Exact Solutions to Nonlocal Linear Dark Energy Models*

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Abstract

A general class of cosmological models driven by a nonlocal scalar field is studied. We show that the considering linear cosmogical model with a nonlocal field can be transformed to models with local scalar fields. This transformation allows to find exact special solutions of the nonlocal Einstein equations. The exact solution in the Friedman–Robertson–Walker and Bianchi I metrics are presented.

1. Introduction

Present cosmological observations [1] do not exclude an evolving dark energy (DE) state parameter w, whose current value is less than -1, which leads to violation of the NEC (see [2, 3] for a review of DE problems).

The purpose of this paper is to present recent results concerning studies of the string field theory (SFT) inspired nonlocal cosmological models (about string cosmology see review [4]). A Distinguished feature of these models [5]–[17] is the presence of an infinite number of higher derivative terms (note also nonlocal models in the Minkowski space-time [18]–[23]). For special values of the parameters these models describe linear approximations to the cubic bosonic or nonBPS fermionic SFT nonlocal tachyon models, or p-adic string models.

Field theories, which violate the NEC, are of interest not only for the construction of cosmological dark energy models with the state parameter $w < -1$, but also for the solution of the cosmological singularity problem. A simple possibility to violate the NEC is just to deal with a phantom field. In the present paper we consider nonlocal linear models with solutions,

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which are linear combinations of local fields. Some of these local fields are phantoms. Namely due to the presence of these ghost excitations such nonlocal models present an interest for cosmology.

The question of stability has to be consider in the full SFT framework and demands further investigations. We believe that due to these string theory origin the corresponding nonlocal cosmological models, which are nonlinear in matter fields, have no problem with instability both in classical and quantum cases. In this paper we consider only the classical case and models, which are linear in the Minkowski space-time.

In [11, 12, 13, 16, 17] nonlocal linear cosmological models have been studied. In paper [16] a systematic method that permits us to transform the initial nonlocal system into infinity set of local systems has been proposed for the SFT inspired nonlocal actions. In this paper we generalize this method on nonlocal action with an arbitrary analytic function $\mathcal{F}\left(-\Box_g\right)$, using formulas for nonlocal energy density and pressure proposed in [12]. The choice of a local system is equivalent to the choice of a special solution of the nonlocal system. We demonstrate that it is possible to find exact special solutions to nonlocal equations in the Friedmann–Robertson–Walker (FRW) and Bianchi I metrics.

2. Nonlocal linear models

In this paper we consider a model of gravity coupling with a nonlocal scalar field, which induced by string field theory

$$S=\int d^4x\sqrt{-g}\left(\frac{M_P^2}{2}R+\frac{M_s^2}{2g_4}\left(\phi\,\mathcal{F}\left(-\Box_g\right)\phi-\Lambda\right)\right), \qquad (1)$$

where $g_{\mu\nu}$ is the metric tensor (we use the signature $(-,+,+,+)$), $\Box_g=\frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu$, M_P is a mass Planck, M_s is a characteristic string scale related with the string tension α': $M_s=1/\sqrt{\alpha'}$, ϕ is a dimensionless scalar field, g_4 is a dimensionless four dimensional effective coupling constant related with the ten dimensional string coupling constant g_o and the compactification scale. We use dimensionless coordinates, which are initial coordinate, multiplied on M_s. The cosmological constant is $\Lambda M_s^2/(2g_4)$.

The string field theory inspired form of the function $\mathcal{F}$:

$$\mathcal{F}(z)=-\xi^2z+1-c\,e^{-2z}, \qquad (2)$$

where ξ and c are positive constants, has been considered in [11, 16]. In this paper we consider the case of arbitrary analytic function $\mathcal{F}$, namely, $\mathcal{F}(\Box_g)=\sum\limits_{n=0}^{\infty}c_n\Box_g^n$, where c_n are constants. The Einstein equations are

$$G_{\mu\nu}=\frac{1}{m_p^2}T_{\mu\nu}, \qquad (3)$$

where $G_{\mu\nu}$ is the Einstein tensor. The energy-momentum tensor

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}. \tag{4}$$

Note that the energy-momentum tensor $T_{\mu\nu}$ includes the nonlocal terms, so the Einstein equations are nonlocal ones.

