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# Spinfoams: Simplicity Constraints and Correlation Functions

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# Contents

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# Chapter 0

# Introduction

Searching for a quantum theory of gravity is the major challenge in today's fundamental physics, which will merge the conceptual novelties of general relativity (GR) and quantum mechanics (QM). As the two most great conceptual revolutions in the physics of the twentieth century, GR and QM have reshaped our basic understanding of space and time, and respectively, matter, energy and causality. And at the same time, GR and QM have destroyed the single coherent picture of the world provided by pre-relativistic classical physics, since each was formulated in terms of assumptions contradicted by the other theory. In particular, in GR, the central lesson is *gravity is geometry*, which means that there is no non-dynamical background spacetime in nature, but the conventional quantum field theory relies heavily on the existence of a fixed, non-dynamical background spacetime. We are still far from having found the novel consistent picture of the world.

To find this consistent picture, if one takes the central lesson of GR seriously: gravity is geometry, it leads to the direction of research investigated by background independent approaches to quantum gravity, particularly, loop quantum gravity (LQG) [1–5].

There are several different approaches to LQG. In the canonical quantization approach [1–4], the Hilbert space of kinematical states and the field operators that act on it are obtained from classical GR following a rather standard canonical quantization strategy, which provide a mathematically well-defined and physically compelling description of the kinematics of quantum gravity. However, the Hilbert space of dynamical states, where the Hamiltonian solves, is not easy to obtain because of the non-polynomial structure of the current Hamiltonian operator.

This problem can be solved by spin-foam formalism [5–13], which circumvents the complexity of the canonical LQG Hamiltonian operator: the dynamics can be expressed by a vertex amplitude of a spin-foam. Research in this direct is currently moving ahead fast [14–18].

The topic of this thesis is some themes of spinfoam formalism. The focus is on the imposing of simplicity constraint and the computing the Lorentzian propagator. The former is to find the way to connect different aspects related to the EPRL spinfoam model, and the later is to test the resulting model and try to extract physics from that. The rest of the thesis is organized as follows. We first give a brief introduction to the spinfoam formalism and kinematical space in chapter 1, then we derive the spinfoam model from simplicity constraints and obtain the kinematical space in chapter 2 and 3. In chapter 4, we calculate the two-point correlation function of gravity from this model in the physical relevant Lorentzian signature. We conclude in chapter 5.

# Chapter 1

# Spinfoam formalism and kinematics

This chapter is divided into three sections. In section 1.1, we give a compact presentation of spinfoam formalism, which describe the dynamics of quantum gravity. In section 1.2, we give the kinematics of loop quantum gravity, which is obtained from canonical quantization. In section 1.3, we give the truncated kinematics from polyhedral geometry and a outline of this thesis in the end.

## 1.1 Spinfoam formalism: a simple presentation

In this section, we will give a brief presentation of spinfoam formalism.

Let us start with the notion of *spinfoam*. Firstly, a *foam* is defined by an oriented two-complex with or without boundary. We take a combinatorial definition of an oriented 2-complex. An oriented 2-complex $\mathcal{K} := (V(\mathcal{K}), E(\mathcal{K}), F(\mathcal{K})$ consists of sets of *vertices* $v \in V(\mathcal{K})$, *edges* $e \in E(\mathcal{K})$ and *faces* $f \in F(\mathcal{K})$, equipped with a *boundary relation* $\partial$ associating an ordered pair of vertices $(s(e), t(e))$ ("source" and "target") to each edge $e$ and a finite sequence of edges $\{e_k^{\epsilon_{e_k f}}\}_{k=1,\ldots,n}$ to each face $f$, with $t(e_k) = s(e_{k+1})$, $t(e_n) = s(e_1)$ and $\epsilon_{ef} = \pm 1$; here we call $e^{-1}$ the edge with reversed order of $e$. We let $\partial f$ denote the cyclically ordered set of edges that bound the face $f$, or (if it is clear from the context) the cyclically ordered set of vertices that bound the boundary edges of $f$. We also write $\partial v$ to indicate the set of edges bounded by $v$, and of faces that have $v$ in their boundary. Similarly, we write $\partial e$ to indicate the set of the faces bounded by $e$. When $e \in \partial f$, we define $\epsilon_{ef} = 1$ if the orientation of $e$ is consistent with the one induced by the face $f$ and $\epsilon_{ef} = -1$

if it is not. This oriented two-complex $\mathcal{K}$ is called a *foam*. A *spinfoam* is then a foam "colored" with *spins* and *intertwiners*. To color the foam, let us consider the group $SU(2)$ and associate an irreducible representation $j_f$ (a spin) to each face of the two-complex $\mathcal{K}$, and an $SU(2)$ intertwiner $i_e$ to each edge of the two complex. The triple $\sigma = (\mathcal{K}, j_f, i_e)$ is called a *spinfoam*.

The dynamics is given by the spinfoam amplitude

$$Z_\sigma = \sum_{j_f, i_e} \prod_f A_f(j_f) \prod_v A_v(j_f, i_e) \tag{1.1.1}$$

where the sum is over an assignment of an irreducible representation $j_f$ of $SU(2)$ to each face and of an intertwiner $i_e$ to each edge of the two-complex; $A_v(j_f, i_e)$ is the *vertex amplitude* and $A_f(j_f)$ is the *face amplitude*. In [19], Bianchi et al. argue that the face amplitude is uniquely determined for any spinfoam amplitude of the form (1.1.1) by three inputs: (a) the choice of the boundary Hilbert space, (b) the requirement that the composition law holds when gluing two-complexes; and (c) a particular "locality" requirement, or, more precisely, a requirement on the local composition of group elements. For most models so far considered, these requirements are implemented when face amplitude $A_f(j_f)$ is fixed to be the $SU(2)$ dimension

$$A_f(j_f) = 2j_f + 1. \tag{1.1.2}$$

However, in section 2.6, we find some exceptions when the new degree of freedom emerges. The vertex amplitude $A_v(j_f, i_e)$ is determined by the specific spinfoam model, however, in this thesis, we concentrate on the most recent and studied EPRL vertex, which we will show in detain in section 2.

## 1.2 The Hilbert space of kinematical states

In this section, we introduce the Hilbert space of kinematical states, obtained from the canonical approach of LQG, which is closely related to spinfoam formalism. In fact, the first spinfoam models ever constructed [20–22] were indeed directly inspired or derived from the formalism of canonical loop quantum gravity. The spinfoam formalism and canonical loop quantum gravity can ideally be viewed as the covariant and the canonical versions, respectively, of a background-independent quantum theory of gravity [13]. This scenario is nicely realized in three dimensions [23], and there are recent attempts to implement it in quantum cosmology [24–28].

General relativity admits two different kind of formulations: the Lagrangian formulation and the Hamiltonian formulation. Each formulation has its own merits. The Lagrangian formulation makes the symmetries of the theory manifest, while the Hamiltonian formulation yields insights into the nature of the dynamics. At the classical level, in the Lagrangian formulation, general relativity can be presented as a theory of metrics, while in the Hamiltonian formulation can also be recast as a dynamical theory of connections. Such a reformulation brings general relativity closer to gauge theories in the sense that, in the Hamiltonian framework, they now share the same quantization procedure to obtain the quantum kinematics. Following this standard quantization procedure, we obtain a quantum theory of gravity at the kinematical level, however not a solution to the Hamiltonian constraint.

Now let us come to give an outline of the main features of canonical loop quantum gravity. We refer to the beautiful textbooks [1, 2] and some other existing reviews [3, 4] for more details.

Consider a four-dimensional manifold $\mathcal{M}$, which has the product topology $\mathcal{M} = \mathbb{R}\times\, \Sigma$, with $\Sigma$ a compact three dimensional manifold without boundary . The action of the pure gravity is given by

$$S[e,\omega] = \int_{\mathcal{M}} \mathrm{Tr}\Big(\big({}^*e\wedge e + \frac{1}{\gamma} e\wedge e\big)\wedge F(\omega)\Big), \tag{1.2.1}$$

where $\gamma$ is the Barbero-Immirzi parameter; the tetrad $e^I_\alpha$ denotes metric via $g_{\alpha\beta} = \eta_{IJ} e^I_\alpha e^J_\beta$; $F$ is the curvature of $so(1,3)$-valued connection $\omega$. Upon the standard 3+1 decomposition, the phase space in these variables is parameterized by the pull back to $\Sigma$ of $\omega$ and $e$. In local coordinates we can express them in terms of the 3-dimensional connection $A^i_a$ and the triad field $E^a_i$:

$$\begin{aligned} A^i_a &= \gamma K^i_a + \Gamma^i_a \\ E^a_i &= \frac{1}{2}\epsilon^{abc}\epsilon_{ijk} e^j_b e^k_c \end{aligned} \tag{1.2.2}$$

where $a = 1,2,3$ are space coordinate indices and $i, j = 1,2,3$ are $su(2)$ indices, with $K^i_a$ extrinsic curvature of $\Sigma$ and $\Gamma$ a spin connection. The symplectic structure is defined by

$$\{A^i_a(x), E^b_j(y)\} = \delta^b_a \delta^i_j \delta(x,y). \tag{1.2.3}$$

In terms of $(A, E)$, the action (1.2.1) can be rewritten as

$$S[A,E] = \int \mathrm{d}t \int_\Sigma E^a_i \dot{A}^i_a - N\mathcal{H} - N^a\mathcal{H}_a - \Lambda^i \mathcal{G}_i \tag{1.2.4}$$

with the constraints given by

$$\mathcal{G}_i = D_a E^a_i$$
$$\mathcal{H}_a = E^b_k F^k_{ab}$$
$$\mathcal{H} = \frac{\epsilon_k{}^{ij} E^a_i E^b_j}{|\det e|}(F^k_{ab} - (1+\gamma^2)\epsilon^k_{mn} K^m_a K^n_b), \tag{1.2.5}$$

which are, respectively, the $SU(2)$ Gauss constraint, imposing the local gauge invariance on the states, the diffeomorphism constraint, generating 3-dimensional diffeomorphisms on $\Sigma$, and the Hamiltonian constraint, representing the evolution of $\Sigma$ in the (unphysical) coordinate time. They are the first-class constraints.

Now we have derived the classical Hamiltonian formulation of general relativity, with the canonical variables and the first-class constraints on the classical phase space. At the quantum level the canonical variables will be replaced by operators acting on the state space of the theory and the full dynamical content of general relativity will be encoded in the action of the first-class constraints on the states. Consequently let us focus on these states.

We introduce the notion of graph $\Gamma$ and show that a Hilbert space $\mathcal{H}(\Gamma)$ can be associated to it. The kinematic Hilbert space $\mathcal{H}_{\mathrm{LQG}}$ of LQG is defined in terms of this graph Hilbert space $\mathcal{H}(\Gamma)$. We take a combinatorial [1] definition of an oriented graph. An oriented graph is an oriented 1-complex $\Gamma := (N(\Gamma), L(\Gamma))$ consists of sets of *nodes* $n \in N(\Gamma)$ and *links* $l \in L(\Gamma)$, equipped with a *boundary relation* $\partial$ associating an ordered pair of nodes $(s(l), t(l))$ ("source" and "target") to each link $l$.

Now let us consider the group $SU(2)$ and associate an irreducible representation $j_l$ (a spin) to each link $l$ of the graph $\Gamma$, and an $SU(2)$ intertwiner $i_n$ to each node $n$ of the graph. The triple $s = (\Gamma, j_l, i_n)$ defines a *spin network*. To be more explicit, now let us briefly recall the notion of *intertwiner*. Consider[2] $L$ links entering a node $n$ . Intertwiners between $L$ representations of $SU(2)$ form a finite dimensional

[1] One can also start by defining a graph embedded in the manifold $\mathcal{M}$, in the final picture, however, one can get rid of the background manifold as well. And then the fundamental theory can be formulated combinatorially. In fact, one can understand the states associated with combinatorial graphs to be solutions to the diffeomorphism constraint on the states associated with the embedded graph. One takes the diffeomorphism constraint into account by considering the equivalence class of embedded spin networks, which we will introduce later, under the action of $Diff(M)$. Then the stats, purely algebraic and combinatorial objects, are defined regardless of any embedding.

[2] The notion of graph, including link and node, is not necessary to define a intertwiner. However, we consider intertwiner on a node here, for labelling simplicity.

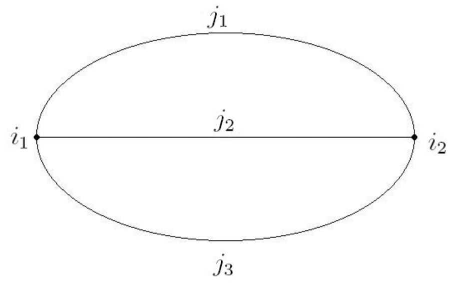



Figure 1.1: A simple spin network.

vector space. Consider $L$ particles of spin $j_1, .., j_L$ . Simultaneous eigenstates of the operators $J_l^2$ and $J_l^z$ with $l = 1, .., L$,

$$|m_1, .., m_L\rangle = \bigotimes_{l=1}^{L} |j_l, m_l\rangle, \tag{1.2.6}$$

provide an orthonormal basis of the Hilbert space $\mathcal{H}_n = \otimes_{l\in\partial n}\mathcal{H}_{j_l}$ of the system. Now we focus on *singlet states*, i.e. on the subspace $\mathcal{H}_n^o$ of $\mathcal{H}_n$ of states that are invariant under rotations. More explicitly, if one expands an element $|i\rangle$ of an orthonormal basis of $\mathcal{H}_n^o$ on the basis $|m_1, ...m_L\rangle$ of $\mathcal{H}_n$ and call $i^{m_1...m_L}$ its coefficients,

$$|i\rangle = \sum_{m_1,...,m_L} i^{m_1...m_L}|m_1, ..., m_L\rangle \tag{1.2.7}$$

then the tensors $i^{m_1...m_L}$ are invariant under the action of $SU(2)$ on all their indices:

$$R^{(j_1)m_1}{}_{m_1'}(h_1)...R^{(j_L)m_L}{}_{m_L'}(h_L)i^{m_1'...m_L'} = i^{m_1...m_L}.\forall h_1, ..., h_L \in SU(2) \tag{1.2.8}$$

where $D^{(j)}(h)$ is the $j$-representation matrix of the $SU(2)$ group element $h$. Here the invariant tensors $i^{m_1...m_L}$ intertwine the representations $j_1, ..., j_L$ labeling the links that enter the node $n$, where the upper indices $m_1, ...m_L$ denote the node $n$ is the target of all the links $l$, while lower indices (if any) denote $n$ the source, since we have the oriented links. Now we have a spin network $s$ as a triple $(\Gamma, j_l, i_n)$ given by: a one-dimensional oriented (with a suitable orientation of the links) complex $\Gamma$, a labeling $j_l$ of each link $l$ of $\Gamma$ by an irreducible representation $j_l$ of $SU(2)$, and a labeling $i_n$ of each node $n$ of $\Gamma$ by an intertwiner $i_n$.

A *spin-network state* $\psi_s(h_l)$, labeled by a spin-network $s$, is defined as

$$\psi_s(h_l) = \Big(\bigotimes_{n\in\Gamma} i_n\Big) \cdot \Big(\bigotimes_{l\in\Gamma} \sqrt{2j_l + 1}D^{(j_l)}(h_l)\Big) \tag{1.2.9}$$

The linear space generated by spin-network states on the graph $\Gamma$ can be promoted to a Hilbert space $\mathcal{H}(\Gamma)$ introducing the following scalar product:

$$(\psi_1, \psi_2) = \int_{SU(2)^L} \mathrm{d}h_1 ... \mathrm{d}h_L \overline{\psi_1(h_1, ..., h_L)} \psi_2(h_1, ..., h_L) \tag{1.2.10}$$

where $\mathrm{d}h$ is the Haar measure on $SU(2)$. The resulting (gauge-invariant) graph Hilbert space $\mathcal{H}(\Gamma)$ is the linear space of square integrable functions on $SU(2)^L/SU(2)^N$,

$$\mathcal{H}(\Gamma) = L^2(SU(2)^L/SU(2)^N, \mathrm{d}\mu_{\mathrm{Haar}}), \tag{1.2.11}$$

where the denominator means that the states in $\mathcal{H}(\Gamma)$ are invariant under the local $SU(2)$ gauge transformation on the nodes

$$\psi(h_l) \mapsto \psi(g_{t_l} h_l g_{s_l}^{-1}). \tag{1.2.12}$$

Spin network states on a graph $\Gamma$ provide an orthonormal basis of the Hilbert space $\mathcal{H}(\Gamma)$.

On the set of graph Hilbert spaces $\{\mathcal{H}(\Gamma)\}$, it is possible to define a uniform measure and complete the construction to get the LQG Hilbert space $\mathcal{H}_{\mathrm{LQG}} = \lim_{\Gamma \to \infty} \mathcal{H}_\Gamma$.

## 1.3 Polyhedral quantum geometry

In this section we consider the truncation of the LQG Hilbert space $\mathcal{H}_{\mathrm{LQG}}$ constructed in section 1.2. We restrict ourself to a single graph Hilbert space $\mathcal{H}(\Gamma)$ and decompose it in terms of $SU(2)$-invariant spaces $\mathcal{H}_n$ associated to each node $n$. Here we will briefly review the state that this node space $\mathcal{H}_n$ is the quantization of the space of shapes of the geometry of solids figures (tetrahedra [29], or more general polyhedra [30]).

Let us start with the classical phase space of shapes of a flat polyhedron in $\mathbb{R}^3$ with fixed area. A (classical) flat three-dimensional polyhedron can be described by a set of $L$ vectors $\vec{A}_l$, $l = 1...L$, satisfying the following closure constraint:

$$\vec{G} = \sum_{l=1...L} \vec{A}_l = 0 \ . \tag{1.3.1}$$

Here the $L$ vectors $\vec{A}_l$ can be interpreted as the vectorial areas of the $L$ triangles in the boundary of the polyhedron, in the sense that the norm $a_l = |A_l|$ is the area of the polygon $l$ and normalized vector $\vec{n}_l = \vec{A}_l/|A_l|$ is the normal when embedded in to a $\mathbb{R}^3$ Euclidean space.

To introduce a symplectic structure, one can associate to each normal $A^i_l$ a generator of the algebra of $SO(3)$, s.t.:

$$\{A^i_l, A^j_{l'}\} = \delta_{f,f'}\epsilon^{ij}{}_k\, A^k_l \tag{1.3.2}$$

A quantum representation of this Poisson algebra is precisely defined by the generators of $SU(2)$ on the space $\mathcal{H}_n$ for a 4-valent node $n$. The operator corresponding to the area $a_l = |\vec{A}_l|$ is the Casimir of the representation $j_l$, therefore the space "quantizes" the space of the shapes of the tetrahedron with areas $j_l(j_l+1)$. Furthermore, the Hamiltonian flow of $\vec{G}$ in (1.3.1), generates the rotations of the tetrahedron in $R^3$. By imposing equation (1.3.1) and factoring out the orbits of this flow, one obtains the intertwiner space $\mathcal{K}_n$.

In this way, one gives an intertwiner a geometrical interpretation in terms of quantum polyhedron. In chapter 2 and chapter 3, we will show the relation among spinfoam formalism, kinematical Hilbert space and polyhedral quantum geometry. They are presented in a coherent picture. In chapter 2, we show the boundary space of simplicial EPRL spinfoam model can be obtained from simplicity constraints, which is the simplicial truncation of LQG kinematical Hilbert space and the boundary state has a geometrical interpretation in terms of quantum tetrahedron geometry. This consistent picture is generalized into arbitrary-valence spinfoam formalism in chapter 3. In chapter 4, we compute the two-point correlation function of Lorentian EPRL spinfoam model and show it matches the one from Regge geometry.

# Chapter 2

# The simplicial Spinfoam models

In this chapter [1] we will present the detail of simplicial spinfoam models: we obtain the spinfoam model by imposing the simplicity constraints to the $BF$ model. Here we consider the most recent and most studied spinfoam model, which is called Engle-Pereira-Rovelli-Livine (EPRL) model [14–16, 33]. This model, as well as the so-called Freidel-Krasnov-Livine-Speziale (FKLS) model[2] [17, 18], can be considered as a refined Barrett-Crane (BC) model, based on the vertex amplitude introduced by Barrett and Crane [34, 35]. Both of the EPRL and FKLS models are motivated by a desire to modify the BC model. The key problem of the BC model is the fact that *intertwiner* quantum numbers are fully constrained by imposing the simplicity constraints, which are second class, as strong operator equations. But imposing second class constraints strongly may lead to the incorrect elimination of physical degrees of freedom. It is therefore natural to try to free intertwiner degrees of freedom by imposing the simplicity constraints more weakly: they must be imposed in the quantum theory in such a way that in the classical limit the constraints hold, but all physical degrees of freedom remain free.

Among the several procedures proposed to impose these constraints, are the master constraint procedure [14–16, 33] originally used for construction the EPRL model, and the coherent state procedure [17, 18] used to derive the FKLS model. In addition, the EPRL model can be also obtained using the coherent state procedure [36, 37] developed in the FKLS model. Here one presents the Gupta-Bleuler procedure, namely asking the matrix elements of the simplicity constraints to vanish on

[1] This chapter is partly based on work done together with Carlo Rovelli. The results have been published in [31, 32].

[2] The EPRL and the FKLS models are very closed to each other, in the sense that they are equal to each other when Barbero-Immirzi parameter $\gamma$ smaller than 1 in the Euclidean signature.

physical boundary states [31, 32, 38]. In the Gupta-Bleuler procedure, we construct a candidate physical boundary Hilbert space, and then we prove that in this space the matrix elements of all the constraints vanish. In this sense, constraints are imposed *weakly*, rather than strongly as in the BC theory. The fact that the matrix elements of the constraints vanish assures that the constraints hold in the classical limit.

Following this procedure, one shows that the physical boundary space satisfies the simplicity constraints weakly, and matches the kinematical state space of the canonical approach, which we briefly introduced in section 1.2. A natural map between the two state spaces can be obtained by identifying eigenstates of the same physical quantities. We consider this boundary space is *physical*, since it matches the on of loop quantum gravity, with the same physical degree of freedom. Surprisingly, however, this physical state space is *not* the maximal solution, where the simplicity constraints hold weakly. This is just a subspace of the solution space where the simplicity constraints hold weakly, which means there exist other states in the complementary space of this physical boundary space, where the simplicity constraints also hold weakly. In other words, one can obtain a larger (weak) solution space to the simplicity constraints, which we call it *enlarged* boundary space. But still, one is not sure if this enlarged space is the maximal solution. The enlarged space includes an additional degree of freedom, described by a new quantum number, which affects non-trivially both the face amplitude and the vertex amplitude of the spinfoam model. We comment more on this in section 2.6.

The model we construct contains in fact a slight modification with respect to the original EPRL one introduced in [14–16, 33] (corresponding to a slightly different factor ordering of the constraints). The same modification was already considered by Alexandrov in [39]. We show that with the modification the matrix elements vanish *exactly*, and not just in the large quantum number limit, as in previous constructions. However, we still call this model EPRL if there is no confusion rise.

Surprisingly, however, the state space defined by imposing the simplicity constraint weakly is larger than the one of quantum gravity. It includes one additional degree of freedom, described by a new quantum number $r_f$. The quantum number $r_f$ affects non-trivially both the face amplitude and the vertex amplitude of the model. The quantum number $r_f$ is frozen if in addition to the weak imposition of the (linear) simplicity constraint, we also impose strongly a diagonal quadratic constraint. With a suitable operator ordering of this constraint, the state space can be reduced back down to the LQG state space.

Does the $r_f$ quantum number have physical relevance? If we take the principle that the quantum theory we are seeking has the same number of degrees of freedom as the classical theory, then the answer is negative. This principle indicates that the appropriate way of imposing the constraints is the one that gets rids of the extra states. However, we think it is nevertheless interesting to keep in mind the existence of these additional solutions to the weak simplicity constraints. We comment more on this in section 2.6.

We work only on a fixed triangulation, and assume that the Barbero-Immirzi parameter $\gamma$ is positive. The candidate boundary space is given in section **??**, both in the Euclidean and the Lorentzian signatures. In section 2.3 the Gupta-Bleuler procedure is used to show that this boundary space does solve all the constraints in the weak sense. In sections 2.4.1 and 2.4.2, the physical boundary space is shown to be isomorphic to that of canonical approach and used to derive the (modified) EPRL vertex amplitude from the $BF$ amplitude. In section 2.6, the enlarged boundary space is obtained and the affection of the new degree of freedom is discussed.

## 2.1 The EPRL spinfoam models

In this section one present the EPRL spinfoam models, both in the Euclidean and the Lorentzian signatures. One uses $s = \pm 1$ to denote the signatures: $s = 1$ for the Euclidean case and $s = -1$ for the Lorentzian case. The structure groups in the two cases are the spin group $Spin(4)$ (if $s = 1$) and the Lorentz group $SL(2, \mathbb{C})$ (if $s = -1$) respectively. Throughout this thesis, $SL(2, \mathbb{C})$ refers to the 6-dimensional real Lie group of $2 \times 2$ complex matrices with unit determinant, and is called simply the Lorentz group. It covers the group of proper orthochronous Lorentz transformations, $SO^+(3, 1)$, which is the component of the group $O(3, 1)$ connected to the identity.

Consider a fixed 4-dimensional triangulation $\Delta$, which is formed by oriented 4-simplices, tetrahedra, triangles, segments and points. The cellular complex $\Delta^*$ dual to this triangulation $\Delta$, is made by faces $f$, edges $e$ and vertices $v$, dual respectively to triangles $f$, tetrahedra $t$ and 4-simplices $v$ of $\Delta$. The new model is defined on the 2-skeleton of $\Delta^*$, by a standard spin-foam partition function:

$$Z = \sum_{j_f, i_e} \prod_f (2j_f + 1) \prod_v A_v(j_f, i_e), \tag{2.1.1}$$

where the sum is over an assignment of an irreducible representation $j_f$ of $SU(2)$

to each face $f$, and over an assignment of an element $i_e$ of a basis in the space of intertwiners to each edge $e$. The face amplitude is given by the $SU(2)$ dimension $2j+1$, which is determined in [19] by the structure of the boundary Hilbert space and the condition that amplitudes behave appropriately under compositions. We recall that an intertwiner is an element of the $SU(2)$ invariant subspace of the tensor product of the four Hilbert spaces carrying the four representations associated to the four faces adjacent to a given $e$. We use the usual basis given by the spin of the virtual link, under a fixed pairing of the four faces.

In the Euclidean theory ($s$=1), the amplitude $A_v(j_f, i_e)$ associated to each vertex $v$ is given by

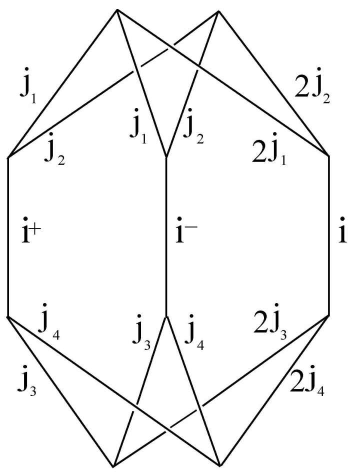
If $0 < \gamma < 1$

$$A^{<}(j_f, i_e) = \sum_{i_a^+ i_a^-} \Big(\prod_e Y^{i_e}_{i_e^+ i_e^-}(j_f)\Big) 15j_{Spin(4)}\left(\left(\frac{1+\gamma}{2}j, \frac{1-\gamma}{2}j\right); (i_n^+, i_n^-)\right)$$

If $\gamma > 1$

$$A^{>}(j_f, i_e) = \sum_{i_a^+ i_a^-} \Big(\prod_e Y^{i_e}_{i_e^+ i_e^-}(j_f)\Big) 15j_{Spin(4)}\left(\left(\frac{\gamma+1}{2}j + \frac{\gamma-1}{2}, \frac{\gamma-1}{2}(j+1)\right); (i_n^+, i_n^-)\right) \tag{2.1.2}$$

where the coefficients $Y^i_{i^+ i^-}(j_l)$ are given by the evaluation of the spin network




In the Lorentzian theory ($s = -1$), the amplitude $A_v(j_f, i_e)$ associated to each vertex $v$ is given by

$$A_v(j_f, i_e) = \sum_{k_e} \int \mathrm{d}p_e (k_e^2 + p_e^2) \left( \prod_e f^{i_e}_{k_e p_e}(j_f) \right) 15j_{SL(2,\mathbb{C})} \left( (j_f, \gamma(j_f+1)); (k_e, p_e) \right), \tag{2.1.3}$$

where the sum and the integral are over an assignment of an irreducible unitary representation $(k, p)$ of $SL(2,\mathbb{C})$ , with $k$ a nonnegtive integer and $p$ real [40, 41]; $15j_{SL(2,\mathbb{C})}$ is the Wigner 15j symbol of the group $SL(2,\mathbb{C})$; $f^{i_e}_{k_e p_e}(j_f)$ is the fusion coefficient obtained contracting $SU(2)$ intertwiners and $SL(2,\mathbb{C})$ intertwiners.

The boundary Hilbert space, satisfying all the kinematical constraints in a weak sense, play a very important role in the construction of the vertex amplitudes (2.1.2) and (2.1.3). Let us now come to give the boundary Hilbert space.

## 2.2 The physical boundary Hilbert space

### 2.2.1 The physical boundary Hilbert space: Euclidean theory

Given a 3-surface $\Sigma$ intersecting no vertices of $\Delta^*$, let $\gamma_\Sigma := \Delta^* \cap \Sigma$. We start from the Hilbert space associated with $\Sigma$:

$$\mathcal{H}^{\mathrm{Eu}}_\Sigma = L^2 \left( Spin(4)^{|L(\gamma_\Sigma)|}, \mathrm{d}\mu_{\mathrm{Haar}} \right), \tag{2.2.1}$$

where we replace $SO(4)$ with its covering group $Spin(4) = SU(2) \times SU(2)$ and $\mu_{\mathrm{Haar}}$ is the Haar measure on the group $Spin(4)$; $|L(\gamma_\Sigma)|$ denotes the number of links in $\gamma_\Sigma$. Let $\hat{J}_f(t)^{IJ}$ denote the right-invariant vector fields, determined by the basis $J^{IJ}$ of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$, on the copy of $Spin(4)$ associated with the link $l = f \cap \Sigma$ determined by $f$, with orientation such that the node $n = t \cap \Sigma$ is the source of $l$.

By Peter-Weyl theorem, $\mathcal{H}_\Sigma$ can be decomposed as follows

$$\mathcal{H}_\Sigma = \bigoplus_{j_l} \bigotimes_l \left( \mathcal{H}^*_{j_l} \otimes \mathcal{H}_{j_l} \right), \tag{2.2.2}$$

where $j_l$ is an assignment of a $Spin(4)$ representation to each link $l$ and $\mathcal{H}_j$ is the carrier space of the representation $j$. The two Hilbert spaces associated to the link $l$ are naturally associated to the two nodes that bound the link $l$, because they transform under the action of a gauge transformation at one end of the link.

Regrouping the four Hilbert spaces associated to each node $n$, the last equation can be rewritten in the form

$$\mathcal{H}_\Sigma = \bigoplus_{j_l} \bigotimes_n \mathcal{H}_n. \tag{2.2.3}$$

Here the Hilbert space associated to a node $n$ is

$$\mathcal{H}_n = \bigotimes_{a=1}^{4} \mathcal{H}_{j_a}, \tag{2.2.4}$$

where $a = 1, 2, 3, 4$ runs here over the four edges that join at the node $n$ (that is, the four faces of the boundary tetrahedron), and we have identified the Hilbert space carrying a representation and its dual. We restrict our attention to a single boundary tetrahedron $t$, and its associated Hilbert space $\mathcal{H}_n$, which we call simply $\mathcal{H}$ in the following.

