 >$$

and

$$C|\rho^0 > \;=\; - |\rho^0 > .$$

This means, then, that

$$\begin{aligned} C|K^{\pm} > &= + |K^{\mp}> \\ C|\pi_0^{\pm} > &= + |\pi_0^{\mp}> \\ C|K^0 > &= + |\bar{K}^0> \end{aligned} \qquad (5.25)$$

and

$$\begin{aligned} C|K^{*\pm} > &= - |K^{*\mp}> \\ C|\rho_0^{\pm} > &= - |\rho_0^{\mp}> \\ C|K^{*0} > &= - |\bar{K}^{*0}> \quad . \end{aligned} \qquad (5.26)$$

These phases of charge-conjugation are consistent with the definition (5.6) of the charge-conjugation of the $\underline{143}$ representation of U(6,6).

Now since the initial state in the process under consideration is an eigenstate of C, the final state will also be an eigenstate. For the initial ${}^1S_o$ state the total angular momentum is zero. Hence the two pseudoscalars in the final state have to be in an S state. Hence the parity is + of this state and therefore parity conservation does not allow ${}^1S_o$ decays into two pseudoscalars. For the ${}^3S_1$ state the total angular momentum is 1 and the final state is then in a P state and the parity is − . Consider the $\pi^+\pi^-$ system in the final state.

$$C|\pi^+\pi^- > \quad = \quad | \pi^-\pi^+ >$$

and to get back to original state interchange $\pi^+\pi^-$ giving

$$(-1)^{\ell}|\pi^+\pi^- \; > \; = \; - \; |\pi^+\pi^- > \qquad (\text{as} \, \ell = 1)$$

The $(-1)^{\ell}$ factor arises from space reflection. Hence the C-parity of the final state is negative and this matches up with the C-parity of ${}^3S_1$ initial state. So this process can proceed.

Considering just the SU(3) parts of the final state we can construct eigenstates of C easily. For two pseudoscalars e.g.

$$C|\pi^+\pi^- \; \pm \pi^-\pi^+ > = \; \pm \; |\pi^+\pi^- \; \pm \pi^-\pi^+ > \qquad (5.27)$$

where the ± correspond to D and F octet couplings respectively.

(Since the meson-meson coupling is $\underline{143}_D$ the only SU(3) couplings are octet and singlet). Thus, we see that for conservation of C-parity we need an F coupling of the two mesons. This is exactly what we have for this process by using inhomogeneous U(6,6) (5.24). We also note incidentally that the C-parity of $\bar{p}\not{q}p$ is $-$, where p represents the spinor for the proton. The numerical results from this amplitude are presented later along with the results for VP and VV modes.

For VP the eigenstates of C are as follows, using K, K^* as examples,

$$C|K^+K^{*-} \pm K^-K^{*+}\rangle = \mp |K^+K^{*-} \pm K^-K^{*+}\rangle \quad \text{etc.} \qquad (5.28)$$

using (5.25) and (5.26). Since there are only octet and singlet couplings we just pick out the different C-parity states by looking for D and F couplings.

For VV the eigenstates of C are given by

$$C|K^{*+}K^{*-} \pm K^{*-}K^{*+}\rangle = \pm|K^{*+}K^{*-} \pm K^{*-}K^{*+}\rangle \qquad (5.29)$$

using (5.26).

b) Evaluation of amplitude for annihilation into a vector and a pseudoscalar meson.

The amplitude for this is given by (5.22). Using (5.28) we see that the d^{ijk} term is for $^3S_1$ and the f^{ijk} term is for ^1S_o modes respectively. We have to combine the expressions

(5.22) and (5.20) to evaluate the trace. The Lorentz traces here are nastier than those arising in the PP case. We give the results of these calculations only, as their evaluation is only a matter of some lengthy algebra.