From action (1) we also obtain the following equation of motion

$$\mathcal{F}(-\Box_g)\phi = 0 \tag{5}$$

Classical solutions to the above equations in the FRW metric were studied and analyzed in [11, 12, 16, 17]. In this paper we consider solutions, which depend only on time, in the FRW and Bianchi I metrics. So, for our propose it is sufficient to consider the energy-momentum tensor in the form of a perfect fluid

$$T_{\mu\nu} = g_{\mu\nu}\,\mathrm{diag}(\varrho, -p, -p, -p), \tag{6}$$

where

$$\begin{aligned}\varrho &= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^{n-1}c_n\left(\sum_{l=0}^{n-1}\partial_t(\Box_g)^l\phi\partial_t(\Box_g)^{n-1-l}\phi - \sum_{l=0}^{n}(\Box_g)^l\phi(\Box_g)^{n-l}\phi\right),\\ p &= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^{n-1}c_n\left(\sum_{l=0}^{n-1}\partial_t(\Box_g)^l\phi\partial_t(\Box_g)^{n-1-l}\phi + \sum_{l=0}^{n}(\Box_g)^l\phi(\Box_g)^{n-l}\phi\right).\end{aligned} \tag{7}$$

In the FRW metric the Einstein equations have the following form:

$$\left\{\begin{aligned} H^2 &= \frac{1}{3m_p^2}\varrho,\\ \dot{H} &= -\frac{1}{2m_p^2}(\varrho + p),\end{aligned}\right. \tag{8}$$

where $H = \dot{a}/a$ and dot denotes the time derivative. Hereafter we use the dimensionless parameter $m_p^2 = g_4 M_P^2/M_s^2$. The consequence of (8) is the following equation:

$$\dot{\varrho} + 3H(\varrho + p) = 0. \tag{9}$$

The main idea of finding a solution to the equations of motion is start with equation (5) and try to find a function ϕ, which is an eigenfunction of the box operator. If $\Box_g\phi = -\alpha^2\phi$, then such a function ϕ is a solution to (5) if and only if

$$\mathcal{F}(\alpha^2) = 0. \tag{10}$$

Note that values of roots do not depend on $H(t)$ and, therefore, coincide with roots in the Minkowski space-time. The analysis is more complicated in the case, when the function $\mathcal{F}(\alpha^2)$ has multiple roots. We skip this possibility for simplicity. Since equation (5) is linear one can take the following function as a solution

$$\phi = \sum_{k=1}^{N} \phi_k, \tag{11}$$

where $\Box_g \phi_k = -\alpha_k^2 \phi_k$ and $\mathcal{F}(\alpha_k^2) = 0$ for any k. Without loss of generality we assume that for any k_1 and $k_2 \neq k_1$ the conditions $\alpha_{k_1}^2 \neq \alpha_{k_2}^2$ are satisfied. The straightforward calculations allows to get from (7):

$$\begin{aligned} \varrho(\phi) &= \frac{1}{2}\sum_{k=1}^{N} \mathcal{F}'(\alpha_k^2)\left(\dot\phi_k^2 - \alpha_k^2\phi_k^2\right), \\ p(\phi) &= \frac{1}{2}\sum_{k=1}^{N} \mathcal{F}'(\alpha_k^2)\left(\dot\phi_k^2 + \alpha_k^2\phi_k^2\right), \end{aligned} \tag{12}$$

where $\mathcal{F}'(\alpha^2) \equiv \frac{d\mathcal{F}(\alpha^2)}{d\alpha^2}$.
System (8) is a nonlocal and nonlinear system of equation. At the same time using formulas (12) for the energy density and pressure it is possible to generate from (8) local systems, which correspond to particular solutions of the initial nonlocal system. Note that this method allows to find only solutions, which correspond to simple roots, because multiple roots correspond to $\mathcal{F}'(\alpha_k^2) = 0$, so the these energy density and pressure are not including in (12). If $\mathcal{F}(\alpha_k^2)$ are simple real roots, then positive and negative values of $\mathcal{F}'(\alpha_k^2)$ alternate, so we can obtain phantom fields.

Let us assume that the field ϕ has the form (11), in other words we assume that equations:

$$\Box_g \phi_k = -\alpha_k^2 \phi_k \tag{13}$$

are satisfied. Therefore, we can rewrite system (8) in the following form:

$$\left\{ \begin{aligned} 3H^2 &= \frac{1}{2m_p^2}\left(\Lambda + \sum_{k=1}^{N} \mathcal{F}'(\alpha_k^2)\left(\dot\phi_k^2 - \alpha_k^2\phi_k^2\right)\right), \\ \dot H &= -\frac{1}{2m_p^2}\sum_{k=1}^{N} \mathcal{F}'(\alpha_k^2)\dot\phi_k^2. \end{aligned} \right. \tag{14}$$

It is easy to check that equations (13) and (14) coincide with the Einstein equation for the following action:

$$S_{loc} = \int d^4x\sqrt{-g}\left(\frac{m_p^2 R}{2} + \frac{1}{2}\sum_{k=1}^{N} \mathcal{F}'\left(\alpha_k^2\right)\left(g^{\mu\nu}\partial_\mu\phi_k\partial_\nu\phi_k + \alpha_k^2\phi_k^2\right) - \Lambda\right).$$

Therefore, we can state that if the function F has a finite number of roots (denote this number as N) then in such way we can transform it in a local system with N local scalar or phantom scalar fields. In the opposite case, when the function F has an infinite number of roots, we obtain that our model with one nonlocal scalar fields generate an infinity number of local models.

