# Multiparticle dynamics in the E-$\phi$ tracking code ESME

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**Abstract.** ESME has developed over a twenty year period from its origins as a program for modeling rf gymnastics to a rather general facility for that fraction of beam dynamics of synchrotrons and storage rings which can be properly treated in the two dimensional longitudinal phase space. The features of this program which serve particularly for multiparticle calculations are described, some uderlying principles are noted, and illustrative results are given.

## INTRODUCING ESME

Multiparticle tracking has established utility for modeling evolution of longitudinal phase space distributions of particles in accelerators as they respond to the rf in acceleration or bunch manipulation. ESME has been primarily developed for design studies of machine cycles and rf gymnastics using single particle equations of motion.

One goes from single particle to multiparticle dynamics by calculating the beam current every time step and including its effect on the single particle motion. However, the number of macroparticles needed and bandwidth required for quantities in the frequency domain need careful attention. It is very easy to generate spectacular spurious instabilities by long time steps or too few macroparticles. Considerable attention has been given in recent years to these considerations and to development of facilities for multiparticle dynamics.

### Space Charge Model

The self-impedance from the direct interparticle forces is derived from an electrostatic calculation in the beam rest frame transformed to the lab frame.[1] The force at the $n$-th time step arises from the gradient of the azimuthal charge distribution $\Lambda_n(\Theta)$, which can be evaluated from its Fourier series for the frequency domain or directly for time domain. The impedance representing the self force is

$$\frac{Z_m^{sc}}{m} = -i\frac{Z_0 g}{2\beta\gamma^2} \; , \tag{1}$$

[1] Work supported by the U. S. Department of Energyunder contract number DE-AC02-76CH03000.

where $Z_0 = \sqrt{\mu_0/\varepsilon_0} = 377\Omega$ and $g$ is a factor containing the dependence on beam and vacuum chamber transverse dimensions. For a uniform cylindrical beam of radius $a$ centered in a smooth beampipe of radius $b$

$$g = 1 + 2\log(\frac{b}{a}) \; . \tag{2}$$

In ESME this factor is scaled with beam momentum to account for the change in beam radius and rolled off at very high frequency to approximate the exact solution at high mode numbers.

### Frequency Domain Facilities

ESME is fundamentally time domain, but beam current and image currents can be Fourier analyzed at harmonics of synchronous circulation frequency $\Omega_{s,n}$. At the end of the $n$-th time step the beam has an azimuthal charge distribution $\Lambda_n(\Theta)$. Assuming that the distribution is practically unchanged in a single time step (usually one beam turn), it gives rise to a beam current

$$I_{b,n}(\Omega_{s,n}t) = \frac{e\Omega_{s,n}}{2\pi}\sum_m \Lambda_{m,n} e^{i(-m\Omega_{s,n}t+\psi_{m,n})} \; , \tag{3}$$

here expressed as a sum of phasors.

The current induces voltage through the total longitudinal coupling impedance $Z_{||}(\omega)$; this quantity evaluated at $m\Omega_{s,n}$ is denoted by the phasor $Z_{m,n}e^{i\chi_{m,n}}$. The beam-induced voltage is applied to each particle at time $t_n$ when the synchronous particle is at the gap; that voltage depends on the relative phase between the particle and the current. The synchronous particle is defined to have phase 0. Thus, the energy increment for the $i$-th particle on the $n$-th turn resulting from the beam current is

$$eV_{i,n}^b = -\frac{Ne^2\Omega_{s,n}}{2\pi}\sum_m \Lambda_{m,n} Z_{m,n} e^{i(m\Theta_{i,n}+\psi_{m,n}+\chi_{m,n})} \; . \tag{4}$$

The apparent absence of frequencies other than circulation harmonics is not a defect in this formulation. Synchrotron sidebands are generated in the tracking the same way they are generated in a synchrotron — by the phase modulation.

## Time Domain Facilities

The gradient of $\Lambda$ is calculated at each particle location by cubic spline interpolation over three bins of a charge histogram. For simple resonances, a Green's function solution is used which permits turn-to-turn accumulation of the wakefield. The wakefield for arbitrary distribution and arbitrary $Z_{\|}$ is calculated using the response to triangular unit current pulse calculated by TCBI or the like convolved with the current distribution. One may freely inter-mix potentials expressed in time domain and frequency domain. The calculation of bunch disruption occurring with heavy transient beam loading corrected by feed forward and feedback around the cavities plus global phase and voltage feedback loops, for example, is an application which would draw on nearly all of these features.

