CANONICAL QUANTIZATION OF CONSTRAINED SYSTEMS*

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The canonical formalism for quantizing theories with constraints (Dirac formalism) is presented. The method is extended to superspace and equivalence with action principle quantization is shown.

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1. *Introduction*

Canonical quantization in a theory consists of obtaining the Poisson bracket relations between any two physical variables and carrying them over to the quantum commutation or anticommutation relation with an $i\hbar$ prescription, namely,

$$[A_{\mathrm{op}}, B_{\mathrm{op}}]_{\pm} = i\hbar\{A, B\}. \tag{1.1}$$

This method works quite well in quantum mechanics where for example

$$[q_{\mathrm{op}}, p_{\mathrm{op}}] = i\hbar\{q, p\} = i\hbar. \tag{1.2}$$

However, in physical theories of quantum fields, which often contain constraints, this method leads to inconsistencies. A quick way of seeing this is to suppose that there exists a constraint in our theory given by

$$\Gamma(q, p) = 0. \tag{1.3}$$

According to our prescription, in passage to the quantum theory, this must map to the null operator.

$$\Gamma(q, p) \to \phi_{\mathrm{op}}. \tag{1.4}$$

It follows, therefore, that

$$[A_{\mathrm{op}}, \Gamma_{\mathrm{op}}]_{\pm} = [A_{\mathrm{op}}, \phi_{\mathrm{op}}]_{\pm} = i\hbar\{A, \Gamma\}. \tag{1.5}$$

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The left hand side of this expression clearly vanishes as it is the commutator or anti-commutator of an operator with the null operator. The classical Poisson bracket on the right hand side, however, is not in general zero. There is an inconsistency.

In the case of constrained systems, therefore, the naive quantization procedure has to be modified. Dirac [1] recognized that the consistent way to quantize such a theory is to modify the naive Poisson brackets such that the new brackets (known as the Dirac brackets) between a physical variable and a constraint vanish. Consequently one can write the quantum relations as

$$[A_{\text{op}}, B_{\text{op}}]_{\pm} = i\hbar\{A, B\}_{\text{D}} \tag{1.6}$$

and the inconsistency is overcome.

Dirac has given a detailed and systematic way of handling constrained systems [2] and we will discuss this in Section 2 where we will also work out the particular cases of Maxwell's theory and the nonlinear sigma model. In the next section we will define the Poisson brackets for systems containing anticommuting coordinates [3]. The natural geometrical framework for a supersymmetric theory [4] is the superspace where in addition to the usual bosonic coordinates, there are anticommuting coordinates. Furthermore, supersymmetric theories inherently contain constraints as we will explain in Section 4. Canonical quantization in superspace must, therefore, follow Dirac's procedure. As we will show in Section 4, Dirac's method carries over to superspace only if it is not extended blindly [5]. We discuss the results of superspace quantization only for the supersymmetric quantum mechanics [5] although we have obtained results for the SUSY nonlinear σ model and the chiral superfield [6] as well. Finally in Section 5 we discuss very briefly about the relation between Dirac's method and that of action principle quantization [7] with concluding remarks in Section 6.

2. *Dirac quantization*

Let us consider a Lagrangian

$$L = L(q_i, \dot{q}_i) \qquad i = 1, 2, \ldots, N, \tag{2.1}$$

where q_i and $\dot{q}_i$ represent N coordinates and velocities. The canonical momentum is defined by

$$p^i = \frac{\partial L}{\partial \dot{q}_i}. \tag{2.2}$$

If we now want to go over to the Hamiltonian formalism we are looking at the transformation

$$(q_i, \dot{q}_i) \to (q_i, p^i). \tag{2.3}$$

The Jacobian of this transformation is determined by the matrix

$$\frac{\partial p^i}{\partial \dot{q}_j} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}. \tag{2.4}$$

If this matrix is nonsingular, then the transformation is unique and the naive canonical quantization procedure goes through.

However, in most physical cases in quantum field theory, this matrix is singular. Consequently the transformation and hence the Hamiltonian for the system becomes nonunique.

If we now analyze Eq. (2.4), then

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\right) = 0 \tag{2.5}$$

implies that some of the momenta are not independent variables. Let the rank of the matrix $\left(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\right)$ be R where $R < N$. Then we can solve for R velocities as

$$\dot{q}^a = f^a(q, p^b, \dot{q}^\alpha) \quad a, b = 1, \ldots, R; \quad \alpha = R+1, \ldots, N. \tag{2.6}$$

Reinserting this into the definition of the momenta we obtain

$$p_i = \frac{\partial L}{\partial \dot{q}^i}(q, f^a, \dot{q}^\alpha) = g_i(q, p^a, \dot{q}^\alpha).$$

Of course, for $a = 1, \ldots, R$

$$p_a = g_a$$

but for $\alpha = R+1, \ldots, N$ one can show that

$$p_\alpha = g_\alpha(q, p_a) \quad a = 1, \ldots, R.$$

Thus we define

$$\Gamma_\alpha = p_\alpha - g_\alpha(q, p_a) \quad \alpha = R+1, \ldots, N; \quad a = 1, \ldots, R \tag{2.7}$$

and the equations

$$\Gamma_\alpha = 0 \quad \alpha = R+1, \ldots, N$$

which are called the primary constraints define a $2N-(N-R) = N+R$ dimensional hypersurface Γ_c in the phase space Γ. We call two functions A, B on Γ weakly equal, $A \approx B$, if they are equal on Γ_c, namely,

$$(A-B)|_{\Gamma_c} = 0.$$

Let us define the canonical Hamiltonian H_c as

$$H_c(q, p_a \dot{q}^\alpha) = \sum_{b=1}^{R} p_b f^b(q, p_a, \dot{q}^\alpha) + \sum_{\alpha=R+1}^{N} \dot{q}^\alpha g_\alpha(q, p_a) - L(q, f^a, \dot{q}^\alpha)$$

$$a = 1, \ldots, R; \quad \alpha = R+1, \ldots, N \tag{2.8}$$

which has the properties

$$\frac{\partial H_c}{\partial \dot{q}^\alpha} = 0 \qquad \alpha = R+1, \dots, N.$$

That is, $H_c = H_c(q, p_a)$ and more importantly

$$\dot{q}^i \approx \frac{\partial}{\partial p_i}\left(H_c + \sum_{\alpha=R+1}^{N} \dot{q}^\alpha \Gamma_\alpha\right), \quad \frac{\partial L}{\partial q^i} \approx -\frac{\partial}{\partial q^i}\left(H_c + \sum_{\alpha=R+1}^{N} \dot{q}^\alpha \Gamma_\alpha\right).$$

For solutions of the Euler Lagrange equation $\dot{p}^i = \frac{\partial L}{\partial q^i}$, therefore, we have (summation convention implied)

$$\dot{q}^i \approx \{q^i, H_c + \dot{q}^\alpha \Gamma_\alpha\}, \qquad \dot{p}_i \approx \{p_i, H_c + \dot{q}^\alpha \Gamma_\alpha\}. \tag{2.9}$$

The $\dot{q}^\alpha$ $\alpha = R+1, \dots, N$ remain undetermined (since their Heisenberg equations of motions reduce to identities) and we shall from now on denote them by λ_α and define the primary Hamiltonian as $H_p = H_c + \lambda_\alpha \Gamma_\alpha$ with the undetermined coefficient functions λ_α. Note that H_p contains only the $m = N - R$ primary constraints Γ_α. Because of Eq. (2.9) H_p gives the time evolution for any phase space function $A(q, p)$ not explicitly dependent on time as

$$\dot{A}(q, p) \approx \{A(q, p), H_p\}. \tag{2.10}$$

Furthermore, we want the constraints to have no dynamical evolution which requires

$$\dot{\Gamma}_\alpha \approx \{\Gamma_\alpha, H_p\} \approx \{\Gamma_\alpha, H_c\} + \lambda_\beta \{\Gamma_\alpha, \Gamma_\beta\} \approx 0. \tag{2.11}$$

As is obvious Eq. (2.11) may either determine some of the unknown Lagrange multipliers or may give rise to more functional relations between momenta and coordinates known as secondary constraints. One continues this process until all the constraints are determined to be evolution free.

