ON THE ERNST ELECTRO-VACUUM EQUATIONS AND ERGOSURFACES

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The question of smoothness at the ergosurface of the space-time metric constructed out of solutions $(\mathscr{E},\varphi)$ of the Ernst electro-vacuum equations is considered. We prove smoothness of those ergosurfaces at which $\Re\mathscr{E}$ provides the dominant contribution to $f=-(\Re\mathscr{E}+|\varphi|^2)$ at the zero-level-set of f. Some partial results are obtained in the remaining cases: in particular we give examples of leading-order solutions with singular isolated "ergocircles".

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1. Introduction

In recent work [1] we have shown that a vacuum space-time metric is smooth near a "Ernst ergosurface" $E_{\mathscr{E}}=\{\Re\mathscr{E}=0,\rho\neq 0\}$ if and only if the Ernst potential $\mathscr{E}$ is smooth near $E_{\mathscr{E}}$ and does not have zeros of infinite order there. It is of interest to enquire whether a similar property holds for electro-vacuum metrics. While we have not been able to obtain a complete answer to this question, in this note we present a series of partial results, amongst which:

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Theorem 1.1 *Consider a smooth solution* $(\mathscr{E},\varphi)$ *of the electro-vacuum Ernst equations (2.2)–(2.3) below, and let the Ernst ergosurface* $E_{\mathscr{E},\varphi}$ *be defined as the set*

$$E_{\mathscr{E},\varphi} := \{\mathscr{E} + \overline{\mathscr{E}} + 2\overline{\varphi}\varphi = 0\,, \quad \rho \neq 0\}\,. \tag{1.1}$$

Suppose that $\mathscr{E} + \overline{\mathscr{E}}$ *has a zero of finite order at* $E_{\mathscr{E},\varphi}$*. If the* φ *terms contribute subleading terms to* $\mathscr{E}+\overline{\mathscr{E}}+2\overline{\varphi}\varphi$ *at* $E_{\mathscr{E},\varphi}$*, then there exists a neighborhood of* $E_{\mathscr{E},\varphi}$ *on which the tensor field (2.1) obtained by solving (2.5)–(2.6) is smooth and has Lorentzian signature.*

Theorem 1.1 is proved in Section 3.

To make things clear, consider a point p at which

$$f := -\frac{1}{2}(\mathscr{E} + \overline{\mathscr{E}} + 2\overline{\varphi}\varphi)$$

vanishes. Expanding $\mathscr{E}$ and φ in a Taylor series at p, let m be the order of the leading Taylor polynomial of $\Re\mathscr{E} - \Re\mathscr{E}(p)$, and let k be the corresponding order for $\varphi - \varphi(p)$. Then we say that the φ terms contribute subleading terms to f if $2k > m$.

Under the remaining conditions of Theorem 1.1, the condition of a zero of finite order is *necessary and sufficient*, as smoothness of the metric near $E_{\mathscr{E},\varphi}$ implies analyticity of $\mathscr{E}$ and φ.

It follows from the analysis in [1] that, in vacuum, a generic point on $E_{\mathscr{E},\varphi}$ will be a zero of $\mathscr{E}$ of order one. One expects this result to remain true in electro-vacuum, so that Theorem 1.1 should cover generic situations.

A significant application of Theorem 1.1, to solutions obtained by applying a Harrison transformation to a vacuum solution, is given in Section 4 below.

Some partial results, presented in Section 5, are obtained in the cases not covered by Theorem 1.1: We describe completely the leading-order behavior of φ at those ergosurfaces at which φ provides the dominant contribution to f. We show that there exist Taylor polynomials solving the Ernst equation at leading order which result in singularities of the space-time metric on $E_{\mathscr{E},\varphi}$. This result does not, however, prove that there exist smooth solutions of the electro-vacuum Ernst equations which lead to metrics which are singular at the ergosurface because it is not clear that the "leading-order solutions" that we construct correspond to solutions of the full, non-truncated equations.

2. Preliminaries

We use the same parameterisation of the metric as in [1]:

$$ds^2 = f^{-1}\left[h\left(d\rho^2 + d\zeta^2\right) + \rho^2 d\phi^2\right] - f\left(dt + a d\phi\right)^2\,, \tag{2.1}$$

with all functions depending only upon ρ and ζ. In electro-vacuum the Ernst equations form a system of two coupled partial differential equations for two complex valued functions $\mathscr{E}$ and φ [5], which we assume to be smooth:

$$\left(\mathscr{E}+\bar{\mathscr{E}}+2\overline{\varphi}\varphi\right) L\mathscr{E} = \left(\frac{\partial \mathscr{E}}{\partial \bar z}+2\overline{\varphi}\frac{\partial \varphi}{\partial \bar z}\right)\frac{\partial \mathscr{E}}{\partial z}+\left(\frac{\partial \mathscr{E}}{\partial z}+2\overline{\varphi}\frac{\partial \varphi}{\partial z}\right)\frac{\partial \mathscr{E}}{\partial \bar z}\,, \tag{2.2}$$

$$\left(\mathscr{E}+\bar{\mathscr{E}}+2\overline{\varphi}\varphi\right) L\varphi = \left(\frac{\partial \mathscr{E}}{\partial \bar z}+2\overline{\varphi}\frac{\partial \varphi}{\partial \bar z}\right)\frac{\partial \varphi}{\partial z}+\left(\frac{\partial \mathscr{E}}{\partial z}+2\overline{\varphi}\frac{\partial \varphi}{\partial z}\right)\frac{\partial \varphi}{\partial \bar z}\,, \tag{2.3}$$

where

$$L=\frac{\partial^2}{\partial z\partial \overline{z}}+\frac{1}{2(z+\bar z)}\Big(\frac{\partial}{\partial z}+\frac{\partial}{\partial \bar z}\Big)\,,$$

with $z=\rho+\mathrm{i}\zeta$. The metric functions are determined from[1]

$$f = -\frac{1}{2}(\mathscr{E}+\bar{\mathscr{E}}+2\overline{\varphi}\varphi)\,, \tag{2.4}$$

