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The Maulik–Okounkov R-matrix from the Ding–Iohara–Miki algebra

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The integrability of 4d $\mathcal{N} = 2$ gauge theories has been explored in various contexts, e.g., the Seiberg–Witten curve and its quantization. Recently, Maulik and Okounkov proposed that an integrable lattice model is associated with the gauge theory, through an R-matrix, which we refer to as the MO R-matrix in this paper, constructed in the instanton moduli space. In this paper, we study the R-matrix using the Ding–Iohara–Miki (DIM) algebra. We provide a concrete boson realization of the universal R-matrix in DIM and show that the defining conditions for the MO R-matrix can be derived from this free boson oscillator expression. Several consistency checks for the oscillator expression are also performed.

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1. Introduction

It has been more than 30 years since the discovery of the integrable nature of 4d $\mathcal{N} = 2$ super Yang–Mills theories (Refs. [1–4]). After intensive studies, it turned out that at least for a rather large family of 4d $\mathcal{N} = 2$ theories, the class $\mathcal{S}$ theories (Ref. [5]), each gauge theory in the Coulomb branch is associated to a Hitchin integrable system (Ref. [6]). The quantization of such an integrable model is described by the introduction of Ω-background (Ref. [7]), which has two deformation parameters. When one parameter is tuned to be zero, which is referred to as the Nekrasov–Shatashvili limit, the instanton partition functions have new connections with integrable quantum mechanical systems such as the Toda-chain, Calogero system, and Bethe ansatz equation (Ref. [8]). For the general choice of the parameters, such an integrable structure is promoted to a quantum field-theoretical one described by the *qq*-character, which corresponds to infinite-dimensional quantum algebras (Ref. [9]). The explicit algebraic realization was demonstrated by Refs. [10–13], where the relevant symmetries were identified with the (quantum) W-algebra (Ref. [14]), Ding–Iohara–Miki (DIM) algebra (Refs. [15,16]), and SH^c algebra for a degenerate limit (Refs. [17–19]).

A more direct link with the quantum integrability structure of such systems was explored in Ref. [20], where an R-matrix was constructed in the cohomology ring of the instanton moduli space. We will refer to it as the Maulik–Okounkov (MO) R-matrix in this paper. From the construction of the R-matrix, the SH^c algebra was identified as the $\widehat{\mathfrak{gl}}_1$ Yangian. A remarkable property of the MO R-matrix is that it acts on the tensor product of two Fock spaces instead of the finite-dimensional spin degrees of freedom. By composing several Fock spaces with R-matrices, we can realize the Hilbert space of $\mathfrak{u}(1) \times \mathcal{W}_N$ algebras as a Yangian. It was essential in Ref. [17] to prove the AGT conjecture (Refs. [21,22]). While the appearances of the algebras are totally different, the correspondence between the representations is exactly even for the degenerate representations (Ref. [23]).

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To push forward in the direction of understanding these integrable models and especially the physical meaning of the MO R-matrix in gauge theories, we consider a generalization to the 5d version of AGT (Ref. [24]). The compactification on the S^1 circle implies that the corresponding $\mathcal{W}$-algebra should be q-deformed. As the representations of the W-algebra are universally covered by the SH^c (or Yangian), the q-deformed version is described by DIM (Ref. [25]), and the 5d AGT relation can be proved with its help (see, e.g., Ref. [13]). As its 4d sibling, the DIM algebra also possesses an R-matrix (Ref. [26]). This fact can be understood from the aspect that DIM is a quantum-group analogy for the Yangian, and thus is equipped with a universal R-matrix (Ref. [27]). The matrix elements of the R-matrix at the first several levels were computed in Refs. [28,29].

From the viewpoint of string theory, one merit of considering the q-deformed version is that the duality structure is more manifest. It is known that the DIM has an $SL(2,\mathbb{Z})$ automorphism (Ref. [16]). Since it interchanges the topological string amplitudes including D and NS branes through the topological vertex (Refs. [30,31]) realized as an intertwinner of DIM (Ref. [32]), it may be identified with the duality symmetry of a type IIB superstring. Thus the integrability property begins to have a new perspective in the relevance of nonperturbative duality in the brane-web (see, e.g., Refs. [33–35]).

The purpose of this paper is to explore an oscillator realization of the R-matrix for DIM. While there are some works on this subject (see, e.g., Refs. [26,28,36,37]), the bosonic representation along the MO line is new. Since the MO R-matrix gives the bosonized Hamiltonians of the (generalized) Calogero–Sutherland system, our result gives those for the Ruijsenaars–Schneider model (Refs. [38–42]). These Hamiltonians describe the structure of the cohomology ring for the (K-theoretical) moduli space of instanton moduli spaces.

We organize this paper as follows. In Sect. 2, we briefly describe the DIM algebra and its bosonic (horizontal) representation. In Sect. 3, we first review the R-matrix proposed by Maulik–Okounkov (Ref. [20]) and explain how conserved charges can be constructed out of it. Next, we use the notion of the universal R-matrix to derive the property that should be obeyed by the R-matrix for DIM. In Sect. 4, we expand the R-matrix in terms of the spectral parameter and derive the recursion relation among coefficients. We also give the first two terms in the expansion. In Sect. 5, we derive some conserved charges from the R-matrix. Finally in Sect. 6, we calculate the action of the R-matrix on the generalized Macdonald basis and show consistency with the previous result (Refs. [28,29]).

Notation conventions

Let us explain our convention on notation here. There are two parameters introduced for the Ω-background, q and t, but it is more convenient for us to introduce a third one p, which is related to q and t by

$$p = \frac{q}{t}.$$

There are also several types of bosons used in this article and since it seems confusing, we list all of them here:

- First, b_n satisfying $[b_m, b_n] = \frac{1}{m}(1-q^{-m})(1-t^m)(1-p^m)(\hat{\gamma}^m - \hat{\gamma}^{-m})\hat{\gamma}^{-|m|}\delta_{m+n,0}$ ($\hat{\gamma}$ is a central element in the DIM algebra) is a generator in the DIM algebra. We note that in different representations of DIM it takes different forms, e.g., in the vertical representation it is a diagonal operator, and in the horizontal representation it is mapped to the q-boson a_n.

- The q-boson a_n satisfies the commutation relation $[a_m, a_n] = m\frac{1-q^{|m|}}{1-t^{|m|}}\delta_{m+n,0}$. It is used in the horizontal representation of the DIM algebra. We note that it has no definite coproduct structure.
- Finally, α_n is the normal boson oscillator defined by $[\alpha_n, \alpha_m] = n\delta_{n+m,0}$. It is used only in the degenerate limit $q \to 1$. We note that the vertex operator $\alpha(z)$ has nothing to do with α_n, especially since the latter does not give the mode expansion of $\alpha(z)$.

We have to work on tensor products of Fock spaces/representation spaces. We use the superscript $^{(i)}$ to denote the operator living in the ith Fock space/representation space. For example, $\alpha_n^{(2)}$ is the boson operator α_n in the second Fock space, and $x^{+(0)}(z)$ is the generator $x^+(z)$ defined in the 0th (auxiliary) copy of the DIM algebra. We also use short notation to express some special combination of bosons in several Fock spaces:

$$\alpha_n^{\pm} := \frac{1}{\sqrt{2}}\left(\alpha_n^{(1)} \pm \alpha_n^{(2)}\right), \qquad a_n^{\pm} := \frac{1}{\sqrt{2}}\left(a_n^{(1)} \pm a_n^{(2)}\right), \qquad a_n^{p-} := a_n^{(2)} - p^{\frac{|n|}{2}} a_n^{(1)}.$$

For a state $|\psi\rangle$ in the ith representation space, however, we use the notation $|\psi\rangle_i$.

2. The DIM algebra and its universal R-matrix

In this section, we review the defining relations of the DIM algebra and the universal R-matrix. We adopt the same conventions of notation as in our previous paper (Ref. [35]).