Let us say, at this point, that the total number of constraints in the system is n, $n < 2N$, and are given by

$$\Gamma_\alpha \approx 0 \qquad \alpha = 1, 2, \dots, n. \tag{2.12}$$

As is evident

$$\{\Gamma_\alpha, H_p\} \approx 0.$$

Next, let us divide the constraints into two classes.

(*i*) Those constraints which have weakly vanishing Poisson brackets with every other constraint are called first class constraints, ψ_α, $\alpha = 1, \dots, n_1$.

(*ii*) Those which have at least one nonvanishing Poisson bracket with the other constraints are known as second class, ϕ_α, $\alpha = 1, \dots n_2$; such that $n_1 + n_2 = n$.

The first class constraints ψ_α are associated with local gauge invariances and one chooses gauge fixing conditions γ_β as additional constraints such that the first class constraints become second class. Therefore, after gauge fixing all constraints become second class and Dirac has shown that they must be even in number. Let us denote them now collectively as

$$\Gamma_\alpha \approx 0 \quad \alpha = 1, 2, \ldots, 2p, 2p < N. \tag{2.13}$$

Since these are all second class constraints, one can define the matrix of their Poisson brackets as

$$C_{\alpha\beta} \approx \{\Gamma_\alpha, \Gamma_\beta\}. \tag{2.14}$$

Note that the matrix $C_{\alpha\beta}$ is antisymmetric and Dirac had shown that it is nonsingular so that its inverse $C^{-1}_{\alpha\beta}$ exists.

Next let us define a modified Poisson bracket (Dirac bracket) between two variables A and B as

$$\{A, B\}_{\mathrm{D}} = \{A, B\} - \{A, \Gamma_\alpha\} C^{-1}_{\alpha\beta} \{\Gamma_\beta, B\}. \tag{2.15}$$

Note that the Dirac bracket is defined such that any variable has a weakly vanishing Dirac bracket with any constraint, i.e.,

$$\{A, \Gamma_\alpha\}_{\mathrm{D}} = \{A, \Gamma_\alpha\} - \{A, \Gamma_\beta\} C^{-1}_{\beta\gamma} \{\Gamma_\gamma, \Gamma_\alpha\}$$

$$\approx \{A, \Gamma_\alpha\} - \{A, \Gamma_\beta\} C^{-1}_{\beta\gamma} C_{\gamma\alpha} = \{A, \Gamma_\alpha\} - \{A, \Gamma_\alpha\} = 0. \tag{2.16}$$

Although we have chosen not to do so, it can be shown using the method of Lagrange brackets that the Dirac bracket is indeed the Poisson bracket when evaluated subject to the constraints in Eq. (2.13).

It is now straightforward to go over to the quantum theory using

$$[A_{\mathrm{op}}, B_{\mathrm{op}}]_\pm = i\hbar \{A, B\}_{\mathrm{D}}. \tag{2.17}$$

A few comments are in order. First of all a very useful property of the Dirac brackets is the iterative property. That is, if there are a large number of constraints, one does not have to invert a large matrix but rather one can focus on a subset of all the constraints and define an intermediate Dirac bracket and so on. Secondly although we have worked out everything for a finite dimension, the method extends readily to continuum field theory. One must, however, recognize that integration over intermediate variables is implied in relations such as Eq. (2.15) and appropriate boundary conditions might be required (e.g. to render C^{-1} unique).

Let us now apply the method to two simple theories.

a) Maxwell's theory:

$$L = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu},$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad \mu, \nu = 0, 1, 2, 3. \tag{2.18}$$

Our metric conventions are those of Bjorken and Drell.

The canonical momenta are defined as

$$\Pi^{\mu}(x) = \frac{\partial L}{\partial \dot{A}_{\mu}(x)}$$

from which we obtain

$$\vec{\Pi}(x) = -(\dot{\vec{A}}+\vec{\nabla}A^0), \quad \Pi^0(x) \approx 0. \tag{2.19}$$

According to Dirac's procedure, we now modify the canonical Hamiitonian as

$$\begin{aligned} H_p &= H_c + \int d^3x \lambda_1(x)\Pi^0(x) \\ &= \int d^3x (\tfrac{1}{2}\vec{\Pi}^2 + \tfrac{1}{2}\vec{B}^2 + \lambda_1(x)\Pi^0(x) - A^0(x)\vec{\nabla}\cdot\vec{\Pi}(x)). \end{aligned} \tag{2.20}$$

Here we have introduced the magnetic field defined by

$$\vec{B} = \vec{\nabla}\times\vec{A}. \tag{2.21}$$

With the naive Poisson bracket relations

$$\begin{aligned} \{A^{\mu}(x), \Pi_{\nu}(y)\} &= \delta^{\mu}_{\nu}\delta^3(x-y), \\ \{A^{\mu}(x), A_{\nu}(y)\} &= 0 = \{\Pi^{\mu}(x), \Pi_{\nu}(y)\}, \end{aligned} \tag{2.22}$$

we can calculate the time evolution of our primary constraint.

$$\dot{\Pi}^0(x) = \{\Pi^0(x), H_p\} = \vec{\nabla}\cdot\vec{\Pi} \approx 0. \tag{2.23}$$

Furthermore, this constraint can be easily seen to have no dynamical evolution. Thus the constraints of our system are

$$\Pi^0(x) \approx 0, \quad \vec{\nabla}\cdot\vec{\Pi}(x) \approx 0 \tag{2.24}$$

and they are first class constraints. This is expected because of the local U(1) invariance of the Maxwell's theory. Correspondingly we have to choose a gauge and we choose the Coulomb gauge

$$\vec{\nabla}\cdot\vec{A}(x) \approx 0. \tag{2.25}$$

The constraint on the time evolution of this condition leads further to the condition

$$A^0(x) \approx 0. \tag{2.26}$$

Thus the entire set of constraints in this case can be written as

$$\Gamma_1 = \Pi^0(x) \approx 0, \quad \Gamma_2 = \vec{\nabla}\cdot\vec{\Pi}(x) \approx 0, \quad \Gamma_3 = A^0(x) \approx 0, \quad \Gamma_4 = \vec{\nabla}\cdot\vec{A}(x) \approx 0. \tag{2.27}$$