The irreducible unitary representations of $Spin(4)$ are labelled by a couple of spins $(j^+, j^-)$ and are given by the tensor product of two $SU(2)$ irreducibles. That is $\mathcal{H} := \mathcal{H}_n$ has the structure

$$\mathcal{H} = \bigotimes_{a=1}^{4} \mathcal{H}_{(j_a^+, j_a^-)} = \bigotimes_{a=1}^{4} \left(\mathcal{H}_{j_a^+} \otimes \mathcal{H}_{j_a^-}\right). \tag{2.2.5}$$

The physical intertwiner state space $\mathcal{K}_{\text{ph}}$ is a subspace of this space, where the constraints hold in a suitable sense.

As a first step to give the boundary space, let us restrict the representations $(j^+, j^-)$ to the ones that satisfy [39]

$$\begin{aligned} &j^+ > j^- \\ &(1+\gamma)j^- = (1-\gamma)j^+ \quad \text{if } 0 < \gamma < 1 \\ &(\gamma+1)j^- = (\gamma-1)(j^+ + 1) \quad \text{if } \gamma > 1 \end{aligned} \tag{2.2.6}$$

Note that this relation is different from the original construction of the EPRL model when $\gamma > 1$, which is key for our structure. We call $\gamma$-simple the $Spin(4)$ representations that satisfy this relation. One might worry that this relation restricts the value of Barbero-Immirzi parameter $\gamma$. However, one can find $\gamma$ free in the physically relevant Lorentzian theory.

Next, the Clebsch-Gordan decomposition for the single component of $\mathcal{H}$ associated with a single boundary face $f$ gives

$$\mathcal{H}_{j^+ \otimes j^-} = \mathcal{H}_{j^+} \bigotimes \mathcal{H}_{j^-} = \bigoplus_{p=|j^+ - j^-|}^{j^+ + j^-} \mathcal{H}_p. \tag{2.2.7}$$

Consider the highest spin term in each factor for $\gamma < 1$ and the lowest for $\gamma > 1$ respectively; this selects the "extremum" subspace

$$\begin{aligned}\mathcal{H}^{\max} &= \bigotimes_{a=1}^{4} \mathcal{H}_{j^+ + j^-}, \quad \text{for } \gamma < 1; \\ \mathcal{H}^{\min} &= \bigotimes_{a=1}^{4} \mathcal{H}_{j^+ - j^-}, \quad \text{for } \gamma > 1. \end{aligned} \tag{2.2.8}$$

The final physical intertwiner space $\mathcal{K}_{\text{ph}}$ is given by the $SU(2)$-invariant subspace of $\mathcal{H}^{\text{ext}}$:

$$\mathcal{K}_{\text{ph}} = \text{Inv}_{\text{SU(2)}}[\mathcal{H}^{\text{ext}}]. \tag{2.2.9}$$

The total physical boundary space $\mathcal{H}_{\text{ph}}$ of the theory is then obtained as the span of spin-networks in $L^2[Spin(4))^L/Spin^N]$ with $\gamma$-simple representations on edges and with intertwiners in the spaces $\mathcal{K}_{\text{ph}}$ at each node.

In section 2.3, we will show the physical boundary space $\mathcal{H}_{\text{ph}}$ in the both signatures solve all the kinematic constraints in a suitable sense.

### 2.2.2 The physical boundary Hilbert space: Lorentzian theory

Now let us come to give the physical boundary Hilbert space in the Lorentzian signature. Given a 3-surface $\Sigma$ intersecting no vertices of $\Delta^*$, let $\gamma_\Sigma := \Delta^* \cap \Sigma$. We start from the Hilbert space associated with $\Sigma$ [16, 33]:

$$\mathcal{H}_\Sigma = L^2\left(SL(2,\mathbb{C})^{|L(\gamma_\Sigma)|}, \mathrm{d}\mu_{\text{Haar}}\right), \tag{2.2.10}$$

where $\mu_{\text{Haar}}$ is the Haar measure on the group $SL(2,\mathbb{C})$; $|L(\gamma_\Sigma)|$ denotes the number of links in $\gamma_\Sigma$. We fix the orientation such that the node $n = e \cap \Sigma$ is the source of the link $l = f \cap \Sigma$.

By Peter-Weyl theorem, $\mathcal{H}_\Sigma$ can be decomposed as follows

$$\mathcal{H}_\Sigma = \bigoplus_{\chi_l} \bigotimes_{l} \left(\mathcal{H}^*_{\chi_l} \otimes \mathcal{H}_{\chi_l}\right), \tag{2.2.11}$$

where $\chi_l$ is an assignment of an $SL(2,\mathbb{C})$ representation to each link $l$ and $\mathcal{H}_\chi$ is the carrier space of the representation $\chi$. The two Hilbert spaces associated to the link $l$ are naturally associated to the two nodes that bound the link $l$, because they transform under the action of a gauge transformation at one end of the link. Regrouping the four Hilbert spaces associated to each node $n$, the last equation can be rewritten in the form

$$\mathcal{H}_\Sigma = \bigoplus_{\chi_l} \bigotimes_{n} \mathcal{H}_n. \tag{2.2.12}$$

The Hilbert space associated to a node $n$ is

$$\mathcal{H}_n = \bigotimes_{a=1}^{4} \mathcal{H}_{\chi_a}, \tag{2.2.13}$$

where $a = 1, 2, 3, 4$ runs here over the four links that join at the node $n$ (that is, the four faces of the boundary tetrahedron $t$), and we have identified the Hilbert space carrying a representation and its dual. Here the nodes $n$ label the tetrahedra $t$ in the boundary. We restrict our attention to a single boundary tetrahedron, and its associated Hilbert space $\mathcal{H}_n$, which we call simply $\mathcal{H}$ in the following.

Consider the irreducible unitary representations of the principal series of $SL(2, \mathbb{C})$ (for details see [40, 41]), $\mathcal{H} := \mathcal{H}_n$ has the structure

$$\mathcal{H} = \bigotimes_{a=1}^{4} \mathcal{H}_{(k_a, p_a)}, \tag{2.2.14}$$

with $k$ a nonnegative integer and $p$ real. The physical intertwiner state space $\mathcal{K}_{\text{ph}}$ is a subspace of this space, where the constraints hold in a suitable sense.

As a first step to give the physical boundary space, let us restrict the representations to the ones that satisfy [39]

$$p = \gamma(k + 1). \tag{2.2.15}$$

We call $\gamma$-simple the $SL(2, \mathbb{C})$ representations that satisfy this relation. With this relation, the continuous label $p$ becomes quantized, because $k$ is discrete. It is because of this fact that any continuous spectrum depending on $p$ comes out effectively discrete on the subspace satisfying the relation (2.2.15). Notice that the relation here is slightly different from the one used in the literature [16, 33], which is $p = \gamma k$. We will show later that this difference is very important for our construction.

Next, fix an $SU(2)$ subgroup of $SL(2, \mathbb{C})$, then the $(k, p)$ representation for the single component of $\mathcal{H}$ associated with a single boundary face $f$ splits into the irreducible representations $\mathcal{H}_j$ of the $SU(2)$ subgroup as

$$\mathcal{H}_{(k,p)} = \bigoplus_{j=k}^{\infty} \mathcal{H}_j, \tag{2.2.16}$$

with $j$ increasing in steps of 1. Consider the lowest spin term in each factor, where $j$ in the decomposition (3.2.23) is reduced to

$$j = k; \tag{2.2.17}$$

this selects the "minimal" subspace

$$\mathcal{H}^{\min} = \bigotimes_{a=1}^{4} \mathcal{H}_{k_a}. \tag{2.2.18}$$

The final physical intertwiner space $\mathcal{K}_{\rm ph}$ is given by the $SU(2)$-invariant subspace of $\mathcal{H}^{\min}$:

$$\mathcal{K}_{\rm ph} = \mathrm{Inv}_{\rm SU(2)}[\mathcal{H}^{\min}]. \tag{2.2.19}$$

The total physical boundary space $\mathcal{H}_{\rm ph}$ of the theory is then obtained as the span of spin-networks in $L^2[SL(2,\mathbb{C})^L/SL(2,\mathbb{C})^N]$ with $\gamma$-simple representations on edges and with intertwiners in the spaces $\mathcal{K}_{\rm ph}$ at each node.

In section 2.3, we will show the physical boundary space $\mathcal{H}_{\rm ph}$ in the both signatures solve all the kinematic constraints in a suitable sense.

## 2.3 Kinematic constraints

Now let us come to introduce the kinematic constraints, including the simplicity constraints and the closure constraint, and show all of them are satisfied on the physical boundary space $\mathcal{H}_{\rm ph}$. We start from the classical formula. We study both of the Euclidean and the Lorentzian theories. We use $G$ to denote the structure groups: $G = Spin(4)$ in Euclidean case and $G = SL(2,\mathbb{C})$ in Lorentzian case. The corresponding Lie algebra is denoted by $\mathcal{G}$.

### 2.3.1 The classical discrete constraints

Following [15, 16], we start with a Regge geometry [42] on a fixed triangulation. Consider a 4d triangulation, which is formed by oriented 4-simplices, tetrahedra, triangles, segments and points. We call $v, t$ and $f$ respectively the 4-simplices, the tetrahedra and the triangles of the triangulation. For each simplex $v$, we introduce a variable $e^I_\mu(v)$: a right-handed tetrad one-form, constant over a coordinate patch covering the simplex $v$, with the determinant $\det(e) > 0$ positive. Here $\mu = (0,a)$ and $a = 1,2,3$ are spacetime indices, while $I = (0,i)$ and $i = 1,2,3$ are internal indices (the value 0 instead of 4 is for later convenience and does always not indicate a Lorentzian metric, since we consider the both signatures here). Without loss of generality, we can choose a linear coordinate system with basis vectors $\overrightarrow{X}_\mu$ parallel with four edges of $v$ emanating from the same point, and where the (coordinate) length of the four segments is 1. Consider in particular the tetrahedron $t$ spanned by the three vectors $\overrightarrow{X}_a$. To each triangle $f_a$ (coordinate-)normal to the coordinate basis vector $\overrightarrow{X}_a$, we associate a bivector ${}^*\Sigma_a(t)$ defined by:

$$ {}^*\Sigma_a^{IJ} = \frac{1}{2}\epsilon_a{}^{bc}e_b^I e_c^J, \tag{2.3.1} $$

$\Sigma_f(t)$ can be seen as elements in the algebra $\mathcal{G}$, say $\mathfrak{so}(4)$ in the Euclidean case, and $\mathfrak{sl}(2,\mathbb{C})$ in the Lorentzian case; and $*$ stands for the Hodge dual in the internal indices, the completely antisymmetric objects $\epsilon^{IJKL}$ defined as $\epsilon^{0123} = 1$. If we choose $\Sigma_f(t)$ as independent variables instead of the tetrads, and $n_I$ denotes the normal to the tetrahedron $t$, the simplicity constraints on $\Sigma_f(t)$, which assure that a tetrad field exist, can be stated as follows [15, 16]:

$$ C_f^J := n_I \ ({}^*\Sigma_f(t))^{IJ} = 0. \tag{2.3.2} $$

The usual quadratic diagonal

$$ C_{ff} := {}^*\Sigma_f(t) \cdot \Sigma_f(t) = 0 \tag{2.3.3} $$

and off-diagonal

$$ C_{ff'} := {}^*\Sigma_f(t) \cdot \Sigma_{f'}(t) = 0 \tag{2.3.4} $$

simplicity constraints can be easily shown to follow from (2.3.2). Here the dot stands for the scalar product in the Lie algebra. In addition, we should impose the closure constraint

$$ \sum_{f\in\partial t} \Sigma_f(t) = 0. \tag{2.3.5} $$

Here the sum is over the four tetrahedra that bound the tetrahedron. The new linear simplicity constraint (2.3.2) selects the solution of the quadratic constraints where $\Sigma_f = \int_f {}^*(e \wedge e)$. This reformulation is central for the new model [14–16, 33]. In particular, if we choose a "time" gauge where $n^I = (1, 0, 0, 0)$, the simplicity constraint (2.3.2) turns out to be

$$ {}^*\Sigma_f^{0i}(t) = 0. \tag{2.3.6} $$

The classical discrete action is [16, 37]

$$
\begin{aligned}
S = &- \sum_{f \in int\Delta} \mathrm{Tr}\left[\Sigma_f(t) U_f(t) + \frac{1}{\gamma}\, {}^*\Sigma_f(t) U_f(t)\right] \\
&- \sum_{f \in \partial\Delta} \mathrm{Tr}\left[\Sigma_f(t) U_f(t,t') + \frac{1}{\gamma}\, {}^*\Sigma_f(t) U_f(t,t')\right],
\end{aligned}
\tag{2.3.7}
$$

where $U_f(t,t')$ is the group element of $G$, giving the parallel transport across each triangle $f$ bounding $t$ and $t'$ and $U_f(t) := U_f(t,t)$ is the holonomy around the full link, starting at $t$. We use here unites where $2\kappa = 16\pi G = 1$ and $\gamma$ the Barbero-Immirzi parameter. This action, plus the simplicity and closure constraints defines a discretization of general relativity [15, 16]. From the action, we can read off the boundary variables as $\Sigma_f(t) \in \mathcal{G}$, $U_f(t,t') \in G$. One can also see that the variable conjugate to $U_f(t,t')$ is

$$
J_f(t) := \Sigma_f(t) + \frac{1}{\gamma} {}^*\Sigma_f(t), \tag{2.3.8}
$$

inverting which gives

$$
{}^*\Sigma_f(t) = \frac{\gamma^2}{s-\gamma^2}\Big(\frac{s}{\gamma} J_f(t) - {}^* J_f(t)\Big). \tag{2.3.9}
$$

Here $s$ denotes the signature: $s = 1$ in Euclidean theory and $s = -1$ in Lorentzian theory.

Thus to each boundary triangle $f$ in the boundary of the triangulation, we have a Lie group element $U_f$ and, as conjugate variable a Lie algebra element $J_f$. It is convenient to think these variables as associated with the links of the graph formed by the one-skeleton of the cellular complex dual to the boundary triangulation. Notice that these define precisely the same boundary phase space as the one of lattice Yang-Mills theory. As in Yang-Mills theory, the symplectic structure can be taken to be

$$
\begin{aligned}
\{U_f, U_{f'}\} &= 0, \\
\{(J_f)^{IJ}, U_{f'}\} &= \delta_{ff'}\ U_f\ \tau^{IJ}, \\
\{(J_f)^{IJ}, (J'_f)^{KL}\} &= \delta_{ff'}\ \lambda^{IJKL}_{MN} (J_f)^{MN},
\end{aligned}
\tag{2.3.10}
$$

where $\tau^{IJ}$ and $\lambda^{IJKL}_{MN}$ are, respectively, the generators and the structure constants of $G$.

In terms of the momentum variable $J$, the constraints (2.3.2) and (2.3.5) read respectively:

$$\text{simplicity constraints :}\quad C_l^J = n_I\left(({}^*J_l)^{IJ} - \frac{s}{\gamma}J_l^{IJ}\right) = 0, \tag{2.3.11}$$

$$\text{closure constraints :}\quad G^{IJ} = \sum_{a=1}^{4} J_{l_a}^{IJ} = 0. \tag{2.3.12}$$

Here we shift the subscript $f$ into $l$, which denote the link dual to the triangle $f$ in the boundary. These constraints will give the solution $B_f = \int_f {}^*\big(e(t)\wedge e(t)\big)$, where $e(t)$ is a tetrad one-form covering the tetrahedron $t$, and $J_f = B_f + \frac{1}{\gamma}{}^*B_f$, with triangle $f$ dual to the link $l$. The usual

$$\text{quadratic diagonal}\quad C_{ll} := \Big(1-\frac{1}{\gamma^2}\Big){}^*J_l\cdot J_l + \frac{2}{\gamma}J_l\cdot J_l = 0 \tag{2.3.13}$$

$$\text{and off}-\text{diagonal}\quad C_{ll'} := \Big(1-\frac{1}{\gamma^2}\Big){}^*J_l\cdot J_{l'} + \frac{2}{\gamma}J_l\cdot J_{l'} = 0 \tag{2.3.14}$$

simplicity constraints can be easily shown to follow from (2.3.11). This reformulation is central for the new models [14–16, 33]. In particular, if we choose a "time" gauge where $n_I = (0,0,0,1)$, the simplicity constraint (2.3.11) turns out to be

$$C_l^i = J_l^{0i} + \gamma\,{}^*J_l^{0i} = 0, \tag{2.3.15}$$

which leads to the key constraint of the new model

$$C_l^i = K_l^i + s\gamma\ L_l^i = 0, \tag{2.3.16}$$

where $L_l^j := \frac{1}{2}\epsilon^j{}_{kl}J_l^{kl}$ and $K_l^j := J_l^{0j}$ are respectively the generators of the $SU(2)$ subgroup that leaves $n_I$ invariant, and the generators of the corresponding boosts; again $s$ denotes the signature with $s=1$ for Euclidean theory and $s=-1$ for Lorentzian theory. In terms of these generators, the closure constraint (3.2.16) becomes

$$G_L^i = \sum_{a=1}^{4} L_l^i = 0 \tag{2.3.17a}$$

$$\text{and}\quad G_K^i = \sum_{a=1}^{4} K_l^i = 0. \tag{2.3.17b}$$

To quantize the constraints (3.2.15) (2.3.17), one just need to replace the generators with the associated operators.

### 2.3.2 The physical boundary space as a week solution to the kinematic constraints: Euclidean theory

Given a carrier space $\mathcal{H}_{(j^+,j^-)}$, the canonical basis is given by the basis diagonalizing simultaneously the operators $J^+, J^-, L\cdot L$ and $L^3$ , which is noted as $|(j^+,j^-);j,m\rangle$ or simply as $|j,m\rangle$ if no confusion arises. Here $J^\pm$ denotes the self-dual/anti-self-dual decomposition of $J_f^{IJ}$:

$$J_f^{(\pm)i} := \frac{1}{2}(L_f^i \pm K_f^i). \tag{2.3.18}$$

On this canonical basis, the generators act in the following way [3] [10]:

[3]This action can be obtained indirectly from the Lorentzian action (3.2.48). In fact, comparing the Euclidean :

$$\begin{aligned}
&[L_3, L_\pm] = \pm L_\pm \qquad [L_+, L_-] = 2L_3 \\
&[L_+, K_+] = [L_-, K_-] = [L_3, K_3] = 0 \\
&[K_3, L_\pm] = \pm K_\pm \qquad [L_\pm, K_\mp] = \pm 2K_3 \qquad [L_3, K_\pm] = \pm K_\pm \\
&[K_3, K_\pm] = \pm L_\pm \qquad [K_+, K_-] = 2L_3,
\end{aligned} \tag{2.3.19}$$

and the Lorentzain commutators:

$$\begin{aligned}
&[L_3, L_\pm] = \pm L_\pm \qquad [L_+, L_-] = 2L_3 \\
&[L_+, K_+] = [L_-, K_-] = [L_3, K_3] = 0 \\
&[K_3, L_\pm] = \pm K_\pm \qquad [L_\pm, K_\mp] = \pm 2K_3 \qquad [L_3, K_\pm] = \pm K_\pm \\
&[K_3, K_\pm] = \mp L_\pm \qquad [K_+, K_-] = -2L_3,
\end{aligned} \tag{2.3.20}$$

one can find the only difference is the last two equations up to a negative sign. Considering the Casimirs, one can obtain the action (2.3.21) from (3.2.48).

$$
\begin{aligned}
L^3|j,m\rangle =&m|j,m\rangle,\\
L^+|j,m\rangle =&\sqrt{(j+m+1)(j-m)}|j,m+1\rangle,\\
L^-|j,m\rangle =&\sqrt{(j+m)(j-m+1)}|j,m-1\rangle,\\
K^3|j,m\rangle =&\alpha_{(j)}\sqrt{j^2-m^2}|j-1,m\rangle+\gamma_{(j)}m|j,m\rangle-\alpha_{(j+1)}\sqrt{(j+1)^2-m^2}|j+1,m\rangle,\\
K^+|j,m\rangle =&\alpha_{(j)}\sqrt{(j-m)(j-m-1)}|j-1,m+1\rangle\\
&+\gamma_{(j)}\sqrt{(j-m)(j+m+1)}|j,m+1\rangle\\
&+\alpha_{(j+1)}\sqrt{(j+m+1)(j+m+2)}|j+1,m+1\rangle,\\
K^-|j,m\rangle =&-\alpha_{(j)}\sqrt{(j+m)(j+m-1)}|j-1,m-1\rangle\\
&+\gamma_{(j)}\sqrt{(j+m)(j-m+1)}|j,m-1\rangle\\
&-\alpha_{(j+1)}\sqrt{(j-m+1)(j-m+2)}|j+1,m-1\rangle,
\end{aligned}
\tag{2.3.21}
$$

where

$$
L^{\pm}=L^1\pm iL^2, \qquad K^{\pm}=K^1\pm iK^2
$$

$$
\begin{aligned}
\text{and}\quad \alpha_{(j)}&=\frac{1}{j}\sqrt{\frac{(j^2-(j^++j^--1)^2)(j^2-(j^+-j^-)^2)}{4j^2-1}},\\
\gamma_{(j)}&=\frac{j^+(j^++1)-j^-(j^-+1)}{j(j+1)}.
\end{aligned}
\tag{2.3.22}
$$

From the action (2.3.21) of the generators on the canonical basis states, one can find that the action of $L$ is proportional to the second term of the corresponding action of $K$, explicitly,

$$
\begin{aligned}
(K^3-\gamma_{(j)}L^3)|j,m\rangle =&\alpha_{(j)}\sqrt{j^2-m^2}|j-1,m\rangle-\alpha_{(j+1)}\sqrt{(j+1)^2-m^2}|j+1,m\rangle,\\
(K^+-\gamma_{(j)}L^+)|j,m\rangle =&\alpha_{(j)}\sqrt{(j-m)(j-m-1)}|j-1,m+1\rangle|j,m+1\rangle\\
&\alpha_{(j+1)}\sqrt{(j+m+1)(j+m+2)}|j+1,m+1\rangle,\\
(K^--\gamma_{(j)}L^-)|j,m\rangle =&-\alpha_{(j)}\sqrt{(j+m)(j+m-1)}|j-1,m-1\rangle|j,m-1\rangle\\
&-\alpha_{(j+1)}\sqrt{(j-m+1)(j-m+2)}|j+1,m-1\rangle.
\end{aligned}
\tag{2.3.23}
$$

We can see from (2.3.23) that the action of $K-\gamma_{(j)}L$ on the states $|j,m\rangle$ in $\mathcal{H}_j$ results in states orthogonal to $\mathcal{H}_j$. Namely,

$$
\langle j,m'|K^i-\gamma_{(j)}L^i|j,m\rangle=0 \tag{2.3.24}
$$

Now let us come to show the physical Hilbert space $\mathcal{H}_{\rm ph}$ derived in section 2.2.1 solves indeed the constraint operators associated to the simplicity constraints (3.2.15) and the closure constraints (2.3.17). Namely, we will show

(i) the simplicity constraints (3.2.15) are satisfied in the "extreme" $\gamma$-simple representation $\mathcal{H}^{\text{ext}}$,

(ii) the closure constraints (2.3.17) are satisfied in the intertwiner space $\mathcal{K}_{\text{ph}}$.

To show (i), let us consider the states in the "extreme" space $\mathcal{H}^{\text{ext}}$ in equation (2.2.8). Using the relation (2.2.6), $\gamma_{(j)}$ in equation (2.3.22) turns out to be the Barbero-Immirzi parameter $\gamma$. (In fact, this is the reason we use $\gamma_{(j)}$ to denote this parameter in the action [4]. ) Hence the matrix elements of the l.h.s of (3.2.15) hence vanish on the "minimal" $\gamma$-simple space:

$$\langle j, m' | C^i | j, m \rangle = \langle j, m' | \big( K^i - \gamma L^i \big) | j, m \rangle = 0. \tag{2.3.25}$$

Notice that the slight difference of our relation (3.2.15) from the old one plays a key role here. Notice also that what we obtain is that the matrix elements vanish *exactly*, and not just in the large spin limit.

To show (ii), observe that the l.h.s. of (3.2.16a) is the generator of $SU(2)$ transformations at the node and vanishes strongly on (2.2.19) by definition; the l.h.s. of (3.2.16b) is proportional to the one of (3.2.16a) by (2.3.29) and therefore vanishes weakly. Thus $\mathcal{K}_{\text{ph}}$ is the intertwiner space as a solution of *all* the constraints: all the constraints hold weakly.

Notice that the intertwiner space $\mathcal{K}_{\text{ph}}$ is not $Spin(4)$-invariant, but only $SU(2)$-invariant, since we impose the closure constraint weakly, instead of strongly. One can impose the closure constraint strongly to get an $Spin(4)$-invariant Hilbert space, as in [43], but we still prefer this construction, since the resulting space is naturally isometric to the LQG one, while its projection on the $Spin(4)$-invariant states is not [44]. (Canonical quantization in a fixed gauge, as the one used in LQG, is generally reliable for determining the correct Hilbert space.)

Summarizing, we have introduced the kinematic constraints, in the Euclidean theory, and shown that all of them are satisfied on the physical boundary space $\mathcal{H}_{\text{ph}}$ derived in section 2.2.1. Since we have not proven that the physical Hilbert space considered is the *maximal* space where the constraints hold weakly, one might worry that the physically correct quantization of the degrees of freedom of general relativity could need a larger space. Also, it has been pointed out that imposing

[4] From the expression of $\gamma(j)$ in equation (2.3.22), we will see the requirement of $j^+ > j^-$ in $\gamma$-simple relation (2.2.6). In fact, the sign of $\gamma$ represents the sign of $j^+ - j^-$.

From (2.3.22) to obtain $\gamma$-simple relation, one can rewrite $j^+(j^+ + 1) - j^-(j^- + 1)$ into $(j^+ - j^-)(j^+ + j^- + 1)$.

second class constraints weakly might lead to inconsistencies in some cases [39]. And unfortunately, there does exist a larger space of the weaker solution, which we will discuss in section 2.6.

### 2.3.3 The physical boundary space as a week solution to the kinematic constraints: Lorentzian theory

Given a carrier space $\mathcal{H}_{(k,p)}$, the canonical basis is given by the basis diagonalizing simultaneously the Casimir operators $J \cdot J, {}^*J \cdot J, L \cdot L$ and $L^3$ , which is noted as $|(k,p);j,m\rangle$ or simply as $|j,m\rangle$. On this canonical basis, the generators act in the following way [41]:

$$
\begin{aligned}
L^3|j,m\rangle =& m|j,m\rangle,\\
L^+|j,m\rangle =& \sqrt{(j+m+1)(j-m)}|j,m+1\rangle,\\
L^-|j,m\rangle =& \sqrt{(j+m)(j-m+1)}|j,m-1\rangle,\\
K^3|j,m\rangle =& -\alpha_{(j)}\sqrt{j^2-m^2}|j-1,m\rangle - \beta_{(j)}m|j,m\rangle + \alpha_{(j+1)}\sqrt{(j+1)^2-m^2}|j+1,m\rangle,\\
K^+|j,m\rangle =& -\alpha_{(j)}\sqrt{(j-m)(j-m-1)}|j-1,m+1\rangle\\
& -\beta_{(j)}\sqrt{(j-m)(j+m+1)}|j,m+1\rangle\\
& -\alpha_{(j+1)}\sqrt{(j+m+1)(j+m+2)}|j+1,m+1\rangle,\\
K^-|j,m\rangle =& \alpha_{(j)}\sqrt{(j+m)(j+m-1)}|j-1,m-1\rangle\\
& -\beta_{(j)}\sqrt{(j+m)(j-m+1)}|j,m-1\rangle\\
& +\alpha_{(j+1)}\sqrt{(j-m+1)(j-m+2)}|j+1,m-1\rangle,
\end{aligned}
\tag{2.3.26}
$$

where

$$
\begin{aligned}
& L^\pm = L^1 \pm iL^2, \qquad K^\pm = K^1 \pm iK^2\\
\text{and}\quad & \alpha_{(j)} = \frac{i}{j}\sqrt{\frac{(j^2-k^2)(j^2+p^2)}{4j^2-1}}, \qquad \beta_{(j)} = \frac{kp}{j(j+1)}.
\end{aligned}
\tag{2.3.27}
$$

Now let us go to show the physical Hilbert space $\mathcal{H}_{\text{ph}}$ derived last subsection solves indeed the constraint operators associated to the simplicity constraints (3.2.15) and the closure constraints (2.3.17). Namely, we will show

(i) the simplicity constraints (3.2.15) are satisfied in the "minimal" $\gamma$-simple representation $\mathcal{H}^{\text{min}}$,

(ii) the closure constraints (2.3.17) are satisfied in the intertwiner space $\mathcal{K}_{\text{ph}}$.

To show (i), let us consider the states in the "minimal" space $\mathcal{H}^{\min}$ in equation (2.2.18). For these lowest spin states, equation (2.2.17) implies that the states are of the form $|(k,p);k,m\rangle$, or simply as $|k,m\rangle$. The action (3.2.48) of the generators on these states reads:

$$\begin{aligned}
\big(K^3 + \beta_{(k)}L^3\big)|k,m\rangle &= \alpha_{(k+1)}\sqrt{(k+1)^2-m^2}|(k+1,m)\rangle, \\
\big(K^+ + \beta_{(k)}L^+\big)|k,m\rangle &= -\alpha_{(k+1)}\sqrt{(k+m+1)(k+m+2)}|(k+1,m+1)\rangle, \\
\big(K^- + \beta_{(k)}L^-\big)|k,m\rangle &= \alpha_{(k+1)}\sqrt{(k-m+1)(k-m+2)}|(k+1,m-1)\rangle.
\end{aligned}$$

It is straightforward to obtain

$$\langle k,m'|\big(K^i + \beta_{(k)}L^i\big)|k,m\rangle = 0. \tag{2.3.28}$$

Using the relation (2.2.15), $\beta_{(k)}$ turns out to be the Barbero-Immirzi parameter $\gamma$ and the matrix elements of the l.h.s of (3.2.15) hence vanish on the "minimal" $\gamma$-simple space:

$$\langle k,m'|C^i|k,m\rangle = \langle k,m'|\big(K^i + \gamma L^i\big)|k,m\rangle = 0. \tag{2.3.29}$$

Notice that the slight difference of our relation (3.2.15) from the old one plays a key role here. Notice also that what we obtain is that the matrix elements vanish *exactly*, and not just in the large spin limit.

To show (ii), observe that the l.h.s. of (3.2.16a) is the generator of $SU(2)$ transformations at the node and vanishes strongly on (2.2.19) by definition; the l.h.s. of (3.2.16b) is proportional to the one of (3.2.16a) by (2.3.29) and therefore vanishes weakly. Thus $\mathcal{K}_{\rm ph}$ is the intertwiner space as a solution of *all* the constraints: all the constraints hold weakly.

Notice that the intertwiner space $\mathcal{K}_{\rm ph}$ is not $SL(2,\mathbb{C})$-invariant, but only $SU(2)$-invariant, since we impose the closure constraint weakly, instead of strongly. One can impose the closure constraint strongly to get an $SL(2,\mathbb{C})$-invariant Hilbert space, as in [43], but we still prefer this construction, since the resulting space is naturally isometric to the LQG one, while its projection on the $SL(2,\mathbb{C})$-invariant states is not [44]. (Canonical quantization in a fixed gauge, as the one used in LQG, is generally reliable for determining the correct Hilber space.) As we shall see in the next section, Lorentz invariance is fully implemented by the transition amplitudes.