$$\frac{\partial h}{\partial z} = (z+\bar z)h\left(\frac{1}{2}\left(\frac{\partial \mathscr{E}}{\partial z}+2\overline{\varphi}\frac{\partial \varphi}{\partial z}\right)\left(\frac{\partial \bar{\mathscr{E}}}{\partial z}+2\varphi\frac{\partial \overline{\varphi}}{\partial z}\right)f^{-2}+2\frac{\partial \overline{\varphi}}{\partial z}\frac{\partial \varphi}{\partial z}f^{-1}\right), \tag{2.5}$$

$$\frac{\partial a}{\partial z} = \frac{1}{4}(z+\overline{z})\left(\frac{\partial \mathscr{E}}{\partial z}+2\overline{\varphi}\frac{\partial \varphi}{\partial z}-\frac{\partial \bar{\mathscr{E}}}{\partial z}-2\varphi\frac{\partial \overline{\varphi}}{\partial z}\right)f^{-2}\,. \tag{2.6}$$

The equations are singular at the *Ernst ergosurface* $E_{\mathscr{E},\varphi}$ defined by (1.1).

Let $\lambda\in\mathbb{C}$, $\mu\in\mathbb{R}$, then the following transformation maps solutions of (2.2)–(2.3) into solutions, *without changing* the right-hand sides of (2.4)–(2.6)

$$\mathscr{E}\to\mathscr{E}+2\bar\lambda\varphi-|\lambda|^2+\mathrm{i}\mu\,,\qquad \varphi\to\varphi-\lambda\,. \tag{2.7}$$

This is easiest seen by noting, first, that both f and $d\mathscr{E}+2\bar\varphi d\varphi$ are left unchanged by (2.7).

3. $\mathscr{E}$-dominated ergosurfaces

Suppose that $E_{\mathscr{E},\varphi}\neq\emptyset$ and that $\mathscr{E}$ and φ are smooth in a neighborhood of $E_{\mathscr{E},\varphi}$. Let $z_0=\rho_0+\mathrm{i}\zeta_0\in E_{\mathscr{E},\varphi}$, we can choose μ and λ so that the potentials transformed as in (2.7) satisfy

$$\mathscr{E}(z_0)=0\,,\quad \varphi(z_0)=0\,. \tag{3.1}$$

Assume first,

$$Df(z_0)\neq 0\,.$$

[1] Note that $\mathscr{E}$ here is minus $\mathscr{E}$ in [1].

Performing a Taylor expansion of $\mathscr{E}$ and φ at z_0 and inserting into (2.2)–(2.3), a Singular [2] calculation (and, as a cross-check, a Maple one) shows[2] that either

$$\partial_z\varphi(z_0) \;=\; \partial_z\mathscr{E}(z_0)=0\,, \tag{3.2}$$

$$0\neq\partial_{\bar z}\mathscr{E}(z_0) \;=\; 4\rho_0\partial_z\partial_{\bar z}\mathscr{E}(z_0)=4\rho_0\overline{\partial_z^2\mathscr{E}}(z_0)\,, \tag{3.3}$$

$$\partial_z^2\mathscr{E}(z_0)\partial_z\partial_{\bar z}\varphi(z_0) \;=\; \partial_z^2\varphi(z_0)\partial_z\partial_{\bar z}\mathscr{E}(z_0)\,, \tag{3.4}$$

$$\partial_z^2\mathscr{E}(z_0)\overline{\partial_z^2\varphi}(z_0) \;=\; \overline{\partial_z\partial_{\bar z}\varphi}(z_0)\partial_z\partial_{\bar z}\mathscr{E}(z_0)\,, \tag{3.5}$$

or that (3.2)–(3.5) is satisfied by the complex conjugates of $(\mathscr{E},\varphi)$. In the latter case the linear part of the Taylor expansion of $(\mathscr{E},\varphi)$ is a holomorphic function of z, while it is anti-holomorphic in the former. In the calculations proving smoothness across $E_{\mathscr{E},\varphi}\cap\{df\neq 0\}$ the equations (3.4)–(3.5) are not used.

Using (3.3) in (2.6) one finds

$$\lim_{z\to z_0} f^2\partial_z\left(a+\frac{\rho}{f}\right)=\lim_{z\to z_0}\partial_z\left[f^2\partial_z(a+\frac{\rho}{f})\right]=\lim_{z\to z_0}\partial_{\bar z}\left[f^2\partial_z(a+\frac{\rho}{f})\right]=0\,. \tag{3.6}$$

It follows as in the proof of Theorem 4.1 of [1] that the function $a+\rho/f$ is smooth across $E_{\mathscr{E},\varphi}\cap\{df\neq 0\}$.

The same argument with $a-\rho/f$ instead of $a+\rho/f$ applies if the complex conjugate solution is used.

A similar calculation with (2.5) shows that

$$\lim_{z\to z_0} f^2\partial_z\ln(|h/f|)=\lim_{z\to z_0}\partial_z(f^2\partial_z\ln(|h/f|))=\lim_{z\to z_0}\partial_{\bar z}(f^2\partial_z\ln(|h/f|))=0\,. \tag{3.7}$$

The remaining arguments of the proof of Theorem 4.1 of [1] apply and we conclude that the metric (2.1) extends smoothly across $E_{\mathscr{E},\varphi}\cap\{df\neq 0\}$, and has Lorentzian signature in a neighborhood of this set.

Suppose, next, that f has a zero of higher order at $z_0\in E_{\mathscr{E},\varphi}$. Since φ enters quadratically in f and in the right-hand sides of (2.5)–(2.6), and through cubic terms in the right-hand sides of (2.2)–(2.3), one would hope that φ will only contribute to subleading terms in Taylor expansions of those equations. But then the analysis of the leading-order behavior of f near $E_{\mathscr{E},\varphi}$ is reduced to the analysis already done in [1], which would prove smoothness of the space-time metric at the Ernst ergosurface without any provisons.