These constraints are all second class and consequently one can calculate the matrix of Poisson bracket of constraints and it turns out to be

$$C_{\alpha\beta}(x, y) = \{\Gamma_\alpha(\vec{x}, t), \Gamma_\beta(\vec{y}, t)\}$$
$$= \begin{Bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\vec{\nabla}_x^2 \\ +1 & 0 & 0 & 0 \\ 0 & +\vec{\nabla}_x^2 & 0 & 0 \end{Bmatrix} \delta^3(x-y). \tag{2.28}$$

We would assume the boundary condition that all fields vanish at infinite separations and this determines the inverse of the matrix to be

$$C_{\alpha\beta}^{-1}(x, y) = \begin{Bmatrix} 0 & 0 & +\delta^3(x-y) & 0 \\ 0 & 0 & 0 & -\dfrac{1}{4\pi|\vec{x}-\vec{y}|} \\ -\delta^3(x-y) & 0 & 0 & 0 \\ 0 & +\dfrac{1}{4\pi|\vec{x}-\vec{y}|} & 0 & 0 \end{Bmatrix}. \tag{2.29}$$

With this we can now calculate the fundamental Dirac brackets to be

$$\{A^\mu, A^\nu\}_{\mathrm{D}} = 0 = \{\Pi^\mu, \Pi^\nu\}_{\mathrm{D}}$$

$$\{A^\mu, \Pi^\nu\}_{\mathrm{D}} = (g^{\mu\nu} - g^{\mu 0} g^{\nu 0})\delta^3(x-y) + \partial_x^\mu \partial_y^\nu \frac{1}{4\pi|\vec{x}-\vec{y}|}. \tag{2.30}$$

b) Nonlinear sigma model:

The model is defined in $1+1$ dimensions as

$$L = \tfrac{1}{2}\, \partial_\mu \phi^i \partial^\mu \phi^i \qquad i = 1, 2, \ldots, n$$

with the constraint

$$\phi^i \phi^i = 1. \tag{2.31}$$

We can incorporate the constraint into the Lagrangian and write

$$L = \tfrac{1}{2}\, \partial_\mu \phi^i \partial^\mu \phi^i + \frac{\lambda}{2}(\phi^i \phi^i - 1). \tag{2.32}$$

Here λ is a Lagrange multiplier field. The canonical momenta are given by

$$\Pi^i = \frac{\partial L}{\partial \dot{\phi}^i} = \dot{\phi}^i, \quad \Pi_\lambda = \frac{\partial L}{\partial \dot{\lambda}} \approx 0. \tag{2.33}$$

Thus we see that the primary constraint of the theory is

$$\Pi_\lambda \approx 0. \tag{2.34}$$

Consequently according to Dirac's procedure, we define the primary Hamiltonian to be

$$
\begin{aligned}
H_p &= H_c + \int dx u \Pi_\lambda \\
&= \int dx \left(\tfrac{1}{2} \Pi^i \Pi^i + \tfrac{1}{2} \vec{\nabla}\phi^i \cdot \vec{\nabla}\phi^i - \frac{\lambda}{2}(\phi^i\phi^i - 1) + u\Pi_\lambda \right).
\end{aligned}
\tag{2.35}
$$

Note that

$$
\dot{\Pi}_\lambda = \{\Pi_\lambda, H_p\} = \tfrac{1}{2}(\phi^i\phi^i - 1) \approx 0. \tag{2.36}
$$

Requiring this constraint to be invariant in time leads to

$$
\{\phi^i\phi^i - 1, H_p\} = 2\phi^i\Pi^i \approx 0. \tag{2.37}
$$

Furthermore, the time evolution equation of this constraint implies

$$
\{\phi^i\Pi^i, H_p\} \approx \lambda + \Pi^i\Pi^i + \phi^i\vec{\nabla}^2\phi^i \approx 0. \tag{2.38}
$$

It is easy to check that the invariance of this constraint in time does not lead to any new constraints. Thus we can write all of our constraints as

$$
\begin{aligned}
&\Gamma_1 = \phi^i\phi^i - 1 \approx 0, \quad \Gamma_2 = \phi^i\Pi^i \approx 0, \quad \Gamma_3 = \Pi_\lambda \approx 0, \\
&\Gamma_4 = \lambda + \Pi^i\Pi^i + \phi^i\vec{\nabla}^2\phi^i \approx 0.
\end{aligned}
\tag{2.39}
$$

These constraints are all second class and the matrix of the Poisson bracket of the constraints is obtained to be

$$
C_{\alpha\beta}(x, y) \approx \{\Gamma_\alpha(x, t), \Gamma_\beta(y, t)\} = \left\{
\begin{matrix}
0 & 2 & 0 & 0 \\
-2 & 0 & 0 & \lambda(x) + 3\Pi^i(x)\Pi^i(x) - \phi^i(x)\phi^i(y)\vec{\nabla}_y^2 \\
0 & 0 & 0 & -1 \\
0 & -(\lambda(x) + 3\Pi^i(x)\Pi^i(x) - \phi^i(x)\phi^i(y)\vec{\nabla}_y^2) & 1 & 0
\end{matrix}
\right\} \delta(x-y). \tag{2.40}
$$

The inverse is then determined to be

$$
C_{\alpha\beta}^{-1}(x, y) = \tfrac{1}{2} \left\{
\begin{matrix}
0 & -1 & -(\lambda(x) + 3\Pi^i(x)\Pi^i(x) - \phi^i(x)\phi^i(y)\vec{\nabla}_y^2) & 0 \\
1 & 0 & 0 & 0 \\
\lambda(x) + 3\Pi^i(x)\Pi^i(x) - \phi^i(x)\phi^i(y)\vec{\nabla}_y^2 & 0 & 0 & 2 \\
0 & 0 & -2 & 0
\end{matrix}
\right\} \delta(x-y). \tag{2.41}
$$

We can now calculate the fundamental Dirac brackets as

$$\{\phi^i(x), \phi^j(y)\}_D = 0,$$

$$\{\phi^i(x), \Pi^j(y)\}_D = (\delta^{ij} - \phi^i\phi^j)\delta(x-y),$$

$$\{\Pi^i(x), \Pi^j(y)\}_D = -(\phi^i\Pi^j - \phi^j\Pi^i)\delta(x-y). \tag{2.42}$$

This completes our discussion of Dirac quantization method when applied to regular field theories.