### *Scaling Concept*

By inspection of the single particle equations of motion, it appears that the phase space motion can be accelerated by scaling the phase slip factor $\eta$ and the potential by the same factor, hereafter $\lambda$.[2] Both the single particle and collective potential are scaled the same. Indeed, particle distributions have been practically identical when time $t$ in an un-scaled calculation is compared to time $t/\lambda$ in a scaled calculation for a variety of test calculations. The obvious gain is a factor $\lambda^{-1}$ in computing time by speeding up the clock in the scaled calculation. However, more importantly, scaling of the time means that frequencies like the rf frequency and resonance frequencies in $Z_{\|}$ are also scaled. The frequencies of Fourier components are scaled, and, for $\lambda > 1$, fewer harmonics are needed to span the range of a given $Z_{\|}(\omega)$. For a particularly interesting example consider the space charge equivalent impedance; it is nearly linear in frequency so that just a few Fourier harmonics would appear necessary to characterize it. The evaluation is not compromised by widely spaced frequency sampling. Because the number of macroparticles can be scaled by $\lambda^{-3}$ when the number of bins is reduced by $\lambda^{-1}$ with same level of numerical noise, there is a potential gain from scaling of $\lambda^4$ in many cases. This cubic rule for macroparticle number has been shown rigorously for space charge in time domain[3] and heuristically in frequency domain for any smooth $Z_{\|}(\omega)$[4]

# DEMONSTRATION APPLICATIONS

The two examples following are chosen for their suggestivity rather than as exhibits of results of special importance in themselves. Nonetheless, the first example, self bunching of a coasting beam, may surprise even some experienced beam physicists.

## Coasting Beam Self-Bunching

Excitation of a passive resonator by coasting beam and beam response to the resonant voltage is an archetypal multiparticle dynamics problem. One expects that beam will self-bunch above threshold, with the bunches decelerated from the initial beam energy. This example illustrates that bunching is dependent on the relation of resonant frequency to nearest harmonic of beam circulation frequency. Animations using parameters of the Los Alamos PSR with addition of an h=3 resonator ($R_{\text{shunt}} = 300$ Ohm, $Q = 100$) show a distinctly different qualitative behavior depending whether the resonant frequency is one percent above or below three times the beam circulation frequency. The distributions after 160 ms are shown in Figs. 1 and 2; animated GIF's may be viewed at `www-ap.fnal.gov/~jmaclach` as psr3hi.gif and psr3lo.gif. In either case, mean energy is decelerated equally, in buckets when the resonance is above the beam circulation harmonic or by phase displacement with empty buckets when it is below. The analogy to Robinson instability (h=1) is suggestive.

## Negative Mass Instability

Attempts to compare the Hardt[5] analysis of negative mass instability (NMI) to numerical tracking have run up against particle statistics problems deriving from computer speed and memory limitations. The relaxation of these constraints because of computer developments invites another look. Furthermore, time scaling should ameliorate both time and storage problems. In addition, numerically quiet distributions of a reasonable number of macroparticles can be (at least initially) even quieter than the real beam distribution.

Several cases based on the Fermilab Main Injector (FMI) have been calculated to explore the effects of computational parameters; some results are given in Table 1. The maximum growth rate occurs near 60 GHz regardless of time/bandwidth scaling, much below the 120 GHz predicted by Hardt's model. The two orders of magnitude difference between the growth for 0.2 and 0.3 eVs is consistent with the threshold of just below 0.3 eVs calculated with Hardt's formulas. The amount of emittance growth is comparable but less with scaling factor $\lambda > 1$, a discrepancy not observed in previous uses of time scaling. Fig. 3 shows the spectrum of the beam current over the frequency range 21 to 74 GHz taken at intervals of 88 $\mu$s. The rapid shift of the peak to lower frequency may relate to the discrepancy between the analytical prediction and the lower peak frequency observed, *viz.*, the shift may progress substantially before the instability is visible on a linear amplitude scale. Computers are now fast enough and big enough that a calculation could be done for a physically interesting case with the actual number of particles in a bunch, thereby eliminating any question about scaling or numerical noise.