3. Exact solutions in the FRW and Bianchi I metrics

3.1. A simple root at the zero root

Let us assume that $F(0) = 0$ and the zero point is a simple root. In this case the simplest solutions of nonlocal equations are solutions of the Einstein equations for the following local action:

$$S_{local} = \int d^4x\sqrt{-g}\left(\frac{m_p^2 R}{2} - \frac{1}{2}\mathcal{F}'(0)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \Lambda\right). \qquad (15)$$

In dependence of form of the function $\mathcal{F}(\alpha^2)$ the constant $B \equiv \mathcal{F}'(0)$ maybe either less or more than zero. Here we present exact solutions, which correspond to this root in the FRW and Bianchi I metrics. Note that a type of solutions essentially depends on signs of B and Λ. Some of solutions in the FRW metric have been found in [13, 16].

3.2. Solutions in the FRW metric

The Friedmann equations are as follows:

$$\left\{\begin{aligned} 3H^2 &= \frac{B}{2m_p^2}\dot\phi^2 + \frac{\Lambda}{m_p^2},\\ \dot H &= -\frac{B}{2m_p^2}\dot\phi^2. \end{aligned}\right. \qquad (16)$$

At $\Lambda = 0$ and $B > 0$

$$\phi(t) = \pm\sqrt{\frac{2m_p^2}{3B}}\ln(3(t-t_0)) + C_2, \qquad H(t) = \frac{1}{3(t-t_0)}, \qquad (17)$$

where t_0 and C_2 are arbitrary constants.

If $\Lambda > 0$, then we obtain a real solution:

$$H_0(t) = \sqrt{\frac{\Lambda}{3m_p^2}}\tanh\left(\sqrt{\frac{3\Lambda}{m_p^2}}(t-t_0)\right), \qquad (18)$$

where t_0 is an arbitrary complex number. Note that there exist such complex t_0 that $H_0(t)$ is real. For example, at $t_0 = \tilde{t}_0 + \frac{\pi}{2}i$, where $\tilde{t}_0$ is a real number, we obtain the following real solutions

$$\tilde{H}_0(t) = \sqrt{\frac{\Lambda}{3m_p^2}}\coth\left(\sqrt{\frac{3\Lambda}{m_p^2}}(t-\tilde{t}_0)\right). \qquad (19)$$

It is easy to see that $\dot{H}_0(t) > 0$ for any real t (if t_0 is real), hence, from the second equation of (16) we obtain that $\phi(t)$ can be real field only if it is a phantom one, namely, at $B < 0$. The explicit form of $\phi(t)$ is as follows:

$$\phi_0(t) = \pm\sqrt{-\frac{2m_p^2}{3B}}\arctan\left(\sinh\left(\sqrt{\frac{3\Lambda}{m_p^2}}(t-t_0)\right)\right) + C_2, \qquad (20)$$

where C_2 is an arbitrary constant.

The function $\tilde{H}_0(t)$ corresponds to the solution

$$\tilde{\phi}_0(t) = \pm\sqrt{\frac{2m_p^2}{3B}}\left(\ln\left(e^{\sqrt{3\Lambda/m_p^2}t}+1\right) - \ln\left(e^{\sqrt{3\Lambda/m_p^2}t}-1\right)\right) + C_2, \qquad (21)$$

which is real at $B > 0$.

Note that we have found two-parameter set of exact solutions at any $\Lambda > 0$. In other words, at any $\Lambda > 0$ we have found the general solution of (16), which correspond to $\alpha = 0$.