3. *Poisson brackets for a system containing anticommuting variables*

Let us consider a classical Lagrangian $L(x, \dot{x}, \theta, \dot{\theta})$ which depends not only on the bosonic coordinates x_i and velocities $\dot{x}_i$ but also on the fermionic coordinates θ_α and velocities $\dot{\theta}_\alpha$. Because of the anticommuting nature of the fermionic quantities, there exists an arbitrariness in the definition of derivatives with respect to these coordinates and we choose to use a left derivative so that in

$$\frac{\partial\phi(\theta)}{\partial\theta_\alpha}$$

the θ derivative acts from the left and

$$\delta\phi(\theta) = \delta\theta_\alpha \frac{\partial\phi(\theta)}{\partial\theta_\alpha}. \tag{3.1}$$

We can now define the canonical momenta as

$$p^i = \frac{\partial L}{\partial \dot{x}_i}, \quad \Pi^\alpha = \frac{\partial L}{\partial \dot{\theta}_\alpha}, \tag{3.2}$$

so that the Euler-Lagrange equations can be written as

$$\frac{dp^i}{dt} = \frac{\partial L}{\partial x_i}, \quad \frac{d\Pi^\alpha}{dt} = \frac{\partial L}{\partial \theta_\alpha}. \tag{3.3}$$

The Hamiltonian for the system is simply given by

$$H = p^i\dot{x}_i + \dot{\theta}_\alpha \Pi^\alpha - L, \tag{3.4}$$

so that

$$\begin{aligned} \delta H &= \delta p^i \dot{x}_i + \dot{\theta}_\alpha \delta\Pi^\alpha - \delta x_i \frac{\partial L}{\partial x_i} - \delta\theta_\alpha \frac{\partial L}{\partial \theta_\alpha} \\ &= \delta p^i \dot{x}_i + \dot{\theta}_\alpha \delta\Pi^\alpha - \delta x_i \dot{p}^i - \delta\theta_\alpha \dot{\Pi}^\alpha. \end{aligned} \tag{3.5}$$

Hamilton's equations are now obtained as

$$\frac{\partial H}{\partial x_i} = -\dot{p}^i, \quad \frac{\partial H}{\partial \theta_\alpha} = -\dot{\Pi}^\alpha,$$

$$\frac{\partial H}{\partial p^i} = \dot{x}_i, \quad \frac{\partial H}{\partial \Pi^\alpha} = -\dot{\theta}_\alpha. \tag{3.6}$$

It is worth noting here the signs of the fermionic equations compared to those of the bosonic equations and we would like to emphasize that it is due to our particular choice of convention for the fermionic derivatives.

It is now clear that given any physical quantity $A(x, p, \theta, \Pi)$ which is bosonic in nature and depends on the bosonic as well as the fermionic variables, its change with time can be written as

$$\frac{dA}{dt} = \frac{\partial A}{\partial x_i}\dot{x}_i + \frac{\partial A}{\partial p_i}\dot{p}_i + \dot{\theta}_\alpha \frac{\partial A}{\partial \theta_\alpha} + \dot{\Pi}_\alpha \frac{\partial A}{\partial \Pi_\alpha}$$

$$= \frac{\partial A}{\partial x_i}\frac{\partial H}{\partial p^i} - \frac{\partial A}{\partial p^i}\frac{\partial H}{\partial x_i} + \frac{\partial A}{\partial \theta_\alpha}\frac{\partial H}{\partial H^\alpha} + \frac{\partial A}{\partial \Pi^\alpha}\frac{\partial H}{\partial H_\alpha} = \{A, H\}. \tag{3.7}$$

This, therefore, defines for us the fundamental Poisson bracket between two bosonic quantities as

$$\{B_1, B_2\} = \frac{\partial B_1}{\partial x_i}\frac{\partial B_2}{\partial p^i} - \frac{\partial B_1}{\partial p^i}\frac{\partial B_2}{\partial x_i} + \frac{\partial B_1}{\partial \theta_\alpha}\frac{\partial B_2}{\partial \Pi^\alpha} + \frac{\partial B_1}{\partial \Pi^\alpha}\frac{\partial B_2}{\partial \theta_\alpha}. \tag{3.8}$$

The Poisson bracket relations between a bosonic and a fermionic quantity as well as between two fermionic quantitites is now simply obtained from Eq. (3.8) as

$$\{B_1, F_1\} = \frac{\partial B_1}{\partial x_i}\frac{\partial F_1}{\partial p^i} - \frac{\partial B_1}{\partial p^i}\frac{\partial F_1}{\partial x_i} + \frac{\partial B_1}{\partial \theta_\alpha}\frac{\partial F_1}{\partial \Pi^\alpha} + \frac{\partial B_1}{\partial \Pi^\alpha}\frac{\partial F_1}{\partial \theta_\alpha}, \tag{3.9}$$

$$\{F_1, F_2\} = \frac{\partial F_1}{\partial x_i}\frac{\partial F_2}{\partial p^i} - \frac{\partial F_1}{\partial p^i}\frac{\partial F_2}{\partial x_i} - \frac{\partial F_1}{\partial \theta_\alpha}\frac{\partial F_2}{\partial \Pi^\alpha} - \frac{\partial F_1}{\partial \Pi^\alpha}\frac{\partial F_2}{\partial \theta_\alpha}. \tag{3.10}$$

Note that all these relations are true if a left derivative is used for the fermions. In particular, in our convention, the canonical Poisson bracket relations become

$$\{x_i, x_j\} = 0 = \{p_i, p_j\}, \quad (x_i, p_j\} = \delta_{ij},$$

$$\{\theta_\alpha, \theta_\beta\} = 0 = \{\Pi_\alpha, \Pi_\beta\}, \quad \{\theta_\alpha, \Pi_\beta\} = -\delta_{\alpha\beta}. \tag{3.11}$$

The continuation to a classical field theory where the fields depend continuously on the coordinates x and θ is straightforward. In particular if we have a field theory described by $L(\phi,$ *derivatives of* $\phi)$, where $\phi(x, \theta)$ is a bosonic field, then one can define the canonical momentum as

$$\Pi(x,\theta) = \frac{\partial L}{\partial\dot{\phi}(x,\theta)}. \tag{3.12}$$

The canonical Poisson bracket relations would then be given by

$$\{\phi(x,\theta), \phi(x',\theta')\}_{x_0=x_0'} = 0,$$

$$\{\Pi(x,\theta), \Pi(x',\theta')\}_{x_0=x_0'} = 0,$$

$$\{\phi(x,\theta), \Pi(x',\theta')\}_{x_0=x_0'} = \delta(x-x')\delta(\theta-\theta'). \tag{3.13}$$

In the quantum theory, therefore, the quantization condition would correspond to the fundamental commutation relations

$$[\phi(x,\theta), \phi(x',\theta')]_{x_0=x_0'} = 0 = [\Pi(x,\theta), \Pi(x',\theta')]_{x_0=x_0'}$$

$$[\phi(x,\theta), \Pi(x',\theta')]_{x_0=x_0'} = i\hbar\delta(x-x')\delta(\theta-\theta'). \tag{3.14}$$

4. *Supersymmetric quantum mechanics*

Let us consider the superfield $\phi^i(t,\theta,\bar{\theta})$ which not only depends on time but also on the anticommuting coordinates θ and $\bar{\theta}$ [8]. Here i is an internal symmetry index. Further, let us define the derivatives

$$D = \frac{\partial}{\partial\bar{\theta}} - i\theta\frac{\partial}{\partial t}, \quad \bar{D} = \frac{\partial}{\partial\theta} - i\bar{\theta}\frac{\partial}{\partial t}. \tag{4.1}$$

These are called the covariant derivatives in the extended space spanned by t, θ, $\bar{\theta}$ also known as the superspace.