[2] See the Singular file `em1.in` and the Maple file `em1.mw` at `http://th.if.uj.edu.pl/~szybka/CS/`

It turns out that this is not the case: we shall see in the next section that there exist leading-order Taylor polynomials satisfying the leading-order equations for which the φ terms are *not* dominated by $\mathscr{E}$. Nevertheless, the argument just given establishes that *if* the φ terms are dominated by $\mathscr{E}$, then the analysis of [1] proves smoothness of the metric across $E_{\mathscr{E},\varphi}$, and Theorem 1.1 is proved.

REMARK 3.1 *Consider a $\mathscr{E}$-dominated zero z_0 of f, after shifting $\Im\mathscr{E}$ by a real constant we can assume that $\mathscr{E}(z_0) = 0$. It then follows from [1, Proposition 5.1] that the order of the zero of $\mathscr{E}$ at z_0 coincides with the order of the zero of $\Re\mathscr{E}$.*

4. Harrison–Neugebauer–Kramer transformations

It is of interest to enquire what happens with Ernst ergosurfaces under Neugebauer–Kramer transformations [5, Equation (34.8e)] (see also [4]) of $(\mathscr{E}, \varphi)$:

$$\begin{aligned} \mathscr{E}' &= \mathscr{E}(1 - 2\bar{\gamma}\varphi - \gamma\bar{\gamma}\mathscr{E})^{-1}\,, \\ \varphi' &= (\varphi + \gamma\mathscr{E})(1 - 2\bar{\gamma}\varphi - \gamma\bar{\gamma}\mathscr{E})^{-1}\,. \end{aligned} \tag{4.1}$$

Under (4.1) f is transformed to

$$f' = \frac{f}{|1 - 2\bar{\gamma}\varphi - \gamma\bar{\gamma}\mathscr{E}|^2}\,, \tag{4.2}$$

so that $E_{\mathscr{E},\varphi}$ is mapped into itself. The same remains of course true under Harrison [3] transformations [5, Equation (34.12)], which are a special case of (4.1) when the initial φ vanishes:

$$\mathscr{E}' = \mathscr{E}(1 - \gamma\bar{\gamma}\mathscr{E})^{-1}\,, \qquad \varphi' = \gamma\mathscr{E}(1 - \gamma\bar{\gamma}\mathscr{E})^{-1}\,. \tag{4.3}$$

As a significant corollary of Theorem 1.1, we obtain

COROLLARY 4.1 *Let $(\mathscr{E}', \varphi')$ be obtained by a Harrison transformation from a smooth solution $(\mathscr{M}, g)$ of the vacuum equations with a non-empty ergosurface, then the conclusion of Theorem 1.1 holds.*

PROOF: As discussed in [1], the Ernst potential $\mathscr{E}$ is analytic near $E_{\mathscr{E},\varphi}$, hence has a zero of finite order. Clearly, the order of zero of $|\varphi'|^2$ as defined by (4.3) is higher than the order of zero of $\mathscr{E}'$; the latter is the same as the order of zero of $\Re\mathscr{E}'$ by the results in [1]. □

Somewhat more generally, consider $p \in E_{\mathscr{E},\varphi}$, as explained above we can always introduce a gauge so that $\varphi(p) = 0$. In this gauge, let $(\mathscr{E}', \varphi')$ be obtained by a Neugebauer–Kramer transformation from a solution satisfying the hypotheses of Theorem 1.1 near p, then the conclusion of Theorem 1.1 holds near p for the metric constructed by using $(\mathscr{E}', \varphi')$. This follows immediately from (4.1).

5. Some remaining possibilities

It remains to consider the case where the φ terms dominate in f, and the case where all terms are of the same order. The latter case will be referred to as *balanced*.

5.1. Balanced leading-order solutions with singular ergocircles

The simplest such possibility is $Df(z_0) = 0$, $DDf(z_0) \neq 0$ and $\mathscr{E}(z_0) = \varphi(z_0) = 0$. It is easy to completely analyze the first few leading-order equations with the ansatz

$$\partial_z \mathscr{E}(z_0) = \partial_{\bar z} \mathscr{E}(z_0) = \partial_z^2 \mathscr{E}(z_0) = \partial_{\bar z}^2 \mathscr{E}(z_0) = 0\,. \tag{5.1}$$

A MAPLE–assisted calculation[3] then shows that the leading-order equations do not introduce any constraints on $\partial_z \varphi(z_0)$, and that if we set

$$\alpha := \partial_z \varphi(z_0) \neq 0\,,$$

then one has

$$\begin{aligned} |\partial_{\bar z} \varphi(z_0)|^2 &= |\alpha|^2\,, \\ \partial_z \partial_{\bar z} \mathscr{E}(z_0) &= -4|\alpha|^2\,. \end{aligned} \tag{5.2}$$

Recall that (2.5)–(2.6) leads to the following equations for the metric function a

$$\frac{f^2}{\rho}\partial_z\left(a + \frac{\rho}{f}\right) = \underbrace{\left(\frac{\partial \mathscr{E}}{\partial z} + 2\overline{\varphi}\frac{\partial \varphi}{\partial z} + \frac{f}{z+\bar z}\right)}_{=:\mathring{\sigma}_1}, \tag{5.3}$$

$$\frac{f^2}{\rho}\partial_z\left(a - \frac{\rho}{f}\right) = \underbrace{-\left(\frac{\partial \overline{\mathscr{E}}}{\partial z} + 2\varphi\frac{\partial \overline{\varphi}}{\partial z} + \frac{f}{z+\bar z}\right)}_{=:\mathring{\sigma}_2}. \tag{5.4}$$