Summarizing, we have introduced the kinematic constraints, in the Lorentzian theory, and shown that all of them are satisfied on the physical boundary space $\mathcal{H}_{\rm ph}$ derived in the last subsection. Again we have not proven that the physical Hilbert space considered is the *maximal* space where the constraints hold weakly, and a enlarged space is considered in section 2.6.

## 2.4 Dynamics

### 2.4.1 Dynamics: Euclidean theory

From the construction of the physical boundary space in section 2.2.1, we have the remarkable result that $\mathcal{K}_{\rm ph}$ is naturally isomorphic to the $SU(2)$ intertwiner space, and therefore the constrained boundary space $\mathcal{H}_{\rm ph}$ can be identified with the $SU(2)$ LQG state space $\mathcal{H}_{SU(2)}$ associated to the graph which is dual to the boundary of the triangulation, namely the space of the $SU(2)$ spin networks on this graph. In this section, we exhibit this isomorphism by the embedding of the Hilbert space of LQG $\mathcal{H}_{SU(2)}$ into the boundary Hilbert space $\mathcal{H}_{\rm ph}$; we also use this embedding and the BF amplitude of a single 4-simplex $v$ to derive the (modified) Euclidean EPRL vertex amplitude (2.1.2).

The way we construct the boundary space gives a projection, which maps simple $Spin(4)$ spin-network states to $SU(2)$ spin-network states. The corresponding embedding $Y$ is defined as the hermitian conjugate of this projection, which is given by

$$\begin{aligned}\text{If } 0<\gamma<1 \quad Y^{<}_{(j)}: \qquad & \mathcal{H}_j \longrightarrow \mathcal{H}_{(\frac{1+\gamma}{2}j,\frac{1-\gamma}{2}j)}, \\ & |j,m\rangle \longmapsto \left|\left(\frac{1+\gamma}{2}j,\frac{1-\gamma}{2}j\right);j,m\right\rangle\end{aligned} \tag{2.4.1}$$

$$\begin{aligned}\text{If } \gamma>1 \quad Y^{>}_{(j)}: \qquad & \mathcal{H}_j \longrightarrow \mathcal{H}_{(\frac{\gamma+1}{2}j+\frac{\gamma-1}{2},\frac{\gamma-1}{2}(j+1))}, \\ & |j,m\rangle \longmapsto \left|\left(\frac{\gamma+1}{2}j+\frac{\gamma-1}{2},\frac{\gamma-1}{2}(j+1)\right);j,m\right\rangle\end{aligned} \tag{2.4.2}$$

for the representations and

- If $0<\gamma<1$

$$\begin{aligned}Y^{<}_{(j_l)}: \mathrm{Inv}_{SU(2)}(\otimes_{a=1}^4\mathcal{H}_{j_a}) &\longrightarrow \mathrm{Inv}_{Spin(4)}(\otimes_{a=1}^4\mathcal{H}_{(\frac{1+\gamma}{2}j,\frac{1-\gamma}{2}j)}), \\ i^{m_1m_2m_3m_4} &\longmapsto \int_{Spin(4)} \mathrm{d}g\left(\prod_{a=1}^4 D^{(\frac{1+\gamma}{2}j,\frac{1-\gamma}{2}j)}(g)^{(j'_a,m'_a)}{}_{(j_a,m_a)}\right)i^{m_1m_2m_3m_4},\end{aligned} \tag{2.4.3}$$

- If $\gamma > 1$

$$Y^{>}_{(j_l)} : \mathrm{Inv}_{SU(2)}(\otimes_{a=1}^{4}\mathcal{H}_{j_a}) \longrightarrow \mathrm{Inv}_{Spin(4)}(\otimes_{a=1}^{4}\mathcal{H}_{(\frac{\gamma+1}{2}j+\frac{\gamma-1}{2},\frac{\gamma-1}{2}(j+1))}),$$
$$i^{m_1m_2m_3m_4} \longmapsto \int\limits_{Spin(4)} \mathrm{d}g \left(\prod_{a=1}^{4} D^{(\frac{\gamma+1}{2}j+\frac{\gamma-1}{2},\frac{\gamma-1}{2}(j+1))}(g)^{(j'_a,m'_a)}{}_{(j_a,m_a)}\right) i^{m_1m_2m_3m_4}, \tag{2.4.4}$$

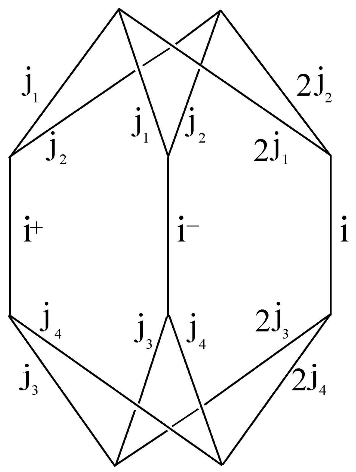
for intertwiners, where $D^{(k,p)}(g)^{(j',m')}{}_{(j,m)}$ denote the matrix elements of the irreducible representation $(j^+, j^-)$, with indices $(j, m)$. The $Spin(4)$ action can be factorized into two $SU(2)$ group elements, one acting one the selfdual, and the other on the antiselfdual representations. One of the two factors can be eliminated by virtue of the $SU(2)$ invariance of the trivalent intertwiners and $i$. What remains is an $SU(2)$ integration over just one of the representations. Using the relation

$$\int\limits_{SU(2)} \mathrm{d}g\, D(g)^{m_1}{}_{m'_1} D(g)^{m_2}{}_{m'_2} D(g)^{m_3}{}_{m'_3} D(g)^{m_4}{}_{m'_4} = \sum_i, i^{m_1m_2m_3m_4} i_{m'_1m'_2m'_3m'_4}, \tag{2.4.5}$$

one can obtain

$$Y_{(j_l)}|i\rangle = \sum_{i^+i^-} Y^i_{i^+i^-}(j_l)|i^+i^-\rangle, \tag{2.4.6}$$

where the coefficients $Y^i_{i^+i^-}(j_l)$ are given by the evaluation of the spin network




If we piece these maps at each node, we obtain the map $Y : \mathcal{H}_{SU(2)} \to \mathcal{H}_{\mathrm{ph}}$ of the entire LQG space into the state space of the new theory. In the spin network basis we obtain

If $0<\gamma<1$

$$Y^{<}_{(j_l)}: \quad |j_l, i_n\rangle \mapsto \sum_{i_n^+, i_n^-} Y^{i_n}_{i_n^+, i_n^-}(j_l) \left| \left(\frac{1+\gamma}{2}j, \frac{1-\gamma}{2}j\right); i_n^+, i_n^- \right\rangle$$

If $\gamma>1$

$$Y^{>}_{(j_l)}: \quad |j_l, i_n\rangle \mapsto \sum_{i_n^+, i_n^-} Y^{i_n}_{i_n^+, i_n^-}(j_l) \left| \left(\frac{\gamma+1}{2}j + \frac{\gamma-1}{2}, \frac{\gamma-1}{2}(j+1)\right); i_n^+, i_n^- \right\rangle \tag{2.4.7}$$

Now let us use this embedding and the BF amplitude to give the new amplitude (2.1.2). The BF amplitude of a single 4-simplex $v$ for a given boundary state $|\Psi\rangle$ reads

$$A(\Psi) = \int\limits_{Spin(4)^{10}} \prod_f \mathrm{d}g_f \ \Psi(g_f) \int\limits_{Spin(4)^5} \prod_e \mathrm{d}V_e \prod_f \delta(V_{e_f}\, g_f\, V_{e'_f}^{-1}), \tag{2.4.8}$$

where $V_{e_f}, V_{e'_f}$ are the two group elements around the perimeter of $f$, which is in the 4-simplex $v$ and not in the boundary. The integral over $g_f$ gives

$$A(\Psi) = \int\limits_{Spin(4)^5} \prod_e \mathrm{d}V_e \ \Psi(V_{e_f}^{-1} V_{e'_f}). \tag{2.4.9}$$

In the new theory, for any boundary state $\Psi \in \mathcal{H}_{\mathrm{ph}}$, according to the embedding (2.4.7), there exist a LQG state $\psi \in \mathcal{H}_{\mathrm{LQG}}$, such that $\Psi = Y(\psi)$. Let us consider the specific case when $\psi$ is a spin-network state $|\psi\rangle = |j_f, i_e\rangle$ on the boundary. The amplitude is then given explicitly by

If $0<\gamma<1$

$$A^{<}(j_f, i_e) = \sum_{i_a^+ i_a^-} \big(\prod_e Y^{i_e}_{i_e^+ i_e^-}(j_f)\big) 15j_{Spin(4)} \left( \left(\frac{1+\gamma}{2}j, \frac{1-\gamma}{2}j\right); (i_n^+, i_n^-) \right)$$

If $\gamma>1$

$$A^{>}(j_f, i_e) = \sum_{i_a^+ i_a^-} \big(\prod_e Y^{i_e}_{i_e^+ i_e^-}(j_f)\big) 15j_{Spin(4)} \left( \left(\frac{\gamma+1}{2}j + \frac{\gamma-1}{2}, \frac{\gamma-1}{2}(j+1)\right); (i_n^+, i_n^-) \right) \tag{2.4.10}$$

### 2.4.2 Dynamics: Lorentzian theory

We have the remarkable result that $\mathcal{K}_{\mathrm{ph}}$ is naturally isomorphic to the $SU(2)$ intertwiner space, and therefore the constrained boundary space $\mathcal{H}_{\mathrm{ph}}$ can be identified with the $SU(2)$ LQG state space $\mathcal{H}_{SU(2)}$ associated to the graph which is dual to the boundary of the triangulation, namely the space of the $SU(2)$ spin networks on this graph. For completeness, let us repeat some materials in [16, 33] to exhibit this isomorphism by the embedding of the Hilbert space of LQG $\mathcal{H}_{SU(2)}$ into the boundary Hilbert space $\mathcal{H}_{\mathrm{ph}}$ of the new model; we also use this embedding and the BF amplitude of a single 4-simplex $v$ to derive the EPRL vertex amplitude.

The way we construct the boundary space gives a projection, which maps simple $SL(2,\mathbb{C})$ spin-network states to $SU(2)$ spin-network states. The corresponding embedding $f$ is defined as the hermitian conjugate of this projection, which is given by

$$\begin{aligned} f_{(j)}: \qquad \mathcal{H}_j &\longrightarrow \mathcal{H}_{(j,\gamma(j+1))}, \\ |j,m\rangle &\longmapsto |(j,\gamma(j+1));j,m\rangle \end{aligned} \tag{2.4.11}$$

for the representations and

$$\begin{aligned} f_{(j_l)}: \mathrm{Inv}_{SU(2)}(\otimes_{a=1}^4 \mathcal{H}_{j_a}) &\longrightarrow \mathrm{Inv}_{SL(2,\mathbb{C})}(\otimes_{a=1}^4 \mathcal{H}_{(j_a,\gamma(j_a+1))}), \\ i^{m_1m_2m_3m_4} &\longmapsto \int\limits_{SL(2,\mathbb{C})} \mathrm{d}g \left(\prod_{a=1}^4 D^{(j_a,\gamma(j_a+1))}(g)^{(j'_a,m'_a)}{}_{(j_a,m_a)}\right) i^{m_1m_2m_3m_4}, \end{aligned} \tag{2.4.12}$$

for intertwiners [16, 33], where $D^{(k,p)}(g)^{(j',m')}{}_{(j,m)}$ denote the matrix elements of the irreducible representation $(k,p)$, with indices $(j,m)$. Let indices $(j,m)\equiv\alpha$, $\chi^{\alpha_1\alpha_2\alpha_3\alpha_4}$ denote the $SL(2,\mathbb{C})$ intertwiner defined by a virtual link carring the representation $\chi=(k,p)$, and $\mathrm{d}\chi$ the Plancherel measure on the spectrum. Then using the relation

$$\int\limits_{SL(2,\mathbb{C})} \mathrm{d}g\, D^{(\chi_1)}(g)^{\alpha_1}{}_{\alpha'_1} D^{(\chi_2)}(g)^{\alpha_2}{}_{\alpha'_2} \overline{D^{(\chi_3)}}(g)^{\alpha_3}{}_{\alpha'_3} \overline{D^{(\chi_4)}}(g)^{\alpha_4}{}_{\alpha'_4} = \int \mathrm{d}\chi\, \chi^{\alpha_1\alpha_2\alpha_3\alpha_4} \overline{\chi_{\alpha'_1\alpha'_2\alpha'_3\alpha'_4}}, \tag{2.4.13}$$

one can obtain

$$f_{(j_l)}|i\rangle = \int \mathrm{d}\chi\, f^i_\chi(j_l)|\chi\rangle, \tag{2.4.14}$$

where the coefficients $f^i_\chi(j_l)$ are given by

$$f^i_\chi(j_l) = i^{m_1 m_2 m_3 m_4} \overline{\chi}_{(j_1,m_1)(j_2,m_2)(j_3,m_3)(j_4,m_4)}. \tag{2.4.15}$$

If we piece these maps at each node, we obtain the map $f : \mathcal{H}_{SU(2)} \to \mathcal{H}_{\mathrm{ph}}$ of the entire LQG space into the state space of the new theory. In the spin network basis we obtain

$$f_{(j_l)} : |j_l, i_n\rangle \mapsto \int \mathrm{d}\chi_n \, f^{i_n}_{\chi_n}(j_l) |(j_l, \gamma(j_l+1)), \chi_n\rangle. \tag{2.4.16}$$

Now let us use this embedding and the BF amplitude to give the new amplitude (2.1.3). The BF amplitude of a single 4-simplex $v$ for a given boundary state $|\Psi\rangle$ reads

$$A(\Psi) = \int\limits_{SL(2,\mathbb{C})^{10}} \prod_f \mathrm{d}g_f \; \Psi(g_f) \int\limits_{SL(2,\mathbb{C})^5} \prod_e \mathrm{d}V_e \prod_f \delta(V_{e_f}\, g_f\, V^{-1}_{e'_f}), \tag{2.4.17}$$

where $V_{e_f}, V_{e'_f}$ are the two group elements around the perimeter of $f$, which is in the 4-simplex $v$ and not in the boundary. The integral over $g_f$ gives

$$A(\Psi) = \int\limits_{SL(2,\mathbb{C})^5} \prod_e \mathrm{d}V_e \; \Psi(V^{-1}_{e_f} V_{e'_f}). \tag{2.4.18}$$

In the new theory, for any boundary state $\Psi \in \mathcal{H}_{\mathrm{ph}}$, according to the embedding (2.4.16), there exist a LQG state $\psi \in \mathcal{H}_{\mathrm{LQG}}$, such that $\Psi = f(\psi)$. Let us consider the specific case when $\psi$ is a spin-network state $|\psi\rangle = |j_f, i_e\rangle$ on the boundary. The amplitude is then given explicitly by

$$A(j_f, i_e) = \int \mathrm{d}\chi_e \big( \prod_e f^{i_e}_{\chi_e}(j_f) \big) 15j((j_f, \gamma(j_f+1)), \chi_e). \tag{2.4.19}$$

In terms of $(k,p)$, the Plancherel measure $\mathrm{d}\chi$ can be expressed as $(k^2+p^2)\mathrm{d}p$, which gives the expression (2.1.3).

## 2.5 Geometrical observables

The *kinematics* of canonical loop quantum gravity is rather well understood; in particular, the properties of the geometrical operators, including the area and the volume operators [45–47] are well established. In this section, we study the geometrical operators in the new spinfoam model and their relation with the $SU(2)$ ones in LQG.

### 2.5.1 The area operator

The area operator of the new spinfoam model has been derived in [16, 37] and shown to match the LQG one. Classically, the area $A(f)$ of a triangle $f$ is given by $A(f)^2 = \frac{1}{2}({}^*B_f)^{IJ} \cdot ({}^*B_f)_{IJ}$. If we fix the time gauge, we have $A_3(f)^2 = \frac{1}{2}({}^*B_f)^{ij} \cdot ({}^*B_f)_{ij}$. These two quantities are equal up to a constrained term. As shown in [16, 37], using the constraints, the operator related to $A_3(f)^2$ can be obtained as $A_3(f)^2 = \kappa^2\gamma^2 L_f^2$, which matches three-dimensional area as determined by LQG, including for the correct Barbero-Immirzi parameter proportionality factor.

### 2.5.2 The volume operator in Euclidean theory

The volume of the tetrahedron $t$ is given by

$$V(t) = \frac{1}{6}\det(e(v)). \tag{2.5.1}$$

In terms of the variables ${}^*B$ defined in (2.3.1), the volume of a boundary tetrahedron $t$ reads $V$ related to the tetrahedra $t$ as

$$V(t) = \sqrt{\frac{1}{27}\epsilon^{abc}\mathrm{Tr}[{}^*B_a{}^*B_b{}^*B_c]} \tag{2.5.2}$$

To see this, let the gauge-fixed simplicity constraint (2.3.6) hold, then the ${}^*B_f^{0i}(t)$ vanish and the above quantity is equal to

$$V_3(t) = \sqrt{\frac{1}{27}\epsilon^{abc*}B_a^{ij*}B_b^{jk*}B_c^{ki}} = \frac{1}{6}\det(e), \tag{2.5.3}$$

which is exactly the expression (2.5.1) of the discrete volume. Note that the $SO(4)$ volume $V_{SO(4)}(t)$ is gauge invariant, hence we can obtain eq (2.5.2) by the gauge-fixed version (2.5.3) without loss of generality. Going to the variables $J$, and using (2.3.9), the volume reads

$$V(t) = \sqrt{\frac{1}{27}\Big(\frac{\gamma^2}{1-\gamma^2}\Big)^3\epsilon^{abc}\mathrm{Tr}\Big[\Big(\frac{1}{\gamma}J_a -^* J_a\Big)\Big(\frac{1}{\gamma}J_b -^* J_b\Big)\Big(\frac{1}{\gamma}J_c -^* J_c\Big)\Big]} \tag{2.5.4}$$

The volume operator $\hat{V}(t)$ of the tetrahedron $t$ is then formally given by (2.5.12) with $J^{IJ}$ replaced by the corresponding operators:

$$\hat{V}(t) = \sqrt{\frac{1}{27}\Big(\frac{\gamma^2}{1-\gamma^2}\Big)^3\epsilon^{abc}\mathrm{Tr}\Big[\Big(\frac{1}{\gamma}\hat{J}_a -^* \hat{J}_a\Big)\Big(\frac{1}{\gamma}\hat{J}_b -^* \hat{J}_b\Big)\Big(\frac{1}{\gamma}\hat{J}_c -^* \hat{J}_c\Big)\Big]}. \tag{2.5.5}$$

However, the physical volume should be defined on the physical boundary space $\mathcal{H}_{\rm ph}$, satisfying the constraints. Since the volume operator does not change the graph of the spin network sates, nor the coloring of the links, its action can be studied on the Hilbert space associated to a single node. Consider the matrix element of the square of the volume operator between two states in the physical Hilbert space (we drop the hats):

$$\langle i|V(t)^2|j\rangle = \frac{1}{27}\big(\frac{\gamma^2}{1-\gamma^2}\big)^3\epsilon^{abc}\langle i|\big(\frac{1}{\gamma}J^{ij}_a - {}^*J^{ij}_a\big)\big(\frac{1}{\gamma}J^{jk}_b - {}^*J^{jk}_b\big)\big(\frac{1}{\gamma}J^{ki}_c - {}^*J^{ki}_c\big)|j\rangle. \tag{2.5.6}$$

Writing this in terms of $L$ and $K$ components gives

$$\langle i|V(t)^2|j\rangle = \frac{1}{27\cdot 8}\big(\frac{\gamma^2}{1-\gamma^2}\big)^3\epsilon^{abc}\epsilon^{ij}{}_m\epsilon^{jk}{}_n\epsilon^{ki}{}_p\langle i|\big(\frac{1}{\gamma}L^m_a - K^m_a\big)\big(\frac{1}{\gamma}L^n_b - K^n_b\big)\big(\frac{1}{\gamma}L^p_c - K^p_c\big)|j\rangle. \tag{2.5.7}$$

Notice that the intertwiner space is the subspace of the product of the space $\mathcal{H}_a$ associated to the link $a$, and the action of $(K_a,\, L_a)$ is in fact on $\mathcal{H}_a$. Hence we can use the form (2.3.29) of the simplicity constraint to simplify Eq. (2.5.7), although the r.h.s seems a polynomial. Using the form (2.3.29) of the constraint, we can rewrite it as

$$\langle i|V(t)^2|j\rangle = \frac{1}{27\cdot 8}\big(\frac{\gamma^2}{1-\gamma^2}\big)^3\big(\frac{1}{\gamma}-\gamma\big)^3\epsilon^{abc}\epsilon_{ijk}\langle i|L^i_aL^j_bL^k_c|j\rangle \tag{2.5.8}$$

and a little algebra gives

$$\langle i|V(t)^2|j\rangle = \gamma^3\ \langle i|\epsilon^{abc}\epsilon_{ijk}L^i_aL^j_bL^k_c|j\rangle. \tag{2.5.9}$$

That is

$$V(t) = \gamma^{\frac{3}{2}}\sqrt{\epsilon^{abc}\epsilon_{ijk}L^i_aL^j_bL^k_c} \tag{2.5.10}$$

Now, the operator on the r.h.s. is precisely the LQG volume operator $V_{\rm LQC}$, as it acts on $\mathcal{K}_{\rm ph}$ including the correct dependence on the Barbero-Immirzi parameter $\gamma$.

### 2.5.3 The volume operator in Lorentzian theory

Let us now turn to study the volume operator in Lorentzian theory. Following last subsection, the volume of a boundary tetrahedron $t$ is $V(t) = \sqrt{|V^2(t)|}$ where

$$V^2(t) = \frac{1}{27}\epsilon^{abc}\mathrm{Tr}[{}^*B_a{}^*B_b{}^*B_c], \tag{2.5.11}$$

which in terms of $J$ turns out to be

$$V^2(t) = \frac{1}{27}\Big(\frac{\gamma^2}{1+\gamma^2}\Big)^3 \epsilon^{abc}\mathrm{Tr}\Big[\Big(\frac{1}{\gamma}J_a + {}^*J_a\Big)\Big(\frac{1}{\gamma}J_b + {}^*J_b\Big)\Big(\frac{1}{\gamma}J_c + {}^*J_c\Big)\Big] \tag{2.5.12}$$

The volume operator $\hat{V}(t)$ of the tetrahedron $t$ is then formally given by (2.5.12) with $J^{IJ}$ replaced by the corresponding operators:

$$\widehat{V^2}(t) = \frac{1}{27}\Big(\frac{\gamma^2}{1+\gamma^2}\Big)^3 \epsilon^{abc}\mathrm{Tr}\Big[\Big(\frac{1}{\gamma}\hat{J}_a + {}^*\hat{J}_a\Big)\Big(\frac{1}{\gamma}\hat{J}_b + {}^*\hat{J}_b\Big)\Big(\frac{1}{\gamma}\hat{J}_c + {}^*\hat{J}_c\Big)\Big]. \tag{2.5.13}$$

Since the volume operator does not change the graph of the spin network sates, nor the coloring of the links, its action can be studied on the Hilbert space associated to a single node. Consider the matrix element of the square of the volume operator between two states in the physical intertwiner space (we drop the hats):

$$\langle i|\widehat{V^2}|j\rangle = \frac{1}{27}\big(\frac{\gamma^2}{1+\gamma^2}\big)^3 \epsilon^{abc}\langle i|\big(\frac{1}{\gamma}J^{ij}_a + {}^*J^{ij}_a\big)\big(\frac{1}{\gamma}J^{jk}_b + {}^*J^{jk}_b\big)\big(\frac{1}{\gamma}J^{ki}_c + {}^*J^{ki}_c\big)|j\rangle. \tag{2.5.14}$$

Writing this in terms of $L$ and $K$ components gives

$$\langle i|\widehat{V^2}|j\rangle = \frac{1}{27}\big(\frac{\gamma^2}{1+\gamma^2}\big)^3 \epsilon^{abc}\epsilon^{ij}{}_m\epsilon^{jk}{}_n\epsilon^{ki}{}_p\langle i|\big(\frac{1}{\gamma}L^m_a - K^m_a\big)\big(\frac{1}{\gamma}L^n_b - K^n_b\big)\big(\frac{1}{\gamma}L^p_c - K^p_c\big)|j\rangle. \tag{2.5.15}$$

Notice that the intertwiner space is the subspace of the product of the space $\mathcal{H}_a$ associated to the link $a$, and the action of $(K_a, L_a)$ is in fact on $\mathcal{H}_a$. Hence we can use the form (3.2.15) of the simplicity constraint to simplify Eq. (2.5.7), although the r.h.s seems a polynomial. Using the form (3.2.15) of the constraint, we can rewrite it as

$$\langle i|\widehat{V^2}|j\rangle = \frac{1}{27}\big(\frac{\gamma^2}{1+\gamma^2}\big)^3\big(\frac{1}{\gamma} + \gamma\big)^3 \epsilon^{abc}\epsilon_{ijk}\langle i|L^i_aL^j_bL^k_c|j\rangle \tag{2.5.16}$$

and a little algebra gives

$$\langle i|\widehat{V^2}|j\rangle = \Big(\frac{\gamma}{3}\Big)^3 \epsilon^{abc}\epsilon_{ijk}\langle i|L^i_aL^j_bL^k_c|j\rangle. \tag{2.5.17}$$

That is

$$\hat{V}(t) = \Big(\frac{\gamma}{3}\Big)^{\frac{3}{2}}\sqrt{\left|\epsilon^{abc}\epsilon_{ijk}L^i_aL^j_bL^k_c\right|}. \tag{2.5.18}$$

Now, the operator on the r.h.s. is precisely the LQG volume operator $V(t)_{\mathrm{LQC}}$ of the tetrahedron, as it acts on $\mathcal{K}_{\mathrm{ph}}$ including the correct dependence on the Barbero-Immirzi parameter $\gamma$.

## 2.6 New degree of freedom

In this section, let us come to revisit the way we solve the simplicity constraints. Since we have not proven that the physical Hilbert space considered is the *maximal* space where the constraints hold weakly, one might worry that the physically correct quantization of the degrees of freedom of general relativity could need a larger space. And a little bit unexpectedly, there does exist a larger space of the weaker solution, which we will discuss in this section.

### 2.6.1 The weaker $\gamma$-simple relation

Let us come to the Euclidean theory first. To solve the simplicity constraints weakly, a sufficient condition is $\gamma(j)$ in equation (2.3.22) equals the Barbero-Immirzi parameter $\gamma$:

$$\gamma j(j+1) = j^+(j^+ + 1) - j^-(j^- + 1) \tag{2.6.1}$$

To satisfy this condition, it is not necessary that one only select the "extreme" section of the the Clebsch-Gordan decomposition for the single component of $\mathcal{H}$ associated with a single boundary face $f$

$$\mathcal{H}_{j^+ \otimes j^-} = \mathcal{H}_{j^+} \bigotimes \mathcal{H}_{j^-} = \bigoplus_{p=|j^+-j^-|}^{j^++j^-} \mathcal{H}_p. \tag{2.6.2}$$

In other words, $j$ can runs from $j^+ - j^-$ to $j^+ + j^-$, but not only $j = j^+ + j^-$ for $\gamma < 1$ and $j = j^+ - j^-$ for $\gamma > 1$. Let us introduce a new quantum number $r$ where

$$\begin{aligned} j =& j^+ + j^- - r \quad \text{when } 0 < \gamma < 1 \\ j =& j^+ - j^- + r \quad \text{when } \gamma > 1. \end{aligned} \tag{2.6.3}$$

Here one find $r$ is restricted as

$$0 \leq r \leq 2j^-. \tag{2.6.4}$$

There is a weaker $\gamma$-simple relation than (2.2.6):

- $j^+ > j^-$
- For $0 < \gamma < 1$,

$$\begin{aligned} 2j^+ &= j + r + \frac{\gamma j(j+1)}{j+r+1} \\ 2j^- &= j + r - \frac{\gamma j(j+1)}{j+r+1} \end{aligned} \tag{2.6.5}$$

- For $\gamma > 1$,

$$2j^+ = \frac{\gamma j(j+1)}{j-r} + (j-r)$$
$$2j^- = \frac{\gamma j(j+1)}{j-r} - (j-r) \tag{2.6.6}$$

One can check the weaker $\gamma$-simple relation (2.6.5) (2.6.6) satisfy equation (2.6.1), hence for each $r_a$, there exist an invariant subspace where the simplicity constraint and the closure constraint hold weakly:

$$\mathfrak{I}^{<}_{\{j_a,r_a\}} = \mathrm{Inv}_{SU(2)}\left[\bigotimes_{a=1}^{4} \mathcal{H}_{j_a}^{\frac{1}{2}(j+r+\frac{\gamma j(j+1)}{j+r+1}, j+r-\frac{\gamma j(j+1)}{j+r+1})}\right] \quad \text{for } 0 < \gamma < 1$$
$$\mathfrak{I}^{>}_{\{j_a,r_a\}} = \mathrm{Inv}_{SU(2)}\left[\bigotimes_{a=1}^{4} \mathcal{H}_{j_a}^{\frac{1}{2}(\frac{\gamma j(j+1)}{j-r}+(j-r), \frac{\gamma j(j+1)}{j-r}-(j-r))}\right] \quad \text{for } \gamma > 1 \tag{2.6.7}$$

If $r$ is restricted to vanish, this subspace goes back to the physical boundary space $\mathcal{K}_{\mathrm{Ph}}$ in equation (2.2.9). Here, however, one obtain an enlarged boundary space:

$$\mathcal{K}^{\mathrm{Eu}}_{\mathrm{Ph}} = \bigoplus_{\{j_a,r_a\}} \mathfrak{I}_{\{j_a,r_a\}} \tag{2.6.8}$$

This is the enlarged boundary space in the Euclidean thoery where the simplicity constraint and the closure constraint hold weakly. Now let us come to the Lorentzian theory.

For the Lorentzian case, the weaker $\gamma$-simple relation is

$$k = j - r$$
$$p = \frac{\gamma j(j+1)}{j-r}, \tag{2.6.9}$$

where the new quantum number $r \geq 0$. Again when $r = 0$, this weaker $\gamma$-simple relation goes back to (2.2.15) before. For each $r_a$, there exist an invariant subspace where the simplicity constraint and the closure constraint hold weakly:

$$\mathfrak{I}_{\{j_a,r_a\}} = \mathrm{Inv}_{SU(2)}\left[\bigotimes_{a=1}^{4} \mathcal{H}_{j_a}^{(\frac{\gamma j(j+1)}{j-r}, j-r)}\right] \tag{2.6.10}$$

If $r$ is restricted to vanish, this subspace goes back to the physical boundary space $\mathcal{K}_{\mathrm{Ph}}$ in equation (2.2.19). Here, however, one obtain an enlarged boundary space:

$$\mathcal{K}^{\mathrm{Lo}}_{\mathrm{Ph}} = \bigoplus_{\{j_a,r_a\}} \mathfrak{I}_{\{j_a,r_a\}} \tag{2.6.11}$$

### 2.6.2 Dynamics from the enlarged boundary space

The weaker $\gamma$-simple relation gives different embedding maps of the LQG space into the enlarged state space and then the vertex amplitude as well. What's more, it also gives different face amplitudes.