## PARTING THOUGHTS

Particle tracking of the evolution of a distribution is not in general superior or inferior to solving the Vlasov equation — simply different. Numerical modeling in dynamics is especially helpful to look for gross oversights, carry practical calculations beyond threshold, evaluate the result of the simultaneous action of several processes, and to get a quick check on expectations. Graphical output from macroparticle models can communicate results easily to non-experts. Also it's nice to get an easy reality check during a protracted calculation; reasonably well-tested codes are a useful source of comparisons or benchmarks.

**Table 1.** Emittance growth $\Delta\varepsilon/\varepsilon$ and frequency of strongest Fourier component $\hat{f}$ [GHz] in NMI of $4.5 \cdot 10^{10}$ proton FMI bunches of full emittance $\varepsilon$ [eVs]. Model parameters are number of macroparticles $n_{\mathrm{p}}[10^6]$, number of charge bins $n_{\mathrm{b}}$, and time scaling factor $\lambda$. See text for other parameters.

| $\varepsilon$ | $n_{\mathrm{p}}$ | $n_{\mathrm{b}}$ | $\lambda$ | $\Delta\varepsilon/\varepsilon$ | $\hat{f}$ |
|---|---|---|---|---|---|
| 0.2 | 128 | 4096 | 1 | 0.222 | 58.3 |
| 0.2 | 16 | 2048 | 2 | 0.112 | 58.3 |
| 0.3 | 128 | 4096 | 1 | 0.003 | 42.4 |
| 0.5 | 128 | 4096 | 1 | 0.000 | none |

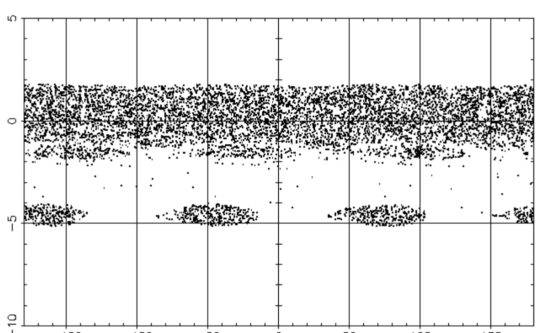


**Figure 1.** Coasting beam response with parameters like Los Alamos Proton Storage Ring with added passive cavity tuned 1% *above* the third harmonic of the circulation frequency; see text for parameter details

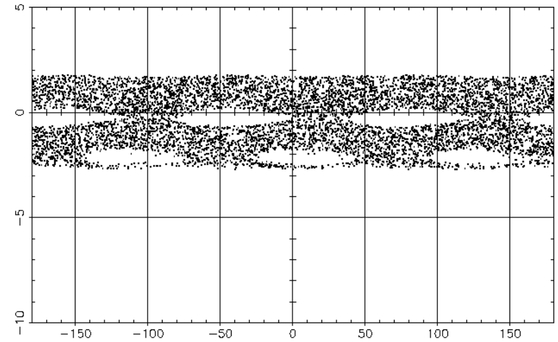


**Figure 2.** Coasting beam response to the same cavity used for the Fig. 2 result, except tuned 1% *below* the third harmonic of the circulation frequency

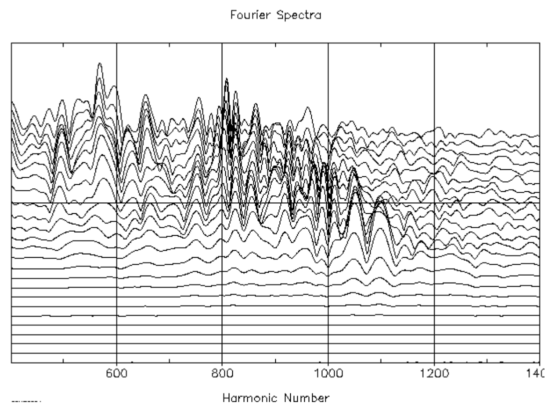


**Figure 3.** Fourier spectrum at times separated by 88 $\mu$s for 0.2 eVs FMI bunch of $4.5 \cdot 10^{10}$ protons just above transition ($\eta = 8.9 \cdot 10^{-5}$) The abscissa is labeled in harmonics of the 53 MHz rf; the frequency span shown is 21 – 74 GHz.

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