A general action in superspace has the form

$$S = \int dt d\theta d\bar{\theta} L(\phi^i, D\phi^i, \bar{D}\phi^i) \tag{4.2}$$

and is invariant under the supersymmetry transformations

$$\delta\phi^i(t,\theta,\bar{\theta}) = \xi\left(\frac{\partial}{\partial\bar{\theta}} + i\theta\frac{\partial}{\partial t}\right)\phi^i(t,\theta,\bar{\theta}) \tag{4.3}$$

and

$$\bar{\delta}\phi^i(t,\theta,\bar{\theta}) = \bar{\xi}\left(\frac{\partial}{\partial\theta} + i\bar{\theta}\frac{\partial}{\partial t}\right)\phi^i(t,\theta,\bar{\theta}), \tag{4.4}$$

ξ and $\bar{\xi}$ here are constant anticommuting parameters.

From various considerations one can choose the general form of the Lagrangian to be [8]

$$L(\phi^i, D\phi^i, \bar{D}\phi^i) = -\tfrac{1}{2}\bar{D}\phi^i D\phi^i - V(\phi^i), \tag{4.5}$$

where $V(\phi^i)$ is a polynomial in ϕ^i satisfying the internal symmetry invariance. If we Taylor expand the superfield $\phi^i(t, \theta, \bar{\theta})$ in the anticommuting parameters θ and $\bar{\theta}$, it takes the form

$$\phi^i(t, \theta, \bar{\theta}) = q^i(t)+\theta\psi^i(t)+\bar{\psi}^i(t)\bar{\theta}+\bar{\theta}\theta d^i(t). \tag{4.6}$$

Using the Berezin integration rules [9] for integration of the anticommuting variables, we can write the action in component form as

$$S = \int dt\left(\tfrac{1}{2}\dot{q}^{i2}+\tfrac{1}{2}d^{i2}+\frac{i}{2}(\bar{\psi}^i\dot{\psi}^i-\dot{\bar{\psi}}^i\psi^i)\right.$$
$$\left.-d^i\frac{dv(q)}{\partial q^i}-\tfrac{1}{2}(\psi^i\bar{\psi}^i-\bar{\psi}^j\psi^j)\frac{\partial^2 v(q)}{\partial q^i\partial q^j}\right). \tag{4.7}$$

The supersymmetry transformations under which this action is invariant are given by

$$\begin{aligned}
\delta q^i &= -\xi\bar{\psi}^i, \quad \bar{\delta} q^i = +\bar{\xi}\psi^i,\\
\delta\psi^i &= -\xi(d^i+i\dot{q}^i), \quad \bar{\delta}\psi^i = 0,\\
\delta\bar{\psi}^i &= 0, \quad \bar{\delta}\bar{\psi}^i = \bar{\xi}(-d^i+i\dot{q}^i),\\
\delta d^i &= i\xi\dot{\bar{\psi}}^i, \quad \bar{\delta} d^i = i\bar{\xi}\dot{\psi}^i.
\end{aligned} \tag{4.8}$$

Let us, however, continue working in superspace. From the Lagrangian in Eq. (3.5) we see that the canonical momenta are given by

$$\Pi^i(t, \theta, \bar{\theta}) = \frac{\partial L}{\partial\dot{\phi}^i(t, \theta, \bar{\theta})} = \frac{i}{2}(\bar{\theta}D\phi^i+\bar{D}\phi^i\theta). \tag{4.9}$$

It is clear from the discussions of Sec. 3 that the canonical Poisson bracket relations in this case can be written as

$$\begin{aligned}
\{\phi^i(t, \theta, \bar{\theta}), \phi^j(t, \theta', \bar{\theta}')\} &= 0 = \{\Pi^i(t, \theta, \bar{\theta}), \Pi^j(t, \theta', \bar{\theta}')\},\\
\{\phi^i(t, \theta, \bar{\theta}), \Pi^j(t, \theta', \bar{\theta}')\} &= \delta^{ij}\delta(\bar{\theta}-\bar{\theta}')\delta(\theta-\theta') = \delta^{ij}\delta^2(\theta-\theta').
\end{aligned} \tag{4.10}$$

However, in quantizing such a theory we must be careful because there are constraints inherent in such theories and hence we should use Dirac brackets rather than the Poisson brackets. To see the constraints, let us look at the canonical momentum in Eq. (4.9). From the form of the canonical momentum and the anticommuting nature of θ and $\bar{\theta}$, it follows that

$$\begin{aligned}
\zeta^i &= \delta^2(\theta)\Pi^i(t, \theta, \bar{\theta}) \approx 0,\\
\eta_1^i &= \delta^2(\theta)\frac{\partial}{\partial\theta}\left(\Pi^i+\frac{i}{2}\phi^i\right) \approx 0,\\
\eta_2^i &= \delta^2(\theta)\frac{\partial}{\partial\bar{\theta}}\left(-\Pi^i+\frac{i}{2}\phi^i\right) \approx 0.
\end{aligned} \tag{4.11}$$

Here we have used $\delta^2(\theta) = \bar{\theta}\theta$ and these are our primary constraints. Note that the first equation in component language expresses the fact that the auxiliary field has no canonical momentum associated with it. The last two constraints in Eq. (4.11) simply express the relation between the canonical momentum and field variables in the case of fermions.

The primary Hamiltonian in this case is given by

$$H_p = \int d^2\theta \left(\tfrac{1}{2} \frac{\partial \Pi^i}{\partial \bar{\theta}} \frac{\partial \Pi^i}{\partial \theta} - \tfrac{1}{2} \frac{\partial \phi^i}{\partial \bar{\theta}} \frac{\partial \phi^i}{\partial \theta} + V(\phi^i) + \lambda_1^i \eta_1^i + \lambda_2^i \eta_2^i + \lambda_3^i \zeta^i \right). \tag{4.12}$$

It can be now simply checked that the requirement that the constraints do not evolve in time leads to one more constraint

$$\delta^2(\theta) \left[\frac{\partial^2 \phi^i}{\partial\theta \partial\bar{\theta}} - \frac{\partial V}{\partial \phi^i} \right] \approx 0 \tag{4.13}$$

which in the component language corresponds to the equation for the auxiliary fields.

Let us now write the constraints as

$$\zeta_1^i = \delta^2(\theta) \left[\frac{\partial^2 \phi^i}{\partial\theta \partial\bar{\theta}} - \frac{\partial V}{\partial \phi^i} \right] \approx 0, \quad \zeta_2^i = \delta^2(\theta) \Pi^i \approx 0,$$

$$\eta_1^i = \delta^2(\theta) \frac{\partial}{\partial\bar{\theta}} \left(\Pi^i + \frac{i}{2} \phi^i \right) \approx 0, \quad \eta_2^i = \delta^2(\theta) \frac{\partial}{\partial\bar{\theta}} \left(-\Pi^i + \frac{i}{2} \phi^i \right) \approx 0. \tag{4.14}$$

One can now check that the constraints do not evolve in time. Consequently these are all the constraints in the theory and they are all second class.

We would now calculate the Dirac brackets iteratively. First note that the Poisson bracket involving the upper constraints is given by

$$\{\zeta_1^i(t, \theta, \bar{\theta}), \zeta_1^j(t, \theta', \bar{\theta}')\} = 0 = \{\zeta_2^i(t, \theta, \bar{\theta}), \zeta_2^j(t, \theta', \bar{\theta}')\},$$

$$\{\zeta_1^i(t, \theta, \bar{\theta}), \zeta_2^j(t, \theta, \bar{\theta}')\} = \delta^{ij} \delta^2(\theta) \delta^2(\theta'). \tag{4.15}$$

Thus the matrix representing the Poisson brackets of these two constraints has the form

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \delta^{ij} \delta^2(\theta) \delta^2(\theta'). \tag{4.16}$$

This matrix is antisymmetric. However, the more interesting thing to note is the fact that it is singular. This simply reflects the fact that only certain components of the constraint superfield have nonvanishing Poisson bracket. This feature is different from the conventional treatment of the constrained systems where the matrix of Poisson brackets of the second class constraints is nonsingular as has been shown by Dirac.