The modified embedding maps to equation (2.4.7) (2.4.16) are given as

- In the Euclidean theory, if $0 < \gamma < 1$

$$Y^{<}_{(j_l)} : |j_l, i_n\rangle \mapsto \sum_{i_n^+, i_n^-} Y^{i_n}_{i_n^+, i_n^-}(j_l) \left| \left( \frac{j+r}{2} + \frac{\gamma j(j+1)}{2(j+r+1)}, \frac{j+r}{2} - \frac{\gamma j(j+1)}{2(j+r+1)} \right); i_n^+, i_n^- \right\rangle \tag{2.6.12}$$

- In the Euclidean theory, if $\gamma > 1$

$$Y^{>}_{(j_l)} : |j_l, i_n\rangle \mapsto \sum_{i_n^+, i_n^-} Y^{i_n}_{i_n^+, i_n^-}(j_l) \left| \left( \frac{\gamma j(j+1)}{2(j-r)} + \frac{j-r}{2}, \frac{\gamma j(j+1)}{2(j-r)} - \frac{j-r}{2} \right); i_n^+, i_n^- \right\rangle \tag{2.6.13}$$

- In the Lorentzian theory,

$$f_{(j_l)} : |j_l, i_n\rangle \mapsto \int \mathrm{d}\chi_n \, f^{i_n}_{\chi_n}(j_l) |(j_l - r_l, \frac{\gamma j_l(j_l+1)}{j_l - r_l}), \chi_n\rangle. \tag{2.6.14}$$

Using these modified embeddings and BF amplitude, one can obtain the new vertex amplitude:

- In the Euclidean theory, if $0 < \gamma < 1$

$$A^{<}(j_f, i_e) = \sum_{i_a^+ i_a^-} \big( \prod_e Y^{i_e}_{i_e^+ i_e^-}(j_f) \big)$$
$$15j_{Spin(4)} \left( \left( \frac{j+r}{2} + \frac{\gamma j(j+1)}{2(j+r+1)}, \frac{j+r}{2} - \frac{\gamma j(j+1)}{2(j+r+1)} \right); (i_n^+, i_n^-) \right) \tag{2.6.15}$$

- In the Euclidean theory, if $\gamma > 1$

$$A^{>}(j_f, i_e) = \sum_{i_a^+ i_a^-} \big( \prod_e Y^{i_e}_{i_e^+ i_e^-}(j_f) \big)$$
$$15j_{Spin(4)} \left( \left( \frac{\gamma j(j+1)}{2(j-r)} + \frac{j-r}{2}, \frac{\gamma j(j+1)}{2(j-r)} - \frac{j-r}{2} \right); (i_n^+, i_n^-) \right) \tag{2.6.16}$$

- In the Lorentzian theory,

$$A(j_f, i_e) = \int \mathrm{d}\chi_e \big( \prod_e f^{i_e}_{\chi_e}(j_f) \big) 15j((j_l - r_l, \frac{\gamma j_l(j_l+1)}{j_l - r_l}), \chi_e). \tag{2.6.17}$$

Now let us come to see the face amplitudes and partition functions. It is argued in [19] that the face amplitude of a spinfoam model is determined by three inputs: (a) the choice of the boundary Hilbert space, (b) the requirement that the composition law holds when gluing two complexes $\mathcal{K}$ and $\mathcal{K}'$, (c) a particular locality requirement (see [19] for the details of the three assumptions). These requirements are implemented if the partition function has the form:

By inserting the vertex amplitudes that we have defined into this expression, we complete the definition of an Euclidean and a Lorentzian model.

Expanding the delta function in representation, we obtain

$$Z_{E,L}(\mathcal{K}) \;\; = \sum_{j_f, r_f, i_e} \prod_f d^{E,L}(j_f, r_f) \; \prod_v A^{E,L}_v(j_f, r_f, i_e)$$

where the face amplitude is

- In the Euclidean theory, if $0 < \gamma < 1$

$$d^{E<}(j_f, r_f) = \Big( j + r + \frac{\gamma j(j+1)}{(j+r+1)} + 1 \Big) \Big( j + r - \frac{\gamma j(j+1)}{(j+r+1)} + 1 \Big) \tag{2.6.18}$$

- In the Euclidean theory, if $\gamma > 1$

$$d^{E>}(j_f, r_f) = \Big( \frac{\gamma j(j+1)}{(j-r)} + j - r + 1 \Big) \Big( \gamma j(j+1)(j-r) - j - r + 1 \Big) \tag{2.6.19}$$

- In the Lorentzian theory,

$$d^L(j_f, r_f) = \frac{\gamma^2 j_f^2 (j_f+1)^2}{(j_f - r_f)^2} + (j_f - r_f)^2. \tag{2.6.20}$$

where the dimension factors $A^E_f := \Big[ 2j^+ + 1 \Big] \Big[ 2j^- + 1 \Big]$ and $A^L_f := \Big[ k_f^2 + \gamma^2 j_f^2 (j_f+1)^2 / k_f^2 \Big]$ are the face amplitudes for the Euclidean and Lorentzian theories. In the Euclidean case, the face amplitudes is different from the one obtained in [19] and coincide with the ones deduced from the BF partition function. In [19] the face amplitude obtained is the dimension of $SU(2)$ unitary irrep i.e. $2j_f + 1$. The origin of the difference is the difference in the boundary Hilbert space. The one here, $\mathcal{H}^E_{\gamma_v}$ or $\mathcal{H}^L_{\gamma_v}$, has additional degree of freedom with respect to the space $L^2(SU(2)^L)$ of [19].

We have studied the quantum theory following from imposing the constraints (3.2.15) and (3.2.16a) weakly, and we have shown that this leads to a new degree of freedom, represented by the quantum number $r_f$. Does this degree of freedom have a physical interpretation relevant for quantum gravity? There are some reasons to suspect a negative answer. Let us consider the Euclidean theory for simplicity.

First, we have seen that $r_f$ does not affect the boundary geometry. We expect all gravitational degrees of freedom to be captured by the geometry. Therefore the theory we have obtained has extra degrees of freedom with respect to general relativity. This can also be seen as following. In the classical theory we have the well known ("left area=right area") relation

$$|\Sigma^+|^2 = |\Sigma^-|^2, \tag{2.6.21}$$

which implies

$$|1-\gamma|j^+ = |1+\gamma|j^- \tag{2.6.22}$$

which in turns implies $r_j = 0$. This might indicate that the quantum theory of gravity that has GR as a classical limit is the one with $r_j = 0$. Alternatively, however, we might require something weaker; for instance, we can still obtain states compatible with GR in the classical limit by demanding that

$$\begin{aligned} &\lim_{j^\pm\to\infty} \frac{r}{j^-} = 0 \quad \text{for} \quad 0<\gamma<1 \\ &\lim_{j^\pm\to\infty} \frac{r}{j^-} = 2 \quad \text{for} \quad \gamma>1 \end{aligned} \tag{2.6.23}$$

in the large-$j$ asymptotic regime. This would make the quantum number relevant for the microphysics and not affecting the classical limit. On the other hand, this choice is a bit artificial.

Furthermore, in the classical theory the area of a face can be equally computed in the time gauge as $A_4 = \sqrt{(\Sigma_f)^{IJ}(\Sigma_f)_{IJ}}$ or as $A_3 = \gamma\sqrt{\Sigma^i_f\Sigma^i_f}$. Classically the two areas $A_4$ and $A_3$ are equal after the simplicity constraint is imposed, and they indeed equal in the large-$j$ limit after quantization [16]. Let us denote the condition $A_4 = A_3$ the *consistency constraint*. If we ask $A_4$ and $A_3$ to be equal as operators in the quantum level on the boundary Hilbert space (as in the case of [16]), then again this fixes $r_f$.

Notice that with an appropriate factor ordering $r_f$ could be fixed, but to to a value different from zero. Doing so will reduce the role of $r_f$ from that of a quantum number (different in each state) to that of a single fixed parameter in the definition of the theory. The actual value of $r_f$ fixed would depend on how the operators corresponding to $A_4$ and $A_3$ are ordered. In this sense $r_f$ is related to the operator-ordering ambiguities of the consistency constraint. Once an order is chosen, there is no more independent quantum number $r_f$ in the theory. With a suitable ordering, we can fix $r_f = 0$

For these consideration, it may be reasonable to suspect that the weak imposition of the constraints (3.2.15) and (3.2.16a) alone may in fact be too weak to properly define quantum general relativity, in the same sense in which the strong imposition of these constraints in the old Barrett-Crane model was too strong. There is a simple way out, which is to impose the (non-commuting) simplicity constraints weakly, and the diagonal simplicity constraint (for instance in the form (2.6.21)) strongly. With this choice of constraints, properly ordered, we obtain $r_f = 0$, the boundary space isometric to the LQG state space in the boundary [5], and precisely the new models amplitudes. Finally, the gluing conditions gives the SU(2) face amplitude. Thus, we recover precisely the quantum gravity theory described for instance in [11].

Note that one could also take the point of view that the quantum numbers $r_f$ label different possible definitions of the spin-foam models. In each of these spin-foam models, the boundary Hilbert space solves the simplicity constraint weakly. And for different choices of $r_f$ the boundary Hilbert spaces are isometric to each other.

[5]Note that the boundary space with $r_f = 0$ of Euclidean theory is only isomorphic to a subspace of the kinematical Hilbert space $\mathcal{H}_{Kin}$ of canonical LQG and cannot completely describe all the quantum states for the fields on the boundary $\mathcal{S}$, since the spins in the summation cannot cover all the $SU(2)$ spins, for some values of Barbero-Immirzi parameter $\gamma$. However, this situation only appears in the Euclidean theory but disappears in the Lorentzian theory.

# Chapter 3

# The generalized spinfoam models

The spinfoam theory introduced in [14–18, 33] can be derived starting from the Plebanski formulation of GR [48] (including the Barbero-Immirzi parameter $\gamma$), and defined as a $BF$ theory discretized on a simplicial cellular complex and constrained by the so called simplicity constraint. The constraint can be imposed using the master-constraint technique [15, 16, 49–54], or, more simply, using the Gupta–Bleuler procedure, namely asking the matrix elements of the constraint to vanish on physical states [31, 32], which we introduce in chapter 2. The resulting model has remarkable properties: (i) the boundary states have a geometrical interpretation in terms of quantum tetrahedral geometry [29, 34, 35, 55]; (ii) there are strong indications that the semiclassical behavior of the theory matches classical general relativity [28, 56–58], thus correcting difficulties of earlier models [59–62]; and (iii) the boundary kinematics is strictly related to that of LQG, as we have shown in chapter 2.

The relation with LQG, however, is limited by the fact that the simplicial-spinfoam boundary states include only four-valent spin networks. This is a drastic reduction of the LQG state space. In [63], Kamiński, Kisielowski, and Lewandowski (KKL) have considered a generalization of the spinfoam formalism to spin networks of arbitrary valence, and have constructed a corresponding vertex amplitude. This generalization provides truncated transition amplitudes between *any* two LQG states [11, 13], thus correcting the limitation of the relation between the model and LQG. This generalization, on the other hand, gives rise to several questions. The KKL vertex is obtained via a "natural" mathematical generalization of the simplicial Euclidean vertex amplitude. Is the resulting vertex amplitude still related to constrained BF theory (and therefore to GR)? In particular, do KKL states satisfy the simplicity constraint? Can we associate to these states a geometrical interpretation similar to the one of the simplicial case? Can the construction be extended to the physically relevant Lorentzian case?

In this chapter [1] we answer several of these questions. We show that it is possible to start form a discretization of BF theory on a general 2-cell complex, and impose the same boundary constraints that one impose in the simplicial case (simplicity and closure). Remarkably, on the one hand, they reduce the BF vertex amplitude to a (generalization of) the KKL vertex amplitude, in the Euclidean case studied by KKL. On the other hand, a theorem by Minkowski [64] guarantees that these constraints are precisely those needed to equip the classical limit of each truncation of the boundary state space to a finite graph, with a geometrical interpretation, which turns out to be in terms of polyhedra [30].

These results reinforce the overall coherence of the generalized spinfoam formalism. Also, we consider the new quantum number in this spinfoam formalism.

An outline for the chapter is as follows. In Section 3.1, we review the spinfoam representation of the BF partition function on a general complex, and we discuss the structure of the boundary Hilbert space of BF theory. In Section 3.2, we implement the geometric constraint to the BF boundary Hilbert space. After solving the constraint weakly, two new boundary Hilbert space are constructed for both the Euclidean and the Lorentzian theory. We also show that the new boundary Hilbert space carries a representation of quantum polyhedral geometry. In Section 3.3, we derive the new spinfoam vertex amplitude and face amplitude from the new boundary Hilbert space. In the last Section, we conclude and point out the open issues. We assume that the Barbero-Immirzi parameter $\gamma$ is positive.

## 3.1 spinfoam Representation of BF Theory

We start with a brief review of the construction of the BF spinfoam partition function and the structure of its boundary Hilbert space [65], which is the starting point of the definition of the theory. The BF partition function is formally defined by the path integral

$$Z_{BF} := \int DA\, DB \quad \exp\big(i\int_M \mathrm{Tr}(B\wedge F[A])\big) \tag{3.1.1}$$

[1] This chapter is based on work done together with Muxin Han and Carlo Rovelli. The results have been published in [38].

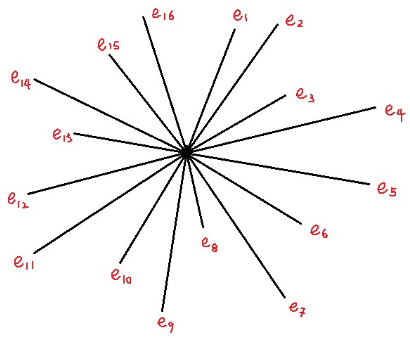



Figure 3.1: A generalized spinfoam vertex.

where $B$ is a 2-form field on the manifold $M$, with values in the Lie algebra $\mathfrak{g}$ of a group $G$ and $F$ is the curvature of the $G$-connection $A$. Here we take the internal gauge group $G$ to be either $G = Spin(4)$ (for the Euclidean case) or $G = SL(2, \mathbb{C})$ (for the Lorentzian case). A formal integration over $B$ gives

$$Z_{BF} = \int DA \prod_{x \in M} \delta(F[A]) \tag{3.1.2}$$

which is an integration over the flat connections. In order to make sense of the formal path integral (3.1.2), we discretize it. However, instead of discretizing the path integral on an oriented 2-complex dual to a simplicial decomposition of the manifold $M$ as is usually done, we introduce here an arbitrary oriented 2-complex $\mathcal{K}$ (as in [44, 63]) with or without boundary.

We take a combinatorial definition of an oriented 2-complex. An oriented 2-complex $\mathcal{K} := (V(\mathcal{K}), E(\mathcal{K}), F(\mathcal{K})$ consists of sets of *vertices* $v \in V(\mathcal{K})$, *edges* $e \in E(\mathcal{K})$ and *faces* $f \in F(\mathcal{K})$, equipped with a *boundary relation* $\partial$ associating an ordered pair of vertices $(s(e), t(e))$ ("source" and "target") to each edge $e$ and a finite sequence of edges $\{e_k^{\epsilon_{e_k f}}\}_{k=1,\ldots,n}$ to each face $f$, with $t(e_k) = s(e_{k+1})$, $t(e_n) = s(e_1)$ and $\epsilon_{ef} = \pm 1$; here we call $e^{-1}$ the edge with reversed order of $e$. We let $\partial f$ denote the cyclically ordered set of edges that bound the face $f$, or (if it is clear from the context) the cyclically ordered set of vertices that bound the boundary edges of $f$. We also write $\partial v$ to indicate the set of edges bounded by $v$, and of faces that have $v$ in their boundary. Similarly, we write $\partial e$ to indicate the set of the faces bounded by $e$. When $e \in \partial f$, we define $\epsilon_{ef} = 1$ if the orientation of $e$ is consistent with the one induced by the face $f$ and $\epsilon_{ef} = -1$ if it is not.

The boundary graph $\gamma = \partial\mathcal{K}$ is a 1-cell subcomplex of $\mathcal{K}$. An edge $e \in E(\mathcal{K})$

is an edge of the boundary graph $\gamma$ if and only if it is contained in only one face, otherwise it is an internal edge. A vertex $v \in V(\mathcal{K})$ is a vertex of the boundary graph $\gamma$ if and only if it is contained in exactly one internal edge of $\mathcal{K}$, otherwise it is an internal vertex of $\mathcal{K}$. We assume boundary vertices and boundary edges to form a graph, which is the boundary of the two-complex.

We introduce also the notion of the boundary graph $\gamma_v$ of a single vertex $v$. This is the graph whose nodes are the edges $e$ in $\partial v$ and whose links are the faces $f$ in $\partial v$. The boundary relation defining the graph is the relation $e \in \partial f$ and the orientation of the links is the one induced by the faces. The graph $\gamma_v$ can be visualized as the intersection between the two complex and a small sphere surrounding the vertex.

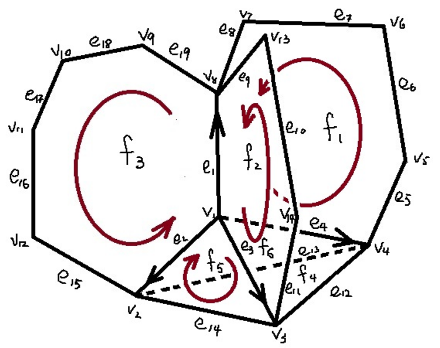


Figure 3.2: An oriented 2-cell complex $\mathcal{K} := (F(\mathcal{K}), E(\mathcal{K}), V(\mathcal{K}))$, where $F(\mathcal{K}) = \{f_1, \cdots, f_6\}$, $E(\mathcal{K}) = \{e_1, \cdots, e_{19}\}$, $V(\mathcal{K}) = \{v_1, \cdots, v_{14}\}$. $v_1$ is internal vertex, and $e_1, e_2, e_3, e_4$ are internal edges, while all other edges and vertices belong to the boundary graph $\gamma = \partial\mathcal{K}$.

We discretize the BF partition function on the oriented 2-cell complex $\mathcal{K}$, by replacing the continuous field $A$ with the assignment of an element of $G$ to each edge. By convention, $g_{e^{-1}} := g_e^{-1}$. Then equation (3.1.2) becomes

$$Z_{BF}(\mathcal{K}) = \int dg_e \prod_f \delta(\prod_{e \in \partial f} g_e^{\epsilon_{ef}}), \tag{3.1.3}$$

where $dg_e$ is the product over all the edges of the Haar measure, the product over $f$ is over all the faces of $\mathcal{K}$ and the product over $e$ is the product over the edges bounding the face $f$ of the group element associated to these edges, ordered by the orientation of the face. This is the partition function of BF theory.

We now express this partition function as a sum over representations and intertwiners. For this, it is convenient to treat the Euclidean and Lorentzian cases separately.

### 3.1.1 Spin(4) BF Theory

Consider the Euclidean case $G = Spin(4)$. The delta function on $Spin(4)$ can be expanded in irreducible representations

$$\delta(U) = \sum_{\rho} \dim(\rho)\chi^{\rho}(U) \tag{3.1.4}$$

where $\rho = (j^+, j^-)$ labels the unitary irreducible representation of Spin(4), $\dim(\rho) = (2j^+{+}1)(2j^-{+}1)$ is the dimension of the representation space, and $\chi_\rho$ is the character of the representation $\rho$. Irreducible representations can also be conveniently labelled with the two half integers $k = j^+ + j^-$ and $p = j^+ - j^-$.

Expanding the delta function in representations, (3.1.2) becomes

$$\begin{aligned} Z_{BF}(\mathcal{K}) &= \int dg_e \prod_f \left( \sum_\rho \dim(\rho)\ \chi^\rho(U_f) \right) \\ &= \sum_{\rho_f} \int dg_e \prod_f \dim(\rho_f)\ \chi^{\rho_f}(U_f). \end{aligned} \tag{3.1.5}$$

This is the expression for the spinfoam amplitude in the group element basis. Let us now translate this into the more common representations-intertwiners basis.

This can be obtained by performing the integrals, precisely as in the simplicial case. We have one integration per edge, of the form

$$K_{\mathbf{M},\mathbf{N}} = \int dg_e \prod_{f\in\partial e} \Pi^{\rho_f}_{M_f N_f}(g_e^{\epsilon_{ef}}) \tag{3.1.6}$$

where $\Pi^\rho_{MN}(g)$ is the matrix element of the Spin(4) representation $\rho$; $\mathbf{M} = M_{f_1}, ..., M_{f_n}$ is a multi-index; and the product is over the $n$ faces bounded by $e$ (including repeated faces). In the case where $\mathcal{K}$ is dual to a simplicial complex, $n{=}4$. It is immediate to see that $K_{\mathbf{M},\mathbf{N}}$ is the operator in the tensor product $(\bigotimes_{f_{out}} \rho_f) \otimes (\bigotimes_{f_{in}}\ \rho_f^\dagger)$ of the $\rho_f$ representation spaces (where $f_{in}$ are the faces with the same orientation as $e$ and $f_{out}$ are the faces with opposite orientation.) that projects on its invariant subspace

$$\mathcal{H}_e = \mathrm{Inv}\big[(\bigotimes_{f_{out}} \rho_f) \otimes (\bigotimes_{f_{in}}\ \rho_f^\dagger)\big]. \tag{3.1.7}$$

Let $I$ label an orthonormal basis in $\mathcal{H}_e$. (These are called intertwiners.) Then

$$K_{\mathbf{M},\mathbf{N}} = \sum_I\ I_{\mathbf{M}}\ I^\dagger_{\mathbf{N}}. \tag{3.1.8}$$

For each internal edge $e$, the two intertwiners are associated to the two vertices bounding the edge (see Figure 3), in the sense that their indices are contracted with the other intertwiners at the same vertex. The result of the integration is therefore

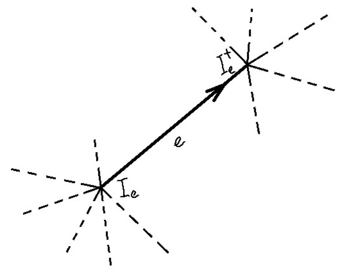



Figure 3.3: Assign $I_e$ to the begin point and assign $I_e^\dagger$ to the end point of an internal edge $e$.

$$Z_{BF}(\mathcal{K}) = \sum_{\rho_f} \prod_f \dim(\rho_f) \sum_{I_e} \prod_v A_v(\rho_f, I_e). \tag{3.1.9}$$

Here the sum over $I_e$ is over the assignment of one intertwiner to each edge $e$ of $\mathcal{K}$. The product over $v$ is over the vertices of $\mathcal{K}$. The vertex amplitude $A_v(\rho_f, I_e)$ is defined as follows. Say at the vertex $v \in V(\mathcal{K})$ there are $n$ outgoing edges $e_{out}$ and $m$ incoming edges $e_{in}$. Then

$$A_v(\rho_f, I_e) := \mathrm{Tr}\left(\bigotimes_{e_{out}} I_{e_{out}} \bigotimes_{e_{in}} I^\dagger_{e_{in}}\right) \tag{3.1.10}$$

The trace in eq.(3.1.10) is precisely the spinfoam trace defined in [44, 63]. The contractions between the intertwiners in the spinfoam trace could be described by the follows: For each edge $e$ each index $M_i$ is associated with a face $f$ bounded by the edge $e$. The trace is defined by contracting the two indices associated with the same face of the two intertwiners corresponding to the two edges bounding $f$. This can be easily seen to give the character $\chi^\rho$ of (3.1.5). In the special case when the complex $\mathcal{K}$ is dual to a simplicial complex, there are 5 internal edges joining at $v$ and each pair of edges determines a 2-face, the spinfoam trace is nothing but the Spin(4) 15-j symbol.

Alternatively, the BF partition function can also be expressed in the form [44, 63]

$$Z_{BF}(\mathcal{K}) = \sum_{\rho_f} \prod_f \dim(\rho_f)\ \mathrm{Tr}\left(\bigotimes_{e \in E(\mathcal{K})} P_e\right) \tag{3.1.11}$$

where $P_e := \sum_{I_e} I_e \otimes I_e^\dagger$ is understood as the projection operator projecting from the product of the representations on the 2-faces bounded by $e$ to its invariant subspace. And the index contractions in $\mathrm{Tr}\left(\otimes_{e\in E(\mathcal{K})} P_e\right)$ are the contractions between intertwiners, as above.

Most gravitational spinfoam theories, constructed as constrained BF, have this same structure (3.1.11).

### 3.1.2 SL(2,$\mathbb{C}$) BF Theory

Let now $G = SL(2,\mathbb{C})$. The derivation of the spinfoam representation of $SL(2,\mathbb{C})$ is as above, with a few differences. $SL(2,\mathbb{C})$ unitary irreps (in the principle series) can be labelled by the same quantum numbers $(k,p)$ as the $SO(4)$ ones, but now $p$ is a real number [40]. The unitary irreps of $SL(2,\mathbb{C})$ are infinite dimensional and can be decomposed into an infinite direct sum of SU(2) irreps, i.e.

$$V^{(k,p)} = \bigoplus_{j=k}^{\infty} V_j^{(k,p)} \tag{3.1.12}$$

where $V_j^{(k,p)} \sim V_j$ is the carrier space of the spin$j$ representation of SU(2). This decomposition provides a convenient basis $|j,m>$ in $V^{(k,p)}$, obtained diagonalizing $L^2$ and $L^z$ of SU(2). In this basis, for $g \in SL(2,\mathbb{C})$, we write the representation matrices on $V^{(k,p)}$ as $\Pi^{(k,p)}_{jm,j'm'}(g)$ where $j \in \{k, k+1, \cdots, \infty\}$ and $m \in \{-j, \cdots, j\}$. As one might expect from the fact that $p$ is a continuous label, the representation "matrix element" $\Pi^{(k,p)}_{jm,j'm'}$ is distributional on the Hilbert space $L^2[SL(2,\mathbb{C})]$ defined by the Haar measure. These matrix elements form a generalized orthonormal basis and define a Fourier-like transform. That is, for any square integrable function $f(g)$ on $SL(2,\mathbb{C})$,

$$\begin{aligned} f(g) &= \frac{1}{8\pi^4}\sum_k \int_{-\infty}^{+\infty} \mathrm{d}p\,(k^2+p^2)\,\mathrm{Tr}\big[F(k,p)\,\Pi^{(k,p)}(g^{-1})\big] \\ F(k,p) &= \int_{SL(2,\mathbb{C})} f(g)\ \Pi^{(k,p)}(g)\ \mathrm{d}\mu_H(g) \end{aligned} \tag{3.1.13}$$

which is known as Plancherel theorem [40]. Accordingly, we have an identity for Fourier decomposition of delta function on $SL(2,\mathbb{C})$

$$\delta(g) = \frac{1}{8\pi^4}\sum_k \int_{-\infty}^{+\infty} \mathrm{Tr}\left[\Pi^{(k,p)}(g)\right](k^2+p^2)\ \mathrm{d}p \tag{3.1.14}$$

in analogy with eq.(3.1.4). Proceeding as in the Euclidean case, we find

$$\begin{aligned} Z_{BF}(\mathcal{K}) &= \int \prod_e \mathrm{d}g_e \prod_f \delta(U_f) && (3.1.15) \\ &= \sum_{k_f} \int \mathrm{d}p_f \prod_f (k_f^2 + p_f^2) \int \mathrm{d}g_e \prod_f \mathrm{Tr}\left[\Pi^{(k_f,p_f)}(U_f)\right] \end{aligned}$$

As in the euclidean case, each $g_e$ integral is of the form

$$K_{\mathbf{jm},\mathbf{j'm'}} = \int \mathrm{d}g_e \prod_{f\in\partial e} \Pi^{(k_f,p_f)}_{j_f m_f, j'_f m'_f}\left(g_e^{\epsilon_{ef}}\right). \tag{3.1.16}$$

Formally, this is still a projector on the invariant component of the tensor product of $n$ irreducibles. However, since now one of the two Casimirs has continuous spectrum $p$, then the trivial representation $p = k = 0$ is not a proper subspace of the tensor product, but only a generalized subspace. This does not forbids us to introduce an orthonormal basis of intertwiners $I$ in this subspace, as we did in the Euclidean case, and write

$$K_{\mathbf{jm},\mathbf{j'm'}} = \sum_I I_{\mathbf{jm}}\, I^\dagger_{\mathbf{j'm'}} \tag{3.1.17}$$

but we have to remember that the intertwiners are generalized vectors. Using this, we can formulate the spinfoam representation of $SL(2,\mathbb{C})$ BF theory in the same way as we did for Spin(4) theory.

- The Fourier decomposition of the $SL(2,\mathbb{C})$ delta function assigns an $SL(2,\mathbb{C})$ irrep labeled by $(k_f, p_f)$ to each face $f$.
- Eq.(3.1.16) assigns an $SL(2,\mathbb{C})$ intertwiner $I^e$ to each source of each edge $e$, and a dual intertwiner $I^{e\dagger}$ to its target.
- At each vertex $v$ with $n$ outgoing edges $e_1^{out}, \cdots, e_n^{out}$ and $m$ incoming edges $e_1^{in}, \cdots, e_m^{in}$, the intertwiners $I^{e^{out}}$ and $I^{e^{in}\dagger}$ are contracting on their $\mathbf{j}, \mathbf{m}$ and $\mathbf{j}', \mathbf{m}'$ indices, according to how the faces neighboring the vertex are bounded by the edges. The result of this contraction gives the spinfoam vertex amplitude

$$A_v\Big((k,p)_f, I_e\Big) := \mathrm{Tr}\left(\left(\bigotimes_{e_{out}} I_e\right) \otimes \left(\bigotimes_{e_{in}} I_e^\dagger\right)\right) \tag{3.1.18}$$

- Finally the partition function of $SL(2,\mathbb{C})$ BF theory is

$$Z_{BF} = \sum_{k_f I_e} \int \mathrm{d}p_f \prod_f (k_f^2 + p_f^2) \prod_v A_v\Big((k,p)_f, I_e\Big) \tag{3.1.19}$$

This expression, however, is ill defined, due to the fact that the intertwiners are generalized vectors, and the trace (3.1.18) may diverge. This issue is addressed and answered in [66, 67], where it is shown that the source o f the divergence is a redundant integral over $SL(2,\mathbb{C})$ in the definition of $A_v$. It is then immediate to regularize $A_v$ by removing one $SL(2,\mathbb{C})$ integration per each vertex. The resulting amplitude is proven in [66, 67] to be finite, except for some particular pathological vertices, which we exclude here for simplicity. In what follows we always assume that the vertex amplitude is so regularized that the redundant integral is removed.