2.1. DIM algebra

The DIM algebra, which will be denoted $U_{q_1,q_2}(\widehat{\mathfrak{gl}_1})$ in this paper, has two parameters $q_1 = q$ and $q_2 = t^{-1}$ and satisfies the following defining relations:

$$[\psi^{\pm}(z), \psi^{\pm}(w)] = 0, \qquad \psi^+(z)\psi^-(w) = \frac{g(\hat{\gamma}w/z)}{g(\hat{\gamma}^{-1}w/z)}\psi^-(w)\psi^+(z), \tag{1}$$

$$\psi^+(z)x^{\pm}(w) = g(\hat{\gamma}^{\mp 1/2}w/z)^{\mp 1}x^{\pm}(w)\psi^+(z), \tag{2}$$

$$\psi^-(z)x^{\pm}(w) = g(\hat{\gamma}^{\mp 1/2}z/w)^{\pm 1}x^{\pm}(w)\psi^-(z), \tag{3}$$

$$x^{\pm}(z)x^{\pm}(w) = g(z/w)^{\pm 1}x^{\pm}(w)x^{\pm}(z), \tag{4}$$

$$[x^+(z), x^-(w)] = \frac{(1-q_1)(1-q_2)}{(1-q_1q_2)}\big(\delta(\hat{\gamma}^{-1}z/w)\psi^+(\hat{\gamma}^{1/2}w) - \delta(\hat{\gamma}z/w)\psi^-(\hat{\gamma}^{-1/2}w)\big), \tag{5}$$

where

$$g(z) := \prod_{i=1,2,3}\frac{1-q_iz}{1-q_i^{-1}z}, \qquad q_3 := (q_1q_2)^{-1},$$

$\hat{\gamma}$ is a central element of the algebra, and $\delta(z) = \sum_{n\in\mathbb{Z}} z^n$ is the delta function. In addition, it also satisfies the Serre relations (Ref. [43]),

$$\left[x_0^+, \left[x_1^+, x_{-1}^+\right]\right] = 0, \qquad \left[x_0^-, \left[x_1^-, x_{-1}^-\right]\right] = 0. \tag{6}$$

The mode expansion of each generator is given by

$$x^{\pm}(z) = \sum_{n\in\mathbb{Z}} x_n^{\pm} z^{-n}, \qquad \psi^{\pm}(z) = \sum_{n\geq 0}\psi_{\pm n}^{\pm} z^{\mp n}. \tag{7}$$

These defining relations are a two-parameter generalization of the Drinfeld realization of the quantum group (Ref. [44]) and thus the DIM algebra is sometimes called the quantum toroidal algebra (associated with $\mathfrak{gl}_1$).

This algebra has the following known coproduct structure:

$$\begin{aligned}
\Delta(\hat{\gamma}^{\pm\frac{1}{2}}) &= \hat{\gamma}^{\pm\frac{1}{2}} \otimes \hat{\gamma}^{\pm\frac{1}{2}},\\
\Delta(\psi^{\pm}(z)) &= \psi^{\pm}(\gamma_{(2)}^{\pm\frac{1}{2}} z) \otimes \psi^{\pm}(\gamma_{(1)}^{\mp\frac{1}{2}} z),\\
\Delta(x^{+}(z)) &= x^{+}(z) \otimes 1 + \psi^{-}(\gamma_{(1)}^{\frac{1}{2}} z) \otimes x^{+}(\gamma_{(1)} z),\\
\Delta(x^{-}(z)) &= x^{-}(\gamma_{(2)} z) \otimes \psi^{+}(\gamma_{(2)}^{\frac{1}{2}} z) + 1 \otimes x^{-}(z),
\end{aligned} \tag{8}$$

where, $\gamma_{(1)}^{\pm\frac{1}{2}} = \hat{\gamma}^{\pm\frac{1}{2}} \otimes 1$ and $\gamma_{(2)}^{\pm\frac{1}{2}} = 1 \otimes \hat{\gamma}^{\pm\frac{1}{2}}$.

Given a quantum group, a special operator called the R-matrix is usually inherited from its quasi-triangular Hopf algebra nature. This R-matrix, which is often called the universal R-matrix, has the following properties:

$$\mathbf{R}\Delta(e)(z)\mathbf{R}^{-1} = \mathcal{P} \circ \Delta(e)(z), \quad \text{(for } e \in \text{the quantum group)}, \tag{9}$$

$$(\Delta \otimes 1)\mathbf{R} = \mathbf{R}_{13}\mathbf{R}_{23}, \qquad (1 \otimes \Delta)\mathbf{R} = \mathbf{R}_{13}\mathbf{R}_{12}, \tag{10}$$

where $\mathcal{P}$ is the operator that exchanges two copies of algebras (quantum groups), i.e., $\mathcal{P}(a \otimes b) = b \otimes a$, and the notation $\mathbf{R}_{ij}$ denotes the R-matrix acting on the ith and jth algebras. Such an R-matrix automatically meets the Yang–Baxter equation. Since the DIM algebra is a generalization to the quantum groups, we expect a similar R-matrix structure in DIM, and indeed in Ref. [26], it was argued that the DIM algebra is a quasi-triangular Hopf algebra and equipped with the universal R-matrix $\mathbf{R}(u)$. This object will be the main character of the paper.

2.2. *Horizontal representation of the DIM algebra and the q-deformed $\mathcal{W}$-algebras*

In addition to $\hat{\gamma}$, there is one more central element ψ_0^+/ψ_0^- in the DIM algebra. The representation of the algebra can thus be parameterized with two numbers (ℓ_1, ℓ_2) such that $\hat{\gamma} = q_3^{\ell_1/2}$ and $\psi_0^+/\psi_0^- = q_3^{\ell_2}$. A large family of the representations is parameterized by $(\ell_1, \ell_2) \in \mathbb{Z} \times \mathbb{Z}$, among which the SL(2,$\mathbb{Z}$) automorphism of the algebra acts as

$$S \cdot (l_1, l_2) = (-l_2, l_1), \qquad T \cdot (l_1, l_2) = (l_1, l_1 + l_2). \tag{11}$$

We often refer to the transformation S, which is a 90° rotation on the representation plane, as the S-duality of the algebra.

The $(1, 0)$ representation is called the horizontal representation of the DIM algebra in the literature. It is characterized by the q-boson representation. Let us introduce the following vertex operators:

$$\eta(z) =: \exp\left(\sum_{n\neq 0} \frac{1 - t^{-n}}{n} a_{-n} z^n\right):,$$

$$\xi(z) =: \exp\left(-\sum_{n\neq 0} \frac{1 - t^{-n}}{n} p^{-\frac{|n|}{2}} a_{-n} z^n\right):,$$

$$\varphi^+(z) = \exp\left(-\sum_{n>0}\frac{1-t^n}{n}(1-p^{-n})p^{\frac{n}{4}}a_n z^{-n}\right),$$
$$\varphi^-(z) = \exp\left(\sum_{n>0}\frac{1-t^{-n}}{n}(1-p^{-n})p^{\frac{n}{4}}a_{-n} z^{n}\right), \tag{12}$$

where $:\bullet:$ denotes the normal ordering, $p = \frac{q}{t}$, and a_n is the q-deformed boson oscillator satisfying

$$[a_m, a_n] = m\frac{1-q^{|m|}}{1-t^{|m|}}\delta_{m+n,0}. \tag{13}$$

The horizontal representation is defined by the vertex operators

$$\rho_u(\hat{\gamma}^{\pm\frac{1}{2}}) = p^{\mp\frac{1}{4}}, \quad \rho_u(\psi^\pm(z)) = \varphi^\pm(z), \quad \rho_u(x^+(z)) = u\eta(z), \quad \rho_u(x^-(z)) = u^{-1}\xi(z), \tag{14}$$

and $u \in \mathbb{C}$ is the weight of the representation.

It is convenient to introduce the boson oscillators b_n as the mode expansion of $\log\psi^\pm$:

$$\psi^+(z) = \psi_0^+ \exp\left(\sum_{n>0} b_n \hat{\gamma}^{\frac{n}{2}} z^{-n}\right), \qquad \psi^-(z) = \psi_0^- \exp\left(-\sum_{n>0} b_{-n}\hat{\gamma}^{\frac{n}{2}} z^{n}\right). \tag{15}$$

From the algebraic relation (1) and the coproduct of $\psi^\pm$, we see that b_n satisfies

$$[b_m, b_n] = \frac{1}{m}(1-q^{-m})(1-t^m)(1-p^m)(\hat{\gamma}^m - \hat{\gamma}^{-m})\hat{\gamma}^{-|m|}\delta_{m+n,0},$$
$$\Delta(b_n) = b_n \otimes \hat{\gamma}^{-|n|} + 1 \otimes b_n. \tag{16}$$

In the horizontal representation, b_n and $\psi_0^\pm$ are respectively mapped to

$$b_n \mapsto -\frac{1-t^n}{|n|}(p^{\frac{|n|}{2}} - p^{-\frac{|n|}{2}})a_n, \qquad \psi_0^\pm \mapsto 1. \tag{17}$$

It is known (Ref. [25]) that one can embed the q-deformed $\mathcal{W}_m$ algebra in the $(m, 0)$ representation of the DIM. The $(m, 0)$ representation can be constructed by taking the coproduct of m $(1, 0)$ representations. For the $m = 2$ case, the representation is isomorphic to the direct sum of U(1) current and q-Virasoro generators. The latter is realized as (Ref. [25])

$$\rho_{u_1,u_2}(\Delta(t(z))) := \rho_{u_1} \otimes \rho_{u_2}(\Delta(t(z))) =: u_1\Lambda_1(z) + u_2\Lambda_2(z), \tag{18}$$

where

$$t(z) = \alpha(z)x^+(z)\beta(z), \tag{19}$$

with

$$\alpha(z) = \exp\left(-\sum_{n=1}^{\infty}\frac{1}{\hat{\gamma}^n - \hat{\gamma}^{-n}}b_{-n}z^n\right),$$
$$\beta(z) = \exp\left(\sum_{n=1}^{\infty}\frac{1}{\hat{\gamma}^n - \hat{\gamma}^{-n}}b_{n}z^{-n}\right). \tag{20}$$