Following Dirac one realizes that the presence of second class constraints simply implies that there are nondynamical degrees of freedom present in the theory. Thus the naive Poisson brackets must be modified so that only the dynamical degrees are involved and, therefore, passage to a quantum theory becomes straightforward. Furthermore, the Poisson brackets must be defined in such a way that a second class constraint gives vanishing Poisson bracket with any dynamical variable when evaluated with this new definition. This is essential so that we can set the constraint equations to zero strongly as operator relations without restricting the Hilbert space any further. In other words, we need a matrix C^{-1} such that

$$\{A(\theta), \zeta^i_\alpha(\theta')\}^* = \{A(\theta), \zeta^i_\alpha(\theta')\}$$

$$- \int d^2\theta'' d^2\theta''' \{A(\theta), \zeta^k_\beta(\theta'')\} C^{-1kl}_{\beta\gamma}(\theta'', \theta''') \{\zeta^l_\gamma(\theta'''), \zeta^i_\alpha(\theta')\} = 0 \tag{4.17}$$

for any dynamical variable $A(\theta)$. It follows from this requirement that the matrix C^{-1} is uniquely given by

$$C^{-1ij} = \delta^{ij}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{4.18}$$

We would like to emphasize here that Eq. (4.17) determines C^{-1} uniquely up to addition of any function of the θ's. However, the additional terms always give vanishing contribution since in a modified Poisson bracket they come multiplied with the constraints which involve delta functions in θ and $\bar{\theta}$. Thus although the matrix of Poisson brackets is singular, we have determined uniquely the matrix C^{-1}, needed to define the modified brackets. We will show later this leads to the same quantization rules as the component calculation would give. The modified Poisson brackets between any two variables is now given by

$$\{A, B\}^* = \{A, B\} - \{A, \zeta^i_\alpha\} C^{-1ij}_{\alpha\beta} \{\zeta^j_\beta, B\}. \tag{4.19}$$

Here integration over intermediate variables is understood. With this definition, then, we can evaluate the modified canonical brackets to be

$$\{\phi^i(t, \theta, \bar{\theta}), \phi^j(t, \theta', \bar{\theta}')\}^* = 0,$$

$$\{\Pi^i(t, \theta, \bar{\theta}), \Pi^j(t, \theta', \bar{\theta}')\}^* = 0,$$

$$\{\phi^i(t, \theta, \bar{\theta}), \Pi^j(t, \theta', \bar{\theta}')\}^* = \delta^{ij}(\delta^2(\theta-\theta') - \delta^2(\theta)) + \delta^2(\theta)\delta^2(\theta') \frac{\partial^2 V}{\partial\phi^i \partial\phi^j}. \tag{4.20}$$

The modified brackets for the other two constraints in Eq. (4.14) are obtained as

$$\{\eta^i_1(t, \theta, \bar{\theta}), \eta^j_1(, \theta', \bar{\theta}')\}^* = 0 = \{\eta^i_2(t, \theta, \bar{\theta}), \eta^j_2(t, \theta', \bar{\theta}')\}^*,$$

$$\{\eta^i_1(t, \theta, \bar{\theta}), \eta^j_2(t, \theta', \bar{\theta}')\}^* = i\delta^{ij}\delta^2(\theta)\delta^2(\theta'). \tag{4.21}$$

Thus the matrix of the modified Poisson brackets of these constraints has the form

$$i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \delta^{ij}\delta^2(\theta)\delta^2(\theta'). \tag{4.22}$$

Following the method outlined above, we see again that the matrix $\tilde{C}^{-1}$ needed to define the Dirac brackets is obtained to be

$$\tilde{C}^{-1ij} = -i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\delta^{ij}. \tag{4.23}$$

The Dirac bracket between two variables A and B is now defined as

$$\{A, B\}_{\mathrm{D}} = \{A, B\}^{*} - \{A, \eta^i_\alpha\}^{*}\tilde{C}^{-1ij}_{\alpha\beta}\{\eta^j_\beta, B\}^{*}, \tag{4.24}$$

where we again assume integration over intermediate variables. Thus we can calculate the fundamental canonical Dirac brackets and they have the form

$$\{\phi^i(t,\theta,\bar{\theta}), \phi^j(t,\theta',\bar{\theta}')\}_{\mathrm{D}} = -i\delta^{ij}(\bar{\theta}\theta' - \bar{\theta}'\theta),$$

$$\{\Pi^i(t,\theta,\bar{\theta}), \Pi^j(t,\theta',\bar{\theta}')\}_{\mathrm{D}} = -\frac{i}{4}\delta^{ij}(\bar{\theta}\theta' - \bar{\theta}'\theta),$$

$$\{\phi^i(t,\theta,\bar{\theta}), \Pi^j(t,\theta',\bar{\theta}')\}_{\mathrm{D}} = -\tfrac{1}{2}\delta^{ij}(\bar{\theta}\theta' + \bar{\theta}'\theta - 2\bar{\theta}'\theta') + \delta^2(\theta)\delta^2(\theta')\frac{\partial^2 V}{\partial\phi^i\partial\phi^j}$$

$$= -\tfrac{1}{2}\delta^{ij}[\delta(\bar{\theta}-\bar{\theta}')\delta(\theta') + \delta(\bar{\theta}')\delta(\theta-\theta')] + \delta^2(\theta)\delta^2(\theta')\frac{\partial^2 V}{\partial\phi^i\partial\phi^j}. \tag{4.25}$$

We can now carry these relations over to the quantum commutation relations. One can ask at this point how these quantization relations compare with the component field calculations. Using the expansion of Eq. (4.6) for the superfield $\phi^i(t,\theta,\bar{\theta})$ and the fact that (see Eq. (4.9))

$$\Pi^i(t,\theta,\bar{\theta}) = \bar{\theta}\theta\dot{q}^i - \frac{i}{2}\theta\psi^i - \frac{i}{2}\bar{\theta}\bar{\psi}^i. \tag{4.26}$$

One can show that the quantization conditions in Eq. (4.25) imply

$$\{q^i, \dot{q}^j\}_{\mathrm{D}} = \delta^{ij}, \quad \{\psi^i, \bar{\psi}^j\}_{\mathrm{D}} = -i\delta^{ij}, \quad \{d^i, \dot{q}^j\}_{\mathrm{D}} = \frac{\partial^2 V(q)}{\partial q^i \partial q^j}. \tag{4.27}$$

All other Dirac brackets vanish. This is, of course, what would have been expected if we had carried out the calculation in component fields in ordinary space [8]. The superspace quantization, therefore, leads to consistent results. We would like to point out here that the negative sign in the Dirac bracket involving the fermions is a consequence of our choice of a left derivative for the fermionic coordinates.