In the vacuum case it was shown that one out of $\mathring{\sigma}_1/f^2$ and $\mathring{\sigma}_2/f^2$ is smooth near $\{f = 0,\ \rho \neq 0\}$, which then implies smoothness of the ergosurface. (An identical analysis applies to $\mathscr{E}$-dominated ergosurfaces.) So one can attempt to repeat the argument here. Letting

$$r_0 := \sqrt{(\rho - \rho_0)^2 + (\zeta - \zeta_0)^2}\,,$$

[3] See the MAPLE file `em2.mw` at `http://th.if.uj.edu.pl/~szybka/CS/`

the leading terms of f, $\mathring{\sigma}_1$, $\mathring{\sigma}_2$ read

$$\begin{aligned} \mathscr{E} &= -4|\alpha z|^2 + O(r_0^3)\,, \\ \varphi &= \alpha z + \bar\gamma \bar z + O(r_0^2)\,, \\ f &= -\alpha\gamma z^2 + 2|\alpha|^2 z\bar z - \bar\gamma\bar\alpha\bar z^2 + O(r_0^3)\,, \\ \mathring{\sigma}_1 &= 2\alpha(\gamma z - \bar\alpha \bar z) + O(r_0^2)\,, \\ \mathring{\sigma}_2 &= -2\alpha(\gamma z - \bar\alpha \bar z) + O(r_0^2)\,, \end{aligned} \tag{5.5}$$

where $\gamma = \overline{\partial_{\bar z}\varphi}(z_0)$. Here, for the typesetting convenience, we used the symbol z for $z - z_0$. Those examples clearly lead to a singularity both in $\mathring{\sigma}_1/f^2$ and in $\mathring{\sigma}_2/f^2$, therefore a different strategy is needed.

Now,

$$f = |\alpha z - \bar\gamma \bar z|^2 + (|\alpha|^2 - |\gamma|^2)|z|^2 + O(r_0^3)\,,$$

so that if $|\alpha| > |\gamma|$ we obtain an isolated zero of f, an "ergocircle". More precisely, the intersection of the set where f vanishes with a neighborhood of z_0 will be $\{z_0\}$. This, at any given value of t, corresponds to an isolated null orbit of the isometry group of the metric generated by ∂_ϕ provided that the metric is non-singular there.

Still assuming $|\alpha| > |\gamma|$, we claim that the metric will be singular at z_0. Indeed, adding (5.3) and (5.4) one finds that ∂a is uniformly bounded near z_0, hence a can be extended by continuity to a Lipschitz continuous function defined on a neighborhood of z_0. But then $g(\partial_\phi, \partial_\phi)$ blows up as ρ_0^2/f at z_0.

5.2. Balanced solutions with radial $\mathscr{E}_{2k}$

The solutions of Section 5.1 are a special case of a family of solutions in which the leading terms in $\mathscr{E}$ take the form

$$\mathscr{E}_{2k} = \mu_1 e^{\mathrm{i}\mu_0}(z - z_0)^k (\bar z - \bar z_0)^k\,, \qquad \mu_0 \in \mathbb{R}\,, \qquad \mu_1 \in \mathbb{R}^*\,. \tag{5.6}$$

Let us write

$$\varphi_k = \sum_{m=0}^{k} \alpha_m (z - z_0)^m (\bar z - \bar z_0)^{k-m}\,, \tag{5.7}$$

where all the α_m's do not vanish simultaneously. Inserting (5.6)–(5.7) into (2.2)–(2.3) one obtains

$$(\mathscr{E}_{2k} + \bar{\mathscr{E}}_{2k})\frac{\partial^2 \mathscr{E}_{2k}}{\partial\bar z \partial z} - 2\frac{\partial \mathscr{E}_{2k}}{\partial \bar z}\frac{\partial \mathscr{E}_{2k}}{\partial z} = 2\bar\varphi_k\left(\frac{\partial\varphi_k}{\partial\bar z}\frac{\partial\mathscr{E}_{2k}}{\partial z} + \frac{\partial\varphi_k}{\partial z}\frac{\partial\mathscr{E}_{2k}}{\partial\bar z}\right) - 2\bar\varphi_k\varphi_k\frac{\partial^2\mathscr{E}_{2k}}{\partial\bar z\partial z}\,, \tag{5.8}$$

$$(\mathscr{E}_{2k} + \bar{\mathscr{E}}_{2k})\frac{\partial^2 \varphi_k}{\partial\bar z \partial z} - \left(\frac{\partial\varphi_k}{\partial\bar z}\frac{\partial\mathscr{E}_{2k}}{\partial z} + \frac{\partial\varphi_k}{\partial z}\frac{\partial\mathscr{E}_{2k}}{\partial\bar z}\right) = 4\bar\varphi_k\frac{\partial\varphi_k}{\partial\bar z}\frac{\partial\varphi_k}{\partial z} - 2\bar\varphi_k\varphi_k\frac{\partial^2\varphi_k}{\partial\bar z\partial z}\,. \tag{5.9}$$

The right-hand side of (5.8) vanishes, and the vanishing of the left-hand side implies $\sin\mu_0 = 0 \Longrightarrow \mu_0 = j\pi$, where $j \in \mathbb{N}$. Changing μ_1 to $-\mu_1$ if necessary we can without loss of generality assume $\mu_0 = 0$. Setting $\alpha_i = 0$ for $i < 0$ or $i > k$, and working out the coefficients of the terms $(z-z_0)^{k-1+l}(\bar z - \bar z_0)^{2k-1-l}$ in (5.9) we obtain for $-k+1 \le l \le 2k-1$

$$\mu_1\alpha_l\left((k-l)^2+l^2\right) = -\sum_{\substack{-m+n+i=l\\ 0\le m,n,i\le k}} 2\bar\alpha_m\alpha_n\alpha_i(k-i)(2n-i)\,. \tag{5.10}$$