For later convenience, we list the expressions of α, β, and $\Lambda_{1,2}$ in the horizontal representation:[1]

$$\rho_u(\alpha(z)) = \exp\left(-\sum_{n=1}^{\infty}\frac{1-t^{-n}}{n}a_{-n}z^n\right),$$

$$\rho_{u_1}\otimes\rho_{u_2}(\Delta(\alpha(z))) = \exp\left(-\sum_{n=1}^{\infty}\frac{1-t^{-n}}{n(p^{\frac{n}{2}}+p^{-\frac{n}{2}})}(p^{\frac{n}{2}}a^{(1)}_{-n}+a^{(2)}_{-n})z^n\right),$$

$$\rho_u(\beta(z)) = \exp\left(\sum_{n=1}^{\infty}\frac{1-t^{n}}{n}a_{n}z^{-n}\right),$$

$$\rho_{u_1}\otimes\rho_{u_2}(\Delta(\beta(z))) = \exp\left(\sum_{n=1}^{\infty}\frac{1-t^{n}}{n(p^{\frac{n}{2}}+p^{-\frac{n}{2}})}(p^{\frac{n}{2}}a^{(1)}_{n}+a^{(2)}_{n})z^{-n}\right),$$

$$\Lambda_1(z) =: \exp\left(\sum_{n\neq 0}\frac{1}{n}\frac{1-t^{-n}}{1+p^{|n|}}z^n\left(a^{(1)}_{-n}-p^{\frac{|n|}{2}}a^{(2)}_{-n}\right)\right):,$$

$$\Lambda_2(z) =: \exp\left(-\sum_{n\neq 0}\frac{1}{n}\frac{1-t^{-n}}{1+p^{|n|}}\left(\frac{z}{p}\right)^n\left(a^{(1)}_{-n}-p^{\frac{|n|}{2}}a^{(2)}_{-n}\right)\right) :=: \Lambda_1^{-1}\left(\frac{z}{p}\right):. \tag{21}$$

3. Relation with the MO R-matrix

In this section, we first review the MO R-matrix (Ref. [20]), which was induced from the instanton moduli space of 4d $\mathcal{N}=2$ super Yang–Mills gauge theories, with a short-cut approach taken in Ref. [45], and then derive the key relation $\mathbf{R}^{\mathrm{MO}}(u)T(Q,u)=T(-Q,u)\mathbf{R}^{\mathrm{MO}}(u)$ satisfied by the MO R-matrix from the horizontal representation of the universal R-matrix of DIM by taking the limit $q\to 1$.

3.1. *A brief review of the MO R-matrix*

The MO R-matrix is defined on the tensor product of two Fock spaces with background charge η_1 and η_2, i.e., the vector spaces are generated by bosons $\alpha_n^{(1)}$ and $\alpha_n^{(2)}$, satisfying $[\alpha_n^{(i)},\alpha_m^{(j)}]=\delta_{ij}n\delta_{n+m,0}$, with the vacuum state of each Fock space being the eigenstate of the boson zero mode, $\alpha_0^{(i)}|\eta_i\rangle=\eta_i|\eta_i\rangle$, $\alpha_n^{(i)}|\eta_i\rangle=0$ for $n>0$ (no summation).

The R-matrix acts only on the subspace spanned by $\alpha_n^-=\frac{1}{\sqrt{2}}\left(\alpha_n^{(1)}-\alpha_n^{(2)}\right)$ and the defining relation it satisfies is

$$\mathbf{R}^{\mathrm{MO}}(\bar{u})L_n(\bar{u},Q)=L_n(\bar{u},-Q)\mathbf{R}^{\mathrm{MO}}(\bar{u}), \tag{22}$$

where $\bar{u}=\frac{1}{\sqrt{2}}(\eta_2-\eta_1)$ and

$$L_n(\bar{u},Q)=\frac{1}{2}\sum_m{}' :\alpha^-_{n+m}\alpha^-_{-m}: + Qn\alpha_n^- - \bar{u}\alpha_n^- =: L_n^{(0)}+Qn\alpha_n^- - \bar{u}\alpha_n^-. \tag{23}$$

[1] Remark: It might be confusing that substituting $\Delta(a_n)=p^{|n|/2}a_n^{(1)}+a_n^{(1)}$ into the above expression of $\rho_u(\alpha(z))$ does not give us $\rho_{u_1}\otimes\rho_{u_2}(\Delta(\alpha(z)))$. We note that the q-boson does not have a consistent coproduct structure, as can be seen from the fact that $\rho_u(x^+(z))=u\exp(\sum_{n\neq 0}\frac{1-t^{-n}}{n}a_{-n}z^n)$ and $\Delta(x^+(z))=x^+(z)\otimes 1+\psi^-(\gamma_{(1)}^{1/2}z)\otimes x^+(\gamma_{(1)}z)$. The correct formula can be obtained only by first taking the coproduct in the expression in terms of the generator of the DIM algebra and then performing the representation map.

Here, $\sum'$ denotes summation without terms involving the boson zero mode. The overall scale of the R-matrix is fixed by its action on the vacuum state,

$$\mathbf{R}^{\mathrm{MO}}(\bar{u})|\eta_1\rangle \otimes |\eta_2\rangle = |\eta_1\rangle \otimes |\eta_2\rangle. \tag{24}$$

In Ref. [45], it was argued that Eq. (22) solves the R-matrix uniquely with the above normalization.

Once we have obtained the explicit expression of the MO R-matrix, we can construct a family of lattice integrable systems by just using the R-matrix as the Lax operator. The transfer matrix in the N-site lattice model is

$$T(\bar{u}) = {}_0\langle\eta|\mathbf{R}^{\mathrm{MO}}_{01}(\bar{u})\mathbf{R}^{\mathrm{MO}}_{02}(\bar{u})\cdots\mathbf{R}^{\mathrm{MO}}_{0N}(\bar{u})|\eta\rangle_0, \tag{25}$$

where we take $\bar{u} = \frac{1}{\sqrt{2}}(\eta_i - \eta)$ for $i = 1, \ldots, N$. For $N = 1$, we find the Hamiltonian of the system matching the Calogero–Sutherland Hamiltonian with an infinite number of particles (Refs. [46,47]),

$$\mathcal{H} = \sum_{n,m>0} (\alpha_{-n}\alpha_{-m}\alpha_{n+m} + \alpha_{-n-m}\alpha_n\alpha_m) + \sqrt{2}Q\sum_{n>0} n\alpha_{-n}\alpha_n, \tag{26}$$

where α_n is the boson on the first site. For larger N, we obtain a coproduct structure for the Calogero–Sutherland Hamiltonian.

The Yangian algebra constructed from the MO R-matrix is isomorphic to the Yangian of $\widehat{\mathfrak{gl}_1}$ (Ref. [20]) (see also Ref. [45] for a short review) and this Yangian algebra can also be obtained in the degenerate limit $q \to 1$ with $t = q^\beta$ for fixed β from the DIM algebra (Ref. [18]).

3.2. *From the R-matrix in DIM to the MO R-matrix*

In this section, we start from the defining condition of the universal R-matrix in DIM,

$$\mathbf{R}\Delta(e)(z)\mathbf{R}^{-1} = \mathcal{P} \circ \Delta(e)(z) \quad \text{for } e \in U_{q_1,q_2}(\widehat{\mathfrak{gl}_1}), \tag{27}$$

to explain the origin of the defining conditions for the MO R-matrix by taking the horizontal representation of the universal R-matrix. The appearances of them are very different and it is not so obvious whether they should be the same. We show, however, that they agree with each other in the degenerate limit, where we set $q = e^\hbar$, $t = q^\beta = e^{\beta\hbar}$, and take the $\hbar \to 0$ limit.