### 3.1.3 Boundary Hilbert Space

Let us rewrite the partition function (3.1.3) in a slightly different form. Split each edge $e$ bounded by the vertices $v$ and $v'$ into two half edges $(ev)$ and $(ev')$, and associate a group element $g_{ev}$ to each half edge (oriented towards the vertex). Then replace each integral $dg_e$ with the two integrals $dg_{ev}$, $dg_{ev'}$. This gives

$$Z_{BF}(\mathcal{K}) = \int dg_{ev} \prod_f \delta\Big(\prod_{e\in\partial g} (g_{ev}^{-1} g_{ev'})^{\epsilon_{ef}}\Big), \tag{3.1.20}$$

where there is one integration per each couple vertex/adjacent-edge. Next, let $v$ be a vertex in the boundary of the face $f$. For each such couple $fv$, introduce a group variable $g_{fv}$. Then (3.1.20) can be rewritten in the form

$$Z_{BF}(\mathcal{K}) = \int dg_{fv} dg_{ev} \prod_f \delta\Big(\prod_{v\in\partial f} g_{fv}\Big) \prod_{fv} \delta(g_{fv}^{-1} g_{ev} g_{e'v}^{-1}) \tag{3.1.21}$$

where $e$ and $e'$ are the two edges in the boundary of $f$ that meet at $v$, ordered by the orientation of $f$. This can be rewritten in the form

$$Z_{BF}(\mathcal{K}) = \int dg_{fv} \prod_f \delta\Big(\prod_{v\in\partial f} g_{fv}\Big) \prod_v A_v(g_{fv}) \tag{3.1.22}$$

where the vertex amplitude $A_v(g_f)$ is defined by

$$A_v(g_f) = \int \prod_{e\in\partial v} dg_e \prod_{f\in\partial v} \delta(g_{e_f} g_f g_{e'_f}^{-1}) \tag{3.1.23}$$

is a function of one group element for each face in the boundary of $v$. Here the integral is over one group element per each edge in the boundary of the vertex $v$ and, as before, $e$ and $e'$ are the two edges in the boundary of $f$ that meet at $v$. This is a rewriting of the connection representation of spin-foam models, in terms of group elements and wedges [68], and is called the "holonomy" form of the partition function in [69].

Let $|F_v|$ be the number of links $f$ of the graph $\gamma_v$, namely the number of faces $f$ in $\partial v$. The vertex amplitude (3.1.23) is a function in

$$\mathcal{H}_{\gamma_v} = L_2[G^{|F_\gamma|}]. \tag{3.1.24}$$

We call this the (non-gauge invariant) boundary Hilbert space of the vertex $v$. It is easy to se that the vertex amplitude (3.1.23) is an element of this space. More precisely, it is an element of the (possibly generalized) subspace

$$\mathcal{K}_{\gamma_v} = L_2[G^{|F_\gamma|}/G^{|E_\gamma|}] \tag{3.1.25}$$

where $|E_\gamma|$ is the number of nodes of $\gamma_v$, namely the number of edges in $\partial v$, formed by the states invariant the gauge transformation

$$\psi(g_e) = \psi(\Lambda_{s_e} g_e \Lambda_{t_e}) \tag{3.1.26}$$

where $\Lambda \in G$ and $s_e$ and $t_e$ are the source and target of $e$.

A moment of reflection shows also that (3.1.10) and (3.1.18) are simply the amplitude (3.1.23) expressed in the standard spin network basis of $\mathcal{K}_{\gamma_v}$. Let us now study the boundary space $\mathcal{H}_{\gamma_v}$ in more detail. (It is convenient to consider the non-gauge-invariant Hilbert space $\mathcal{H}_{\gamma_v}$, besides the gauge invariant one because the expressions of geometric constraints will not be gauge invariant, thus they can only be represented as operators on $\mathcal{H}_{\gamma_v}$.)

The natural derivative operator defined on the Hilbert space $L_2[G]$ is the left invariant derivative that generates the right $G$ action:

$$J^{IJ}\psi(g) \;\; = \;\; \frac{\mathrm{d}}{\mathrm{d}\alpha}\psi(e^{\alpha T^{IJ}}g)\Big|_{\alpha=0} \tag{3.1.27}$$

where $T^{IJ}$ $(I, J = 0, \cdots, 3)$ is a standard Lie algebra generator of $Lie(G)$.

Fix an SU(2) subgroup of $G$, and choose a basis in $Lie(G)$ such that the direction $I = 0$ is preserved by $SU(2)$. Then we can split the six generators $T^{IJ}$ of $Lie(G)$ into 3 rotation generators and 3 boost generators, resulting from the choice of canonical embedding of the rotation $SU(2)$ group into $G$, and thus basically corresponding to the time gauge for the embedding vector. Accordingly, we define $(i, j, k = 1, 2, 3)$

$$L^i := \frac{1}{2}\epsilon^i{}_{jk}J^{jk}, \qquad K^i := J^{0i} \tag{3.1.28}$$

which have the standard commutation relations

$$\begin{aligned}
\left[L^i, L^j\right] &= \epsilon^{ij}{}_k L^k, && (3.1.29)\\
\left[K^i, K^j\right] &= s\epsilon^{ij}{}_k L^k, && (3.1.30)\\
\left[K^i, L^j\right] &= \epsilon^{ij}{}_k K^k && (3.1.31)
\end{aligned}$$

where $s = +1$ for $Spin(4)$ and $s = -1$ for $SL(2, \mathbb{C})$.

We denote by $J_f^{IJ}$ the left invariant derivative operator acting on the variable $g_f$ of $\psi(g_f) \in \mathcal{H}_{\gamma_v}$. Notice that the right invariant vector field

$$R^{IJ}\psi(g) = \frac{\mathrm{d}}{\mathrm{d}\alpha}\psi(ge^{\alpha T^{IJ}})\Big|_{\alpha=0} \tag{3.1.32}$$

satisfies $R^{IJ}\psi(g) = J^{IJ}\psi(g^{-1})$. Therefore

$$J_{f^{-1}}^{IJ} = R_f^{IJ}. \tag{3.1.33}$$

The bivector operators $J_f^{IJ}$ have a physical interpretation in terms of the $BF$ theory we started from. They are the quantum operators that quantize the discretized version of the 2-form field $B^{IJ}$, restricted to a 3-dimensional boundary. The reason for this is the follows: Classically the Hamiltonian analysis of BF theory can be carried out [70]. The resulting non-vanishing Poisson bracket reads

$$\left\{\epsilon^{abc}B_{abIJ}(x), A_d^{KL}(x')\right\} = \delta_d^c \delta_{[I}^K \delta_{J]}^L \delta^3(x, x') \tag{3.1.34}$$

where $a, b, c = 1, 2, 3$, $x$ and $x'$ belong to a 3-dimensional spatial manifold $\mathcal{S}$. These canonical conjugate variables can be discretized in analogy with Hamiltonian lattice gauge theory. Given a graph $\gamma$ imbedded in $\mathcal{S}$, there exists a 2-cell complex dual to the graph $\gamma$, such that given a link $f$ in the graph there is a unique 2-face $S_f$ dual to the link $f$. This 2-cell complex defines a polyhedral decomposition of the spatial manifold $\sigma$. With this setting, we associate a group variable $g_f \in G$ to each link $f$, and associate a Lie algebra variable $B_f^{IJ}$ to each $S_f$ (the Lie algebra variables are also labeled by $f$ because of the 1-to-1 correspondence between links and 2-faces). The Poisson algebra of these discretized variables has the following standard expression

$$\begin{aligned}
\left\{g_f, g_{f'}\right\} &= 0\\
\left\{B_f^{IJ}, g_{f'}\right\} &= \delta_{ff'}T^{IJ}g_f\\
\left\{B_f^{IJ}, B_{f'}^{KL}\right\} &= \delta_{ff'}f^{IJ,KL}{}_{MN}B_f^{MN} && (3.1.35)
\end{aligned}$$

where $f^{IJ,KL}{}_{MN}$ denotes the structure constant of $Lie(G)$. In our case, if we consider our boundary graph $\gamma_v$ and abstractly define the above Poisson algebra on $\gamma_v$, we find that the bivector operator $J_f^{IJ}$ for each oriented link $f$ (as a right invariant vector) is the quantum operator representing the Lie algebra variable $B_f^{IJ}$ (up to $-i\hbar$), because of the commutation relation between $J_f^{IJ}$ and $g_f$ on the boundary Hilbert space.

## 3.2 Boundary Quantum Geometry

We now consider a modification of BF theory. The modification is obtained by restricting the boundary space $\mathcal{H}_{\gamma_v}$ by imposing a certain constraint. Let us first define this constraint and then discuss the consequences and the motivation of imposing it.

### 3.2.1 Geometric Constraints

Consider a vertex $v$ and its boundary graph $\gamma_v$. For each link $f$, consider the Lie algebra element $\Sigma$ given by

$$B_f = {}^*\Sigma_f + \frac{1}{\gamma}\Sigma_f \tag{3.2.1}$$

where the star indicates the Hodge dual in the Lie algebra. Consider a node $e$ of the boundary graph $\gamma_v$, and let $f \in \partial e$ be all oriented away from $e$. Then define

1. **Simplicity Constraint:** There exists a unit vector $(n_e)_I$ for each $e$ such that, for all $f \in \partial e$

$$(n_e)_I {}^*\Sigma_f^{IJ} = 0. \tag{3.2.2}$$

2. **Closure Constraint:**

$$\sum_{f\in\partial e} \Sigma_f^{IJ} = 0. \tag{3.2.3}$$

These are the two constraints on which we focus. The main motivation for considering these constraints is the fact that the action of general relativity in the Holst formulation can be written in the form

$$S_{GR}[e,\omega] = \int B \wedge F[\omega] \tag{3.2.4}$$

where $\omega$ is an $SL(2,\mathbb{C})$ connection,

$$B = {}^*\Sigma + \frac{1}{\gamma}\Sigma \tag{3.2.5}$$

and

$$\Sigma^{IJ} = e^I \wedge e^J \tag{3.2.6}$$

where $e^I$ is the tetrad one form. The restriction $\Sigma_f^{IJ}\big|_{\mathcal{B}}$ of $\Sigma$ to any space-like boundary $\mathcal{B}$ satisfies the conditions:

$$n_I \Sigma^{IJ}\big|_{\mathcal{B}} = 0 \tag{3.2.7}$$

where $n_I$ is the normal to the boundary and

$$d\Sigma = 0. \tag{3.2.8}$$

Equations (3.2.1), (3.2.2) and (3.2.3) can be seen as a discrete consequence of equations (3.2.5), (3.2.7) and (3.2.8). Here, however, we take the discretized equations (3.2.1), (3.2.2) and (3.2.3) as our starting point, and study their consequences. A full discussion on the relation of these equations with continuum general relativity will be considered elsewhere.[2]

The key consequences of these constraints is that they allow $\Sigma$ to determine a classical polyhedral geometry at each node $e$ of the boundary graph $\gamma_v$ (See also [30]). The following is a straight-forward application of Minkowski's theorem [64]

**Proposition 3.2.1.** *Given an F-valent node e in $\gamma_v$, let F bivectors $\Sigma_f$ satisfy (3.2.2) and (3.2.3). Then there exists a (possibly degenerate) flat convex polyhedron in $\mathbb{R}^3$ with F faces, whose face area bivectors coincide with $\Sigma_f^{IJ}$. The resulting polyhedron is unique up to rotation and translation.*

**Proof:** Without loss of generality, we fix the unit vector $(n_e)_I = (1,0,0,0)$ (we call this the time-gauge). The simplicity constraint eq.(3.2.2) reduces to

$$\Sigma_f^{0i} = 0. \tag{3.2.9}$$

[2] The Plebanski simplicity constraint implies the constraints given here. However the reverse is not true in general, unless "shape-matching" conditions [30] are imposed on each face shared by two polyhedra. We do not demand such shape-matching conditions here. There is some evidences from the large-$j$ behavior of the generalized spinfoam model that non-shape-matching amplitudes are suppressed in the large-$j$ asymptotic [71].

Hence the surviving components of $\Sigma_f^{IJ}$ are $\Sigma_f^{ij}$. We denote these nonvanishing components simply by $\Sigma_f^i = \frac{1}{2}\epsilon^i{}_{jk}B^{jk}$ or $\vec{\Sigma}_f$, in terms of which the closure constraint (3.2.3) reads

$$\sum_f \vec{\Sigma}_f = 0. \tag{3.2.10}$$

Consider $\vec{\Sigma}_f$ as vectors in $\mathbb{R}^3$. Call $|\Sigma_f|$ the length of the 3-vector $\vec{\Sigma}_f$, and let $\vec{n}_f := \vec{\Sigma}_f/|\Sigma_f|$. We first suppose the unit vectors $\vec{n}_f$ are non-coplanar. Then we recall Minkowski's Theorem [64], which states that whenever there are $F$ non-coplanar unit 3-vectors $\vec{n}_f$ and $F$ positive numbers $A_f$ satisfying the condition

$$\sum_f A_f \vec{n}_f = 0, \tag{3.2.11}$$

then there exists a convex polyhedron in $\mathbb{R}^3$, whose faces have outward normals $\vec{n}_f$ and areas $A_f$. And the resulting polyhedron is unique up to rotation and translation.[3]

When we apply Minkowski's theorem to our case, we see that the existence of the unit 3-vectors $\vec{n}_f$ and the lengths $|\Sigma_f|$, as well as the closure constraint eq.(3.2.10), together imply that there is a convex polyhedron in $\mathbb{R}^3$, unique up to translation and rotation, such that each $\vec{n}_f$ is an outward normal of a face and each $|\Sigma_f|$ is an area of a face. Such a polyhedron can be concretely constructed via Lasserre reconstruction algorithm [72]. Let $e^i$ the natural triad in $\mathbb{R}^3$, then the 3-vector $\vec{\Sigma}_f$ can be expressed as an oriented area:

$$\Sigma_f^{ij} = \int_f e^i \wedge e^j. \tag{3.2.12}$$

Finally, the case of coplanar unit 3-vectors $\vec{n}_f$ can be obtained as a limit of non-coplanar case, yielding degenerate polyhedra. □

This geometrical interpretation equips the variables $e$ and $f$ with a further new meaning: they represent, respectively, polyhedra in a 4d space and faces of these polyhedra. See Table 1.

The geometrical interpretation in terms of tetrahedra (and now polyhedra) has raised a lively discussion and it is sometimes unpalatable to the more canonical-oriented part of the community. Part of this discussion is based on misunderstanding. The precise claim here is that if we take the Hilbert space of the theory and

[3] Imagine the polyhedron immersed in a homogeneous fluid. Eq.(3.2.11) multiplied by the pressure is the sum of the pressure forces acting on the faces, which obviously vanishes.

| | 2-complex $\mathcal{K}$ | boundary graph $\gamma_v$ | boundary 3d geometry |
|---|---|---|---|
| $e$ | edge | node | polyhedron |
| $f$ | face | link | face of polyhedron |

Table 3.1: The different geometrical interpretations of the labels $e$ and $f$.

we *truncate* it to a finite graph (so that the observable algebra is also truncated), then the truncated Hilbert space (with its observables algebra) has a classical limit, and this classical limit can be naturally interpreted as describing a collection of polyhedra. This is well consistent with classical general relativity, because classical general relativity as well admits truncations where the geometry is discretized. Also, this is not inconsistent with the continuous picture for the same reason for which the fact that the truncation of Fock space to an $n$ particle Hilbert space describes discrete particles, is not inconsistent with the fact that Fock space itself describes a (quantized) field.

Let us now see how the constraints translate on the variable $B$ given in (3.2.1). We have easily:

**Simplicity Constraint:**

$$C_f^J = n_I \left( {}^*B_f^{IJ} - \frac{s}{\gamma} B_f^{IJ} \right) = 0, \tag{3.2.13}$$

**Closure Constraint:**

$$G_e^{IJ} = \sum_{f \in e} B_f^{IJ} = 0, \tag{3.2.14}$$

where $s = +1$ for $Spin(4)$ and $s = -1$ for $SL(2,\mathbb{C})$.

Consider a single polyhedron $e$, with the time-gauge $(n_e)_I = (1,0,0,0)$, and introduce the rotation $L_f^j := \frac{1}{2}\epsilon^j{}_{kl} B_f^{kl}$ and boost $K_f^j := B_f^{0j}$ components of $B_f^{IJ}$. Then the simplicity constraint (3.2.13) becomes simply

$$\vec{K}_f = s\gamma \, \vec{L}_f; \tag{3.2.15}$$

the rotation generators are proportional to the boost generators. The closure constraint (3.2.14) can be written as

$$\sum_{f \in \partial e} \vec{L}_f = 0, \tag{3.2.16a}$$

$$\text{and} \quad \sum_{f \in \partial e} \vec{K}_f = 0. \tag{3.2.16b}$$

where the second, eq.(3.2.16b), is redundant, by eq.(3.2.15).

Let us now move to the quantum theory, and impose the two constraints (3.2.15) and (3.2.16a) weakly [14–16, 31, 32] on the quantum states. This gives

**Simplicity Constraint:**

$$\left\langle \psi, \vec{K}_f \psi' \right\rangle = s\gamma \ \left\langle \psi, \vec{L}_f \psi' \right\rangle . \tag{3.2.17}$$

**Closure Constraint:**

$$\begin{aligned} \sum_{f\in\partial e} \left\langle \psi, \vec{L}_f \psi' \right\rangle &= 0 \\ \sum_{f\in\partial e} \left\langle \psi, \vec{K}_f \psi' \right\rangle &= 0, \end{aligned} \tag{3.2.18}$$

These equations give a subspace $\mathcal{H}^E_{\gamma_v}$ (respectively $\mathcal{H}^L_{\gamma_v}$ in Lorentzian case) of the boundary Hilbert space $\mathcal{H}_{\gamma_v}$ of BF theory, where the constraints hold weakly. That is, we define $\mathcal{H}^E_{\gamma_v}$ as the subspace where these equations hold for any two states $\psi$ and $\psi'$ in a dense domain, for all nodes $s$ of $\gamma_v$. We do not mean $\mathcal{H}^E_{\gamma_v}$ is selected as the maximal weak solution to the geometric constraints; in fact, it may not.

### 3.2.2 New Boundary Hilbert Space: Euclidean Theory

Let us now construct $\mathcal{H}^E_{\gamma_v}$. Here we first define $\mathcal{H}^E_{\gamma_v}$ and then prove that it solves the geometric constraint. We begin with some preliminaries on the structure of the BF boundary Hilbert space. In the Euclidean theory, this space has the following decomposition

$$\mathcal{H}_{\gamma_v} = \bigotimes_f L^2[Spin(4)] = \bigotimes_f \left[ \bigoplus_{\rho_f} V_{\rho_f} \otimes V^*_{\rho_f} \right] . \tag{3.2.19}$$

where $V_\rho$ denote the representation space for the Spin(4) irrep $\rho$ and $V^*_\rho$ is the representation space for the Spin(4) adjoint irrep $\rho^*$. For each face $f$, $V_{\rho_f}$ and $V^*_{\rho_f}$ transforms in a gauge transformation (3.1.26) under the action of $\Lambda_{s_f}$ $\Lambda_{t_f}$, where $s_f$ and $t_f$ are the initial and final points of the link $f$. By regrouping all representations space that transform under the action of the same $\Lambda_e$, namely by regrouping the representation spaces associated to the same vertex $e$ of $\gamma_v$ we can rewrite the decomposition in the form

$$\mathcal{H}_{\gamma_v} = \bigoplus_{\{\rho_f\}} \bigotimes_{e} \bigotimes_{f \in \partial e} V_{\rho_f}^{(e,f)} \tag{3.2.20}$$

where

$$\begin{aligned} V_{\rho_f}^{(s_f,f)} &:= V_{\rho_f} \\ V_{\rho_f}^{(t_f f)} &:= V_{\rho_f}^{*} \end{aligned} \tag{3.2.21}$$

Therefore the sum over colorings $\rho_f$ associates a representation space

$$\bigotimes_{f \in \partial e} V_{\rho_f}^{(e,f)} \tag{3.2.22}$$

to each vertex $e$. This space can be seen as the quantization of the shapes of a polyhedron with faces having fixed areas, determined by the coloring $\rho_f$ [30].

Since $Spin(4) \sim SU(2)_+ \times SU(2)_-$, a unitary irrep of $Spin(4)$ is given by a tensor product of two SU(2) irreps. $V_\rho = V_{j^+} \otimes V_{j^-}$ with spins $j^+$ and $j^-$. The $SU(2)_\pm$ subgroups of $Spin(4)$ are its canonical self-dual and anti-self dual components, generated by $\vec{L} \pm \vec{K}$, and should not be confused with the (non-canonical) $SU(2)$ subgroup generated by $\vec{L}$, used to pick a time gauge. If we decompose $V_\rho = V_{j^+,j^-}$ in irreducibles of $SU(2)$, we have

$$V_{j^+,j^-} = V_{j^+} \otimes V_{j^-} = \bigoplus_{j=j^+-j^-}^{j^++j^-} V_j^{j^+,j^-}. \tag{3.2.23}$$

We now define $\mathcal{H}^E_{\gamma_v}$. Firstly, we introduce a new quantum number, which is denoted by a non-negative integer $r$, related to $j^-$ by $0 \le r \le 2j^-$. In the representation space $V_{j^+,j^-}$, pick the $V_j^{j^+,j^-}$ subspace (in the decomposition above), where $j$ is defined by the modified $\gamma$-simple relation:

$$\begin{aligned} j^+ &= \frac{j+r}{2} + \frac{\gamma j(j+1)}{2(j+r+1)}, \qquad & j^- &= \frac{j+r}{2} - \frac{\gamma j(j+1)}{2(j+r+1)} \qquad & &\text{if } 0 < \gamma < 1 \\ j^+ &= \frac{\gamma j(j+1)}{2(j-r)} + \frac{j-r}{2}, \qquad & j^- &= \frac{\gamma j(j+1)}{2(j-r)} - \frac{j-r}{2} \qquad & &\text{if } \gamma > 1 \end{aligned} \tag{3.2.24}$$

By doing so, we obtain the subspace $V_j^{j^+(j,r),j^-(j,r)}$ in each $V_{j^+,j^-}$. By restricting in this manner all the $V_{\rho_f}$ subspaces in (3.2.19) we obtain a subspace of $\mathcal{H}_{\gamma_v}$. We define the non-gauge-invariant new boundary space to be this subspace. That is

$$\bigoplus_{\{j_f,r_f\}} \bigotimes_e \bigotimes_{f\in e} (V_{j_f}^{j_f^+(j,e),j_f^-(j,r)})^{(e,f)} \tag{3.2.25}$$

where the sum is over non-negative half-integers $j_f$ and $r_f$, and $(j^+(j,r), j^-(j,r))$ depend on $j$ and $r$ as equation (3.2.24) show. The possible coloring in $\mathcal{H}^E_{\gamma_v}$ are labelled by the two non-negative half-integer quantum numbers $j_f$ and $r_f$. The quantum number $j_f$ characterizes the SU(2) spin of the representation and is easily identified with the corresponding LQG quantum number which is associated to each link of the graph. $r_f$ is a new quantum number, also associated to each link of the graph.

Notice also that (3.2.24) restricts also the possible values of $j$ and $r$ to those for which $\frac{\gamma j(j+1)}{2(j+r+1)}$ or $\frac{\gamma j(j+1)}{2(j+r+1)}$ is half integer. This awkward feature of the Euclidean case disappears in the Lorentzian theory.

Next, we define the gauge invariant new boundary space. Consider the diagonal actions of $h \in SU(2)$ on each product representation space eq.(3.2.22) at each $e$. We denote the invariant subspaces under this actions by

$$\mathfrak{I}_e^{\{j_f\}} = \mathrm{Inv}_{SU(2)} \left[ \bigotimes_{f\in e} (V_{j_f}^{j_f^+(j,r),j_f^-(j,r)})^{(e,f)} \right] \tag{3.2.26}$$

Explicitly,

- $0 < \gamma < 1$

$$\mathfrak{I}_e^{\{<\}} = \mathrm{Inv}_{SU(2)} \left[ \bigotimes_{f\in e} (V_{j_f}^{\frac{j+r}{2}+\frac{\gamma j(j+1)}{2(j+r+1)},\frac{j+r}{2}-\frac{\gamma j(j+1)}{2(j+r+1)}})^{(e,f)} \right] \tag{3.2.27}$$

- $\gamma > 1$

$$\mathfrak{I}_e^{\{>\}} = \mathrm{Inv}_{SU(2)} \left[ \bigotimes_{f\in e} (V_{j_f}^{\frac{\gamma j(j+1)}{2(j-r)}+\frac{j-r}{2},\frac{\gamma j(j+1)}{2(j-r)}-\frac{j-r}{2}})^{(e,f)} \right] \tag{3.2.28}$$

The gauge invariant new boundary Hilbert space is defined by

$$\mathcal{H}^E_{\gamma_v} := \bigoplus_{\{j_f,r_f\}} \bigotimes_e \mathfrak{I}_e^{\{j_f\}}. \tag{3.2.29}$$

An orthonormal basis in $\mathcal{H}^E_{\gamma_v}$ can be constructed as follows. Given a polyhedron $e$ with $F$ faces, we assign at $e$ an $F$-valent $SU(2)$ intertwiner $i_e^{A_1\cdots A_F}$ associated with $F$ $SU(2)$ irreps $j_f$, $f = 1, \cdots F$. An orthonormal basis is then defined by the following functions on $[Spin(4)]^{|E(\gamma_v)|}$

- $0 < \gamma < 1$

$$
\begin{aligned}
T^{E<}_{\gamma_v, j_f, r_f, i_e}(g_f) = & \prod_f \sqrt{[j_f + r_f + \frac{\gamma j_f(j_f+1)}{j_f+r_f+1} + 1][j + r - \frac{\gamma j(j+1)}{j+r+1} + 1]} \\
& \prod_e \left[ i_e^{A_{e1}\cdots A_{eF}} \prod_{f\in e} C^{m^+_{ef} m^-_{ef}}_{A_{e1}} \right] \prod_f \left[ \epsilon^{n^+_{ef} n^+_{e'f}} \epsilon^{n^-_{ef} n^-_{e'f}} \right] \\
& \prod_{(e,f)} \left[ D^{\frac{j+r}{2} + \frac{\gamma j(j+1)}{2(j+r+1)}}_{m^+_{ef} n^+_{ef}}(g^+_{ef}) D^{\frac{j+r}{2} - \frac{\gamma j(j+1)}{2(j+r+1)}}_{m^-_{ef} n^-_{ef}}(g^-_{ef}) \right]
\end{aligned}
\tag{3.2.30}
$$

- $\gamma > 1$

$$
\begin{aligned}
T^{E>}_{\gamma_v, j_f, r_f, i_e}(g_f) = & \prod_f \sqrt{[\frac{\gamma j_f(j_f+1)}{j_f+r_f+1} + j_f + r_f + 1][\frac{\gamma j(j+1)}{j+r+1} - (j_f + r_f) + 1]} \\
& \prod_e \left[ i_e^{A_{e1}\cdots A_{eF}} \prod_{f\in e} C^{m^+_{ef} m^-_{ef}}_{A_{e1}} \right] \prod_f \left[ \epsilon^{n^+_{ef} n^+_{e'f}} \epsilon^{n^-_{ef} n^-_{e'f}} \right] \\
& \prod_{(e,f)} \left[ D^{\frac{\gamma j(j+1)}{2(j-r)} + \frac{j-r}{2}}_{m^+_{ef} n^+_{ef}}(g^+_{ef}) D^{\frac{\gamma j(j+1)}{2(j-r)} - \frac{j-r}{2}}_{m^-_{ef} n^-_{ef}}(g^-_{ef}) \right]
\end{aligned}
\tag{3.2.31}
$$

where $g_{ef} = (g^+_{ef}, g^-_{ef}) \in Spin(4)$, $D^j(g)$ is the representation matrix of the $SU(2)$ irrep $j$, and $C^{m^+_{ef} m^-_{ef}}_{A_{ef}}$ denotes the Clebsch-Gordan coefficient ($A_f = -k_f, \cdots, k_f$)

- $0 < \gamma < 1$

$$
\begin{aligned}
& \Big\langle \frac{j+r}{2} + \frac{\gamma j(j+1)}{2(j+r+1)},\ \frac{j+r}{2} - \frac{\gamma j(j+1)}{2(j+r+1)};\ j_f,\ A_{ef} \Big| \\
& \quad \Big| \frac{j+r}{2} + \frac{\gamma j(j+1)}{2(j+r+1)},\ m^+_{ef};\ \frac{1-\gamma}{2} j_f + r_{ef},\ m^-_{ef} \Big\rangle .
\end{aligned}
\tag{3.2.32}
$$

- $\gamma > 1$

$$
\begin{aligned}
& \Big\langle \frac{\gamma j(j+1)}{2(j-r)} + \frac{j-r}{2},\ \frac{\gamma j(j+1)}{2(j-r)} - \frac{j-r}{2};\ j_f,\ A_{ef} \Big| \\
& \quad \Big| \frac{\gamma j(j+1)}{2(j-r)} + \frac{j-r}{2},\ m^+_{ef};\ \frac{\gamma j(j+1)}{2(j-r)} - \frac{j-r}{2},\ m^-_{ef} \Big\rangle .
\end{aligned}
\tag{3.2.33}
$$

$\epsilon^{n^\pm_{ef} n^\pm_{e'f}}$ are the unique 2-valent $SU(2)$ intertwiners with representations

- $0 < \gamma < 1$

$$
j^+_f = \frac{j+r}{2} + \frac{\gamma j(j+1)}{2(j+r+1)}, \qquad j^- = \frac{j+r}{2} - \frac{\gamma j(j+1)}{2(j+r+1)}
$$

- $\gamma > 1$

$$j^+ = \frac{\gamma j(j+1)}{2(j-r)} + \frac{j-r}{2}, \qquad j^- = \frac{\gamma j(j+1)}{2(j-r)} - \frac{j-r}{2}$$

respectively. Thus $T^E_{(\gamma_v, j_f, r_f, i_e)}$ is essentially a function over $g_f = g_{ef} g_{fe'}$. Note that if we ask the quantum numbers $r_f$ to be some fixed integers, then the spin-network functions $T^E_{(\gamma_v, j_f, r_f, i_e)}$ can be equivalently considered as an $SU(2)$ spin-network functions, thus the boundary Hilbert space is spanned by $SU(2)$ spin-networks, as the case of LQG kinematical Hilbert space.

We are now ready to prove our first main result.

**Theorem 3.2.2.** *The Hilbert space $\mathcal{H}^E_{\gamma_v}$ solves the geometric constraint (3.2.17-3.2.18), with $s = 1$.*

**Proof:** The closure constraint (3.2.18) follows immediately since the states in $\mathcal{H}^E_{\gamma_v}$ is invariant under the diagonal $SU(2\partial$ gauge transformation $(g^+_{ef}, g^-_{ef}) \mapsto (h_e g^+_{ef}, h_e g^-_{ef})$ at each $e$ (the constraint is even solved strongly). The nontrivial proof is for the simplicity constraint (3.2.17). Define the self-dual/anti-self-dual operators:

$$\vec{J}^\pm_f := \frac{1}{2}(\vec{L}_f \pm \vec{K}_f) \tag{3.2.34}$$

then (3.2.17) reads

$$(1-\gamma)\left\langle \psi, \vec{J}^+_f \psi' \right\rangle_E - (1+\gamma)\left\langle f, \vec{J}^-_\psi \psi' \right\rangle = 0. \tag{3.2.35}$$

The operators $\vec{J}^\pm_f$ on $L^2(Spin(4))$ act on individual $V^{(e,f)}_{\rho_f}$ (see, e.g. Sec.32.2 of [2]). Therefore we only need to show that in each Clebsch-Gordan subspace $V^{\rho=(j^+,j^-)}_j$, with $\left(j^+ \equiv \frac{j+r}{2} + \frac{\gamma j(j+1)}{2(j+r+1)},\ j^- \equiv \frac{j+r}{2} - \frac{\gamma j(j+1)}{2(j+r+1)}\right)_{<0\gamma<1}$ and $\left(j^+ \equiv \frac{\gamma j(j+1)}{2(j-r)} + \frac{j-r}{2},\ j^- \equiv \frac{\gamma j(j+1)}{2(j-r)} - \frac{j-r}{2}\right)_{\gamma>1}$, the following relation holds for all pairs $\Phi, \Psi$ of vectors

$$(1-\gamma)\langle\Psi|\vec{J}^+|\Phi\rangle - (1+\gamma)\langle\Psi|\vec{J}^-|\Phi\rangle = 0 \tag{3.2.36}$$

where $\langle\ |\ \rangle$ is the Hermitian inner product on the $Spin(4)$ irrep $V_{\rho=(j^+,j^-)}$.