We expect that a complete description of such solutions should be possible (for example, it immediately follows for $2k-1>k$ (*i.e.*, $k>1$) that $\bar\alpha_0\alpha_k\alpha_{k-1}=0$), but we have not attempted to do that. Instead we list here all such leading-order solutions for $k=2$ and $k=3$, as calculated[4] using Maple:

$$\begin{aligned}
k=2\,,\ \mathscr{E}_4 &= -|\alpha|^2|z|^4: &&\varphi_2 = \alpha|z|^2\,, &&\alpha\in\mathbb{C}^*\,,\\
\mathscr{E}_4 &= -4|\alpha|^2|z|^4: &&\varphi_{=}\alpha z^2+\bar\gamma\bar z^2\,, &&\alpha,\gamma\in\mathbb{C}^*\,,\ |\alpha|=|\gamma|\,,\\
k=3\,,\ \mathscr{E}_6 &= -\frac{4}{5}|\alpha|^2|z|^6: &&\varphi_3=\alpha z|z|^2 \text{ or } \varphi_3=\alpha\bar z|z|^2\,, &&\alpha\in\mathbb{C}^*,\\
\mathscr{E}_6 &= -4|\alpha|^2|z|^6: &&\varphi_3=\alpha z^3+\bar\gamma\bar z^3\,, &&\alpha,\gamma\in\mathbb{C}^*\,,\ |\alpha|=|\gamma|\,.
\end{aligned}$$

As before, for typesetting convenience, we used the symbol z for $z-z_0$. (We have not included the solutions with $\varphi_k=0$, as they are not balanced.)

The above suggests the following solutions, for all $k\ge 1$,

$$\mathscr{E}_{2k} = -4|\alpha|^2|z|^{2k}: \quad \varphi_k=\alpha z^k+\bar\gamma\bar z^k\,,\quad \alpha,\gamma\in\mathbb{C}^*\,,\ |\alpha|=|\gamma|\,, \tag{5.11}$$

$$\mathscr{E}_{4k} = -|\alpha|^2|z|^{4k}: \quad \varphi_{2k}=\alpha|z|^{2k}\,,\qquad \alpha\in\mathbb{C}^*\,, \tag{5.12}$$

$$\begin{aligned}\mathscr{E}_{4k+2} &= -\frac{2\,k\,(k+1)\,|\alpha|^2}{2\,k^2+2\,k+1}|z|^{4k+2}:\\ &\varphi_{2k+1}=\alpha z|z|^{2k} \text{ or } \varphi_{2k+1}=\alpha\bar z|z|^{2k}\,,\qquad \alpha\in\mathbb{C}^*\,.\end{aligned} \tag{5.13}$$

Those can be verified by a direct calculation.

Regularity of the metric can be established by showing that $g_{\phi t}=-af$, $\ln g_{\zeta\zeta}=\ln g_{\rho\rho}=\ln(hf^{-1})$, $g_{\phi\phi}=\left(\rho^2-(af)^2\right)/f$ are smooth across $\{f=0,\ \rho>0\}$ and that af does not vanish whenever f does. All solutions with leading-order behavior (5.12), if any, have a zero of f which is of order higher than $4k$. Thus f vanishes to higher order there, and any analysis of the metric near $\{f=0\}$ requires knowledge of the higher-order Taylor coefficients of $\mathscr{E}$ and φ there.

[4] See the Maple file `em3.mw` at `http://th.if.uj.edu.pl/~szybka/CS/`

On the other hand, the solution $\mathscr{E}_6 = -4/5|\alpha|^2|z|^6$, $\varphi_3 = \alpha z|z|^2$ leads to a singularity in the metric. (The same is true for its conjugate pair, namely $\bar{\mathscr{E}}$, $\bar{\varphi}$.) For this solution we have, using (2.4)–(2.6),

$$f = -\frac{1}{5}|\alpha|^2 z^3\bar{z}^3 + \dots\,, \tag{5.14}$$

$$\frac{1}{h}\frac{\partial h}{\partial z} = -56\,\frac{\rho_0}{z^2} + \dots\,, \tag{5.15}$$

$$\frac{\partial a}{\partial z} = 25\frac{\rho_0}{|\alpha|^2 z^4\bar{z}^3} + \dots\,. \tag{5.16}$$

(Eq. (5.14) shows that f vanishes at an isolated point in the (ρ,ζ) plane, leading again to an ergocircle.) Integrating we obtain

$$\ln(-h) = 112\rho_0\frac{\rho-\rho_0}{(\rho-\rho_0)^2+(\zeta-\zeta_0)^2} + \dots\,, \tag{5.17}$$

$$a = \frac{-25}{3|\alpha|^2}\frac{\rho_0}{((\rho-\rho_0)^2+(\zeta-\zeta_0)^2)^3} + \dots\,, \tag{5.18}$$

hence

$$af = \frac{5}{3}\,\rho_0 + \dots\,, \tag{5.19}$$

$$\begin{aligned}\ln(hf^{-1}) &= 112\rho_0\frac{\rho-\rho_0}{(\rho-\rho_0)^2+(\zeta-\zeta_0)^2}\\ &\quad -\ln\left(\frac{1}{5}|\alpha|^2\left((\rho-\rho_0)^2+(\zeta-\zeta_0)^2\right)^3\right) + \dots\,,\end{aligned} \tag{5.20}$$

$$g_{\phi\phi} = \frac{80}{9|\alpha|^2}\frac{\rho_0^2}{((\rho-\rho_0)^2+(\zeta-\zeta_0)^2)^3} + \dots\,. \tag{5.21}$$

Even though af is regular at leading order, the metric is singular at the point (ρ_0,ζ_0). This is not merely a coordinate singularity, since (5.21) shows that the norm $g_{\phi\phi} = g(\partial_\phi,\partial_\phi)$ of the Killing vector ∂_ϕ is unbounded.

5.3. φ-dominated ergocircles

We consider now those solutions where φ dominates in f. It follows immediately from Theorem 5.2 below that such solutions correspond to isolated points of $\{f = 0\}$, hence to ergocircles within the level sets of the coordinate t.

The simplest solutions in this class would have $\mathscr{E}$ vanishing altogether, or vanishing to very high order. In this context, symbolic algebra calculations[5] show that there are no non-trivial solutions such that

[5] See the SINGULAR files `em4a.in`, `em4b.in` at `http://th.if.uj.edu.pl/~szybka/CS/`

- $\varphi = O(|z-z_0|)$ with non-zero gradient at z_0, and $\mathscr{E} = O(|z-z_0|^4)$,
- $\varphi = O(|z-z_0|^2)$ with non-zero Hessian at z_0, and $\mathscr{E} = O(|z-z_0|^9)$.