$$\bar{N}^{[\alpha\beta]\gamma}(\not{q})_\alpha{}^\delta(\not{q})_\beta{}^\varepsilon N_{[\delta\varepsilon]\lambda}\{R,S\}_\gamma{}^\lambda$$

$$= \quad 8q^2(\mu-m)\;(\bar{N}\not{q}\gamma_5\gamma_\mu N - \bar{N}\gamma_\mu\gamma_5\not{q}N)$$

$$\bar{N}^{[\beta\gamma]\alpha}(\not{q})_\alpha{}^\delta(\not{q})_\beta{}^\varepsilon N_{[\lambda\delta]\varepsilon}\{R,S\}_\gamma{}^\lambda$$

$$= \quad 4q^2(\mu-m)\;\{\bar{N}\not{q}\gamma_5\gamma_\mu N - \bar{N}\gamma_\mu\gamma_5\not{q}N\}$$

and

$$\bar{N}^{[\beta\gamma]\alpha}(\not{q})_\alpha{}^\delta(\not{q})_\beta{}^\varepsilon N_{[\varepsilon\lambda]\delta}\{R,S\}_\gamma{}^\lambda$$

$$= \quad 0$$

$$\bar{N}^{[\alpha\beta]\gamma}(\not{q})_\alpha{}^\delta(\not{q})_\beta{}^\varepsilon N_{[\delta\varepsilon]\lambda}[R,S]_\gamma{}^\lambda$$

$$= \quad 8\,q^2(m-\mu)\;(q_\mu+2p_\mu)\quad\bar{N}\gamma_5N$$

$$\bar{N}^{[\beta\gamma]\alpha}(\not{q})_\alpha{}^\delta(\not{q})_\beta{}^\varepsilon N_{[\lambda\delta]\varepsilon}[R,S]_\gamma{}^\lambda$$

$$= + \; 4q^2(m-\mu)\;(q_\mu+2p_\mu)\;\bar{N}\gamma_5N$$

$$\bar{N}^{[\beta\gamma]\alpha}(\not{q})_\alpha{}^\delta(\not{q})_\beta{}^\varepsilon N_{[\varepsilon\lambda]\delta}[R,S]_\gamma{}^\lambda = 0.$$

Collecting all this together and including the SU(3) parts we find that the full VP amplitude is

$$\frac{1}{6\mu^2} q^2(\mu-m)\Big[\Big\{(N\not{q}\gamma_5\gamma_\mu N - \bar{N}\gamma_\mu\gamma_5\not{q}N)^k_{3D+2F} - (\mathrm{Tr}T^k)$$

$$\mathrm{Tr}(\bar{N}\not{q}\gamma_5\gamma_\mu N - \bar{N}\gamma_\mu\gamma_5\not{q}N)\Big\}d^{ijk}$$

$$-(q_\mu+2p_\mu)(\bar{N}\gamma_5N)^k_{3D+2F}\ if^{ijk}\Big]\phi_5{}^i(q_1)\ \phi_\mu^j(q_2) \tag{5.30}$$

(C) <u>Evaluation of matrix element for annihilation into two vector particles</u>

We get the following

$$\bar{N}^{[\alpha\beta]\gamma}N_{[\delta\varepsilon]\lambda}(\not{q})_\alpha{}^\delta(\not{q})_\beta{}^\varepsilon\{P,S\}_\gamma{}^\lambda$$

$$= -8iq^2(\mu-m)\ \bar{N}\{\sigma_{\mu\nu},\not{q}\}N$$

$$\bar{N}^{[\beta\gamma]\alpha}N_{[\lambda\delta]\varepsilon}(\not{q})_\alpha{}^\delta(\not{q})_\beta{}^\varepsilon\{P,S\}_\gamma{}^\lambda$$

$$= -4i\ q^2(\mu-m)\ \bar{N}\{\sigma_{\mu\nu},\not{q}\}N$$

$$\bar{N}^{[\beta\gamma]\alpha}N_{[\varepsilon\lambda]\delta}(\not{q})_\alpha{}^\delta(\not{q})_\beta{}^\varepsilon\{P,S\}_\gamma{}^\lambda$$

$$= 0$$

and

$$\bar{N}^{[\alpha\beta]\gamma} N_{[\delta\varepsilon]\lambda} (\not{q})_\alpha{}^\delta (\not{q})_\beta{}^\varepsilon [P,S]_\gamma{}^\lambda$$

$$= q^2\{-16i(\mu-m)^2\ \bar{N}\ \sigma_{\mu\nu}\ N + 16\ (\mu-m)\ g_{\mu\nu}\bar{N}\not{q}N$$

$$-\ 4i\ \bar{N}\ \not{q}\ \sigma_{\mu\nu}\not{q}N\}$$

$$\bar{N}^{[\beta\gamma]\alpha}\ N_{[\lambda\delta]\varepsilon}\ (\not{q})_\alpha{}^\delta (\not{q})_\beta{}^\varepsilon [P,S]_\gamma{}^\lambda$$