To evaluate these matrix elements, we remind the action of generators of $Spin(4)$

on the canonical basis:

$$
\begin{aligned}
L^3|j,m\rangle =& m|j,m\rangle, \\
L^+|j,m\rangle =& \sqrt{(j+m+1)(j-m)}|j,m+1\rangle, \\
L^-|j,m\rangle =& \sqrt{(j+m)(j-m+1)}|j,m-1\rangle, \\
K^3|j,m\rangle =& \alpha_{(j)}\sqrt{j^2-m^2}|j-1,m\rangle + \gamma_{(j)}m|j,m\rangle - \alpha_{(j+1)}\sqrt{(j+1)^2-m^2}|j+1,m\rangle, \\
K^+|j,m\rangle =& \alpha_{(j)}\sqrt{(j-m)(j-m-1)}|j-1,m+1\rangle \\
&+ \gamma_{(j)}\sqrt{(j-m)(j+m+1)}|j,m+1\rangle \\
&+ \alpha_{(j+1)}\sqrt{(j+m+1)(j+m+2)}|j+1,m+1\rangle, \\
K^-|j,m\rangle =& -\alpha_{(j)}\sqrt{(j+m)(j+m-1)}|j-1,m-1\rangle \\
&+ \gamma_{(j)}\sqrt{(j+m)(j-m+1)}|j,m-1\rangle \\
&- \alpha_{(j+1)}\sqrt{(j-m+1)(j-m+2)}|j+1,m-1\rangle,
\end{aligned} \tag{3.2.37}
$$

where

$$
\begin{aligned}
\alpha_{(j)} &= \frac{1}{j}\sqrt{\frac{(j^2-(j^++j^--1)^2)(j^2-(j^+-j^-)^2)}{4j^2-1}}, \\
\gamma_{(j)} &= \frac{j^+(j^++1)-j^-(j^-+1)}{j(j+1)}.
\end{aligned} \tag{3.2.38}
$$

Using these actions, as in chapter 2, one can check that the modified $\gamma$-simple relation satisfy the simplicity constraint (3.2.17) and closure constraints (3.2.18). □

The proof is the same as the one in chapter 2, which means the weaker $\gamma$-simple relation can be generalized into arbitrary-valence spinfoams.

### 3.2.3 New Boundary Hilbert Space: Lorentzian Theory

Now we turn to the case of $G = SL(2,\mathbb{C})$. In this case the decomposition of the Hilbert space reads

$$
\begin{aligned}
\mathcal{H}_{\gamma_v} &= \bigotimes_f L^2\Big(SL(2,\mathbb{C}), \mathrm{d}\mu_H\Big) \\
&= \bigotimes_f \bigoplus_{k_f=\mathbb{N}/2} \int_{\mathbb{R}}^{\oplus} \mathrm{d}p_f\,\left(p_f^2+k_f^2\right)\; V_{(k_f,p_f)} \otimes V^*_{(k_f,p_f)}
\end{aligned} \tag{3.2.39}
$$

where $k_f$ are still non-negative half-integers but $p_f \in \mathbb{R}$ is now a real number. Here $\int^{\oplus}$ denotes a direct integral decomposition [73] (see also Chapter 30 of [2]). $V_{(k,p)}$

denotes the unitary irrep of $SL(2,\mathbb{C})$ in the principal series, and $V^*_{(k,p)}$ denotes the adjoint irrep. We can then proceed as in the Euclidean theory. The BF boundary Hilbert space reads

$$\mathcal{H}_{\gamma_v} = \bigoplus_{\{k_f\}} \prod_f \int_{\mathbb{R}}^{\oplus} \mathrm{d}p_f \prod_f \left(p_f^2 + k_f^2\right) \bigotimes_e \bigotimes_{f\in e} V^{(e,f)}_{(k_f,p_f)} \tag{3.2.40}$$

The representation space $V_{(k,p)}$ is infinite-dimensional and can be decomposed into $SU(2)$ irreps (irreps of the subgroup generated by $\vec{L}$), i.e.

$$V_{(k,p)} = \bigoplus_{j=k}^{\infty} V_j^{k,p}. \tag{3.2.41}$$

This time we introduce the two parameters $j$ and $r$ by

$$p = \gamma j\, \frac{j+1}{j-r}, \tag{3.2.42}$$

$$k = j - r. \tag{3.2.43}$$

and we define the new boundary space by restricting each $V_{(k,p)}$ to its $V_j^{k,p}$ subspace satisfying (3.2.42). This time $p$ does not need to be half-integer, therefore (3.2.42) can be solved for any $j$. The new quantum numbers associated to each face are $j_f$ and $r_f$, each being a nonnegative half integer.

As before, we consider the diagonal $SU(2)$ action at each $e$ for all $h_e \in SU(2)$. The invariant subspace under this action is

$$\mathfrak{I}_e^{j_f} = \mathrm{Inv}_{SU(2)}\left[\bigotimes_{f\in e}\left(V^{\frac{\gamma j_f(j_f+1)}{j_f-r_f},\, j_f-r_f}\right)^{(e,f)}\right] \tag{3.2.44}$$

The new boundary Hilbert space is defined by a product of these invariant subspaces over all the polyhedra $e$, followed by a sum over all the possible $j_f$ and $r_f$:

$$\mathcal{H}^L_{\gamma_v} := \bigoplus_{\{r_f,j_f\}} \bigotimes_e \mathfrak{I}_e^{j_f} \tag{3.2.45}$$

where $j_f$ and $k_f$ are non-negative half-integers with constraints (1) $j_f \geq r_f$. $\mathcal{H}^L_{\gamma_v}$ is a direct sum over a set of subspaces contained in the fiber Hilbert spaces of $\mathcal{H}_{\gamma_v}$ (see eq.(3.2.39)), thus has well-defined inner product.

An orthonormal basis is constructed as follows. Consider the oriented boundary graph $\gamma_v$. Given a $F$-valent vertex/polyhedron $e$, we assign it an intertwiner $i_e^{A_1\cdots A_F}$ associated with $F$ spins $j_f$, $f = 1, \cdots, F$

$$i_e \in \mathrm{Inv}\left[\bigotimes_{\overrightarrow{(e,f)}\ \text{outgoing}} V_{j_f} \bigotimes_{\overrightarrow{(e,f)}\ \text{incoming}} V^*_{j_f}\right] \tag{3.2.46}$$

An orthogonal basis in $\mathcal{H}^L_{\gamma_v}$ is given by the following functions (distributions) on $SL(2,\mathbb{C})$

$$
\begin{aligned}
&T^L_{(\gamma_v,j_f,r_f,i_e)}(g_f) = \\
&\prod_e i_e^{A_{e1}\cdots A_{eF}} \prod_{(e,e')} \Pi^{(\frac{\gamma j_f(j_f+1)}{j_f-r_f},j_f-r_f)}_{j_f A_{ef}, j_f A_{e'f}}(g_f)
\end{aligned}
\tag{3.2.47}
$$

here $\Pi^{(p,k)}$ denotes the representation matrix in $SL(2,\mathbb{C})$ irrep labeled by $(p,k)$. All the $A_{ef}$ indices of the representation matrices are contracted with the $A_{ef}$ indices of the intertwiners.

The new boundary Hilbert space $\mathcal{H}^L_{\gamma_v}$ is *not* a subspace of the BF boundary Hilbert space $\mathcal{H}_{\gamma_v}$, because $T^L_{(\gamma_v,j_f,r_f,i_e)}$ are constructed by $\Pi^{(k,p)}$ which are distributions. In order to check the geometric constraints Eqs.(3.2.17) and (3.2.18) on $\mathcal{H}^L_{\gamma_v}$, we have to compute the (dual) action of the bivector operator on the distributions $T^L_{(\gamma_v,j_f,k_f,i_e)}$. Fortunately the Hilbert space $L^2\big(SL(2,\mathbb{C})\big)$ has the structure of direct integral decomposition (see eq.(3.2.39)). Then the (dual) action of the bivector operators $\vec{\hat{K}}$ and $\vec{\hat{L}}$ gives the actions of Lie algebra generators $\vec{L}$ and $\vec{K}$ on each fiber Hilbert space $V_{(k,p)}$.

We are now ready to prove our second main result

**Theorem 3.2.3.** *The Hilbert space $\mathcal{H}^L_{\gamma_v}$ solves the geometric constraint (3.2.17,3.2.18), with $s=-1$.*

**Proof:** Closure constraint follows immediately and strongly by the diagonal $SU(2)$ invariance at each polyhedron $e$. We only need to consider a single irrep $V_{(k,p)}$ ($p=\frac{\gamma j(j+1)}{k}$) because $\vec{L}$ and $\vec{K}$ leave it invariant and, different $(p,k)$'s label orthogonal subspaces in $\mathcal{H}^L_{\gamma_v}$.

A canonical basis in $V_{(p,k)}$ is obtained diagonalizing the Casimir operators $J\cdot J, {}^*J\cdot J, L\cdot L$ and $L^3$. The basis can be denoted $|(p,k);j,m\rangle$ or simply as $|j,m\rangle$ since we only consider a single irrep. On this canonical basis, the generators act in the following way [41]:

$$
\begin{aligned}
L^3|j,m\rangle &= m|j,m\rangle, \\
L^+|j,m\rangle &= \sqrt{(j+m+1)(j-m)}|j,m+1\rangle, \\
L^-|j,m\rangle &= \sqrt{(j+m)(j-m+1)}|j,m-1\rangle, \\
K^3|j,m\rangle &= -\alpha_{(j)}\sqrt{j^2-m^2}|j-1,m\rangle - \beta_{(j)}m|j,m\rangle \\
&\quad +\alpha_{(j+1)}\sqrt{(j+1)^2-m^2}|j+1,m\rangle, \\
K^+|j,m\rangle &= -\alpha_{(j)}\sqrt{(j-m)(j-m-1)}|j-1,m+1\rangle \\
&\quad -\beta_{(j)}\sqrt{(j-m)(j+m+1)}|j,m+1\rangle \\
&\quad -\alpha_{(j+1)}\sqrt{(j+m+1)(j+m+2)}|j+1,m+1\rangle, \\
K^-|j,m\rangle &= \alpha_{(j)}\sqrt{(j+m)(j+m-1)}|j-1,m-1\rangle \\
&\quad -\beta_{(j)}\sqrt{(j+m)(j-m+1)}|j,m-1\rangle \\
&\quad +\alpha_{(j+1)}\sqrt{(j-m+1)(j-m+2)}|j+1,m-1\rangle,
\end{aligned}
$$

where

$$L^\pm = L^1 \pm iL^2, \qquad K^\pm = K^1 \pm iK^2 \tag{3.2.48}$$

and

$$\alpha_{(j)} = \tfrac{i}{j}\sqrt{\tfrac{(j^2-k^2)(j^2+p^2)}{4j^2-1}}, \qquad \beta_{(j)} = \tfrac{kp}{j(j+1)} \tag{3.2.49}$$

Using these equations, one can check directly that

$$\langle j,m'|\big(K^i + \beta_{(j)}L^i\big)|j,m\rangle = 0. \tag{3.2.50}$$

which is nothing but

$$\langle j,m'|\big(K^i + \gamma L^i\big)|j,m\rangle = 0. \tag{3.2.51}$$

because $pk = \gamma j(j+1)$.

□

### 3.2.4 Quantum Polyhedral Geometry

In this section we show that the boundary Hilbert space $\mathcal{H}^E_{\gamma_v}$ and $\mathcal{H}^L_{\gamma_v}$ carries a representation of quantum polyhedral geometry, consistent with the classical polyhedral geometry that we have discussed in Section 3.2.1. Recall that we defined two different bivectors $J^{IJ}_{ef}$ and $\Sigma^{IJ}_{ef}$ related by

$$B^{IJ}_f = \left( {}^*\Sigma_f + \frac{1}{\gamma}\Sigma_f \right)^{IJ}_{ef} \tag{3.2.52}$$

Theorem 3.2.1 states that classically, the geometric constraint of $B^{IJ}_f$ implies that $B^{IJ}_f$ is the area bivector of a face $f$ of a polyhedron $e$. On the BF boundary Hilbert space $\mathcal{H}_{\gamma_v}$ the bivector $B^{IJ}_f$ is quantized to be the left invariant vector field $J^{IJ}_f$. Inverting the above equation, we can write the quantum operator corresponding to $\Sigma$ (which we indicate with the same symbol) as

$$\Sigma^{IJ}_f := \frac{\gamma^2}{\gamma^2 - s} \left( {}^*J^{IJ}_{ef} - \frac{1}{\gamma}\, J^{IJ}_{ef} \right) \tag{3.2.53}$$

Give a polyhedron/vertex $e$ of the boundary, if we choose the unit vector $(n_e)_I = (1,0,0,0)$ [4], then the simplicity constraint implies the vanishing of $\Sigma^{0j}_f$ for each face $f$. That is, the matrix elements of the operators $\Sigma^{0i}_f$ vanish on $\mathcal{H}^E_{\gamma_v}$ and $\mathcal{H}^L_{\gamma_v}$, thus we consider them as vanishing operators on $\mathcal{H}^E_{\gamma_v}$ or $\mathcal{H}^L_{\gamma_v}$. The nontrivial operator on $\mathcal{H}^E_{\gamma_v}$ and $\mathcal{H}^L_{\gamma_v}$ is

$$\Sigma^i_f \equiv \frac{1}{2}\epsilon^i{}_{jk}\Sigma^{jk}_f = \frac{\gamma^2}{\gamma^2 - s} \left( \hat{K}^i_f - \frac{1}{\gamma}\hat{L}^i_f \right) \tag{3.2.54}$$

Because of the quantum simplicity constraint (3.2.17), we can identify $\hat{K}^i_{ef}$ with $s\gamma\vec{L}_{ef}$ on the dense domain of the new boundary Hilbert space, as far as the matrix elements of the operators are concerned. Thus, in the sense of their matrix element

$$\vec{\Sigma}_f = s\gamma\ \vec{L}_f \tag{3.2.55}$$

By the $SU(2)$ gauge invariance, then

$$\sum_{f\in\partial e} \hat{\Sigma}_f = 0 \tag{3.2.56}$$

(with all $f$'s oriented out of $e$.) Consider now a family of coherent states that makes the spread of these operators small. These coherent states are then characterized by expectation values of $\vec{\Sigma}_f$ that satisfy the equation above. By Minkowski theorem,

[4] Although it seems the boundary states depend on the normal vectors to the polyhedra, the partition functions are invariant under local gauge transformations in the bulk (see [74]). On the boundary, there exists a manifestly Lorentz covariant formalism, given by a certain class of "projected spin networks" [43].

they determine a polyhedron $e$ at each vertex. $\vec{\Sigma}_{ef}$ represents the normal to face area of the polyhedron $e$, normalized so that its norm is the area of the face [30]. The area operator for a face $f$ (in units that $8\pi\ell_p^2 = 1$ ) is then

$$\hat{A}_f = \gamma\sqrt{\hat{L}^i_{ef}\hat{L}^i_{ef}} = \gamma\sqrt{j_f(j_f+1)}. \tag{3.2.57}$$

It is clear that the area operator doesn't depend on the orientation of the face. Thus the two areas of the two faces of the two polyhedra $e$ and $e'$ that are determined by the same face $f$ are equal. (Recall that the one of the two is determined by the left invariant vector field $J$ and the other by the right invariant vector field $R$, since $R_f = J_{f^{-1}}$.)

At fixed values of the areas, the shapes of the polyhedra is described by the intertwiner spaces at each $e$. We recall that an over-complete basis in these spaces is formed by the Livine-Speziale coherent intertwiners [17]

$$||\vec{j},\vec{n}\rangle := \int_{SU(2)} \mathrm{d}\mu_H(g) \prod_{f\subset e} D^{j_f}(g)|j_f, n_f\rangle \tag{3.2.58}$$

These can be labeled [75] by the elements in $\times_f S^2/SL(2,\mathbb{C})$. Thinking of $S^2$ as the compacted complex plane of $z_f$, a coherent intertwiner is determined by $F$ quantum area $j_f$ and $F-3$ complex cross-ratios $\vec{Z}$

$$Z_k = \frac{(z_{k+3}-z_1)(z_2-z_3)}{(z_{k+3}-z_3)(z_2-z_1)} \tag{3.2.59}$$

which are invariants of $SL(2,\mathbb{C})$. The space of these cross-ratio $\times_f S^2/SL(2,\mathbb{C})$ can be identified [76] with the Kapovich and Millson phase space $\mathcal{S}_F$ [77], which is also *the space of shapes of polyhedra at fixed areas* $j_f$. Thus, we can label the coherent intertwiner by $||\vec{j},\vec{Z}\rangle$, in variables that relate directly to the shape of the polyhedron. The resolution of identity in the intertwiner space can be expressed as a integral over the Kapovich and Millson phase space $\mathcal{S}_F$, i.e.

$$\mathbf{1}_{\mathcal{I}(\vec{j})} = \int_{\mathcal{S}_F} \mathrm{d}\mu(\vec{Z})\ ||\vec{j},\vec{Z}\rangle\ \langle\vec{j},\vec{Z}|| \tag{3.2.60}$$

where the explicit expression of the measure $\mathrm{d}\mu(\vec{Z})$ is given in [75]. Finally the volume operator for a polyhedron can be defined as in [30], in terms of the classical volume of a polyhedron and the coherent intertwiner.

Notice that the quantum polyhedral geometry doesn't depend on the quantum numbers $r_f$. The quantum numbers $r_f$ don't affect the quantum 3-geometry on the boundary.

## 3.3 Amplitudes

### 3.3.1 Vertex Amplitude: Euclidean theory

If we take BF theory and restrict all vertex-boundary spaces to $\mathcal{H}^E_{\gamma_v}$ (or $\mathcal{H}^L_{\gamma_v}$) we obtain a new dynamical model. Here we give explicitly its vertex and face amplitude. Let's start with the Euclidean case. The BF vertex amplitude can be written in the holonomy representation: (each edge joining at $v$ is uniquely determined by a vertex/polyhedron $e$ on the boundary) reads

- $0 < \gamma < 1$

$$\begin{aligned} A^<_v(g_f) &= \sum_{j^\pm_f, i^\pm_e} \prod_f \sqrt{[j_f + r_f + \frac{\gamma j_f(j_f+1)}{j_f + r_f + 1} + 1][j + r - \frac{\gamma j(j+1)}{j+r+1} + 1]} \\ &\quad A_v(j^+_f, j^-_f; i^+_e, i^-_e) T^{BF}_{\gamma_v, j^\pm_f, i^\pm_e}(g_f) \end{aligned} \tag{3.3.1}$$

- $\gamma > 1$

$$\begin{aligned} A^>_v(g_f) &= \sum_{j^\pm_f, i^\pm_e} \prod_f \sqrt{[\frac{\gamma j_f(j_f+1)}{j_f + r_f + 1} + j_f + r_f + 1][\frac{\gamma j(j+1)}{j+r+1} - (j_f + r_f) + 1]} \\ &\quad A_v(j^+_f, j^-_f; i^+_e, i^-_e) T^{BF}_{\gamma_v, j^\pm_f, i^\pm_e}(g_f) \end{aligned} \tag{3.3.2}$$

Here

$$A_v(j^+_f, j^-_f; i^+_e, i^-_e) = \mathrm{Tr}\left(\bigotimes_{e\in v} I^\dagger_e\right) \tag{3.3.3}$$

where $I = (i^+, i^-)$ and we assume the valence of $v$ is $n$. $T^{BF}_{(\gamma_v, j^\pm_f, i^\pm_e)} \in \mathcal{H}_{\gamma_v}$ is a $Spin(4)$ spin-network function on the boundary graph $\gamma_v$

$$T^{BF}_{\gamma_v, j^\pm_f, i^\pm_e}(g_f) := T_{\gamma_v, j^+_f, i^+_e}(g^+_f) T_{\gamma_v, j^-_f, i^-_e}(g^-_f) \tag{3.3.4}$$

where

$$\begin{aligned} T_{\gamma_v, j_f, i_e}(g_f) &= \prod_f \sqrt{2j_f+1} \prod_e \left[(i_e)^{\{m_{ef}\}}\right] \\ &\quad \prod_{(e,f)} \left[D^{j_f}_{m_{ef} n_{ef}}(g_{ef})\right] \bigotimes_f \left[\epsilon^{n_{ef} n_{e'f}}\right] \end{aligned} \tag{3.3.5}$$

The vertex amplitude eq.(3.3.2) is a distribution of the boundary Hilbert space $\mathcal{H}_{\gamma_v}$, i.e. there is a dense domain of $\mathcal{H}_{\gamma_v}$ spanned by the spin-network functions $T^{BF}_{(\gamma_v, j_f^\pm, i_e^\pm)}$, such that $A_v(g_{ee'})$ lives in the algebraic dual of this dense domain. After imposing the geometric constraint, we restrict ourself to the subspace $\mathcal{H}^E_{\gamma_v}$. Such a restriction results in a (dual) projection of the vertex amplitude $A_v$, i.e. we obtain

$$A_v^E(g_f) = \sum_{j_f, r_f, i_e} \left\langle T_{\gamma_v, j_f, r_f, i_e}, A_v \right\rangle T^E_{\gamma_v, j_f, r_f, i_e}(g_f) \tag{3.3.6}$$

where $T_{\gamma_v, j_f, r_f, i_e}$ is a orthonormal basis of $\mathcal{H}^E_{\gamma_v}$ (recall eq.(3.2.31)), and $\langle\ ,\ \rangle$ is the inner product of the BF boundary Hilbert space $\mathcal{H}_{\gamma_v}$. The evaluation of $A_v^E$ is straightforward:

$$\begin{aligned} A_v^E(g_f) = \sum_{j_f, r_f, i_e} \prod_f \sqrt{2j_f^+ + 1}\sqrt{2j_f^- + 1} \\ \sum_{i_e^+, i_e^-} A_v\left(j_f^+,\ j_f^-;\ i_e^+,\ i_e^-\right) \prod_e f^{i_e}_{i_e^+, i_e^-}\ T^E_{\gamma_v, j_f, r_f, i_e}(g_{ee'}) \end{aligned} \tag{3.3.7}$$

where we $j^+, j^-$ depend on $(j, r)$ as equation (3.2.24) and for each $F$-valent boundary polyhedron/vertex

$$\begin{aligned} f^{i_e}_{i_e^+, i_e^-} = \overline{i_e^{A_{e1}\cdots A_{eF}}} C^{m_{e1}^+ m_{e1}^-}_{A_{e1}} \cdots C^{m_{eF}^+ m_{eF}^-}_{A_{eF}} \\ (i_e^+)_{m_{e1}^+ \cdots m_{eF}^+} (i_e^-)_{m_{e1}^- \cdots m_{eF}^-} \end{aligned} \tag{3.3.8}$$

Then in the $(j_f, r_f, i_e)$-spin-network representation, the vertex amplitude is

$$A_v^E(j_f, r_f, i_e) = \sum_{i_e^+, i_e^-} A_v\left(j_f^+, j_f^-;\ i_e^+,\ i_e^-\right) \prod_e f^{i_e}_{i_e^+, i_e^-} \tag{3.3.9}$$

which nontrivially depends on the quantum numbers $r_f$ via the definition of $j_f^-$.

There is another way to write this vertex amplitude in $(j_f, r_f, i_e)$-spin-network representation. Define a map $I_E^{\{r_f\}}$ from $SU(2)$ intertwiners to $Spin(4)$ intertwiners, depending on the quantum numbers $r_f$. Given an $F$-valent $SU(2)$ intertwiner $i_e$ with spins $k_1, \cdots, k_F$, let

- $0 < \gamma < 1$

$$\begin{aligned} I^{r_f}_{E<} :\ i_e \mapsto I^{r_f}_{E<}(i_e) = i_e^{A_{e1}\cdots A_{eF}} C^{n_{e1}^+ n_{e1}^-}_{A_{e1}} \cdots C^{n_{eF}^+ n_{eF}^-}_{A_{eF}} \\ \int \mathrm{d}g^+ \mathrm{d}g^- \prod_{f \in e} D^{\frac{j+r}{2} + \frac{\gamma j(j+1)}{2(j+r+1)}}_{m_{ef}^+ n_{ef}^+}(g^+)\ D^{\frac{j+r}{2} - \frac{\gamma j(j+1)}{2(j+r+1)}}_{m_{ef}^- n_{ef}^-}(g^-) \end{aligned} \tag{3.3.10}$$

- $\gamma > 1$

$$I_{E>}^{r_f}\,:\; i_e \mapsto I_{E>}^{r_f}(i_e) = i_e^{A_{e1}\cdots A_{eF}} C_{A_{e1}}^{n_{e1}^+ n_{e1}^-} \cdots C_{A_{eF}}^{n_{eF}^+ n_{eF}^-}$$
$$\int \mathrm{d}g^+ \mathrm{d}g^- \prod_{f\in e} D_{m_{ef}^+ n_{ef}^+}^{\frac{\gamma j(j+1)}{2(j-r)}+\frac{j-r}{2}}(g^+)\, D_{m_{ef}^- n_{ef}^-}^{\frac{\gamma j(j+1)}{2(j-r)}-\frac{j-r}{2}}(g^-) \qquad (3.3.11)$$

Given an edge $e \in E(\mathcal{K})$, we associate an intertwiner $I_E^{\{r_f\}}(i_e)$ to the initial point of the edge $e$, and a dual intertwiner $I_E^{\{r_f\}}(i_e)^\dagger$ to the final point of $e$. Then the vertex amplitude $A_v^E$ can be written a spinfoam trace of the intertwiners $I_E^{\{r_f\}}(i_e)$

$$A_v^E(k_f, r_f, i_e) = \mathrm{Tr}\left(\bigotimes_{e\in v} I_E^{\{r_f\}}(i_e)^\dagger\right) \qquad (3.3.12)$$

where we have again assumed that all the edges joining at $v$ are oriented towards $v$.

### 3.3.2 Vertex Amplitude: Lorentzian theory

The Lorentzian vertex amplitude can be defined in the same manner. The $SL(2,\mathbb{C})$ BF vertex amplitude is expressed in the holonomy representation as a distribution

$$\begin{aligned} A_v(g_f) &= \sum_{k_f, I_e} \int \prod_f \mathrm{d}p_f \prod_f \left(k_f^2 + p_f^2\right) \\ &\qquad A_v\Big(p_f, k_f; I_e\Big)\; T^{BF}_{\gamma_v,(k,p)_f,(\mathbf{l},\mathbf{n})_e}(g_f) \end{aligned} \qquad (3.3.13)$$

where

$$A_v\Big(p_f, k_f; I_e\Big) = \mathrm{Tr}\left(\bigotimes_e I_e^\dagger\right) \qquad (3.3.14)$$

and

$$T^{BF}_{\gamma_v,p_f,k_f,I_e}(g_f) = \prod_e I_{\{j_{ef}\},\{m_{ef}\};I_e} \prod_f \Pi^{p_f,k_f}_{j_{ef}m_{ef},j_{e'f}m_{e'f}}(g_f)$$

Recall that we always assume the vertex amplitude is associated with an integrable spin-network graph, thus is finite after regularization [66, 67].

We can project $A_v$ on the new boundary Hilbert space $\mathcal{H}^L_{\gamma_v}$, in the same way as the Euclidean case

$$\begin{aligned} A_v^L(g_f) &= \sum_{j_f, r_f, i_e} \prod_f \left(\frac{\gamma^2 j_f^2 (j_f+1)^2}{(j_f - r_f)^2} + (j_f - r_f)^2\right) \\ &\qquad \Big\langle T^L_{\gamma_v, j_f, r_f, i_e}\;,\; A_v\Big\rangle\; T^L_{\gamma_v, j_f, r_f, i_e}(g_{ee'}) \end{aligned} \qquad (3.3.15)$$

where $\langle\ ,\ \rangle$ is the inner product on the BF boundary Hilbert space. The states

$$T^L_{\gamma_v,j_f,r_f,i_e}(g_f) = \prod_e i_e^{A_{e1}\cdots A_{eF}} \prod_{(e,e')} \Pi^{\frac{\gamma j_f(j_f+1)}{j_f-r_f},j_f-r_f}_{j_f A_{ef}, j_f A_{e'f}}(g_f)$$

form an orthogonal basis in $\mathcal{H}^L_{\gamma_v}$. By using the orthogonality relation

$$\int_{SL(2,\mathbb{C})} \mathrm{d}g\ \overline{\Pi^{(p,k)}_{jm,ln}(g)}\ \Pi^{(p',k')}_{j'm',l'n'}(g) = \frac{1}{k^2+p^2}\delta^{kk'}\delta(p-p')\delta_{jj'}\delta_{ll'}\delta_{mm'}\delta_{nn'} \tag{3.3.16}$$

it is straightforward to show that in the $(j_f, r_f, i_e)$-spin-network representation, the resulting vertex amplitude reads

$$\begin{aligned} A^L_v\left(j_f,k_f,i_e\right) &= \left\langle T^L_{(\gamma_v,j_f,k_f,i_e)}\ ,\ A_v\right\rangle \\ &= \sum_{I_e} A_v\Big(\big(\frac{\gamma j_f(j_f+1)}{j_f-r_f}, j_f-r_f\big); I_e\Big) \prod_e f^{i_e}_{I_e} \end{aligned} \tag{3.3.17}$$

where

$$f^{i_e}_{I_e} := \overline{i_e^{\{A_{ef}\}}}\ I_{\{j_f\},\{A_{ef}\}I_e}\left(\frac{\gamma j_f(j_f+1)}{j_f-r_f}, j_f-r_f\right) \tag{3.3.18}$$

As expected, the vertex amplitude $A^L_v$ obtained in this manner is divergent, and we need a regularization procedure. To this aim, rewrite the vertex amplitude in terms of spinfoam trace as we did for the Euclidean theory. We define a formal map $I^{r_f}_L$ from $SU(2)$ intertwiners into $SL(2,\mathbb{C})$ intertwiners, depending on the quantum numbers $r_f$

$$I^{r_f}_L(i_e)_{\{j'_f\},\{A'_f\}} = \int \mathrm{d}g\ \prod_{f\subset e} \Pi^{(\frac{\gamma j_f(j_f+1)}{j_f-r_f},j_f-r_f)}_{j'_f A'_{ef}, j_f A_{ef}}(g)\cdot i_e^{\{A_{ef}\}}$$

which gives $A^L_v$ by a spinfoam trace

$$A^L_v(j_f,r_f,i_e) = \mathrm{Tr}\left(\bigotimes_{f\in e} I^{\{r_f\}}_L(i_{e_f})^\dagger\right) \tag{3.3.19}$$

To regularize the vertex amplitude $A^L_v$ it is sufficient to removing one of the $dg$ integration (which is redundant) at each vertex. With this, the vertex amplitude $A^L_v$ is finite.

### 3.3.3 Face Amplitude and Partition Function

It is argued in [19] that the face amplitude of a spinfoam model is determined by three inputs: (a) the choice of the boundary Hilbert space, (b) the requirement that the composition law holds when gluing two complexes $\mathcal{K}$ and $\mathcal{K}'$, (c) a particular locality requirement (see [19] for the details of the three assumptions). These requirements are implemented if the partition function has the form (3.1.22). By inserting the vertex amplitudes that we have defined into this expression, we complete the definition of an Euclidean and a Lorentzian model.