In other words the assumption that $\varphi = O(|z-z_0|)$ and $\mathscr{E} = O(|z-z_0|^4)$ implies $\varphi = O(|z-z_0|^2)$; similarly $\varphi = O(|z-z_0|^2)$ and $\mathscr{E} = O(|z-z_0|^9)$ implies $\varphi = O(|z-z_0|^3)$. Those results require the analysis of the Taylor series of φ to higher order.

More systematically, let us assume that the leading-order Taylor polynomial φ_k of φ is of order k, with the corresponding Taylor polynomial for $\mathscr{E}$ is of order ℓ, while $\Re\mathscr{E} = O(|z-z_0|^m)$. The following shows that both, for balanced and for φ-dominated solutions the order of $\mathscr{E}$ cannot be smaller than that of $|\varphi|^2$ (compare Remark 3.1):

Proposition 5.1 *Suppose that* $\mathscr{E} = O(|z-z_0|^\ell)$, $\varphi = O(|z-z_0|^k)$, *and* $\Re\mathscr{E} = O(|z-z_0|^m)$ *with* $m \geq 2k$, *then*

$$\ell \geq 2k\,. \tag{5.22}$$

Proof: Assume that $\ell < 2k$, then inspection of (2.2) gives

$$\partial_z \mathscr{E}_\ell \partial_{\bar z} \mathscr{E}_\ell = 0\,.$$

Since $\mathscr{E}_\ell$ is purely imaginary this reads $|d\mathscr{E}_\ell|^2 = 0$, and the result follows. □

Clearly $m \geq \ell$ under the hypotheses of Proposition 5.1, so (5.22) implies $m \geq \ell \geq 2k$. We conclude that at a zero which is balanced we must have $m = \ell$; equivalently the order of $\mathscr{E}$ equals that of $\Re\mathscr{E}$. The same is true for $\mathscr{E}$-dominated solutions by Remark 3.1. It follows that the hypothesis that φ dominates in f is equivalent to

$$2k < \ell\,. \tag{5.23}$$

Supposing that f vanishes at $(\rho_0\,,\zeta_0) = z_0$, (2.3) becomes

$$\overline{\varphi}_k \varphi_k L \varphi_k \;=\; 2\overline{\varphi}_k \frac{\partial \varphi_k}{\partial \bar z}\frac{\partial \varphi_k}{\partial z} + O(r_0^{k+\ell-2}) + O(r_0^{3k-3})\,. \tag{5.24}$$

By (5.23) the second term can be absorbed into the first one. Since the first derivatives part of L contributes terms which vanish faster than the second derivative ones, inspection of the leading-order terms leads to the equation

$$\varphi_k \Delta_2 \varphi_k = 2|d\varphi_k|^2 \quad\Longleftrightarrow\quad \Delta_2 \varphi_k^{-1} = 0\,, \tag{5.25}$$

on the set $\{\varphi_k \neq 0\}$, where Δ_2 is the Laplace operator of the metric $d\rho^2 + d\zeta^2$. (Similarly, $(\mathscr{E}\equiv 0\,,\varphi)$ is a solution of (2.2)-(2.3) if and only if $\Delta_3 \varphi^{-1} = 0$, where Δ_3 is the Laplace operator of the metric $d\rho^2 + d\zeta^2 + \rho^2 d\phi^2$.)

We have the following:

THEOREM 5.2 *Homogeneous polynomial solutions of (5.25) are either holomorphic or anti-holomorphic.*

PROOF: Let φ_k be a homogeneous polynomial of order k solving (5.25), conveniently parameterized as

$$\varphi_k = \sum_{m=0}^{k} \alpha_m (z - z_0)^m (\bar{z} - \bar{z}_0)^{k-m} . \tag{5.26}$$

In complex notation the truncated Ernst–Maxwell equation (5.25) reads

$$\varphi_k \frac{\partial^2 \varphi_k}{\partial z \partial \bar{z}} = 2 \frac{\partial \varphi_k}{\partial z} \frac{\partial \varphi_k}{\partial \bar{z}} . \tag{5.27}$$

Inserting (5.26) into (5.27) we obtain

$$\sum_{1 \le m+j \le 2k-1} (k-m)(m-2j)\alpha_m \alpha_j (z-z_0)^{m+j-1} (\bar{z}-\bar{z}_0)^{2k-m-j-1} = 0 . \tag{5.28}$$

Hence, for $1 \le \ell \le 2k-1$:

$$\sum_{m+j=\ell,\, m \le k} (k-m)(m-2j)\alpha_m \alpha_j = 0 . \tag{5.29}$$

For $\ell \le k$ this equation can be written in the form

$$\sum_{m=0}^{\ell} (k-m)(3m-2\ell)\alpha_m \alpha_{\ell-m} = 0 . \tag{5.30}$$

We consider $\ell \le k$. For $\ell = 1$ we have

$$(k+1)\alpha_0 \alpha_1 = 0 .$$

Assume, first, that $\alpha_0 \neq 0$. Then $\alpha_1 = 0$, and for $\ell = 2$ we obtain

$$2(k+2)\alpha_0 \alpha_2 = 0 ,$$

thus $\alpha_2 = 0$. More generally, if we assume for some ℓ_0 that $\alpha_m = 0$ for $0 < m < \ell_0$ we have from (5.30)

$$\ell_0 (k+\ell_0)\alpha_0 \alpha_{\ell_0} = 0 \quad \Longrightarrow \quad \alpha_{\ell_0} = 0 .$$

We can repeat this argument for $\ell = \ell_0 + 1$ and continue up to $\ell = k$. Therefore, assumption $\alpha_0 \neq 0$ leads to $\alpha_m = 0$ for $0 < m \leq k$ and φ_k is holomorphic. Similarly, replacing above φ_k with its complex conjugate reveals that $\alpha_k \neq 0$ implies anti-holomorphicity of φ_k. Note that for $k = 1$ we are done.