5. Equivalence of Dirac quantization and action principle

An alternate method of quantizing a theory is due to Schwinger and goes by the name of action principle quantization. Without going into details, the idea behind this method is to construct a unitary operator which gives rise to canonical transformations in the

Hilbert space of states. The generator of such transformations is constructed to be

$$G = p^i\delta q^i - \delta p^i q^i \tag{5.1}$$

so that the unitary operator has the form

$$U = e^{-\frac{i}{\hbar}G}. \tag{5.2}$$

The change in any operator A under the canonical transformations is now given by

$$\delta A = -\frac{i}{\hbar}[A, G]. \tag{5.3}$$

In particular if we choose $A = q^i$, then we have

$$\delta q^i = -\frac{i}{\hbar}[q^i, p^j\delta q^j - \delta p^j q^j]\ .$$

which leads to the relations

$$[q^i, p^j] = i\hbar, \quad [q^i, q^j] = 0. \tag{5.4}$$

Similarly, choosing $A = p^i$, we obtain

$$[p^i, p^j] = 0. \tag{5.5}$$

These are, of course, the familiar quantization rules of quantum mechanics.

This method works well in the absence of any constraint. When there are constraints present, namely,

$$\Gamma_\alpha = 0 \tag{5.6}$$

then the change in an operator A under a canonical transformation, consistent with the constraints, is defined to be

$$\delta_c A = -\frac{i}{\hbar}[A, G]_c = -\frac{i}{\hbar}[A, G] - \lambda_{A\alpha}\delta\Gamma_\alpha = \delta A - \lambda_{A\alpha}\delta\Gamma_\alpha. \tag{5.7}$$

Let us make a few remarks here. Γ_α represents all the constraints in the theory and $\lambda_{A\alpha}$ are Lagrange multipliers to be determined from consistency conditions. Intuitively, Eq. (5.7) is clear. The first term on the right hand side of the equation gives the change in the entire phase space whereas the second term subtracts out the normal component and hence forces the variations to lie on the hypersurface defined by the constraints.

As examples of the action principle, let us again calculate the familiar cases of Maxwell's theory and the nonlinear sigma model.

a) Maxwell's theory:

We have seen that in the Coulomb gauge, we can write all the constraints as

$$\Gamma_1 = \Pi^0 \approx 0\,, \quad \Gamma_2 = \vec{\nabla}\cdot\vec{\Pi} \approx 0, \quad \Gamma_3 = A^0 \approx 0, \quad \Gamma_4 = \vec{\nabla}\cdot\vec{A} \approx 0. \tag{5.8}$$

We can write the generators in this case to be

$$G = \int d^3x(\Pi^\mu(t,\vec{x})\delta A_\mu(t,\vec{x}) - \delta\Pi^\mu(t,\vec{x})A_\mu(t,\vec{x}))$$
$$= \int d^3x(\Pi^i\delta A_i - \delta\Pi^i A_i). \tag{5.9}$$

It follows, therefore, that

$$\delta_c A^i(x) = -\frac{i}{\hbar}[A^i(x), G]_c = \delta A^i(x) - \int d^3y\lambda^i_\alpha(x,y)\delta\Gamma_\alpha(y),$$

or

$$-\frac{i}{\hbar}\int d^3y[A^i(x), \Pi^j(y)\delta A_j(y) - \delta\Pi^j(y)A_j(y)]_c$$
$$= \delta A^i(x) - \int d^3y(\lambda^i_1(x,y)\ \Pi^0(y) + \lambda^i_2(x,y)\delta(\vec{\nabla}_y\cdot\vec{\Pi}(y))$$
$$+\lambda^i_3(x,y)\delta A^0(y) + \lambda^i_4(x,y)\delta(\vec{\nabla}_y\cdot\vec{A}(y))). \tag{5.10}$$

Comparison of the left hand side and the right hand side implies

$$\lambda^i_1(x,y) = 0 = \lambda^i_3(x,y),$$
$$[A^i(x), A^j(y)]_c = -i\hbar\partial^j_y\lambda^i_2(x,y),$$
$$[A^i(x), \Pi^j(y)]_c = i\hbar(\eta^{ij}\delta^3(x-y) + \partial^j_y\lambda^i_4(x,y)). \tag{5.11}$$

Note here that all commutators are equal time commutators. Furthermore, consistency with the constraints for the last two relations leads to

$$[A^i(x), \partial_{yj}A^j(y)]_c = 0 = +i\hbar\vec{\nabla}^2_y\lambda^i_2(x,y) \quad \text{or} \quad \lambda^i_2(x,y) = 0$$

and

$$[A^i(x), \partial_{yj}\Pi^j(y)]_c = 0 = i\hbar(\partial^i_y\delta^3(x-y) - \vec{\nabla}^2_y\lambda^i_4(x,y))$$

which gives

$$\lambda^i_4(x,y) = \partial^i_x\frac{1}{4\pi|\vec{x}-\vec{y}|}. \tag{5.12}$$

Putting this back into Eq. (5.11) we obtain

$$[A^i(x), \Pi^j(y)]_c = i\hbar\left(\eta^{ij}\delta^3(x-y) + \partial^i_x\partial^j_y\frac{1}{4\pi|\vec{x}-\vec{y}|}\right). \tag{5.13}$$

This can be compared with Eq. (2.30). Note further, that we can write

$$[A^i(x), G]_c = [A^i(x), G] - i\hbar\int d^3y\lambda^i_4(x,y)\delta\Gamma_4(y)$$

with

$$\lambda^i_4(x,y) = \partial^i_x\frac{1}{4\pi|\vec{x}-\vec{y}|}. \tag{5.14}$$

b) Nonlinear sigma model:

We have seen before (Eq. (2.39)) that in this case there are four constraints.

$$\Gamma_1 = \phi^i\phi^i - 1 \approx 0, \quad \Gamma_2 = \phi^i\Pi^i \approx 0,$$
$$\Gamma_3 = \Pi_\lambda \approx 0, \quad \Gamma_4 = \lambda + \Pi^i\Pi^i + \phi^i\vec{\nabla}^2\phi^i \approx 0. \tag{5.15}$$

The generator of canonical transformations can be written in this case as

$$G = \int dx(\Pi^i(x)\delta\phi^i(x) - \delta\Pi^i(x)\phi^i(x) - \delta\Pi_\lambda(x)\lambda(x)). \tag{5.16}$$

Thus

$$\delta_c\phi^i(x) = -\frac{i}{\hbar}[\phi^i(x), G]_c = \delta\phi^i(x) - \int dy(\lambda^i_\phi(x,y)\ \Gamma_1(y) + \lambda^i_{\phi 2}\delta\Gamma_2(y)$$
$$+ \lambda^i_{\phi 3}\delta\Gamma_3(y) + \lambda^i_4\delta\Gamma_4(y)),$$

or

$$-\frac{i}{\hbar}\int dy[\phi^i(x), \Pi^j(y)\delta\phi^j(y) - \delta\Pi^j(y)\phi^j(y) - \delta\Pi_\lambda(y)\lambda(y)]_c$$
$$= \delta\phi^i(x) - \int dy(2\lambda^i_{\phi 1}\phi^j(y)\delta\phi^j(y) + \lambda^i_{\phi 2}(\phi^j(y)\delta\Pi^j(y) + \delta\phi^j(y)\Pi^j(y))$$
$$+ \lambda^i_{\phi 3}\delta\Pi_\lambda(y) + \lambda^i_{\phi 4}(\delta\lambda(y) + 2\Pi^j\delta\Pi^j(y) + \delta\phi^j(y)\vec{\nabla}^2_y\phi^j(y) + \phi^j(y)\vec{\nabla}^2_y\delta\phi^j(y))). \tag{5.17}$$