To compare with quantities in the MO context, we identify the weight u in the horizontal representation as the q-boson zero mode t^{a_0}. Then using the notation $a_n^\pm = \frac{1}{\sqrt{2}}\big(a_n^{(1)} \pm a_n^{(2)}\big)$, we can rewrite the q-Virasoro operator as

$$\begin{aligned}\rho_{u_1,u_2}(\Delta(t(z))) &= t^{a_0^{(1)}}\Lambda_1(z) + t^{a_0^{(2)}}\Lambda_2(z)\\ &= t^{\frac{1}{\sqrt{2}}a_0^+}\Big(t^{\frac{a_0^-}{\sqrt{2}}}\Lambda_1(z) + t^{-\frac{a_0^-}{\sqrt{2}}}\Lambda_2(z)\Big).\end{aligned} \tag{28}$$

By putting $t^{\frac{a_0^-}{\sqrt{2}}}\Lambda_1(z) = 1 + A(z)\hbar + B(z)\hbar^2 + \mathcal{O}(\hbar^3)$, and using $\Lambda_2(z) =: \Lambda_1^{-1}(\frac{z}{p})$:, we have

$$\begin{aligned}t^{-\frac{a_0^-}{\sqrt{2}}}\Lambda_2(z) &= 1 - A\Big(\frac{z}{p}\Big)\hbar + \Big(: A^2\Big(\frac{z}{p}\Big) : - B\Big(\frac{z}{p}\Big)\Big)\hbar^2 + \mathcal{O}(\hbar^3)\\ &= 1 - A(z)\hbar + (: A^2(z) : - B(z) - C(z))\hbar^2 + \mathcal{O}(\hbar^3),\end{aligned}$$

where, $A\left(\frac{z}{p}\right) =: A(z) + \hbar C(z) + \mathcal{O}(\hbar^2)$. Summarizing the above, we get the following simple result:

$$\rho_{u_1,u_2}(\Delta(t(z))) = t^{\frac{1}{\sqrt{2}}a_0^+}(2 + (:A^2(z): - C(z))\hbar^2 + \mathcal{O}(\hbar^3)). \tag{29}$$

Expanding $\Lambda_1(z)$ with respect to $\hbar$, we see that

$$A(z) = \frac{\sqrt{\beta}}{\sqrt{2}} \sum_{n=-\infty}^{\infty} \alpha_{-n}^- z^n, \qquad C(z) = Q\beta \sum_{n=-\infty}^{\infty} n\alpha_{-n}^- z^n, \tag{30}$$

where $Q = \frac{1}{\sqrt{2}}(\sqrt{\beta} - \frac{1}{\sqrt{\beta}})$. Note that in the $\hbar$-expansion we need to replace a_n with $\frac{1}{\sqrt{\beta}}\alpha_n + \mathcal{O}(\hbar)$, where α_n is an undeformed boson operator. Combined together, we have

$$:A^2(z): - C(z) = \beta\left(\frac{1}{2}\sum_{m,n=-\infty}^{\infty} :a_{-n}^- a_{-m}^-: z^{n+m} - Q \sum_{n=-\infty}^{\infty} n a_{-n}^- z^n\right). \tag{31}$$

At the leading order, the q-Virasoro operator becomes

$$\begin{aligned}\rho_{u_1,u_2}(\Delta(t(z))) &= t^{\frac{1}{\sqrt{2}}a_0^+}\left(2 + \hbar^2\beta \sum_{n=-\infty}^{\infty}(L_{-n}^{(0)} - Qn\alpha_{-n}^- + \alpha_0^-\alpha_{-n}^-)z^n + \mathcal{O}(\hbar^3)\right) \\ &=: t^{\frac{1}{\sqrt{2}}a_0^+}(2 + \hbar^2\beta T(Q,\bar{u}) + \mathcal{O}(\hbar^3)),\end{aligned} \tag{32}$$

where $L_n^{(0)} = \frac{1}{2}\sum_m{}' : \alpha_{n-m}^-\alpha_m^- :$ and we replace α_0^- with its eigenvalue $-\bar{u}$, as explained in the previous section.

Since $\mathcal{P}$ exchanges the dofs at two sites, it flips the sign of α_n^-, i.e., $\mathcal{P}\alpha_n^- = -\alpha_n^-$ and thus turns $T(Q,\bar{u})$ into $T(-Q,\bar{u})$:[2]

$$\begin{aligned}&\mathbf{R}(u)\rho_{u_1,u_2}(\Delta(t(z))) = \left(\mathcal{P}\cdot\rho_{u_1,u_2}(\Delta(t(z)))\right)\mathbf{R}(u) \\ \Rightarrow\quad &\mathbf{R}(u)\left(2 + \hbar^2\beta T(Q,\bar{u}) + \mathcal{O}(\hbar^3)\right) = \left(2 + \hbar^2\beta T(-Q,\bar{u}) + \mathcal{O}(\hbar^3)\right)\mathbf{R}(u),\end{aligned} \tag{33}$$

where the natural choice of the spectral parameter is $u = \frac{u_1}{u_2} = t^{-(1/\sqrt{2\beta})\bar{u}}$. We expand $\mathbf{R}(u) = \sum_{n\geq 0} R^{(n)}(\bar{u})\hbar^n$; then we see that the leading order of the R-matrix satisfies

$$R^{(0)}T(Q,\bar{u}) = T(-Q,\bar{u})R^{(0)}, \tag{34}$$

and it can be identified as the MO R-matrix $\mathbf{R}^{\mathrm{MO}}(\bar{u})$.

Remark The relation $\mathbf{R}\Delta(e) = \mathcal{P}\circ\Delta(e)\mathbf{R}$ is satisfied by all the DIM elements e. The DIM algebra is generated by $x^{\pm}(z)$, b_n, and centers. The centers are not relevant in the R-matrix relations. The other generators of the DIM algebra are $t(z)$, $t^*(z)$, and b_n, where

$$\begin{aligned}t^*(z) &= \alpha(p^{-1}z)^{-1}x^-(p^{-1}\gamma^{-1}z)\beta(\gamma^{-2}p^{-1}z)^{-1}, \\ \rho(\Delta(t^*(z))) &= t^{a_0^{(2)}}\Lambda_2(z) + t^{a_0^{(1)}}\Lambda_1(z) = \rho(\Delta(t(z))).\end{aligned} \tag{35}$$

[2] Rigorously speaking, we should write $\rho_{u_1,u_2}(\mathbf{R}(u))\rho_{u_1,u_2}(\Delta(t(z))) = \left(\mathcal{P}\cdot\rho_{u_1,u_2}(\Delta(t(z)))\right)\rho_{u_1,u_2}(\mathbf{R}(u))$, but to avoid the complexity of the equations, we omit the representation map for $\mathbf{R}(u)$. It should be understood that all $\mathbf{R}(u)$ appearing in this article are in the horizontal representation.

We see that $t^*(z)$ induces the same defining relation for the R-matrix. The remaining relation that we have not considered is $\mathbf{R}\Delta(b_n) = \mathcal{P} \circ \Delta(b_n)\mathbf{R}$, which in the degenerate limit $\hbar \to 0$ reduces to $R^{(0)}\alpha_n^+ = \alpha_n^+ R^{(0)}$. This explains why the MO R-matrix acts only on the diagonal space spanned by α^-.

4. Explicit computation of the R-matrix

In this section, we solve the defining equation of the universal R-matrix in the horizontal representation in terms of the q-boson oscillators at the first two lowest orders.[3] For simplicity, we omit the notation for the horizontal representation map ρ and we use the shorthand notation $\Delta^{\mathrm{op}} = \mathcal{P} \circ \Delta$ in the following.

Unlike in the degenerate case of the MO R-matrix, the constraint $\mathbf{R}\Delta(b_n) = \Delta^{\mathrm{op}}(b_n)\mathbf{R}$, namely $\mathbf{R}(p^{\frac{|n|}{2}} a_n^{(1)} + a_n^{(2)}) = (p^{\frac{|n|}{2}} a_n^{(2)} + a_n^{(1)})\mathbf{R}$ is less trivial. This equation does not depend on the spectral parameter, so we have the following operator, which exchanges $a_n^{(1)}$ and $a_n^{(2)}$, as part of the R-matrix:

$$R_{--} = \exp\left(\pi i \sum_{m=1}^{\infty} \frac{1}{m} \frac{1-t^m}{1-q^m} a_{-m}^- a_m^-\right). \tag{36}$$

With the Campbell–Baker–Hausdorff formula (see Appendix A for a review and derivation), we have

$$R_{--} a_n^{(1)} R_{--}^{-1} = a_n^{(2)}, \quad R_{--} a_n^{(2)} R_{--}^{-1} = a_n^{(1)} \quad (\text{for } n \neq 0). \tag{37}$$

Let us put an ansatz for the R-matrix $\mathbf{R}(u) = \mathbf{R}'(u) R_{--}$; then we have $\mathbf{R}'(u)\big(p^{\frac{|n|}{2}} a_n^{(2)} + a_n^{(1)}\big) = \big(p^{\frac{|n|}{2}} a_n^{(2)} + a_n^{(1)}\big)\mathbf{R}'(u)$. This means that $\mathbf{R}'$ commutes with $p^{\frac{|n|}{2}} a_n^{(2)} + a_n^{(1)}$, so $\mathbf{R}'$ is constructed from only the boson operator $a_n^{p-} := a_n^{(2)} - p^{\frac{|n|}{2}} a_n^{(1)}$.