Next, we assume $k \geq 2$ and we turn to the case $\alpha_0 = 0$, $\alpha_k = 0$. Again, we consider $\ell \leq k$. The equation with $\ell = 1$ has already been shown to be satisfied, but for $\ell = 2$ we have

$$(k-1)\alpha_1^2 = 0\,,$$

thus $\alpha_1 = 0$ since $k \neq 1$. The value of $\ell = 3$ gives no new conditions but for $\ell = 4$

$$(k-2)\alpha_2^2 = 0\,,$$

thus $\alpha_2 = 0$.

More generally, let us assume that $\alpha_m = 0$ for $0 \leq m < m_0 \leq k/2$, then (5.30) for $\ell = 2m_0$ implies

$$(k-m_0)\alpha_{m_0}^2 = 0\,,$$

hence we have a contradiction. We conclude that $\alpha_0 = 0$ implies $\alpha_m = 0$ for $0 \leq m \leq k/2$.

The above result applied to the complex conjugate of φ_k shows that $\alpha_k = 0$ implies $\alpha_m = 0$ for $k/2 \leq m < k$, as desired. □

5.3.1. φ-dominated leading-order solutions with singular ergocircles

We continue our analysis of φ of order $k \geq 1$, with the leading term of $\mathscr{E}$ of order $2k+1$ or higher, so that f is $O(r_0^{2k})$. (Note that some possibilities for $k = 1$ and $k = 2$ have already been eliminated at the beginning of Section 5.3.) Since the Ernst–Maxwell equations are invariant under transformation $\varphi \to c\varphi$, $\mathscr{E} \to \bar{c}c\mathscr{E}$, where c is a complex constant, we can without loss of generality assume that the Taylor development $\tilde{\varphi}$ of φ, as truncated at order $k+1$, takes the form

$$\tilde{\varphi} = (z-z_0)^k + \sum_{m=0}^{k+1} \alpha_m (z-z_0)^m (\bar{z}-\bar{z}_0)^{k+1-m}\,. \tag{5.31}$$

Similarly, we have

$$\mathscr{E}_{2k+1} = \sum_{m=0}^{2k+1} \iota_m (z-z_0)^m (\bar{z}-\bar{z}_0)^{2k+1-m}\,. \tag{5.32}$$

The function f takes the form

$$f = -(z - z_0)^k (\bar{z} - \bar{z}_0)^k + O\left(r_0^{2k+1}\right) . \tag{5.33}$$

The leading terms in the Ernst–Maxwell equations appear in order $4k - 1$ and $3k - 1$, respectively

$$\tilde{\varphi} \frac{\partial^2 \mathscr{E}_{2k+1}}{\partial z \partial \bar{z}} = \frac{\partial \tilde{\varphi}}{\partial z} \frac{\partial \mathscr{E}_{2k+1}}{\partial \bar{z}} , \tag{5.34}$$

$$2\overline{\tilde{\varphi}} \left\{ \tilde{\varphi} \left(\frac{\partial^2 \tilde{\varphi}}{\partial z \partial \bar{z}} + \frac{1}{2(z + \bar{z})} \frac{\partial \tilde{\varphi}}{\partial z} \right) - 2 \frac{\partial \tilde{\varphi}}{\partial z} \frac{\partial \tilde{\varphi}}{\partial \bar{z}} \right\} = \frac{\partial \mathscr{E}_{2k+1}}{\partial \bar{z}} \frac{\partial \tilde{\varphi}}{\partial z} . \tag{5.35}$$

It follows from (5.34) that

$$\frac{\partial \mathscr{E}_{2k+1}}{\partial \bar{z}} = \hat{C}(\bar{z}) \tilde{\varphi} , \tag{5.36}$$

where $\hat{C}(\bar{z})$ is arbitrary function of $\bar{z}$. However, we have assumed that $\mathscr{E}$ has leading term of order $2k + 1$. The comparison of (5.36) with (5.32) gives

$$\frac{\partial \mathscr{E}_{2k+1}}{\partial \bar{z}} = (k + 1) \iota_k (z - z_0)^k (\bar{z} - \bar{z}_0)^k , \tag{5.37}$$

thus, $\iota_m = 0$ for $m \neq k$ and $m \neq 2k + 1$.

(Somewhat more generally, an identical argument proves that if $\mathscr{E} = O(|z - z_0|^\ell)$ and $\varphi = O(|z - z_0|^k)$, with $2k < \ell$, φ holomorphic to leading order, then there exists $c \in \mathbb{C}$ such that $\mathscr{E}_\ell$ takes the form $\mathscr{E}_\ell = c(z - z_0)^k (\bar{z} - \bar{z}_0)^{\ell - k}$.)

The field equations imply

$$\frac{f^2}{\rho} \partial_z \ln \left(\left| \frac{h}{f} \right| \right) = \hat{\kappa} , \tag{5.38}$$

where

$$\begin{aligned} \hat{\kappa} \; := \; & \frac{1}{2} \left(\left(\frac{\partial \overline{\mathscr{E}}}{\partial z} + 2\varphi \frac{\partial \overline{\varphi}}{\partial z} + \frac{2f}{z + \bar{z}} \right) \left(\frac{\partial \mathscr{E}}{\partial z} + 2\overline{\varphi} \frac{\partial \varphi}{\partial z} \right) \right. \\ & + \left(\frac{\partial \mathscr{E}}{\partial z} + 2\overline{\varphi} \frac{\partial \varphi}{\partial z} + \frac{2f}{z + \bar{z}} \right) \left(\frac{\partial \overline{\mathscr{E}}}{\partial z} + 2\varphi \frac{\partial \overline{\varphi}}{\partial z} \right) \\ & \left. - 4 \frac{\partial \overline{\varphi}}{\partial z} \frac{\partial \varphi}{\partial z} \left(\mathscr{E} + \overline{\mathscr{E}} + 2\overline{\varphi}\varphi \right) \right) , \end{aligned} \tag{5.39}$$

and recall that the functions $\mathring{\sigma}_1$ and $\mathring{\sigma}_2$ have been defined in (5.3)–(5.4). We are going to show that if the conditions mentioned at the beginning of this section hold, then (5.35), (5.34) imply that

$$\mathring{\sigma}_2 = d\mathring{\sigma}_2 = \ldots = d^{2k}\mathring{\sigma}_2 = 0$$

and

$$\hat{\kappa} = d\hat{\kappa} = \ldots = d^{4k-2}\hat{\kappa} = 0$$

on $E_{\mathscr{E},\varphi}$ but $d^{4k-1}\hat{\kappa} = 0$ only for special solutions.