It is interesting that the type of solutions essentially depends on sign of Λ. In the case $\Lambda < 0$ the following real solutions have been obtained [16]:

$$\tilde{H} = \sqrt{\frac{-\Lambda}{3m_p^2}}\tan\left(\sqrt{-\frac{3\Lambda}{m_p^2}}(t-t_0)\right), \tilde{\tilde{H}} = \sqrt{\frac{-\Lambda}{3m_p^2}}\cot\left(\sqrt{-\frac{3\Lambda}{m_p^2}}(t-t_0)\right).$$

3.3. Solutions in Bianchi I metric

In Bianchi I metric with the interval

$$ds^2 = -dt^2 + a_1^2(t)dx_1^2 + a_2^2(t)dx_2^2 + a_3^2(t)dx_3^2, \qquad (22)$$

the Einstein equations has the following form:

$$H_1H_2 + H_1H_3 + H_2H_3 = \frac{1}{m_p^2}\varrho = \frac{1}{m_p^2}\left(\frac{B}{2}\dot{\phi}^2 + \Lambda\right), \qquad (23)$$

$$\dot{H}_2 + H_2^2 + \dot{H}_3 + H_3^2 + H_2H_3 = -\frac{1}{m_p^2}p = -\frac{1}{m_p^2}\left(\frac{B}{2}\dot{\phi}^2 - \Lambda\right), \qquad (24)$$

$$\dot{H}_1 + H_1^2 + \dot{H}_2 + H_2^2 + H_1 H_2 = -\frac{1}{m_p^2} p = -\frac{1}{m_p^2}\left(\frac{B}{2}\dot{\phi}^2 - \Lambda\right), \qquad (25)$$

$$\dot{H}_1 + H_1^2 + \dot{H}_3 + H_3^2 + H_1 H_3 = -\frac{1}{m_p^2} p = -\frac{1}{m_p^2}\left(\frac{B}{2}\dot{\phi}^2 - \Lambda\right), \qquad (26)$$

where $H_k \equiv \dot{a}_k/a_k$, $k = 1, 2, 3$.

Our goal is to present exact solutions to system (23)–(26). Of course, there exist isotropic solutions, which coincide with exact solutions in the FRW metric. For those solutions $H_1(t) = H_2(t) = H_3(t)$. At the same time exact anisotropic solutions do exist.

For $\Lambda = 0$ we obtain the following solution:

$$H_1(t) = \frac{C_2 + C_1 + 1}{C_2 t + C_3}, \quad H_2(t) = -\frac{C_1}{C_2 t + C_3}, \quad H_3(t) = -\frac{1}{C_2 t + C_3} \qquad (27)$$

$$\phi(t) = \frac{\sqrt{-2B(C_1C_2 + C_1^2 + C_1 + C_2 + 1)m_p^2}}{BC_2} \ln(C_2 t + C_3) + C_4. \qquad (28)$$

Let us consider the case of positive $\Lambda = m_p^2$. There exist not only isotropic solution

$$H_1(t) = H_2(t) = H_3(t) = \frac{1}{\sqrt{3}} \tanh\left(\sqrt{3}(t - t_0)\right), \qquad (29)$$

but also an anisotropic one

$$\begin{aligned} H_1(t) &= \frac{1}{\sqrt{3}} \tanh\left(\frac{\sqrt{3}}{2}(t - t_0)\right), \\ H_2(t) &= \frac{1}{\sqrt{3}} \coth\left(\frac{\sqrt{3}}{2}(t - t_0)\right), \\ H_3(t) &= \frac{1}{2\sqrt{3}}\left(\tanh\left(\frac{\sqrt{3}}{2}(t - t_0)\right) + \coth\left(\frac{\sqrt{3}}{2}(t - t_0)\right)\right). \end{aligned} \qquad (30)$$

The corresponding scalar field is real at $B > 0$

$$\tilde{\phi}(t) = \frac{2m_p}{3\sqrt{B}}\left(\ln(e^{\sqrt{3}(t-t_0)} + 1) - \ln(e^{\frac{\sqrt{3}}{2}(t-t_0)} - 1) - \ln(e^{\frac{\sqrt{3}}{2}}(t - t_0)) + 1)\right).$$

4. Conclusions

We have studied the SFT inspired linear nonlocal model. This model has an infinite number of higher derivative terms. Roots of the characteristic equation do not depend on the form of the metric and this property allows us to study properties of energy density and pressure. We have found that in an arbitrary metric the energy-momentum tensor for an arbitrary N-mode solution is a sum of the energy-momentum tensors for the corresponding one-mode solutions.

The investigation performed in this paper shows that the general field equations in linear nonlocal models admit an equivalent description in terms of local theory and as a consequences we have representation (12) for the energy density and pressure. This calculation also supports the use of the Ostrogradski representation for our system in the case of arbitrary metric.

We have shown that our linear model with one nonlocal scalar field generates an infinite number of local models. Special exact solutions for the nonlocal model in the FRW and Bianchi I metrics have been obtained.

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