The defining relation now becomes

$$\mathbf{R}' \Delta^{\mathrm{op}'}(t(z)) \mathbf{R}'^{-1} = \Delta^{\mathrm{op}}(t(z)), \tag{38}$$

where, $\Delta^{\mathrm{op}'}(t(z))$ and $\Delta^{\mathrm{op}}(t(z))$ ($^{\mathrm{op}'}$ means that we do not exchange u_1 and u_2 as in Δ^{op}) read

$$\begin{aligned} \Delta^{\mathrm{op}'}(t(z)) &= u_1 \Lambda_1^{\mathrm{op}}(z) + u_2 \Lambda_2^{\mathrm{op}}(z), \\ \Delta^{\mathrm{op}}(t(z)) &= u_2 \Lambda_1^{\mathrm{op}}(z) + u_1 \Lambda_2^{\mathrm{op}}(z). \end{aligned} \tag{39}$$

Now $\Lambda_1^{\mathrm{op}}(z)$, $\Lambda_2^{\mathrm{op}}(z)$ after the action of $\mathcal{P}$ read

$$\begin{aligned} \Lambda_1^{\mathrm{op}}(z) &=: \exp\left(\sum_{n\neq 0} \frac{1}{n} \frac{1-t^{-n}}{1+p^{|n|}} z^n \big(a_{-n}^{(2)} - p^{\frac{|n|}{2}} a_{-n}^{(1)}\big)\right):, \\ \Lambda_2^{\mathrm{op}}(z) &=: \exp\left(-\sum_{n\neq 0} \frac{1}{n} \frac{1-t^{-n}}{1+p^{|n|}} \left(\frac{z}{p}\right)^n \big(a_{-n}^{(2)} - p^{\frac{|n|}{2}} a_{-n}^{(1)}\big)\right) := \Lambda_1^{\mathrm{op}}\left(\frac{z}{p}\right)^{-1}. \end{aligned} \tag{40}$$

Recalling that $u = \frac{u_1}{u_2}$, we have

$$\mathbf{R}'(u) \left(u \Lambda_1^{\mathrm{op}}(z) + \Lambda_2^{\mathrm{op}}(z)\right) \mathbf{R}'^{-1}(u) = \Lambda_1^{\mathrm{op}}(z) + u \Lambda_2^{\mathrm{op}}(z). \tag{41}$$

[3] We note that the formal expression of the universal R-matrix is already given in Ref. [26] and the result we obtain below can be obtained from it directly.

At $u = 0$, the above equation becomes $\mathbf{R}'(0)\Lambda_2^{\rm op}(z)\mathbf{R}'(0)^{-1} = \Lambda_1^{\rm op}(z)$ and the solution is

$$\mathbf{R}'(0) = \exp\left(\sum_{m=1}^{\infty}\left(\hbar(1-\beta)+\frac{\pi i}{m}\right)\frac{1-t^m}{(1-q^m)(1+p^m)}a_{-m}^{p-}a_m^{p-}\right). \tag{42}$$

This follows from the fact $\mathbf{R}'(0)a_{-n}^{p-}\mathbf{R}'^{-1}(0) = -p^n a_{-n}^{p-}$. It also satisfies the property

$$\mathbf{R}'(0)\left(u\Lambda_1^{\rm op}(z)+\Lambda_2^{\rm op}(z)\right)\mathbf{R}'^{-1}(0) = \Lambda_1^{\rm op}(z)+u\Lambda_2^{\rm op}(p^2 z). \tag{43}$$

Since $\mathbf{R}'(u)$ is fixed for small u now, one may expand it in terms of u around $u = 0$. We write $\mathbf{R}'(u)$ as

$$\mathbf{R}'(u) = \left(1+R^{(1)}u+R^{(2)}u^2+\cdots\right)\mathbf{R}'(0); \tag{44}$$

then the equation we need to solve becomes

$$\begin{aligned}(1+R^{(1)}u+R^{(2)}u^2+\cdots)\left(\Lambda_1^{\rm op}(z)+u\Lambda_2^{\rm op}(p^2z)\right) &= (\Lambda_1^{\rm op}(z)+u\Lambda_2^{\rm op}(z))\\ &\quad\times(1+R^{(1)}u+R^{(2)}u^2+\cdots),\end{aligned}$$

which reduces to the recursive equation

$$\left[R^{(n)},\Lambda_1^{\rm op}(z)\right] = \Lambda_2^{\rm op}(z)R^{(n-1)} - R^{(n-1)}\Lambda_2^{\rm op}(p^2z). \tag{45}$$

For $n = 1$, $\left[R^{(1)},\Lambda_1^{\rm op}(z)\right] = \Lambda_2^{\rm op}(z)-\Lambda_2^{\rm op}(p^2z)$. Our claim is that the solution is given by

$$\begin{aligned}uR^{(1)} &= \frac{(1-p)}{(1-q)(1-t^{-1})}\left(\sum_{r\in\mathbb{Z}}x_r^+\otimes x_{-r}^- - u\right)\\ &= \frac{u(1-p)}{(1-q)(1-t^{-1})}\left(:\exp\left(-\sum_{n\neq 0}\frac{1-t^{-n}}{n}p^{-\frac{|n|}{2}}a_{-n}^{p-}w^n\right):\Big|_{\text{zero mode}}-1\right),\end{aligned} \tag{46}$$

where the normalization of the R-matrix is again fixed by $\mathbf{R}(u)|0\rangle\otimes|0\rangle = |0\rangle\otimes|0\rangle$, as we required in the case of the MO R-matrix.

Proof. In general, if $[A,B]$ is a c-number, the equation $e^A e^B = e^{[A,B]}e^B e^A$ holds. Using this, we can show the equation $\left[:e^{A(w)}:,:e^{B(z)}:\right] = \left(e^{[A_+(w),\,B_-(z)]}-e^{[B_+(z),\,A_-(w)]}\right):e^{A(w)+B(z)}:$, where $A(w)$ and $B(z)$ are linear summations of boson oscillators and $\pm$ denotes the positive/negative mode part.

With this identity, we have

$$\begin{aligned}&\left[:\exp\left(-\sum_{n\neq 0}\frac{1-t^{-n}}{n}p^{-\frac{|n|}{2}}a_{-n}^{p-}w^n\right):,\Lambda_1^{\rm op}(z)\right]\\ &\quad=\left(\exp\left(\sum_{n=1}^{\infty}\frac{(1-q^n)(1-t^{-n})}{np^{\frac{n}{2}}}\left(\frac{z}{w}\right)^n\right)-\exp\left(\sum_{n=1}^{\infty}\frac{(1-q^n)(1-t^{-n})}{np^{\frac{n}{2}}}\left(\frac{w}{z}\right)^n\right)\right)\\ &\qquad\times:\exp\left(\sum_{n\neq 0}\frac{1}{n}\frac{1-t^{-n}}{1+p^{|n|}}\left(z^n-(p^{\frac{|n|}{2}}+p^{-\frac{|n|}{2}})w^n\right)a_{-n}^{p-}\right):.\end{aligned} \tag{47}$$

Note that

$$\exp\left(\sum_{n=1}^{\infty}\frac{(1-q^n)(1-t^{-n})}{np^{\frac{n}{2}}}x^n\right)=\frac{(1-qp^{-\frac{1}{2}}x)(1-t^{-1}p^{-\frac{1}{2}}x)}{(1-p^{-\frac{1}{2}}x)(1-p^{\frac{1}{2}}x)}$$
$$=1+\frac{(1-q)(1-t^{-1})}{(1-p)}\left(\frac{1}{1-p^{-\frac{1}{2}}x}-\frac{1}{1-p^{\frac{1}{2}}x}\right)$$

(it only applies around $x\sim 0$), we have

$$\frac{(1-p)}{(1-q)(1-t^{-1})}\left[:\exp\left(-\sum_{n\neq 0}\frac{1-t^{-n}}{n}p^{-\frac{|n|}{2}}a^{p-}_{-n}w^n\right):,\Lambda^{\mathrm{op}}_1(z)\right]$$
$$=\left(\frac{1}{1-p^{-\frac{1}{2}}\frac{z}{w}}-\frac{1}{1-p^{\frac{1}{2}}\frac{z}{w}}-\frac{1}{1-(p^{\frac{1}{2}}\frac{z}{w})^{-1}}+\frac{1}{1-(p^{-\frac{1}{2}}\frac{z}{w})^{-1}}\right)$$
$$\times:\exp\left(\sum_{n\neq 0}\frac{1}{n}\frac{1-t^{-n}}{1+p^{|n|}}(z^n-(p^{\frac{n}{2}}+p^{-\frac{n}{2}})w^n)a^{p-}_{-n}\right):$$
$$=\left(\delta\left(\frac{z}{p^{\frac{1}{2}}w}\right)-\delta\left(\frac{p^{\frac{1}{2}}z}{w}\right)\right):\exp\left(\sum_{n\neq 0}\frac{1}{n}\frac{1-t^{-n}}{1+p^{|n|}}(z^n-(p^{\frac{n}{2}}+p^{-\frac{n}{2}})w^n)a^{p-}_{-n}\right):$$
$$=\delta\left(\frac{z}{p^{\frac{1}{2}}w}\right):\exp\left(-\sum_{n\neq 0}\frac{1}{n}\frac{1-t^{-n}}{1+p^{|n|}}\left(\frac{z}{p}\right)^n a^{p-}_{-n}\right):$$
$$-\delta\left(\frac{p^{\frac{1}{2}}z}{w}\right):\exp\left(-\sum_{n\neq 0}\frac{1}{n}\frac{1-t^{-n}}{1+p^{|n|}}(zp)^n a^{p-}_{-n}\right):$$
$$=\delta\left(\frac{z}{p^{\frac{1}{2}}w}\right)\Lambda^{\mathrm{op}}_2(z)-\delta\left(\frac{p^{\frac{1}{2}}z}{w}\right)\Lambda^{\mathrm{op}}_2(p^2z).$$

The zero mode (w^0-order term) is exactly $\Lambda^{\mathrm{op}}_2(z)-\Lambda^{\mathrm{op}}_2(p^2z)$. □

Higher orders can be determined from the recursive formula, but it is rather complicated to perform the calculation.