Expanding the delta function in representation, we obtain

$$Z_{E,L}(\mathcal{K}) = \sum_{j_f, r_f, i_e} \prod_f d^{E,L}(j_f, r_f) \prod_v A_v^{E,L}(j_f, r_f, i_e)$$

where the Euclidean face amplitude is

- $0 < \gamma < 1$

$$d^{E<}(j_f, r_f) = \left[j + r + \frac{\gamma j(j+1)}{j+r+1} + 1\right]\left[j + r - \frac{\gamma j(j+1)}{j+r+1} + 1\right] \quad (3.3.20)$$

- $\gamma > 1$

$$d^{E>}(j_f, r_f) = \left[\frac{\gamma j(j+1)}{j-r} + (j-r) + 1\right]\left[\frac{\gamma j(j+1)}{j-r} - (j-r) + 1\right] \quad (3.3.21)$$

the Lorentzian one is

$$d^L(j_f, r_f) = \frac{\mathfrak{g}^2 j_f^2 (j_f+1)^2}{(j_f - r_f)^2} + (j_f - r_f)^2. \quad (3.3.22)$$

where the dimension factors $A_f^{E<} := \left[j+r+\frac{\gamma j(j+1)}{j+r+1}+1\right]\left[j+r-\frac{\gamma j(j+1)}{j+r+1}+1\right]$, $A_f^{E>} := \left[\frac{\gamma j(j+1)}{j-r}+(j-r)+1\right]\left[\frac{\gamma j(j+1)}{j-r}-(j-r)+1\right]$ and $A_f^L := \left[k_f^2 + \mathfrak{g}^2 j_f^2 (j_f+1)^2/k_f^2\right]$ are the face amplitudes for the Euclidean and Lorentzian theories. In the Euclidean case, the face amplitudes is different from the one obtained in [19] and coincide with the ones deduced from the BF partition function. In [19] the face amplitude obtained is the dimension of $SU(2)$ unitary irrep i.e. $2j_f + 1$. The origin of the difference is the difference in the boundary Hilbert space. The one here, $\mathcal{H}^E_{\mathfrak{g}_v}$ or $\mathcal{H}^L_{\mathfrak{g}_v}$, has additional degree of freedom with respect to the space $L^2(SU(2)^L)$ of [19].

## 3.4 Summary

By imposing the simplicity constraints on a quantum BF theory defined on an arbitrary cellular complex, we have obtained a theory which: (1) is well defined both in the Euclidean and the Lorentzian context; (2) generalizes the existing spinfoam model to general 2-cell complexes, along the lines suggested by [63]; (3) has boundary state that have a natural interpretation in the semiclassical limit as a polyhedral geometry on the boundary. In particular, we have shown that the KKL extension of the spinfoam formalism still satisfies the simplicity conditions weakly.

The weak simplicity constraint allow a space larger than the one of LQG to emerge. The physical interpretation of the additional degree of freedom is unclear. It can be eliminated by imposing the non-commuting simplicity constraints weakly and the diagonal one strongly.

# Chapter 4

# Correlation functions of Lorentzian spinfoam model

Spinfoam amplitudes provide a covariant definition of the dynamics of Loop Quantum Gravity. A basic test the amplitude has to pass is that, in the semiclassical limit, it reproduces the classical action for gravity. Recently, the Lorentzian vertex amplitude has been shown to correctly determine the Regge action for discrete gravity restricted to a 4-simplex. Given this result, the next test for the theory regards the behavior of small quantum fluctuations around a classical solution.

In this chapter [1] we compute two-point correlation functions for the Penrose metric operator. The setting is the one introduced in [78] and developed in [62, 79–85]. In particular, we restrict to a single spinfoam vertex and compute correlations on a semiclassical state peaked on the spacelike boundary geometry of a Lorentzian 4-simplex.

Our main result is the following. We consider the limit, introduced in [86], where the Barbero-Immirzi parameter is taken to zero $\gamma \to 0$, and the spin of the boundary state is taken to infinity $j \to \infty$, keeping the size of the quantum geometry $A \sim \gamma j$ finite and fixed. This limit corresponds to neglecting Planck scale discreteness and twisting effects, at large finite distances. In this limit, the two-point function we obtain exactly matches the one obtained from Lorentzian Regge calculus [42]. We therefore extend to Lorentzian signature the results of [86].

[1] This chapter is based on work done together with Eugenio Bianchi.

## 4.1 The EPRL amplitude in Lorentzian theory

In this section, we give a brief introduction to the $SL(2,\mathbb{C})$ EPRL amplitude of a coherent spin network. Throughout this thesis, $SL(2,\mathbb{C})$ refers to the 6-dimensional real Lie group of $2\times 2$ complex matrices with unit determinant, and is called simply the Lorentz group. It covers the group of proper orthochronous Lorentz transformations, $SO^+(3,1)$, which is the component of the group $O(3,1)$ connected to the identity.

The principal series of irreducible unitary representations of the Lorentz group $SL(2,\mathbb{C})$ are labeled by two parameters $(k,p)$, with $k$ an integer and $p$ a real number [41]. Given a carrier space $\mathcal{H}_{(k,p)}$, the canonical basis is given by the basis diagonalizing simultaneously the Casimir operators, which is denoted as $|(k,p);j,m\rangle$.

The $SL(2,\mathbb{C})$ EPRL amplitude of a single 4-simplex for a boundary coherent spin network state $|j,\Phi(\vec{n})\rangle$ reads

$$\langle W|j,\Phi(\vec{n})\rangle = \int\limits_{SL(2,\mathbb{C})^5} \prod_a \mathrm{d}g_a \prod_{(ab)} P_{ab}(g), \tag{4.1.1}$$

with

$$P_{ab}(g) = \langle j_{ab}, -\vec{n}_{ab}(\xi)|\, Y^\dagger g_a^{-1} g_b Y|j_{ab}, \vec{n}_{ba}(\xi)\rangle. \tag{4.1.2}$$

Notation is as follows. The indices $a,b=1,...,5$ label the tetrahedra on the boundary of the 4-simplex and $(ab)$ labels the triangles between the corresponding tetrahedra; the integral is over one group element of $SL(2,\mathbb{C})$ per each tetrahedron. We restrict ourselves to the spacelike tetrahedra. We use $g$ to denote the group elements, as well as the corresponding representations. The EPRL embedding map $Y$ embeds the spin-$j$ irreducible representation $\mathcal{H}_j$ of $SU(2)$ to the irreducible unitary representation $\mathcal{H}_{(k,p)}$ of $SL(2,\mathbb{C})$, given by

$$Y|j,m\rangle = |(j,\gamma j);j,m\rangle. \tag{4.1.3}$$

The notation $|j,\vec{n}(\xi)\rangle$ denotes an $SU(2)$ coherent state [18, 87] in the spin-$j$ representation on the boundary, labeled by the spin-$j$ and a unit vector $\vec{n}(\xi)$ defining a direction on the sphere $S^2$, associated to a normalized spinor $\xi\in\mathbb{C}^2$. In fact, one can obtain these coherent states starting from the maximal weight vectors $|j,j\rangle$, when $m=j$ on the basis states $|j,m\rangle$, which minimize the ($SU(2)$ invariant) uncertainty $\Delta \equiv |\langle \vec{J}^2\rangle - \langle\vec{J}\rangle^2|$ in the direction of $J_z$. Starting from the highest weight, an infinite set of coherent states on the sphere $SU(2)/U(1) \sim S^2$ are constructed through the group action, $|j,\vec{n}\rangle = n|j,j\rangle$, where $\vec{n}$ is a unit vector defining a direction on the sphere $S^2$ and $n$ an $SU(2)$ group element rotating the direction $\hat{z} \equiv (0,0,1)$ into the direction $\vec{n}$. Just as $|j,j\rangle$ has direction $z$ with minimal uncertainty, $|j,\vec{n}\rangle$ has direction $\vec{n}$ with minimal uncertainty. One can go further to use a normalized spinor $\xi \in \mathbb{C}^2$ to label an $SU(2)$ group element

$$n(\xi) = \begin{pmatrix} \xi_0 & -\bar{\xi}_1 \\ \xi_1 & \bar{\xi}_0 \end{pmatrix}$$

and the corresponding vector $\vec{n}(\xi)$. The antipodal vector $-\vec{n}(\xi)$ is associated to $J\xi$, i.e. $-\vec{n}(\xi) = \vec{n}(J\xi)$, with

$$J\begin{pmatrix} \xi_0 \\ \xi_1 \end{pmatrix} = \begin{pmatrix} -\bar{\xi}_1 \\ \bar{\xi}_0 \end{pmatrix}. \tag{4.1.4}$$

There is simplicial interpretation of these states: the vectors $\vec{n}$ are associated to unit-normals to triangles of a tetrahedron, and $j$ areas of the triangles. We assume all the normals outward to the tetrahedron, which satisfy

$$j_1\vec{n}_1 + j_2\vec{n}_2 + j_3\vec{n}_3 + j_4\vec{n}_4 = 0, \tag{4.1.5}$$

thus we associate to each triangle $(ab)$ normals $-\vec{n}_{ab}$ and $\vec{n}_{ba}$ when $a$ is target of the triangle and respectively $b$ is the source. That is why we have a minus sign in the bra coherent state in the definition of $P_{ab}$ in equation (4.1.2). The coherent spin network state $|j_{aB}, \Phi_a(\vec{n})\rangle$ of a semiclassical tetrahedron $a$ is given by

$$|j_{aB}, \Phi_a(\vec{n})\rangle = \int_{SU(2)} \mathrm{d}h \bigotimes_{b\neq a} h|j_{ab}, \vec{n}_{ab}\rangle \tag{4.1.6}$$

up to a normalization, where $j_{aB}$ denotes the collection of all the $j_{ab}$'s for the given tetrahedron $a$ and $b \neq a$. And the boundary coherent spin network $|j_{AB}, \Phi_A(\vec{n})\rangle$ is given by the tensor product of coherent spin network states the form (4.1.6), one for each tetrahedron on the boundary:

$$|j_{AB}, \Phi_A(\vec{n})\rangle = \bigotimes_{a=1}^{5} |j_{aB}, \Phi_a(\vec{n})\rangle = \bigotimes_{a=1}^{5} \int_{SU(2)} \mathrm{d}h \bigotimes_{b\neq a} h|j_{ab}, \vec{n}_{ab}\rangle, \tag{4.1.7}$$

where $j_{AB}, \Phi_A$ denote all the collections of $j_{ab}$'s and $\Phi_a$'s, when $a, b$ runs over $1,2,3,4,5$. Note that we use the small letter $a = 1,2,3,4,5$ to label the coherent spin network state $|j_{aB}, \Phi_a(\vec{n})\rangle$ of a semiclassical tetrahedron $a$, and the capital letter $A = \{1,2,3,4,5\}$ to label the boundary coherent spin network $|j_{AB}, \Phi_A(\vec{n})\rangle$, the tensor product of the five coherent spin network states on the boundary. For simplicity, we will omit the subscripts, and denote the boundary coherent spin network $|j_{AB}, \Phi_A(\vec{n})\rangle$ as $|j, \Phi(\vec{n})\rangle$ if there is no confusion arising. And the Hermitian inner product is $\langle z, w\rangle = \bar{z}_o w_o + \bar{z}_1 w_1$.

To see this amplitude (4.1.1) explicitly, one can turn to a representation of the Lorentz group $SL(2,\mathbb{C})$ on the space $\mathcal{H}_{(k,p)}$ of homogeneous functions of the *complex affine plane* $\mathbb{C}^2 - \{0,0\}$,

$$f(a\mathbf{z})^{(k,p)} = a^{-1+ip+k}\bar{a}^{-1+ip-k} f(\mathbf{z})^{(k,p)}, \forall a \in \mathbb{C} - \{0\}, \tag{4.1.8}$$

with the group transformation

$$g: \ f(\mathbf{z}) \mapsto f(g^T\mathbf{z}). \tag{4.1.9}$$

The canonical basis is denoted as $f^j_m(\mathbf{z})^{(k,p)}$. The inner product is given by

$$(f,g) = \int \mathrm{d}\mathbf{z}\, \bar{f} g \tag{4.1.10}$$

with $\mathrm{d}\mathbf{z} \equiv \frac{i}{2}(z_0 \mathrm{d}z_1 - z_1 \mathrm{d}z_0) \wedge (\bar{z}_0 \mathrm{d}\bar{z}_1 - \bar{z}_1 \mathrm{d}\bar{z}_0)$. This integral is invariant under the scaling $\mathbf{z} \to a\mathbf{z}$, according to the homogeneity (4.1.8). To modulo this equivalence relation, one can choose $\varphi(z) = f(z,1)$ associated with each $f(z_o, z_1) \in \mathcal{H}_{(k,p)}$; the functions $\varphi(z)$ forms a realization of $\mathcal{H}_{(k,p)}$, which we can still call $\mathcal{H}_{(k,p)}$. Functions $\varphi(z)$ can be considered as the homogeneous functions on the *complex projective line* $\mathcal{P}$, the subspace of the complex affine plane, modulo the equivalence relation $a\mathbf{z} = \mathbf{z}$. For calculating simplicity, we will still keep the formulae of $f(\mathbf{z})$ on the complex affine plane in the following and reduce to $\varphi(z) \in \mathcal{P}$ when necessary.

In this representationthe of homogeneous functions, $SL(2,\mathbb{C})$ coherent state $|(k,p); k, \vec{n}(\xi)\rangle$ with lowest spin $k$ can be written as [57]

$$f^k_\xi(z)^{(k,p)} = \sqrt{\frac{d_k}{\pi}}\, \langle z, z\rangle^{ip-1-k}\, \langle \bar{z}, \xi\rangle^{2k}. \tag{4.1.11}$$

And hence equation (4.1.2) can be rewritten [88] as

$$
\begin{aligned}
P_{ab} =&\langle j_{ab}, -\vec{n}_{ab}(\xi)|\, Y^{\dagger}\, g_a^{-1} g_b\, Y | j_{ab}, \vec{n}_{ba}(\xi)\rangle \\
=&\langle (j_{ab}, \gamma j_{ab}); j_{ab}, -\vec{n}_{ab}(\xi)|\, g_a^{-1} g_b\, |(j_{ab}, \gamma j_{ab}); j_{ab}, \vec{n}_{ba}(\xi)\rangle \\
=&\int_{\mathbb{CP}^1} \mathrm{d}\mathbf{z}\; \overline{g_a\, f^{j_{ab}}_{J\xi_{ab}}(z)^{(j_{ab},\gamma j_{ab})}}\; g_b\, f^{j_{ab}}_{\xi_{ba}}(z)^{(j_{ab},\gamma j_{ab})} \\
=&\int_{\mathbb{CP}^1} \mathrm{d}\mathbf{z}\; \overline{f^{j_{ab}}_{J\xi_{ab}}(g_a^T z)^{(j_{ab},\gamma j_{ab})}}\; f^{j_{ab}}_{\xi_{ba}}(g_b^T z)^{(j_{ab},\gamma j_{ab})} \\
=&\frac{d_{j_{ab}}}{\pi}\int_{\mathbb{CP}^1} \mathrm{d}\mathbf{z}\; \langle g_a^{\dagger}\bar{z}, g_a^{\dagger}\bar{z}\rangle^{-1-(1+i\gamma)j_{ab}} \langle J\xi_{ab}, g_a^{\dagger}\bar{z}\rangle^{2j_{ab}} \langle g_b^{\dagger}\bar{z}, g_b^{\dagger}\bar{z}\rangle^{-1-(1-i\gamma)j_{ab}} \langle g_b^{\dagger}\bar{z}, \xi_{ba},\rangle^{2j_{ab}} \\
=&-\frac{d_{j_{ab}}}{\pi}\int_{\mathbb{CP}^1} \mathrm{d}\mathbf{z}\; \langle g_a^{\dagger}z, g_a^{\dagger}z\rangle^{-1-(1+i\gamma)j_{ab}} \langle J\xi_{ab}, g_a^{\dagger}z\rangle^{2j_{ab}} \langle g_b^{\dagger}z, g_b^{\dagger}z\rangle^{-1-(1-i\gamma)j_{ab}} \langle g_b^{\dagger}z, \xi_{ba},\rangle^{2j_{ab}} \\
=&\frac{d_{j_{ab}}}{\pi}\int_{\mathbb{CP}^1} \mathrm{d}\tilde{\mathbf{z}}_{ab}\; \left(\frac{\langle Z_{ba}, Z_{ba}\rangle}{\langle Z_{ab}, Z_{ab}\rangle}\right)^{i\gamma j_{ab}} \left(\frac{\langle J\xi_{ab}, Z_{ab}\rangle^2 \langle Z_{ba}, \xi_{ba}\rangle^2}{\langle Z_{ab}, Z_{ab}\rangle\langle Z_{ba}, Z_{ba}\rangle}\right)^{j_{ab}}, \qquad (4.1.12)
\end{aligned}
$$

where $\mathrm{d}\tilde{\mathbf{z}}_{ab} \equiv -(\langle Z_{ab}, Z_{ab}\rangle\langle Z_{ba}, Z_{ba}\rangle)^{-1}\mathrm{d}\mathbf{z}$, $Z_{ab} \equiv g_a^{\dagger}z$ and $Z_{ba} \equiv g_b^{\dagger}z$; $\xi_{ba}$ and $J\xi_{ab}$ are spinors associated respectively with $\vec{n}_{ba}(\xi)$ and $-\vec{n}_{ab}(\xi)$, as introduced in equation (4.1.4); note that $g$ is used to denote the group elements, as well as the corresponding unitary representations; the property of unitary representation is considered in the 3rd step ,equation (4.1.11) used in the 5th step and in the 6th step, the integral variable $z$ is changed into its complex conjugate $\bar{z}$, where the minus sign comes from the integral measure.

Let

$$
K_{ab}(g, \mathbf{z}) = \left(\frac{\langle Z_{ba}, Z_{ba}\rangle}{\langle Z_{ab}, Z_{ab}\rangle}\right)^{i\gamma j_{ab}} \left(\frac{\langle J\xi_{ab}, Z_{ab}\rangle^2 \langle Z_{ba}, \xi_{ba}\rangle^2}{\langle Z_{ab}, Z_{ab}\rangle\langle Z_{ba}, Z_{ba}\rangle}\right)^{j_{ab}}, \qquad (4.1.13)
$$

$P_{ab}$ in equation (4.1.12) is simply expressed as

$$
P_{ab} = \int \frac{d_{j_{ab}}}{\pi} \mathrm{d}\tilde{\mathbf{z}}_{ab}\, K_{ab}. \qquad (4.1.14)
$$

Thus the EPRL amplitude (4.1.1) can be written as

$$
\langle W | j, \Phi(\vec{n})\rangle = \int \prod_{a=1}^{5} \mathrm{d}g_a \int \left(\prod_{a<b} \frac{d_{j_{ab}}}{\pi}\mathrm{d}\tilde{\mathbf{z}}_{ab}\right) e^{S_o}, \qquad (4.1.15)
$$

where the "action" $S_o$ is given by

$$
S_o(g, \mathbf{z}) = \sum_{a<b} j_{ab} \log \frac{\langle J\xi_{ab}, Z_{ab}\rangle^2 \langle Z_{ba}, \xi_{ba}\rangle^2}{\langle Z_{ab}, Z_{ab}\rangle\langle Z_{ba}, Z_{ba}\rangle} + i\gamma j_{ab} \log \frac{\langle Z_{ba}, Z_{ba}\rangle}{\langle Z_{ab}, Z_{ab}\rangle}. \qquad (4.1.16)
$$

This expression, however, is ill defined, due to the fact that the integral may diverge. This issue is addressed and answered in [66, 67], where it is shown that the source of the divergence is a redundant integral over $SL(2,\mathbb{C})$ in the vertex amplitude (4.1.15). It is then immediate to regularize the vertex amplitude by removing one $SL(2,\mathbb{C})$ integration. The resulting amplitude with an integral over $\prod_{a=1}^{4} \mathrm{d}g_a$ is proven in [66, 67] to be finite.

## 4.2 Lorentzian two-point function and its integral formula

Following [86], the connected two-point correlation function $G^{abcd}_{nm}$ on a semiclassical boundary state $|\Psi_o\rangle$ is defined as

$$G^{abcd}_{nm} = \langle E^a_n \cdot E^b_n \; E^c_m \cdot E^d_m \rangle - \langle E^a_n \cdot E^b_n \rangle \, \langle E^c_m \cdot E^d_m \rangle \; , \tag{4.2.1}$$

where $(E^a_n)_i$ is a flux operator through a surface $f_{an}$ dual to the triangle between the tetrahedra $a$ and $n$, parallel transported in the tetrahedron $n$. Here the dynamical expectation value of an operator $\mathcal{O}$ on the state $|\Psi_o\rangle$ is defined via

$$\langle \mathcal{O} \rangle = \frac{\langle W | \mathcal{O} | \Psi_o \rangle}{\langle W | \Psi_o \rangle} \; . \tag{4.2.2}$$

The semiclassical boundary state in the literature [18, 36, 57, 89] is given by a superposition of coherent spin networks:

$$|\Psi_o\rangle = \sum_{j_{ab}} \psi_{j_o,\phi_o}(j) |j, \Phi(\vec{n})\rangle \; , \tag{4.2.3}$$

with coefficients $\psi_{j_o,\phi_o}(j)$ given by a gaussian times a phase,

$$\begin{aligned} \psi_{j_o,\phi_o}(j) = &\exp\Big( - \sum_{ab,cd} \alpha^{(ab)(cd)} \, \frac{j_{ab} - (j_o)_{ab}}{\sqrt{(j_o)_{ab}}} \, \frac{j_{cd} - (j_o)_{cd}}{\sqrt{(j_o)_{cd}}} \Big) \times \\ &\times \exp\Big( -i \sum_{ab} \phi_o^{ab} \left( j_{ab} - (j_o)_{ab} \right) \Big) \; . \end{aligned} \tag{4.2.4}$$

where $\phi_o$ labels the simplicial extrinsic curvature, which is an angle associated to the triangle shared by the tetrahedr; the $10 \times 10$ matrix $\alpha^{(ab)(cd)}$ is assumed to be complex with positive definite real part.

Thus the two-point function (4.2.1) can be also written as a superposition:

$$\begin{aligned} G^{abcd}_{nm} =& \frac{\sum_j \psi_j \langle W|E^a_n\cdot E^b_n\; E^c_m\cdot E^d_m|j,\Phi(\vec{n})\rangle}{\sum_j \psi_j \langle W|j,\Phi(\vec{n})\rangle} \\ & - \quad \frac{\sum_j \psi_j \langle W|E^a_n\cdot E^b_n|j,\Phi(\vec{n})\rangle}{\sum_j \psi_j \langle W|j,\Phi(\vec{n})\rangle}\; \frac{\sum_j \psi_j \langle W|E^c_m\cdot E^d_m|j,\Phi(\vec{n})\rangle}{\sum_j \psi_j \langle W|j,\Phi(\vec{n})\rangle}. \end{aligned} \tag{4.2.5}$$

To see this explicitly, let us go first to derive integral expressions for $\langle W|E^a_n \cdot E^b_n|j,\Phi(\vec{n})\rangle$ and $\langle W|E^a_n \cdot E^b_n\; E^c_m \cdot E^d_m|j,\Phi(\vec{n})\rangle$.

As in [86], if one introduces

$$Q^i_{ab} \equiv \langle j_{ab}, -\vec{n}_{ab}(\xi)|\, Y^\dagger g_a^{-1} g_b Y (E^a_b)^i |j_{ab}, \vec{n}_{ba}(\xi)\rangle, \tag{4.2.6}$$

one can obtain

$$\begin{aligned} \langle W|E^a_n\cdot E^b_n|j,\Phi(\vec{n})\rangle &= \int \prod_{a=1}^{5} \mathrm{d}g_a\, \delta_{ij} Q^i_{na} Q^j_{nb} {\prod_{cd}}' P^{cd} \\ \langle W|E^a_n\cdot E^b_n\; E^c_m\cdot E^d_m|j,\Phi(\vec{n})\rangle &= \int \prod_{a=1}^{5} \mathrm{d}g_a\, \delta_{ij} Q^i_{na} Q^j_{nb} \delta_{kl} Q^k_{mc} Q^l_{md} {\prod_{cd}}' P^{cd}\,, \end{aligned} \tag{4.2.7}$$

where the product $\prod'$ is over couples $(cd)$ different from $(na), (nb), (mc), (md)$.

Now let us come to express the insertion $Q^i_{ab}$ in (4.2.6) as a group integral. Using the invariance properties of the map $Y$

$$Y J^i_{ab} |j_{ab}, m_{ab}\rangle = J^i_{ab} Y |j_{ab}, m_{ab}\rangle \tag{4.2.8}$$

and the fact that the generator $J^i_{ab}$ of $SU(2)$ can be obtained as the derivative

$$i\frac{\partial}{\partial \alpha^i}\Big|_{\alpha^i=0} \big(e^{-i\alpha^i \tau_i}\big) = J^i_{ab\pm}, \tag{4.2.9}$$

we have that

$$\begin{aligned} Q^i_{ab} \equiv& \langle j_{ab}, -\vec{n}_{ab}(\xi)|\, Y^\dagger g_a^{-1} g_b Y (E^a_b)^i |j_{ab}, \vec{n}_{ba}(\xi)\rangle \\ =& i\gamma \frac{\partial}{\partial \alpha^i}\Big|_{\alpha_i=0} \int_{\mathbb{CP}^1} \mathrm{d}\mathbf{z}\; \overline{g_a\, f^{j_{ab}}_{J\xi_{ab}}(z)^{(j_{ab},\gamma j_{ab})}}\; g_b\; e^{-i\alpha_i \tau_i}\; f^{j_{ab}}_{\xi_{ba}}(z)^{(j_{ab},\gamma j_{ab})} \\ =& i\gamma \frac{\partial}{\partial \alpha^i}\Big|_{\alpha_i=0} \int_{\mathbb{CP}^1} \frac{d_{j_{ab}}}{\pi} \frac{-\mathrm{d}\mathbf{z}_{ab}}{\langle Z_{ab}, Z_{ab}\rangle \langle \widetilde{Z}_{ba}, \widetilde{Z}_{ba}\rangle} \\ & \left(\frac{\langle \widetilde{Z}_{ba}, \widetilde{Z}_{ba}\rangle}{\langle Z_{ab}, Z_{ab}\rangle}\right)^{i\gamma j_{ab}} \left(\frac{\langle J\xi_{ab}, Z_{ab}\rangle^2 \langle \widetilde{Z}_{ba}, \xi_{ba}\rangle^2}{\langle Z_{ab}, Z_{ab}\rangle \langle \widetilde{Z}_{ba}, \widetilde{Z}_{ba}\rangle}\right)^{j_{ab}} \end{aligned} \tag{4.2.10}$$

where

$$\widetilde{Z}_{ba} = (g_b\, e^{-i\alpha_i\tau_i})^\dagger z_{ab} = e^{-i\alpha_i\tau_i}\, Z_{ba} \tag{4.2.11}$$

Using

$$\langle \widetilde{Z}_{ba}, \widetilde{Z}_{ba}\rangle = \langle Z_{ba}, Z_{ba}\rangle \tag{4.2.12}$$

equation (4.2.10) turns out to be

$$\begin{aligned}
Q^i_{ab} =& i\gamma\frac{d_{j_{ab}}}{\pi}\int_{\mathbb{CP}^1} \mathrm{d}\tilde{\mathbf{z}}_{ab}\ \left(\frac{\langle Z_{ba}, Z_{ba}\rangle}{\langle Z_{ab}, Z_{ab}\rangle}\right)^{i\gamma j_{ab}} \frac{\partial}{\partial\alpha^i}\Big|_{\alpha_i=0}\left(\frac{\langle J\xi_{ab}, Z_{ab}\rangle^2\langle e^{-i\alpha_i\tau^i}Z_{ba}, \xi_{ba}\rangle^2}{\langle Z_{ab}, Z_{ab}\rangle\langle Z_{ba}, Z_{ba}\rangle}\right)^{j_{ab}}\\
=& i\gamma\frac{d_{j_{ab}}}{\pi}\int_{\mathbb{CP}^1} \mathrm{d}\tilde{\mathbf{z}}_{ab}\ \left(\frac{\langle Z_{ba}, Z_{ba}\rangle}{\langle Z_{ab}, Z_{ab}\rangle}\right)^{i\gamma j_{ab}} \left(\frac{\langle J\xi_{ab}, Z_{ab}\rangle^2\langle Z_b, \xi_{ba}\rangle^2}{\langle Z_{ab}, Z_{ab}\rangle\langle Z_{ba}, Z_{ba}\rangle}\right)^{j_{ab}}\\
&\qquad \frac{2j_{ab}}{\langle Z_{ba}, \xi_{ba}\rangle}\frac{\partial}{\partial\alpha^i}\Big|_{\alpha_i=0}\langle e^{-i\alpha_i\tau^i}\ Z_{ba}, \xi_{ba}\rangle\\
=& \frac{d_{j_{ab}}}{\pi}\int_{\mathbb{CP}^1} \mathrm{d}\tilde{\mathbf{z}}_{ab}\ \left(\frac{\langle Z_{ba}, Z_{ba}\rangle}{\langle Z_{ab}, Z_{ab}\rangle}\right)^{i\gamma j_{ab}} \left(\frac{\langle J\xi_{ab}, Z_{ab}\rangle^2\langle Z_b, \xi_{ba}\rangle^2}{\langle Z_{ab}, Z_{ab}\rangle\langle Z_{ba}, Z_{ba}\rangle}\right)^{j_{ab}} 2\gamma j_{ab}\frac{\langle \tau^i Z_{ba}, \xi_{ba}\rangle}{\langle Z_{ba}, \xi_{ba}\rangle}.
\end{aligned} \tag{4.2.13}$$

Let

$$A^i_{ab} \equiv \gamma j_{ab}\frac{\langle \sigma^i Z_{ba}, \xi_{ba}\rangle}{\langle Z_{ba}, \xi_{ba}\rangle}. \tag{4.2.14}$$

we have

$$Q^i_{ab} = \int \frac{d_{j_{ab}}}{\pi}\mathrm{d}\tilde{\mathbf{z}}_{ab}K_{ab}(A^a_b)^i, \tag{4.2.15}$$

with $K_{ab}$ given by equation (4.1.13).

If let

$$q^{ab}_n \equiv A^a_n \cdot A^b_n, \tag{4.2.16}$$

one has

$$\langle W|E^a_n\cdot E^b_n|j,\Phi(\vec{n})\rangle = \int\prod_{a=1}^4 \mathrm{d}g_a \int\left(\prod_{a'<b'}\frac{d_{j_{a'b'}}}{\pi}\mathrm{d}\tilde{\mathbf{z}}_{a'b'}\right)\ q^{ab}_n\, e^{S_o} \tag{4.2.17}$$

$$\langle W|E^a_n\cdot E^b_n\ E^c_m\cdot E^d_m|\Phi(\vec{n})\rangle = \int\prod_{a=1}^4 \mathrm{d}g_a \int\left(\prod_{a'<b'}\frac{d_{j_{a'b'}}}{\pi}\mathrm{d}\tilde{\mathbf{z}}_{a'b'}\right)\ q^{ab}_n q^{cd}_n\, e^{S_o} \tag{4.2.18}$$

Here we remove a redundant $\delta g_5$ integral, as discussed in Sec. 4.1. Then the two-point function (4.2.5) can be reexpressed in terms of group integrals:

$$G^{abcd}_{nm} = \frac{\sum_j \psi_j \int \delta^4 g\, \delta^{10} z\; q^{ab}_n q^{cd}_m e^{S_o}}{\sum_j \psi_j \int \delta^4 g\, \delta^{10} z\; e^{S_o}} - \frac{\sum_j \psi_j \int \delta^4 g\, \delta^{10} z\; q^{ab}_n e^{S_o}}{\sum_j \psi_j \int \delta^4 g\, \delta^{10} z\; e^{S_o}}\, \frac{\sum_j \psi_j \int \delta^4 g\, \delta^{10} z\; q^{cd}_m e^{S_o}}{\sum_j \psi_j \int \delta^4 g\, \delta^{10} z\; e^{S_o}}, \tag{4.2.19}$$

where the group integral is over $\delta^4 g = \prod_{a=1}^4 \delta g_a$ and $\delta^{10} z$ is short for the integral measure $\prod_{a<b} \frac{d_{j_{ab}}}{\pi} \mathrm{d}\tilde{\mathbf{z}}_{ab}$ over $\mathcal{P}$.