$$= -\ \{\ 2iq^4\bar{N}\ \sigma_{\mu\nu}N + 8q^2(\mu-m)g_{\mu\nu}\ \bar{N}\not{q}N$$

$$+\ 8\ i\ (\mu-m)^2\quad \bar{N}\not{q}\ \sigma_{\mu\nu}\not{q}\ N\}$$

$$\bar{N}^{[\beta\gamma]\alpha}\ N_{[\varepsilon\lambda]\delta}\ (\not{q})_\alpha{}^\delta\ (\not{q})_\beta{}^\varepsilon\ [P,S]_\gamma{}^\lambda$$

$$= 16\ (m-\mu)\ [\ \frac{\mu}{m}\ (p_\mu q_\nu - p_\nu q_\mu] + q_\mu q_\nu]\ N\not{q}N$$

Finally the full VV amplitude is

$$= \frac{1}{24\mu^2}\Big\{-\ 4iq^2(\mu-m)\quad \Big((\bar{N}\{\sigma_{\mu\nu},\not{q}\}N)^k_{3D+2F}$$

$$-\ (TrT^k)Tr(\bar{N}\ \{\sigma_{\mu\nu},\not{q}\}N)\Big)\quad d^{ijk}$$

$$+\ \Big(-32i(m-\mu)^2(m+\mu)\{m(\bar{N}\sigma_{\mu\nu}N)^k_{3D+2F} - \mu(\bar{N}\sigma_{\mu\nu}N)^k_{D+2F}\ \}$$

$$+ \ 32\ (m^2-\mu^2)\ (\mu-m)\, g_{\mu\nu} (\bar{N} \not{q} N)^k_{D+2F} \ - \ 8i\,(m-\mu) \{ m (\bar{N} \not{q} \sigma_{\mu\nu} \not{q} N)^k_{3D+2F}$$

$$+ \ \mu (\bar{N} \not{q}\ \sigma_{\mu\nu}\ \not{q}\ N)^k_{D+2F} \Big\}$$

$$- \ 16(m-\mu) \Big[\frac{\mu}{m} (p_\mu q_\nu - p_\nu\ q_\mu) + q_\mu q_\nu \Big]\ (\bar{N} \not{q} N T^k)$$

$$+ \ (Tr T^k)\ \{ 16(m-\mu) \{ \frac{\mu}{m} (p_\mu q_\nu - p_\nu q_\mu) + q_\mu q_\nu \} Tr (\bar{N} \not{q} N)$$

$$+ \ \{ 32i\ (m^2-\mu^2)^2 (\bar{N}\ \sigma_{\mu\nu} N) + 32\ (m^2-\mu^2)(\mu-m)\, g_{\mu\nu} (\bar{N} \not{q} N)$$

$$+ \ 8i\,(\mu-m)^2 (\bar{N} \not{q} \sigma_{\mu\nu} \not{q} N) \} \Big] ' \Big) \quad f^{ijk} \Big\}\ \phi^i_\mu (q_1)\ \ \phi^j_\nu (q_2) \qquad (5.32)$$

This last expression is very complicated so subsequently we never evaluate it explicitly. We only derive sum rules for the $^3S_1$ VV modes.

(d)

Having now calculated all the relevant amplitudes we give the numerical results [28] achieved from (5.24), (5.30) and (5.32). We denote by $A(ab)$ the amplitude for annihilation to particles a and b.

For $^3S_1$ annihilation into two pseudoscalar mesons we obtain

$$A(\pi^+\pi^-) \ : \ A(K^+K^-) \ : \ (A(K^o\bar{K}^o) = 1:2:1 \ .$$

The cross-section relation, if we neglect $K\pi$ mass difference is

$$\sigma(\pi^+\pi^-):\sigma(K^+K^-):\sigma(K^o\bar{K}^o) = 1:4:1$$

If these irregular amplitudes are computed with $K\pi$ mass differences introduced in external mass factors, $\sigma(\pi\pi)$ is tremendously enhanced over $\sigma(K\bar{K})$. Experimentally (see Table 1)

$$\sigma(\pi^+\pi^-):\sigma(K^+K^-):\sigma(K^o\bar{K}^o) = 3:1:.55$$

So, even where there is no mass difficulty i.e. the ratios for K^+K^- and $K^o\bar{K}^o$ are in disagreement with experiment.