5. Charges of the associated integrable system

Let us compute the first charge from $R^{(1)}$ imitating the calculation in the degenerate limit reviewed in Sect. 3.1. The monodromy operator is $\mathbf{T}_0(u)=\mathbf{R}_{10}(u)\mathbf{R}_{20}(u)\cdots\mathbf{R}_{N0}(u)$. The N-site transfer matrix is given by

$$t_N(u):={}_0\langle 0|\mathbf{T}(u)|0\rangle_0={}_0\langle 0|\mathbf{R}_{10}(u)\mathbf{R}_{20}(u)\cdots\mathbf{R}_{N0}(u)|0\rangle_0. \tag{48}$$

In the one-site case, the transfer matrix reads

$${}_0\langle 0|\mathbf{R}_{10}(u)|0\rangle_0={}_0\langle 0|\mathbf{R}'(0)R_{--}|0\rangle_0+u\,{}_0\langle 0|R^{(1)}\mathbf{R}'(0)R_{--}|0\rangle_0+\cdots.$$

The first term gives rise to some vertex operator, which is an overall factor of the R-matrix. The second term can be obtained from the w^0-order term of

$$\begin{aligned} &x^{+(1)}(w)\ {}_0\langle 0|x^{-(0)}(w)\mathbf{R}'(0)R_{--}|0\rangle_0 \\ &\quad = x^{+(1)}(w)\ {}_0\langle 0| : \exp\left(-\sum_{n\neq 0}\frac{1-t^{-n}}{n}p^{-\frac{|n|}{2}}a^{(0)}_{-n}z^n\right) : \mathbf{R}'(0)R_{--}|0\rangle_0 \\ &\quad = x^{+(1)}(w)\ {}_0\langle 0|\mathbf{R}'(0)R_{--}|0\rangle_0, \end{aligned} \tag{49}$$

where we used

$$\begin{aligned} a^{(0)}_n \mathbf{R}'_{10}(0)R_{--10} &= p^{\frac{n}{2}}\mathbf{R}'_{10}(0)a^{(1)}_n R_{--10} \\ &= p^{\frac{n}{2}}\mathbf{R}'_{10}(0)R_{--10}\, a^{(0)}_n \quad (\text{for } n \geq 1). \end{aligned}$$

Dividing out the overall factor, we obtain the first charge x_0^+, which agrees with the Hamiltonian of the Ruijsenaars–Schneider model (Ref. [48]).

It is easy to find charges in the case of the N-site. Let us decompose the R-matrix as $\mathbf{R}(u) = \sum_i a_i \otimes b_i$; then, e.g., charges in the case of the 2-site can be computed as

$$\begin{aligned} {}_0\langle 0|\mathbf{R}_{10}(u)\mathbf{R}_{20}(u)|0\rangle_0 &= {}_0\langle 0|(\Delta\otimes 1)\mathbf{R}_{12}(u)|0\rangle_0 \\ &= \sum_i \Delta(a_i)\ {}_0\langle 0|b_i^{(0)}|0\rangle_0 \\ &= \Delta\left({}_0\langle 0|R_{10}(u)|0\rangle_0\right), \end{aligned}$$

where in the last line we used the fact that the ${}_0\langle 0|b_i^{(0)}|0\rangle_0$ are merely c-numbers. This result suggests that charges in the 2-site case are the coproduct of charges in the case of the 1-site. The 2-site Hamiltonian[4] is $\Delta(x_0^+) = x^+(z)\otimes 1 + \psi^-(\gamma_{(1)}^{\frac{1}{2}}z)\otimes x^+(\gamma_{(1)}z)\Big|_{\text{zero mode}}$ and the coproduct gives rise to nontrivial interactions between 2-sites. This argument can be generalized to higher N-site cases.

Since x_0^+ is exactly the S-dual of the first nontrivial charge in the vertical representation (or (0,m) representation, for $m \in \mathbb{Z}_{>0}$) (Ref. [26]), it is therefore natural to conjecture that higher-rank charges correspond to the S-dual of the ψ_n^+ ($n \geq 2$) in the vertical representation.

6. Comparison with the results from Refs. [28,29]

In this section, we compute the matrix elements of the oscillator expression obtained for $\mathbf{R}(u)$ to compare it with the known results in the literature (Refs. [28,29]).

Let $|\lambda_1,\lambda_2\rangle$ be the eigenstate[5] of $\Delta(x_0^+)$, i.e., $\Delta(x_0^+)|\lambda_1,\lambda_2\rangle = \kappa_{\lambda_1\lambda_2}|\lambda_1,\lambda_2\rangle$ with the corresponding eigenvalue $\kappa_{\lambda_1\lambda_2}$; then $\mathbf{R}(u)|\lambda_1,\lambda_2\rangle$ will be the eigenstate to $\Delta^{\mathrm{op}}(x_0^+)$:

$$\kappa_{\lambda_1\lambda_2}\mathbf{R}(u)|\lambda_1,\lambda_2\rangle = \mathbf{R}(u)\Delta(x_0^+)|\lambda_1,\lambda_2\rangle = \Delta^{\mathrm{op}}(x_0^+)\mathbf{R}(u)|\lambda_1,\lambda_2\rangle, \tag{50}$$

[4] We note that this is a conjecture. Even though the first charge is computed to have the same expression as x_0^+, we did not prove that it has the same coproduct as x_0^+ as we have already taken the horizontal representation and expressed everything in the q-boson, which has no definite coproduct structure.

[5] This basis is often called the generalized Macdonald polynomial basis. The normalization is determined by $|\lambda_1,\lambda_2\rangle = a^{(1)}_{-\lambda_1}a^{(2)}_{-\lambda_2}|0\rangle_1\otimes|0\rangle_2 + \cdots$.

following from the definition of the universal R-matrix. From this relation, we have $\mathbf{R}(u)|\lambda_1,\lambda_2\rangle = R_{\lambda_1\lambda_2}(u)|\lambda_1,\lambda_2\rangle^{\rm op}$. Here, $R_{\lambda_1\lambda_2}(u)$ is a constant factor and $|\lambda_1,\lambda_2\rangle^{\rm op}$ is the polynomial obtained from $|\lambda_1,\lambda_2\rangle$ by replacing u with $1/u$, and exchanging $a^{(1)}$ and $a^{(2)}$. In Ref. [28], the form of $R_{\lambda_1\lambda_2}(u)$ is conjectured as

$$R_{\lambda_1\lambda_2}(u) = \left(\frac{q}{t}\right)^{\frac{1}{2}(|\lambda_1|+|\lambda_2|)} \frac{G_{\lambda_1\lambda_2(u)}}{G_{\lambda_1\lambda_2(\frac{q}{t}u)}},$$
$$G_{\lambda_1\lambda_2(u)} = \Pi_{(i,j)\in\lambda_1}\left(1-uq^{\lambda_i^{(1)}-j}t^{\lambda_j^{(2)T}-i+1}\right)\Pi_{(i,j)\in\lambda_2}\left(1-uq^{-\lambda_i^{(2)}+j-1}t^{-\lambda_j^{(1)T}+i}\right). \quad (51)$$

It was checked at low levels and was further studied in Ref. [49].

Let us check the consistency with our result. We apply the bosonic representation of the R-matrix obtained in Sect. 4 to $|\lambda_1,\lambda_2\rangle$ in the horizontal representation up to level 2.

At level 1, there are two generalized Macdonald polynomials:[6]

$$|\Box,\emptyset\rangle = a_{-1}^{(1)}|0\rangle_1\otimes|0\rangle_2,$$
$$|\emptyset,\Box\rangle = -\frac{1-p}{(p)^{1/2}(1-u)}a_{-1}^{(1)}|0\rangle_1\otimes|0\rangle_2 + a_{-1}^{(2)}|0\rangle_1\otimes|0\rangle_2.$$

The corresponding eigenvalues are

$$\kappa_{\Box\emptyset} = u_2\left(1+(t^{-1}-p+q)u\right), \qquad \kappa_{\emptyset\Box} = u_1\left(1+(t^{-1}-p+q)u^{-1}\right). \quad (52)$$

The action of $\mathbf{R}(u)$ is evaluated as

$$\begin{aligned}
\left(1+R^{(1)}u+\mathcal{O}(u^2)\right)\mathbf{R}'(0)R_{--}|\Box,\emptyset\rangle &= p^{1/2}\left(1-(1-p)u\right)a_{-1}^{(1)}|0\rangle_1\otimes|0\rangle_2 \\
&\quad + (1-p)ua_{-1}^{(2)}|0\rangle_1\otimes|0\rangle_2 + \mathcal{O}(u^2) \\
&= \frac{p^{-1/2}(u-1)}{u-1/p}|\emptyset,\Box\rangle^{\rm op} + \mathcal{O}(u^2), \\
\left(1+R^{(1)}u+\mathcal{O}(u^2)\right)\mathbf{R}'(0)R_{--}|\emptyset,\Box\rangle &= \left((1+u)p^{1/2}-p^{-1/2}\right)a_{-1}^{(2)}|0\rangle_1\otimes|0\rangle_2 + \mathcal{O}(u^2) \\
&= \frac{p^{-1/2}(u-p)}{u-1}|\Box,\emptyset\rangle^{\rm op} + \mathcal{O}(u^2).
\end{aligned}$$

The coefficients before $|\lambda_1,\lambda_2\rangle^{\rm op}$ obtained here agree with the results in Refs. [28,29] up to $\mathcal{O}(u^2)$. One may also check such consistency with Refs. [28,29] at level 2. The details are explained in Appendix B.