Comparing the left hand side and the right hand side we obtain

$$\lambda^i_{\phi 4}(x,y) = 0,$$
$$[\phi^i(x), \lambda(y)]_c = i\hbar\lambda^i_{\phi 3}(x,y),$$
$$[\phi^i(x), \Pi^j(y)]_c = i\hbar(\delta^{ij}\delta(x-y) - 2\lambda^i_{\phi 1}(x,y)\phi^j(y) - \lambda^i_{\phi 2}(x,y)\Pi^j(y)).$$
$$[\phi^i(x), \phi^j(y)]_c = -i\hbar\lambda^i_{\phi 2}(x,y)\phi^j(y). \tag{5.18}$$

From consistency conditions we obtain

$$[\phi^i(x), \phi^j(y)\phi^j(y)]_c = 0 \Rightarrow \lambda^i_{\phi 2}(x,y) = 0,$$

and

$$[\phi^i(x), \phi^j(y)\Pi^j(y)]_c = 0 \Rightarrow \lambda^i_{\phi 1}(x,y) = \tfrac{1}{2}\,\phi^i(x)\delta(x-y). \tag{5.19}$$

Putting these back into Eq. (5.18) we obtain

$$[\phi^i(x), \phi^j(y)]_c = 0,$$
$$[\phi^i(x), \Pi^j(y)]_c = i\hbar(\delta^{ij} - \phi^i(x)\phi^j(x))\delta(x-y). \tag{5.20}$$

Consistency requirements on the other relation in Eq. (5.18) leads to

$$[\phi^i(x), \lambda(y)+\Pi^j(y)\Pi^j(y)+\phi^j(y)\vec{\nabla}_y^2\phi^j(y)]_c = 0$$
$$\Rightarrow \lambda^i_{\phi 3}(x, y) = -2\Pi^i(x)\delta(x-y),$$

so that

$$[\phi^i(x), \lambda(y)]_c = -2i\hbar\Pi^i(x)\delta(x-y). \tag{5.21}$$

Similarly examining the relation for $\delta_c\Pi^i(x)$, we would further obtain

$$[\Pi^i(x), \Pi^j(y)]_c = -i\hbar(\phi^i(x)\Pi^j(x)-\phi^j(x)\Pi^i(x))\delta(x-y). \tag{5.22}$$

Relations (5.20) and (5.22) should be compared with Eq. (2.42).

Note here that the relations in Eqs (5.20) and (5.21) could have been equally well written as

$$[\phi^i(x), G]_c = [\phi^i(x), G]-i\hbar \int dy(\lambda^i_{\phi 1}\delta\Gamma_1+\lambda^i_{\phi 3}\delta\Gamma_3)$$

with

$$\lambda^i_{\phi 1} = \tfrac{1}{2}\,\phi^i(x)\delta(x-y), \quad \lambda^i_{\phi 3} = -2\Pi^i(x)\delta(x-y). \tag{5.23}$$

We are now ready to make contact between the action principle quantization and Dirac quantization. Note that by construction, Dirac brackets give rise to consistent Poisson bracket relations. Furthermore, the consistent quantum relations must be related to a consistent Poisson bracket relation. Thus

$$[A(x), G]_c = i\hbar\{A(x), G\}_c$$

and that

$$\begin{aligned}\{A(x), G\}_c &= \{A(x), G\}_D = \{A(x), G\}-\int dzdy\{A(x), \Gamma_\beta(z)\}C_\beta^{-1}(z, y)\,\{\Gamma_\alpha(y), G\}\\ &= \{A(x), G\}-\int dzdy\{A(x), \Gamma_\beta(z)\}C_{\beta\alpha}^{-1}(z, y)\delta\Gamma_\alpha(y).\end{aligned} \tag{5.24}$$

On the other hand from Eq. (5.7) we know

$$\{A(x), G\}_c = \{A(x), G\}-\int dy\lambda_{A\alpha}(x, y)\delta\Gamma_\alpha(y). \tag{5.25}$$

Comparing the two relations we obtain

$$\lambda_{A\alpha}(x, y) = \int dz\{A(x), \Gamma_\beta(z)\}C_{\beta\alpha}^{-1}(z, y). \tag{5.26}$$

Let us now check this relation both for Maxwell's theory as well as nonlinear sigma model. First, from the form of the constraints in Eq. (2.27) and the form of $C_{\beta\alpha}^{-1}$ in Eq. (2.29) we see that

$$\begin{aligned}\lambda^i_\alpha(x, y) &= \int d^3x\{A^i(x), \Gamma_\beta(z)\}C_{\beta\alpha}^{-1}(z, y)\\ &= -\int dz\delta^3(x-z)\partial_z^iC_{2\alpha}^{-1}(z, y)\\ &= -\partial_x^iC_{2\alpha}^{-1}(x, y) = \delta_{\alpha 4}\partial_x^i\frac{1}{4\pi|\vec{x}-\vec{y}|}.\end{aligned} \tag{5.27}$$

This is precisely our result for the Maxwell theory as given in Eq. (5.14).

For the nonlinear sigma model again we can write using the constraints in Eq. (2.39) and the matrix $C^{-1}_{\beta\alpha}$ in Eq. (2.41)

$$\begin{aligned} \lambda^i_{\phi\alpha}(x, y) &= \int dz\{\phi^i(x), \Gamma_\beta(z)\}C^{-1}_{\beta\alpha}(z, y) \\ &= \int dz(\{\phi^i(x), \Gamma_2(z)\}C^{-1}_{2\alpha}(z, y)+\{\phi^i(x), \Gamma_4(z)\}C^{-1}_{4\alpha}(z, y)) \\ &= \int dz(\delta(x-z)\phi^i(z)C^{-1}_{2\alpha}(z, y)+2\delta(x-z)\Pi^i(z)C^{-1}_{4\alpha}(z, y)) \\ &= \phi^i(x)C^{-1}_{2\alpha}(x, y)+2\Pi^i(x)C^{-1}_{4\alpha}(x, y) \\ &= \delta_{\alpha 1}\tfrac{1}{2}\phi^i(x)\delta(x-y)-\delta_{\alpha 3}2\Pi^i(x)\delta(x-y). \end{aligned} \tag{5.28}$$

This again confirms our earlier result (Eq. 5.23) for the nonlinear sigma model.

This shows that the Dirac quantization and the action principle quantization are equivalent. One can think of Dirac quantization as providing the Lagrange multipliers of the action principle in a well defined way. On the whole, however, Dirac's method handles the problem of quantization of constrained systems in a systematic and global way.

6. *Conclusion*

We have discussed the canonical quantization of constrained theories in ordinary space-time. We have also shown that the method extends to superspace as well although the extension has to be done carefully. And finally we have shown how Dirac quantization and the action principle quantization are equivalent.

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