No ^1S_o mode exists for two pseudoscalars and hence in particular

$$\Lambda(\chi^o\chi^o) = \Lambda(\pi^o\pi^o) = 0.$$

For $^3S_1$ mode into a vector and a pseudoscalar we obtain :

$$\Lambda(\varphi\pi^o) = \Lambda(\varphi\eta) = \Lambda(\varphi\chi^o) = 0$$

where φ is the physical particle

and

$$\Lambda(\rho^{\pm}_{o}\pi^{\mp}_{o}):\Lambda(\omega\pi):\Lambda(\omega\eta):\Lambda(\rho\eta):\Lambda(\rho\chi):\Lambda(\omega\chi):\Lambda(K^{*-}K^+):\Lambda(K^{*o}\bar{K}^o)$$

$$= 3 : 5 : \sqrt{3} : 5/\sqrt{3} : 10/\sqrt{6} : \sqrt{6} : 4 : -1 .$$

For the ^1S_o mode

$$\Lambda(\rho^+\pi^-):\Lambda(K^{*+}K^-) : \Lambda(K^{*o}\bar{K}^o) = 5:4:-1$$

For the $^3S_1$ two vector meson mode

$$\Lambda(\varphi\omega) = \Lambda(\varphi\varphi) = \Lambda(\varphi\varphi) = \Lambda(\omega\omega) = 0$$

and

$$\Lambda(\rho^+\rho^-) = \Lambda(K^{*+}K^{*-}) - \Lambda(K^{*o}\bar{K}^{*o})$$

(Actually, a ratio of cross-sections could be calculated apart from this sum rule but the calculation is too tedious).

For the ^1S_o mode

$$\Lambda(\rho^+\rho^-) : \Lambda(K^{*+}K^{*-}) : \Lambda(K^{*o}\bar{K}^{*o}) : \Lambda(\omega\omega) : \Lambda(\omega\rho)$$

$$= 3 : 4 : -1 : 6 : 5$$

We have already compared the two pseudoscalar predictions with experiment and have found them to be bad. The most striking feature of the rest of the predictions is the non appearance of any φ's in the annihilation process. This prediction is of course contingent on the particular identification of the physical φ as defined by equation (4.11). The comparison of the rest of the predictions with experimental data is very difficult to deal with. This is because of the problem of handling the rather disparate masses of the different mesons. The calculation upto now has been performed with all the mesons having equal mass and this is consistent with the group theoretical basis which we have used. We are forced to give them the same mass if we are to use the Bargmann-Wigner equations, that is if we specify the baryons and mesons to be s-state bound states of quarks and anti-quarks. This construction as mentioned before uniquely gives us the parity of the particles if we assume that the parity of the quark is positive. We could as a working rule use equal mean masses for the amplitude calculated group theoretically, but use physical masses in the other, kinematical, factors which arise when calculating the cross-sections.

However, there are two particular cases where these mass difficulties would be by-passed and this happens when comparing the ratios $K^{+}K^{*-}/K^{o}\bar{K}^{*o}$ and $K^{*+}K^{*-}/K^{*o}\bar{K}^{*o}$. For the first ratio we get for both $^{3}S_{1}$ and $^{1}S_{o}$ a ratio of 16 for the cross-sections which is in great disagreement with the experimental figures, as given in table 1, where in fact the neutral mode is larger than the charged mode. Similarly, we seem to indicate a larger cross-section for $K^{*+}K^{*-}$ as compared to $K^{*o}\bar{K}^{*o}$ while table 1 shows to the contrary. We could compare the other rates by squaring the matrix elements and taking account of phase space[26]. This is, however, unnecessary because we have found that in the easily comparable cases the theory gives very bad predictions. This is therefore a blow to $SU(6)_{W}$ and we could get out of this by saying that this group is too restrictive. Instead of $SU(6)_{W}$ we could try the collinear $SU(3) \otimes SU(3)$ which bears the same relation to $SL(6,C)$ as does $SU(6)_{W}$ to $U(6,6)$. However, even in this case, Buccella and Gatto[29] have shown that the really bad ratio for the $^3S_1$ $K^{*+}K^-/K^{*o}\bar{K}^o$ still exists. This group does give the relation