7. Conclusion and discussion

We showed in this paper that by taking the $(1,0)$ horizontal representation of the DIM algebra, the MO R-matrix can be obtained from the bosonic realization of the universal R-matrix in the degenerate limit $q\to 1$. The universal R-matrix, by construction, is associated with the algebra, not its representation, and in Sect. 6, we did see that the matrix elements of the R-matrix computed

[6] The label of the generalized Macdonald polynomial is determined so that the eigenvalue takes the form $\kappa_{\lambda_1\lambda_2} = u_1e_{\lambda_1}+u_2e_{\lambda_2}$, with $e_\lambda = 1+(t-1)\sum_i(q^{\lambda^{(i)}}-1)t^{-i}$ for $\lambda = \{\lambda^{(i)}\}$. This formula is quoted from Ref. [29].

in two ways agree with each other. We may also have the MO R-matrix as the degenerate limit of the universal R-matrix in a different representation, e.g., in the $(0,1)$ vertical representation. The degenerate limit of the DIM algebra, i.e., the $\widehat{\mathfrak{gl}_1}$ Yangian or the SH^c algebra, however, does not present a manifest SL(2,$\mathbb{Z}$) symmetry, and it is unclear what is a suitable definition for the MO R-matrix in such a representation. We will leave this problem to future work.

Also, there are many unsolved questions even in the $(1,0)$ representation, due to the complexity of the calculation. To conclude this article, we list them here as future directions:

- We solved only one defining equation among three for the universal R-matrix, i.e.,

$$(\Delta \otimes 1)\mathbf{R} = \mathbf{R}_{13}\mathbf{R}_{23}, \qquad (1 \otimes \Delta)\mathbf{R} = \mathbf{R}_{13}\mathbf{R}_{12} \tag{53}$$

  are left untouched. In fact, if we decompose the action of the R-matrix on the vacuum state as

$$\mathbf{R}(u)|0\rangle_1 \otimes |0\rangle_2 = \sum_i a_i \otimes b_i |0\rangle_1 \otimes |0\rangle_2,$$

  using Eq. (53) we can show that $\mathbf{R}(u)$ can act only trivially on the vacuum, i.e., $\mathbf{R}(u)|0\rangle_1 \otimes |0\rangle_2 = |0\rangle_1 \otimes |0\rangle_2$. We conjecture that by setting such a normalization condition, the solution of the R-matrix obtained in this article satisfies Eq. (53), as there is only one unique solution to Eq. (27) with this specific normalization. We emphasize again that it is not straightforward to check this conjecture even at the leading order in u of the R-matrix. This is due to the fact that we took the horizontal representation of the R-matrix and we cannot assign a coproduct for the q-boson used in the representation.
- It is not clear how the oscillator expression of the universal R-matrix is related to that of the MO R-matrix. The recursive formula obtained in this paper for the universal R-matrix is not directly related to that for the MO R-matrix. As we can see, $R^{(1)}$ starts at order $\mathcal{O}(\hbar)$ in the degenerate limit, which also implies that all $R^{(n)}$ $(n \geq 1)$ are of order $\mathcal{O}(\hbar)$, and the naive limit $\hbar \to 0$ suggests

$$\lim_{\hbar \to 0} \mathbf{R}(u) = 1,$$

  which is clearly wrong. Interestingly, at the leading order of $\hbar$, the recursive equation for the $R^{(n)}$ becomes trivial ($[R^{(n+1)}, \alpha_k^-] = [R^{(n)}, \alpha_k^-]$), and the leading order in $\hbar$ of $R^{(1)}$ ($(1-\beta)\hbar \sum_{n>0} \alpha_{-n}\alpha_n$) survives in all the $R^{(n)}$. We can resum all of them into an $\mathcal{O}(\hbar^0)$ term,

$$\sum_{k=1}^{\infty}(1-\beta)\hbar u^k \sum_{n>0} \alpha_{-n}\alpha_n = \frac{(1-\beta)\hbar u}{1-u}\sum_{n>0}\alpha_{-n}\alpha_n \to \frac{2Q}{\bar{u}}\sum_{n>0}\alpha_{-n}\alpha_n, \tag{54}$$

  which reproduces the first nontrivial term in the expansion of the MO R-matrix (see, e.g., Ref. [45]). In other words, this suggests that $R^{(1)}$ contains almost all pieces of the higher-rank terms of the MO R-matrix as its $\hbar$-expansion, with corrections from the $R^{(n)}$ $(n > 1)$. Unfortunately, at the current stage we do not know how to reproduce the second nontrivial expansion mode in the MO R-matrix in a similar way.
- We have neglected an overall vertex operator $\mathbf{R}'(0)R_{--}$ in the calculation of charges in the related integrable lattice model. It is not clear what kind of role it plays both in the Ruijsenaars–Schneider model and in the degenerate limit.

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Appendix A. Campbell–Baker–Hausdorff formula

Let us check Eq. (37). We use the following Campbell–Baker–Hausdorff formula, where $(\mathrm{ad}A)B = [A, B]$:

$$\mathrm{e}^{A}B\mathrm{e}^{-A} = B + [A, B] + \tfrac{1}{2}[A, [A, B]] + \cdots = \mathrm{e}^{\mathrm{ad}A}B. \tag{A.1}$$

Using the relation $[a^{(-)}_m, a^{(-)}_n] = m\frac{1-q^{|m|}}{1-t^{|m|}}\delta_{m+n,0}$, we have

$$\begin{aligned}
\left[\pi i \sum_{m=1}^{\infty} \frac{1}{m}\frac{1-t^m}{1-q^m} a^{(-)}_{-m}a^{(-)}_{m}, a^{(-)}_{n}\right] &= -\pi i a^{(-)}_{n}, \\
\left[\pi i \sum_{m=1}^{\infty} \frac{1}{m}\frac{1-t^m}{1-q^m} a^{(-)}_{-m}a^{(-)}_{m}, a^{(-)}_{-n}\right] &= \pi i a^{(-)}_{-n} \quad (\text{for } n > 0\ ).
\end{aligned} \tag{A.2}$$

Using this equation and the Campbell–Baker–Hausdorff formula, we have

$$\begin{aligned}
&R_{--}a^{(-)}_n R^{-1}_{--} \\
&\quad = \exp\left(\pi i \sum_{m=1}^{\infty} \frac{1}{m}\frac{1-t^m}{1-q^m} a^{(-)}_{-m}a^{(-)}_{m}\right) a^{(-)}_{n} \exp\left(-\pi i \sum_{m=1}^{\infty} \frac{1}{m}\frac{1-t^m}{1-q^m} a^{(-)}_{-m}a^{(-)}_{m}\right) \\
&\quad = \mathrm{e}^{\pm\pi i} a^{(-)}_{n} \\
&\quad = -a^{(-)}_{n} \quad (\text{for } n \neq 0).
\end{aligned} \tag{A.3}$$

Note that $R_{--}a^{(+)}_n R^{-1}_{--} = a^{(+)}_n$ holds because of the trivial commutation relation $[R_{--}, a^{(+)}_n] = 0$.

Combining these relations between R_{--} and $a^{(\pm)}_n$, we have Eq. (37).