## 4.3 Lorentzian geometry and time reversal

Before going to asympototic analysis of the two-point function, we consider in this section the Lorentz geometry of a simplex and introduce the corresponding time reversal transformation $T$. We also make a slight modification of the boundary coherent spin network (4.1.7) by time reversal $T$, as well as the corresponding vertex amplitude (4.1.1).

Recall that we work on a vertex of a four-simplex, which has all its tetrahedra space-like. For each triangle $(ab)$, there is a corresponding "wedge" composed by the two tetrahedra which meet at the triangle. Since all the tetrahedra are space-like, the tetrahedron comes in two types: the outward normals are either future-pointing or past-pointing. The wedges are then classified into tow types: it is called in [57, 90] *thick wedge* if the incident tetrahedra are of same pointing type, otherwise called *thin wedge*. We introduce $\Pi_{ab}$ to denote the Lorentzian geometry of the 4-simplex in the following way:

$$\Pi_{ab} = \begin{cases} 0 & \text{thick wedge} \\ \pi & \text{thin wedge.} \end{cases}$$

Now let us come to construct a future-pointing boundary coherent spin network $|j, \Upsilon(\vec{n})\rangle$; here we use $\Upsilon(\vec{n})$ to denote the intertwiner instead of $\Phi(\vec{n})$, to show it related to the Lorentzian geometry. Since one has the closure condition for the 4-simplex and all the tetrahedra are space-like, with outward normals, there is at least one tetrahedron with past-pointing outward normal. To disguise the tetrahedra are future-pointing or past-pointing, we use $\bar{a}$ to denote the future-pointing tetrahedra, and $\underline{a}$ for the past-pointing ones. We obtain the new future-pointing boundary spin network $|j, \Upsilon(\vec{n})\rangle$, by time reversing the normals of past-pointing tetrahedra:

$$|j, \Upsilon(\vec{n})\rangle := \bigotimes_{\bar{a}} |j_{\bar{a}B}, \Phi_{\bar{a}}(\vec{n})\rangle \cdot \bigotimes_{\underline{a}} T|j_{\underline{a}B}, \Phi_{\underline{a}}(-\vec{n})\rangle \tag{4.3.1}$$

where the minus sign in the past-pointing tetrahedra $|j_{\underline{a}B}, \Phi_{\underline{a}}(-\vec{n})\rangle$ comes due to the gluing condition of tetrahedra, and the effect of time reversal $T$ given by

$$T|j_{aB}, \Phi_a(\vec{n})\rangle = (-1)^{\sum_{b\neq a} j_{ab}}|j_{aB}, \Phi_a(-\vec{n})\rangle. \tag{4.3.2}$$

Thus one obtains

$$|j, \Upsilon(\vec{n})\rangle = e^{-i\sum_{ab}\Pi_{ab}j_{ab}}|j, \Phi(\vec{n})\rangle. \tag{4.3.3}$$

The future-pointing boundary coherent spin network $|j, \Upsilon(\vec{n})\rangle$ is the original one $|j, \Phi(\vec{n})\rangle$ times a phase $e^{-i\sum_{ab}\Pi_{ab}j_{ab}}$. We will use this new boundary data to study the vertex amplitude and also correlation function. The vertex amplitude $\langle W|j, \Upsilon(\vec{n})\rangle$ for the future-pointing coherent spin network $|j, \Upsilon(\vec{n})\rangle$ is then given by

$$\langle W|j, \Upsilon(\vec{n})\rangle = e^{-i\sum_{ab}\Pi_{ab}j_{ab}}\langle W|j, \Phi(\vec{n})\rangle \tag{4.3.4}$$

The corresponding integral form, analogue to equation (4.1.15) and (4.1.16), is

$$\langle W|j, \Upsilon(\vec{n})\rangle = \int \prod_{a=1}^{5} \mathrm{d}g_a \int \left(\prod_{a<b} \frac{d_{j_{ab}}}{\pi}\mathrm{d}\tilde{\mathbf{z}}_{ab}\right) e^S, \tag{4.3.5}$$

with $S$ is given by

$$S(g, \mathbf{z}) = \sum_{a<b} j_{ab} \log \frac{\langle J\xi_{ab}, Z_{ab}\rangle^2 \langle Z_{ba}, \xi_{ba}\rangle^2}{\langle Z_{ab}, Z_{ab}\rangle\langle Z_{ba}, Z_{ba}\rangle} + i\gamma j_{ab} \log \frac{\langle Z_{ba}, Z_{ba}\rangle}{\langle Z_{ab}, Z_{ab}\rangle} + ij_{ab}\Pi_{ab}, \tag{4.3.6}$$

where $\Pi_{ab}$ is related to Lorentzian geometry by

$$\Pi_{ab} = \begin{cases} 0 & \text{thick wedge} \\ \pi & \text{thin wedge.} \end{cases}$$

The corresponding integral form of correlation function can be obtained by replacing $S_o$ by the new $S$ in (4.2.19):

$$G^{abcd}_{nm} = \frac{\sum_j \psi_j \int \delta^4 g\, \delta^{10} z\; q^{ab}_n q^{cd}_m e^S}{\sum_j \psi_j \int \delta^4 g\, \delta^{10} z\; e^S} - \frac{\sum_j \psi_j \int \delta^4 g\, \delta^{10} z\; q^{ab}_n e^S}{\sum_j \psi_j \int \delta^4 g\, \delta^{10} z\; e^S} \frac{\sum_j \psi_j \int \delta^4 g\, \delta^{10} z\; q^{cd}_m e^S}{\sum_j \psi_j \int \delta^4 g\, \delta^{10} z\; e^S}, \tag{4.3.7}$$

In the next section, we will study the asymptotic expansion of this two-point function.

## 4.4 Asymptotic expansion of the two-point function for large spin

In this section we study the large-$j_o$ asympototics of the correlation function (4.3.7). We use the technique developed in [86]. The idea is rescaling the spins $j_{ab}$ and $(j_o)_{ab}$ by an integer $\lambda$ so that $j_{ab} \to \lambda j_{ab}$ and $(j_o)_{ab} \to \lambda (j_o)_{ab}$, thus the two-point function (4.3.7) can be reexpressed as $G(\lambda)$. To study large-spin limit turns then to study large-$\lambda$ limit, via stationary phase approximation. In Sec. 4.4.1 we give a brief framework of this technique. Then in Sec. 4.4.2-4.4.3 we give the detailed calculation.

### 4.4.1 The rescaled correlation function and stationary phase approximation

As in [86], let $j_{ab} \to \lambda j_{ab}$ and $(j_o)_{ab} \to \lambda (j_o)_{ab}$, we rescale the correlation function (4.3.7) as

$$G^{abcd}_{nm}(\lambda) = \frac{\sum_j \int \; d^4g \, \delta^{10}z \, q^{ab}_n q^{cd}_m e^{\lambda S_{\text{tot}}}}{\sum_j \int \; d^4g \, \delta^{10}z \, e^{\lambda S_{\text{tot}}}} - \frac{\sum_j \int \; d^4g \, \delta^{10}z \, q^{ab}_n e^{\lambda S_{\text{tot}}}}{\sum_j \int \; d^4g \, \delta^{10}z \, e^{\lambda S_{\text{tot}}}} \frac{\sum_j \int \; d^4g \, \delta^{10}z \, q^{cd}_m e^{\lambda S_{\text{tot}}}}{\sum_j \int \; d^4g \, \delta^{10}z \, e^{\lambda S_{\text{tot}}}}, \tag{4.4.1}$$

where the “total action” is defined as $S_{\text{tot}} = \log \psi + S$ or more explicitly as

$$S_{\text{tot}}(j, g, \mathbf{z}) = -\frac{1}{2} \sum_{ab,cd} \alpha^{(ab)(cd)} \frac{j_{ab} - (j_o)_{ab}}{\sqrt{(j_o)_{ab}}} \frac{j_{cd} - (j_o)_{cd}}{\sqrt{(j_o)_{cd}}} - i \sum_{ab} \phi^{ab}_o \left(j_{ab} - (j_o)_{ab}\right) + S(j, g, \mathbf{z}) \tag{4.4.2}$$

Using Euler-Maclaurin formula, one can evaluate the sums over spins $j$ using integrals in the large $\lambda$ limit:

$$\sum_j q^{ab}_n \, e^{\lambda S_{\text{tot}}} = \int \mathrm{d}^{10}j \; q^{ab}_n \, e^{\lambda S_{\text{tot}}} + O(\lambda^{-N}) \qquad \forall N > 0 \,, \tag{4.4.3}$$

so that the rescaled correlation function (4.4.1) can be approximately expressed in the large $\lambda$ limit

$$G^{abcd}_{nm}(\lambda) = \frac{\int \delta^{10}j \;\; d^4g \, \delta^{10}z \, q^{ab}_n q^{cd}_m e^{\lambda S_{\text{tot}}}}{\int \delta^{10}j \;\; d^4g \, \delta^{10}z \, e^{\lambda S_{\text{tot}}}} - \frac{\int \delta^{10}j \;\; d^4g \, \delta^{10}z \, q^{ab}_n e^{\lambda S_{\text{tot}}}}{\int \delta^{10}j \;\; d^4g \, \delta^{10}z \, e^{\lambda S_{\text{tot}}}} \frac{\int \delta^{10}j \;\; d^4g \, \delta^{10}z \, q^{cd}_m e^{\lambda S_{\text{tot}}}}{\int \delta^{10}j \;\; d^4g \, \delta^{10}z \, e^{\lambda S_{\text{tot}}}}. \tag{4.4.4}$$

We will study the large-$\lambda$ asymptotics of expression (4.4.4).

Let us rewrite the two-point function (4.4.4) formally as

$$G(\lambda)=\frac{\int \mathrm{d}x\ p(x)q(x)\,e^{\lambda S(x)}}{\int \mathrm{d}x\ e^{\lambda S(x)}}-\frac{\int \mathrm{d}x\ p(x)\,e^{\lambda S(x)}}{\int \mathrm{d}x\ e^{\lambda S(x)}}\frac{\int \mathrm{d}x\ q(x)\,e^{\lambda S(x)}}{\int \mathrm{d}x\ e^{\lambda S(x)}}\ , \tag{4.4.5}$$

then the asymptotic expansion of $G(\lambda)$ for large $\lambda$ is given by

$$G(\lambda)=\frac{1}{\lambda}\,(H^{-1})^{ij}\,p_i'(x_o)\,q_j'(x_o)\ +\mathcal{O}(\tfrac{1}{\lambda^2})\ . \tag{4.4.6}$$

Here $x_o$ is the *critical point*, i.e. the stationary point where the real part of the action vanishes, $\mathrm{Re}S(x_o)=0$; $p_i'=\partial p/\partial x^i$, $H$ is the Hessian matrix at the critical point $H=S''(x_o)$. Our task is to obtain the critical point, the derivative of the insertions in Sec. 4.4.2 and the Hessian in Sec. 4.4.3.

### 4.4.2 The critical point and the derivative of insertions

The critical point is the one where the real part and the derivatives of the total action $S_{\mathrm{tot}}$ vanish. The real part of the total action (4.4.2) is given by

$$\mathrm{Re}S_{\mathrm{tot}}=\ -\sum_{ab,cd}(\mathrm{Re}\,\alpha)^{(ab)(cd)}\,\frac{j_{ab}-(j_o)_{ab}}{\sqrt{(j_o)_{ab}}}\,\frac{j_{cd}-(j_o)_{cd}}{\sqrt{(j_o)_{cd}}}+\sum_{(ab)}j_{ab}\log\frac{|\langle J\xi_{ab},Z_{ab}\rangle|^2|\langle Z_{ba},\xi_{ba}\rangle|^2}{\langle Z_{ab},Z_{ab}\rangle\langle Z_{ba},Z_{ba}\rangle}\ . \tag{4.4.7}$$

Having assumed that the matrix $\alpha$ in the boundary state has positive definite real part, we have that the real part of the total action is negative or vanishing, $\mathrm{Re}S_{\mathrm{tot}}\leq 0$. In particular the total action vanishes for the configuration of spins $j_{ab}$ and group elements $g$ satisfying

$$j_{ab}=(j_o)_{ab}\ , \tag{4.4.8a}$$

$$J\xi_{ab}=\frac{e^{i\phi_{ab}}}{\parallel Z_{ab}\parallel}Z_{ab},\quad \text{and}\quad \xi_{ba}=\frac{e^{i\phi_{ba}}}{\parallel Z_{ba}\parallel}Z_{ba}, \tag{4.4.8b}$$

where $\parallel Z_{ab}\parallel$ is the norm of $Z_{ab}$ induced by the Hermitian inner product, and $\phi_{ab}$ and $\phi_{ba}$ are phases.

The requirement that the variations of the total action with respect to the spinors $z_{ab}$ and $\bar{z}_{ab}$ vanishes, $\delta_z S_{\mathrm{tot}}=\delta_{\bar{z}}S_{\mathrm{tot}}=0$, lead both to

$$e^{-i\phi_{ab}}\frac{g_a J\xi_{ab}}{\|Z_{ab}\|}=e^{-i\phi_{ba}}\frac{g_b\xi_{ba}}{\|Z_{ba}\|} \tag{4.4.9}$$

evaluated at the maximum point (4.4.8b). For the group variables, $\delta_g\, S_{\rm tot} = 0$ leads to

$$\sum_{b:b\neq a} j_{ab}\mathbf{n}_{ab} = 0 \tag{4.4.10}$$

evaluated at the maximum point (4.4.8b). In fact the normals $\vec{n}_{ab}$ in the boundary state are chosen to satisfy the closure condition at each node. Therefore the critical points in the group variables are given by all the solutions of equation (4.4.8b).

The variations of the total action with respect to the spins $j$ turns out to be

$$\frac{\partial S_{\rm tot}}{\partial j_{ab}} = -\sum_{cd} \frac{\alpha^{(ab)(cd)}(j_{cd} - (j_o)_{cd})}{\sqrt{(j_o)_{ab}}\sqrt{(j_o)_{cd}}} - i\phi_o^{ab} + \frac{\partial S}{\partial j_{ab}}\,. \tag{4.4.11}$$

Imposing the maximal-point equation (4.4.8) and the critical-point equation (4.4.9), equation (4.4.11) is reduced into

$$\left.\frac{\partial S_{\rm tot}}{\partial j_{ab}}\right|_{\rm crit} = -i\phi_o^{ab} + i\mu S_{\rm Regge}, \tag{4.4.12}$$

where the parameter $\mu = \pm 1$ measures the discrepancy of the orientations of the 4-simplex $\sigma$: there are two orientations of the 4-simplex $\sigma$, one inherited from the Minkowski space where $\sigma$ is embedded, the other induced from the boundary data; $\mu = 1$ if these two agree and $\mu = -1$ otherwise. The requirement that equation (4.4.12) vanishes selects $\mu = 1$, which means we only consider the case when the orientation of the boundary data agrees with the one induced from the Minkowski space.

We end this subsection by the first derivative of the insertion $q_n^{ab}(g, z)$ evaluated at the critical point:

$$\left.\delta_{z_{an}} q_n^{ab}\right|_{\rm crit} = 0 \tag{4.4.13}$$

$$\left.\delta_{\bar{z}_{an}} q_n^{ab}\right|_{\rm crit} = \gamma^2 (j_o)_{na}(j_o)_{nb}\left(\frac{g_a\sigma^i\xi_{an}}{\|Z_{an}\|}n^i_{bn} - \frac{g_a\xi_{an}}{\|Z_{an}\|}\vec{n}_{an}\cdot\vec{n}_{bn}\right) \tag{4.4.14}$$

$$\left.\delta^r_{g_a} q_n^{ab}\right|_{\rm crit} = \gamma^2 j_{na}j_{nb}\left(n^i_{bn}\langle\xi_{an}, \vec{L}\sigma_i\xi_{an}\rangle - \frac{i}{2}(\vec{n}_{an}\cdot\vec{n}_{bn})\vec{n}_{an}\right) \tag{4.4.15}$$

$$\left.\delta^b_{g_a} q_n^{ab}\right|_{\rm crit} = \gamma^2 j_{na}j_{nb}\left(n^i_{bn}\langle\xi_{an}\vec{K}\sigma_i\xi_{an}\rangle - \frac{1}{2}(\vec{n}_{an}\cdot\vec{n}_{bn})\vec{n}_{an}\right) \tag{4.4.16}$$

$$\left.\delta_{j_{cd}} q_n^{ab}\right|_{\rm crit} = \gamma^2 \left.\delta_{j_{cd}}\right|_{(j_o)_{ab}}(j_{an}\vec{n}_{an}\cdot j_{bn}\vec{n}_{bn}) \tag{4.4.17}$$

### 4.4.3 Hessian matrix of the total action

Following the stationary phase approximation introduced in section 4.4.1, once the Hessian matrix is obtained, one can get asymptotic expansion of the two-point function (4.3.7) by using equations (4.4.6) and (4.4.13). Now let us come to calculate the Hessian matrix.

The Hessian is defined as the matrix of the second derivatives of the total action where the variable $g_5$ has been gauge fixed to the identity. We split the Hessian matrix into derivatives w.r.t. the spins $j$,w.r.t. the group elements $g$ and w.r.t. $z$. The Hessian will then be a $(10+24+20) \times (10+24+20)$ matrix

$$S''_{\text{tot}} = \begin{pmatrix} Q_{jj} & 0_{10\times 24} & 0_{10\times 20} \\ 0_{24\times 10} & H_{gg} & H_{gz} \\ 0_{20\times 10} & H_{zg} & H_{zz} \end{pmatrix} \tag{4.4.18}$$

as

$$\delta_j \delta_g S_{S_{\text{tot}}} = 0 \quad \delta_j \delta_z S_{S_{\text{tot}}} = 0. \tag{4.4.19}$$

We will now describe the non-vanishing blocks of this matrix. $Q_{jj}$ is a $10 \times 10$ matrix containing only derivatives with respect to the spins $j_{ab}$, with elements

$$Q_{(ab)(cd)} = \delta_{j_{ab}} \delta_{j_{cd}} S_{\text{tot}} \Big|_{\text{crit}} = -\frac{\alpha^{(ab)(cd)}}{\sqrt{(j_o)_{ab}}\sqrt{(j_o)_{cd}}} + \delta_{j_{ab}} \delta_{j_{cd}} \Big|_{\text{crit}} S_{\text{Regge}}. \tag{4.4.20}$$

$H_{gg}$ is a $(4\times 6) \times (4 \times 6)$ matrix containing only derivatives with respect to the group elements $g_a$. Note that due to the form of the action, derivatives with respect to two different group variables will be zero and it will be block diagonal

$$H_{gg} = \begin{pmatrix} H_{11} & 0 & 0 & 0 \\ 0 & H_{22} & 0 & 0 \\ 0 & 0 & H_{33} & 0 \\ 0 & 0 & 0 & H_{44} \end{pmatrix} \tag{4.4.21}$$

Each $H_{aa}$ is a $6 \times 6$ matrix. The variation has been performed by splitting the $SL(2,\mathbb{C})$ element into a boost and a rotation generator. This gives

$$H_{aa} = \begin{pmatrix} H^{rr}_{(ai)(aj)} & H^{br}_{(ai)(aj)} \\ H^{rb}_{(ai)(aj)} & H^{bb}_{(ai)(aj)} \end{pmatrix} \tag{4.4.22}$$

with $3 \times 3$ matrices

$$H^{rr}_{(ai)(aj)} = \frac{1}{2} \sum_{b:b\neq a} j_{ab}(-\delta^{ij} + n^i n^j + i\epsilon^{ij}{}_k n^k) \tag{4.4.23}$$

$$H^{rb}_{(ai)(aj)} = -\frac{i}{2} \sum_{b:b\neq a} j_{ab}(-\delta^{ij} + n^i n^j + i\epsilon^{ij}{}_k n^k) \tag{4.4.24}$$

$$H^{br}_{(ai)(aj)} = -\frac{i}{2} \sum_{b:b\neq a} j_{ab}(-\delta^{ij} + n^i n^j + i\epsilon^{ij}{}_k n^k) \tag{4.4.25}$$

$$H^{bb}_{(ai)(aj)} = 2 \sum_{b:b\neq a} j_{ab}(1 + \frac{i}{2}\gamma)\Big( - \delta^{ij} + n^i n^j + i\epsilon^{ij}{}_k n^k\Big). \tag{4.4.26}$$

$H_{zz}$ is a matrix containing only derivatives with respect to the spinors $z_{ab}$ and $\bar{z}_{ab}$.

$$H_{zz} = \begin{pmatrix} S^{''}_{zz} & S^{''}_{z\bar{z}} \\ S^{''}_{\bar{z}z} & S^{''}_{\bar{z}\bar{z}} \end{pmatrix} \tag{4.4.27}$$

Since each spinor $z_{ab}$ has two compononts $(z_o \quad z_1)$, each block $S^{''}$ of $H_{zz}$ seems to be $20 \times 20$ matrix and thus $H_{zz}$ seems to be $40 \times 40$; however, this $40 \times 40$ matrix is degenerate due to the homogeneity of the representation functions (4.1.8) which we discuss in section 4.1. Remind that although we still keep the formulae of $f(\mathbf{z})$, we have chosen a section for $(z_o \quad z_1)$ as $(z_o/z_1 \quad 1)$. Hence the matrix $H_{zz}$ reduced to be $20 \times 20$ and non-degenerate, by removing the derivatives with respect to the second component $z_1$:

$$H_{zz} = \begin{pmatrix} 0_{10\times 10} & S^{''}_{z\bar{z}} \\ S^{''}_{\bar{z}z} & 0_{10\times 10} \end{pmatrix} \tag{4.4.28}$$

with $S^{''}_{zz}$ and $S^{''}_{\bar{z}\bar{z}}$ vanishing, $S^{''}_{z\bar{z}}$ and $S^{''}_{z\bar{z}}$ diagonal matrices. The diagonal elements are first components of the matrix

$$H_{z_{ab}\bar{z}_{ab}} = j_{ab}\Big(\frac{2(g_a J\xi_{ab})(g_a J\xi_{ab})^\dagger + (i\gamma - 1)g_a g_a^\dagger}{\|Z_{ab}\|} - \frac{(i\gamma + 1)g_b g_b^\dagger}{\|Z_{ba}\|}\Big) \tag{4.4.29}$$

$H_{gz} = \big(S^{''}_{gz} \qquad S^{''}_{g\bar{z}}\big)$ is a $24 \times 20$ matrix, containing derivatives with respect to spinor $z_{ab}$ and group element $g_a$. Again, we consider the reduced matrix, by removing the derivatives with respect to the sencond component of spinor $z_{ab}$. The non-vanish elements are the first components of

$$
\begin{aligned}
H^{rz}_{(ai)(ab)} &= e^{i\phi_{ab}} \frac{j_{ab}}{\|Z_{ab}\|}\Big((1-i\gamma)(L^i g_a J\xi_{ab})^\dagger - i(1+\gamma)\mathbf{n}_{ab}(g_a J_{\xi_{ab}})^\dagger\Big) \\
H^{bz}_{(ai)(ab)} &= e^{i\phi_{ab}} \frac{j_{ab}}{\|Z_{ab}\|}\Big((1-i\gamma)(K^i g_a J\xi_{ab})^\dagger - i(1+\gamma)\mathbf{n}_{ab}(g_a J_{\xi_{ab}})^\dagger\Big) \\
H^{r\bar{z}}_{(ai)(ab)} &= -e^{-i\phi_{ab}} \frac{j_{ab}}{\|Z_{ab}\|}(1+i\gamma)(\mathbf{n}_{ab} + L^i g_a)J\xi_{ab} \\
H^{b\bar{z}}_{(ai)(ab)} &= -e^{-i\phi_{ab}} \frac{j_{ab}}{\|Z_{ab}\|}(1+i\gamma)(\mathbf{n}_{ab} + K^i g_a)J\xi_{ab}
\end{aligned}
$$

Now we obtain the Hessian matrix. Substituting (4.4.13)-(4.4.18) into (4.4.6) we obtain the asymptotic expansion of the correlation function

$$
G^{abcd}_{nm}(\alpha) = (\gamma j_o)^3 (R^{abcd}_{nm}(\alpha) + O(\gamma)) + O(j_o^2) \tag{4.4.30}
$$

with

$$
R^{abcd}_{nm} = \frac{1}{(\gamma j_o)^3} \sum_{p<q,r<s} Q^{-1}_{(pq)(rs)} \frac{\partial q^{ab}_n}{\partial j_{pq}} \frac{\partial q^{cd}_m}{\partial j_{rs}}. \tag{4.4.31}
$$

We consider the limit, introduced in [86], where the Barbero-Immirzi parameter is taken to zero $\gamma \to 0$, and the spin of the boundary state is taken to infinity $j \to \infty$, keeping the size of the quantum geometry $A \sim \gamma j$ finite and fixed. This limit corresponds to neglecting Planck scale discreteness and twisting effects, at large finite distances. In this limit, the two-point function (4.4.30) we obtain exactly matches the one obtained from Lorentzian Regge calculus [42].

# Chapter 5

# Conclusions and perspectives

In this concluding chapter we will take a look back and review what we have been able to understand while at the same time pointing out the problems left open.

In this thesis we studied two themes of spinfoam formalism, the imposing of simplicity constraint and the computing the Lorentzian propagator. The former is to find the way to connect different asides related to the EPRL spinfoam model, and the later is to test the resulting model and try to extract physics from that.

To this end, the first chapter give a brief introduction to the formal structure of spinfoam formalism, and also the Kinematical Hilbert space from the canonical quantization procedure and as well from polyhedral quantum geometry. These three are what we connect in the second and third chapters.

In the second chapter we study the simplicial EPRL spinfoam model. Since the simplicity constraint is secondary class, we use Gupta-Bleuler procedure to impose the simplicity constraint *weakly*, namely asking the matrix elements of the simplicity constraints to vanish on physical boundary states. In this way, we find a weak solution to the simplicity constraint, as the Hilbert space of the boundary state, matching the kinematical Hilbert space obtained from the canonical approach. Also, the boundary states have a geometrical interpretation in terms of quantum tetrahedral geometry. In this way, we connect the three introduced in the first chapter. What's more, we give a slight modification of the vertex amplitude, with respect to the original EPRL spinfoam model, in Euclidean theory when $\gamma > 1$ and in Lorentzian theory, corresponding to a slightly different factor ordering of the constraints. In Euclidean theory when $\gamma < 1$, we obtain the exact EPRL vertex amplitude. With the modification (the matrix elements of) the simplicity constraint hold *exactly*, and not just in the large quantum number limit, as in previous constructions.

Surprisedly, however, the "perfect" solution we find is not the maximal subspace where the simplicity constraint holds weakly in our sense. There exist a larger subspace with a new quantum number as a Gupta-Bleuler solution of simplicity constraint. This new quantum number affects non-trivially both the face amplitude and the vertex amplitude of the spinfoam model. The quantum number is frozen if in addition to the weak imposition of the (linear) simplicity constraint, we also impose strongly a diagonal quadratic constraint. With a suitable operator ordering of this constraint, the state space can be reduced back down to the LQG state space. If we take the principle that the quantum theory we are seeking has the same number of degrees of freedom as the classical theory, then the answer is negative. This principle indicates that the appropriate way of imposing the constraints is the one that gets rids of the extra states. However, we think it is nevertheless interesting to keep in mind the existence of these additional solutions to the weak simplicity constraints.

In the third chapter, we generalize the simplicial spinfoams to the polytopal spinfoams, where there can be arbitrary-valent vertex. Thus we bring the model closer to the canonical kinematics and also the polyhedral quantum geometry. In fact, our polytopal spinfoam model is strictly related to the polyhedral quantum geometry, since the discrete simplicity constraint is obtained from the polyhedral geometry.

As last, we compute the two-point function for the Penrose metric operator from the simplicial EPRL spinfoam model, with a single vertex, in Lorentzian theory. We consider the limit, where the Barbero-Immirzi parameter is taken to zero $\gamma \to 0$, and the spin of the boundary state is taken to infinity $j \to \infty$, keeping the size of the quantum geometry $A \sim \gamma j$ finite and fixed. This limit corresponds to neglecting Planck scale discreteness and twisting effects, at large finite distances. In this limit, the two-point function we obtain exactly matches the one obtained from Lorentzian Regge calculus.

More work is needed to properly understand and test the spin-foam model. The calculation of two-point correlation functions for the metric operator on semiclassical states provides support for the existence of an effective field theory (see chapter 4). It is important to extend these calculations to $n$-point functions, and to compute next-to-leading order corrections. Also, the calculation needs to be extended to the generalized spinfoam models, where the vertex is arbitrary valent (see chapter 3). We leave these questions listed above to future research.

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**Abstract**

In this thesis we study the implementation of simplicity constraints that defines the recent Engle-Pereira-Rovelli-Livine spinfoam model and two-point correlation functions of this model. We define in a simple way the boundary Hilbert space of the theory; then show directly that all constraints vanish on this space in a weak sense. We point out that the general solution to this constraint (imposed weakly) depends on a quantum number in addition to those of loop quantum gravity. We also generalize this construction to Kamiński-Kisielowski-Lewandowski version where the foam is not dual to a triangulation. We show that this theory can still be obtained as a constrained BF theory satisfying the simplicity constraint, now discretized on a general oriented 2-cell complex. Finally, we calculate the two-point correlation function of the Engle-Pereira-Rovelli-Livine spinfoam model in the Lorentzian signature, and show the two-point function we obtain exactly matches the one obtained from Lorentzian Regge calculus in some limit.

**Résumé**

Dans cette thèse, nous étudions l'implémentation des contraintes de simplicité dans le nouveau modèle de mousses de spin d'Engle-Pereira-Rovelli-Livine, ainsi que les fonctions de corrélation à deux points de ce modèle. Nous définissons d'une manière simple l'espace de Hilbert limite de la théorie, puis montrons directement que toutes les contraintes s'annulent faiblement sur cet espace. Nous observons que la solution générale à cette contrainte (imposée faiblement) dépend d'un nombre quantique, en plus de ceux de la gravitation quantique à boucles. Nous généralisons également cette construction pour la version de Kamiński-Kisielowski-Lewandowski, où la mousse n'est pas duale à une triangulation. Nous montrons que cette théorie peut aussi être obtenue comme une théorie BF satisfaisant la contrainte de simplicité, cette fois discrétisée sur un 2-complexe cellulaire orienté. Enfin, nous calculons la fonction de corrélation à deux points du modèle de mousses de spin Engle-Pereira-Rovelli-Livine avec la signature lorentzienne, et nous montrons que la fonction à deux points que nous obtenons correspond dans une certaine limite à celle obtenue à partir du calcul de Regge lorentzien .