Inserting (5.31) and (5.37) into (5.35) gives

$$\sum_{m=0}^{k-1}(k+1-m)(m-2k)\alpha_m(z-z_0)^{k+m-1}(\bar{z}-\bar{z}_0)^{k-m} - k\left(\alpha_k + \frac{k+1}{2}\iota_k - \frac{1}{4\rho_0}\right)(z-z_0)^{2k-1} = 0\,. \tag{5.40}$$

The comparison of the coefficients in front of powers of $(z-z_0)$ and $(\bar{z}-\bar{z}_0)$ allows us to read off that $\alpha_m = 0$ for $m = 0, \ldots, k-1$. Moreover,

$$\alpha_k + \iota_k(k+1)/2 = \frac{1}{4\rho_0}$$

and there are no restrictions in the leading order on α_{k+1}, ι_{2k+1}. Hence

$$\begin{aligned}\tilde{\varphi} &= (z-z_0)^k + \alpha_k(z-z_0)^k(\bar{z}-\bar{z}_0) + \alpha_{k+1}(z-z_0)^{k+1}\,,\\ \mathscr{E}_{2k+1} &= \iota_k(z-z_0)^k(\bar{z}-\bar{z}_0)^{k+1}\,.\end{aligned}$$

Keeping this result in mind, we write down the leading terms of $\mathring{\sigma}_2$:

$$\begin{aligned}\mathring{\sigma}_2 &= -\frac{\partial\bar{\mathscr{E}}_{2k+1}}{\partial z} - 2\tilde{\varphi}\left(\frac{\partial\bar{\tilde{\varphi}}}{\partial z} - \frac{1}{2}\frac{\bar{\tilde{\varphi}}}{z+\bar{z}}\right) + O(r_0^{2k+1})\\ &= -2\left(\sum_{m=0}^{k}(k+1-m)\bar{\alpha}_m(\bar{z}-\bar{z}_0)^m(z-z_0)^{2k-m}\right.\\ &\qquad\left. + \left(\frac{k+1}{2}\bar{\iota}_k - \frac{1}{4\rho_0}\right)(\bar{z}-\bar{z}_0)^k(z-z_0)^k\right) + O(r_0^{2k+1})\\ &= O\left(r_0^{2k+1}\right)\,.\end{aligned} \tag{5.41}$$

Therefore, $\mathring{\sigma}_2$ is at least $O(r_0^{2k+1})$. Moreover, it follows from the identity

$$-2\frac{\partial f}{\partial z} = \mathring{\sigma}_1 - \mathring{\sigma}_2 - \frac{2f}{z+\bar{z}}\,, \tag{5.42}$$

that $\mathring{\sigma}_1$ is $O(r_0^{2k-1})$ *but not better*, because it has to compensate for the lowest terms of $\partial_z f$, see (5.33).

Now, we turn to $\hat{\kappa}$. Firstly, we rewrite (5.39) in terms of $\mathring{\sigma}_1$, $\mathring{\sigma}_2$

$$\hat{\kappa} = -\mathring{\sigma}_1\mathring{\sigma}_2 - \frac{f^2}{(z+\bar{z})^2} + 4\frac{\partial\bar{\varphi}}{\partial z}\frac{\partial\varphi}{\partial z}f\,. \tag{5.43}$$

It follows from our previous results and (5.33) that

$$\hat{\kappa} = -\left(\frac{1}{\rho_0} - 2(k+1)\bar{\iota}_k\right)k(z-z_0)^{2k-1}(\bar{z}-\bar{z}_0)^{2k} + O(r_0^{4k})\,. \tag{5.44}$$

Therefore, $\hat{\kappa}$ is only $O(r_0^{4k-1})$ for any

$$\iota_k \neq (2(k+1)\rho_0)^{-1}$$

and any solution with the above leading-order behavior, if it exists, will lead to a singular space-time metric (note, however, that this could be a coordinate singularity).

On the other hand if $\iota_k = (2(k+1)\rho_0)^{-1}$ then $\alpha_k = 0$ and φ is holomorphic also in the order $k+1$. For such solutions $\hat{\kappa}$ is at least $O(r_0^{4k})$, which is *not* incompatible in an *obvious* way with smoothness of the space-time metric at the ergosurface.

6. Concluding remarks

Our results are far from satisfactory, with the following questions open:

1. Which "solutions at leading order", as constructed above using Taylor series expansions (whether balanced, φ- or $\mathscr{E}$-dominated), *do arise* from real solutions of the Ernst–Maxwell equations which are smooth across the zero-level set of f? Here we mean that the associated harmonic map is smooth, without (in a first step) requesting that the associated space-time metric be smooth as well. The non-existence results mentioned at the beginning of Section 5.3 are instructive: there *do* exist Taylor polynomials solving the leading-order equations with $\varphi = O(|z-z_0|)$ with non-zero gradient at z_0 and with, say, $\mathscr{E} = 0$, and one has to go a few orders more in the Taylor series to show that the coefficients of the leading-order Taylor polynomial are all zero. The same mechanism applies to leading-order solutions with $\varphi = O(|z-z_0|^2)$ with non-zero Hessian at z_0.

2. Can one exhaustively describe the balanced leading-order solutions? The question seems hard. There does not seem, however, to be any good reason to invest a lot of energy therein as long as the previous question remains open.

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