Appendix B. Matrix element of the R-matrix at level 2

At level 2, the generalized Macdonald basis is known as

$$\begin{aligned}
|\,\text{vertical domino}, \emptyset\rangle &= \frac{(1+q)(1-t)}{(1-q)(1+t)} a^{(1)}_{-1}a^{(1)}_{-1}|0\rangle_1 \otimes |0\rangle_2 + a^{(1)}_{-2}|0\rangle_1 \otimes |0\rangle_2, \\
|\,\square\square, \emptyset\rangle &= a^{(1)}_{-1}a^{(1)}_{-1}|0\rangle_1 \otimes |0\rangle_2 - a^{(1)}_{-2}|0\rangle_1 \otimes |0\rangle_2, \\
|\square, \square\rangle &= \frac{(1-p)(2t-(1+q-t+qt))}{2p^{1/2}(1-qu)(u-t)} a^{(1)}_{-1}a^{(1)}_{-1}|0\rangle_1 \otimes |0\rangle_2 \\
&\quad + \frac{(1-p)(1-q)(1+t)u}{2p^{1/2}(qu-1)(u-t)} a^{(1)}_{-2}|0\rangle_1 \otimes |0\rangle_2 + a^{(1)}_{-1}a^{(2)}_{-1}|0\rangle_1 \otimes |0\rangle_2,
\end{aligned}$$

$$
\begin{aligned}
|\emptyset, \square\square\rangle =& \frac{(1-p)\big((1-p)(1+t)-(1+q+t-2pt-pqt-t^2+pt^2)u+(t+qt-t^2-qt^2)u^2\big)}{p(1-u)(1-tu)(1-(u+qu-1)t)} \\
&\times a^{(1)}_{-1}a^{(1)}_{-1}|0\rangle_1\otimes|0\rangle_2 \\
&+\frac{(1-p)\big((1+p)(1+t)-(1-q+3t+2pt+pqt+t^2+pt^2)u+(t-qt+t^2+2pt^2+qt^2)u^2\big)}{p(1-u)(1-tu)(1-t(u+qu-1))} \\
&\times a^{(1)}_{-2}|0\rangle_1\otimes|0\rangle_2 \\
&+\frac{2(1-p)}{p^{1/2}(1-tu)}a^{(1)}_{-1}a^{(2)}_{-1}|0\rangle_1\otimes|0\rangle_2+a^{(2)}_{-1}a^{(2)}_{-1}|0\rangle_1\otimes|0\rangle_2-a^{(2)}_{-2}|0\rangle_1\otimes|0\rangle_2, \\
|\emptyset, \begin{smallmatrix}\square\\\square\end{smallmatrix}\rangle =& \frac{(1-p)(1+q)(1-t)}{p(1-q)(1+t)(1-u)(q-u)((1+t)u-(1+q)t)}\Big((1-q)(1+t)u^2 \\
&-\big(t-qt-q^2(1+t)+p(t-q(1+2t))\big)u-qt(1-p)(1+q)\Big)a^{(1)}_{-1}a^{(1)}_{-1}|0\rangle_1\otimes|0\rangle_2 \\
&-\frac{(1-p)}{p(1-u)(u-q)((1+t)u-(1+q)t)}\Big((1-q(1-t)+t+2pt)u^2 \\
&+((1-t)q^2-t-3qt-pq-pt-2pqt)u+qt(1+p)(1+q)\Big)a^{(1)}_{-2}|0\rangle_1\otimes|0\rangle_2 \\
&+\frac{2q(1-p)(1+q)(1-t)}{p^{1/2}(1-q)(1+t)(u-q)}a^{(1)}_{-1}a^{(2)}_{-1}|0\rangle_1\otimes|0\rangle_2 \\
&+\frac{(1+q)(1-t)}{(1-q)(1+t)}a^{(2)}_{-1}a^{(2)}_{-1}|0\rangle_1\otimes|0\rangle_2+a^{(2)}_{-2}|0\rangle_1\otimes|0\rangle_2.
\end{aligned}
$$

The action of the R-matrix can be computed as

$$
\begin{aligned}
\mathbf{R}(u)|\begin{smallmatrix}\square\\\square\end{smallmatrix},\emptyset\rangle =& \frac{p(1+q)(1-t)(1-(1-p)(1+q)u)}{(1-q)(1+t)}a^{(1)}_{-1}a^{(1)}_{-1}|0\rangle_1\otimes|0\rangle_2 \\
&+p(1-(1-p)(1+q)u)a^{(2)}_{-2}|0\rangle_1\otimes|0\rangle_2 \\
&+\frac{(1-p)(1+q)(1-t)}{1+t}ua^{(2)}_{-1}a^{(2)}_{-1}|0\rangle_1\otimes|0\rangle_2 \\
&+(1-p)(1+q)ua^{(2)}_{-2}|0\rangle_1\otimes|0\rangle_2 \\
&+\frac{2p^{1/2}q(1-p)(1-t)(1+q)}{(1-q)(1+t)}ua^{(1)}_{-1}a^{(2)}_{-1}|0\rangle_1\otimes|0\rangle_2+\mathcal{O}(u^2) \\
=& \frac{(u-1)(qu-1)}{(u-1/p)(qpu-1)}|\emptyset,\begin{smallmatrix}\square\\\square\end{smallmatrix}\rangle^{\mathrm{op}}+\mathcal{O}(u^2), \\
\mathbf{R}(u)|\square\square,\emptyset\rangle =& -\frac{p}{t}((1-p)(1+t)-t)ua^{(1)}_{-2}|0\rangle_1\otimes|0\rangle_2-\frac{u}{t}(1-p)(1+t)a^{(2)}_{-2}|0\rangle_1\otimes|0\rangle_2 \\
&-\frac{p}{t}u((1-p)(1+t)-t)a^{(1)}_{-1}a^{(1)}_{-1}|0\rangle_1\otimes|0\rangle_2 \\
&-\frac{u}{t}(1-p)(1-t)a^{(2)}_{-1}a^{(2)}_{-1}|0\rangle_1\otimes|0\rangle_2 \\
&+\frac{2p^{1/2}}{t}(1-p)ua^{(1)}_{-1}a^{(2)}_{-1}|0\rangle_1\otimes|0\rangle_2+\mathcal{O}(u^2) \\
=& \frac{(u-1)(u-t)}{(u-1/p)(pu-t)}|\emptyset,\square\square\rangle^{\mathrm{op}}+\mathcal{O}(u^2),
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{R}(u)|\square,\square\rangle &= \Big(p - \frac{1}{t}(1-q+qt-p+pq+pt-pqt-p^2t)u\Big)a^{(1)}_{-1}a^{(2)}_{-1}|0\rangle_1\otimes|0\rangle_2\\
&\quad + \frac{(1-p)(1-q)(1+t)}{2p^{1/2}t}ua^{(2)}_{-2}|0\rangle_1\otimes|0\rangle_2\\
&\quad + \frac{1}{sp^{1/2}t}(1-p)(1-(1-t)q-(1-2p)t)ua^{(2)}_{-1}a^{(2)}_{-1}|0\rangle_1\otimes|0\rangle_2 + \mathcal{O}(u^2)\\
&= \frac{(u-p)(u/p-q)}{(qu-1)(u-t)}|\square,\square\rangle^{\mathrm{op}} + \mathcal{O}(u^2),\\
\mathbf{R}(u)|\emptyset,\square\!\square\rangle &= \left(p - (2-q+\frac{q}{t}-2p-\frac{q(1-t)}{pt})u\right)a^{(2)}_{-1}a^{(2)}_{-1}|0\rangle_1\otimes|0\rangle_2\\
&\quad - \left(p - \frac{q(1-p)(1+t)}{pt}u\right)a^{(2)}_{-2}|0\rangle_1\otimes|0\rangle_2 - \frac{u}{t}(1-p)(q-pt)a^{(1)}_{-2}|0\rangle_1\otimes|0\rangle_2\\
&\quad - \frac{2(1-p)(1-t)(q-pt)}{p^{1/2}t}ua^{(1)}_{-1}a^{(2)}_{-1}|0\rangle_1\otimes|0\rangle_2\\
&\quad + \frac{(1-p)(1-t)(q-pt)}{t(1+t)}ua^{(1)}_{-1}a^{(1)}_{-1}|0\rangle_1\otimes|0\rangle_2 + \mathcal{O}(u^2)\\
&= \frac{(u-p)(tu/p-1)}{(u-1)(tu-1)}|\square\!\square,\emptyset\rangle^{\mathrm{op}} + \mathcal{O}(u^2),\\
\mathbf{R}(u)|\emptyset,\begin{smallmatrix}\square\\\square\end{smallmatrix}\rangle &= \left(p - \frac{(1+q)(1-p)}{pt}u\right)a^{(2)}_{-2}|0\rangle_1\otimes|0\rangle_2 + \frac{u}{t}(1-p)(q-pt)a^{(1)}_{-2}|0\rangle_1\otimes|0\rangle_2\\
&\quad - \frac{(1-p)(1-t)(q-pt)}{t(1+t)}ua^{(1)}_{-1}a^{(1)}_{-1}|0\rangle_1\otimes|0\rangle_2\\
&\quad + \frac{2(1-p)(1+q)(1-t)(q-pt)}{p^{1/2}qt(1+t)}ua^{(1)}_{-1}a^{(2)}_{-1}|0\rangle_1\otimes|0\rangle_2\\
&\quad - \frac{(1+q)(1-t)(p^2t+(2tp^2-(1-q)(1-p)-2tp)u)}{p(1-q)t(1+t)}a^{(2)}_{-1}a^{(2)}_{-1}|0\rangle_1\otimes|0\rangle_2\\
&\quad + \mathcal{O}(u^2)\\
&= \frac{(u-p)(u/p-q)}{(u-1)(u-q)}|\begin{smallmatrix}\square\\\square\end{smallmatrix},\emptyset\rangle^{\mathrm{op}} + \mathcal{O}(u^2).
\end{aligned}
$$

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