$$A(^3S_1\ \pi^+\ \pi^-) + A(^3S_1 K^o\bar{K}^o) = A(^3S_1 K^+K^-)$$

instead of the ratio of these rates. This sum rule is satisfied to within 20°/o. They then consider what they call the 'minimum group' (SU(2) ⊗ SU(2) ⊗ $W(\sigma_z y)$ collinear. The generators of the small collinear group obey the algebra of the matrices $\frac{1}{2}\tau_i(1\pm\sigma_2)$ and $\frac{1}{2}\lambda_8\sigma_z$. ($\tau$ and σ are the Pauli matrices in the isospin and spin spaces). This group is a subgroup of $[SU(4)_T \otimes SU(2)_X]_W$, a restriction of $SU(6)_W$ due to SU(3) breaking terms. Looked at in another way, this group is obtained from SU(3) ⊗ $SU(3)_{coll}$ by restricting the internal symmetry to that of isotopic spin and hypercharge conservation. They find that even in this case the bad vector-pseudoscalar triplet mode is still present. So, even the smallest collinear group is in contradiction with experiment.

(iv) <u>Conclusion</u>

We have seen how a covariant theory of strong interactions can be set up, incorporating internal symmetries. We went beyond the Poincare group and insisted on the invariance of the S-matrix under inhomogeneous U(6,6). This meant the introduction of 143 momenta which we then restricted to the four physical momenta. Using these prescriptions we looked for amplitudes where the number of irregular couplings would not be forbidding and found that the proton-antiproton annihilation at rest were suitable cases for treatment. However, the experimental results are in clear contradiction with our predictions. The three point results, on the other hand, are quite encouraging. The first and simple Johnson-Treiman[30] relations are fairly well satisfied while the extended Johnson-Treiman[31] relations have been shown by Jackson[32] to be in violent disagreement with experiment. All this suggests that maybe one should only use higher symmetries to classify particles and apply them only to three point vertices. The scattering problems should be done dynamically using the symmetric vertices and the propagators for the intermediate particles.

The trouble with proton-antiproton annihilation at rest is that we have completely neglected the effects of unitarity. Apart from all the competing open two meson channels, there are other open channels like 3 meson modes and also the effect of the closed baryon-antibaryon channel cannot be ignored as pointed out by Fraser[33]. He has shown that these do have a large effect on the two meson channels, but numerical results are hard to obtain because of lack of data in other channels.

APPENDIX A

Good things to know in SU(3).

We use $T^i = \frac{1}{2}\lambda^i$, with the λ^i defined by Gell-Mann[2].

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

$$\lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$\lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$\lambda^\circ = \sqrt{\frac{2}{3}}\ I$$

$$[T^i, T^j] = i\, f^{ijk} T^k, \quad \{T^i, T^j\} = d^{ijk} T^k$$

$$T_r\ (T^i\, T^j) = \tfrac{1}{2}\,\delta^{ij}\ , \quad f^{ojk} = 0\ , \quad d^{ijk} = \sqrt{\tfrac{2}{3}}\ \delta^{jk}$$

f^{ijk} are real and totally antisymmetric
d^{ijk} " " " " symmetric.

$$f^{123} = 1$$
$$f^{345} = -f^{367} = \tfrac{1}{2}$$
$$f^{458} = f^{678} = \frac{\sqrt{3}}{2}$$
$$f^{147} = -f^{156} = f^{246} = f^{257} = \tfrac{1}{2}$$

$$d^{118} = d^{228} = d^{338} = \frac{1}{\sqrt{3}}$$

$$d^{344} = d^{355} = -d^{366} = -d^{377} = \tfrac{1}{2}$$

$$d^{448} = d^{558} = d^{668} = d^{778} = -\frac{1}{2\sqrt{3}}$$

$$d^{888} = -\frac{1}{\sqrt{3}}$$

The meson and baryon matrices are

$$\begin{pmatrix} \frac{\pi^o}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & \frac{-\pi^o}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^o \\ K^- & \bar{K}^o & \frac{-2\eta}{\sqrt{6}} \end{pmatrix} = \begin{pmatrix} \frac{\phi_5^3}{\sqrt{2}} + \frac{\phi_5^8}{\sqrt{6}} & \frac{\phi_5^1 - i\phi_5^2}{\sqrt{2}} & \frac{\phi_5^4 - i\phi_5^5}{\sqrt{2}} \\ \frac{\phi_5^1 + \phi_5^2}{\sqrt{2}} & -\frac{\phi_5^3}{\sqrt{2}} + \frac{\phi_5^8}{\sqrt{6}} & \frac{\phi_5^6 - i\phi_5^7}{\sqrt{2}} \\ \frac{\phi_5^4 + i\phi_5^5}{\sqrt{2}} & \frac{\phi_5^6 + i\phi_5^7}{\sqrt{2}} & -\frac{2\phi_5^8}{\sqrt{6}} \end{pmatrix}$$

$= \sqrt{2}\ T^i \phi_5^i$ (where the subscript 5 denotes pseudoscalar field)

χ^o is the SU(3) pseudoscalar singlet.

The matrix for vector mesons is similar. Replace $\pi \to \rho$, $K \to K^*$, $\eta \to \phi_\mu^8$, $\chi \to \phi_\mu^o$ and instead of pseudoscalar fields have vector fields.

The $\frac{1}{2}^+$ baryon matrix is

$$\begin{pmatrix} \frac{\Sigma^o}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & \Sigma^+ & p \\ \Sigma^- & \frac{-\Sigma^o}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & n \\ \Xi^- & \Xi^o & \frac{-2\Lambda}{\sqrt{6}} \end{pmatrix}$$

Commutation relations for U(6,6) generators[9]. From the fundamental representation $F_r^i \equiv \Gamma_r T^i$ we get the following commutators.

$$[F^i, F^j] = i f^{ijk} F^k$$
$$[F^i, F_5^j] = i f^{ijk} F_5^k$$
$$[F_5^i, F_5^j] = -i f^{ijk} F^k$$
$$[F^i, F_{\mu\nu}^j] = i f^{ijk} F_{\mu\nu}^k$$
$$[F_5^i, F_{\mu\nu}^j] = \tfrac{1}{2} i f^{ijk} \varepsilon_{\mu\nu\kappa\lambda} F_{\kappa\lambda}^k$$
$$[F_{\kappa\lambda}^i, F_{\mu\nu}^j] = i d^{ijk} (g_{\kappa\lambda}\, g_{\lambda\mu}\, F_{\kappa\nu}^k - g_{\kappa\mu}\, F_{\lambda\nu}^k$$

$$-g_{\lambda\nu}F^k_{\kappa\nu} + if^{ijk}\{(g_{\kappa\mu}g_{\lambda\nu}-g_{\lambda\mu}g_{\kappa\nu})F^k - \varepsilon_{\kappa\lambda\mu\nu}F^k_5\}$$

$$[F^i_\mu, F^j_\nu] = if^{ijk}g_{\mu\nu}F^k - id^{ijk}F^k_{\mu\nu}$$

$$[F^i_\mu, F^j_{\nu 5}] = id^{ijk}g_{\mu\nu}F^k_5 + \tfrac{1}{2}\, if^{ijk}\varepsilon_{\mu\nu\kappa\lambda}F^k_{\kappa\lambda}\ ,$$

$$[F^i_{\mu 5}, F^j_{\nu 5}] = -if^{ijk}g_{\mu\nu}F^k - id^{ijk}\, F^k_{\mu\nu}$$

$$[F^i, F^j_\mu] = if^{ijk}F^k_\mu$$

$$[F^i, F^j_{\mu 5}] = if^{ijk}F^k_{\mu 5}$$

$$[F^i_5, F^j_\mu] = id^{ijk}F^k_{\mu 5}$$

$$[F^i_5, F^j_{\mu 5}] = id^{ijk}F^k_\mu$$

$$[F^i_\lambda, F^j_{\mu\nu}] = id^{ijk}(g_{\lambda\mu}F^k_\nu - g_{\lambda\nu}F^k_\mu) - if^{ijk}\varepsilon_{\lambda\mu\nu\kappa}F^k_{\kappa 5}$$

$$[F^i_{\lambda 5}, F^j_{\mu\nu}] = id^{ijk}\,(g_{\lambda\mu}F^k_{\nu 5} - g_{\lambda\nu}F^k_{\mu 5}) + if^{ijk}\varepsilon_{\lambda\mu\nu\kappa}F^k_\kappa$$

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