Particles as Field Singularities in the Unified Algebraic Dynamics

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Nonlinear generalization of Cauchy-Riemann equations to the algebra of biquaternions is considered. In a particular case the latters reduce to the "iversal generating equations" which deal with the 2-spinor and the gauge fields and form the basis of a unified algebraic field theory. For every solution to universal generating equations the components of spinor field satisfy both the eikonal and the wave equations while the strengths of gauge field - both Maxwell and Yang-Mills equations. Making use of their specific ("ak") gauge symmetry, we we reduce universal equations to the equations of shear-free null congruence and, applying the Kerr theorem, integrate them in twistor variables . Particles are considered as bounded singularities of effective metric and electromagnetic fields. For fundamental unisingular solution electric charge of (point- or ring-like) singular source is fixed in magnitude (generally quantized), and related Kerr-Schild metric is of Reisner-Nördstrem or Kerr-Newman type respectively. The value of quadrupole electric moment for the electron is also predicted. Multisingular solutions are presented and briefly discussed.

1. Algebrodymamical approach to field theory and universal generating equations

In general framework of *algebrodynamical* paradigm (see, e.g., [1, 9, 10, 14] and references therein) it was proposed to regard the set of equations

$$d\xi = A(x) * dX * \xi(x), \tag{1}$$

as the basis of some unified non-Lagrangian field theory. In formula (1) the asterisk denotes multiplication in *the algebra of biquaternions* $\mathbb{B}$ (isomorphic to the full 2×2 complex matrix algebra), and X represents 2×2 Hermitian matrix of space-time coordinates. The two-column complex variable $\xi(x)$ can be identified as a fundamental spinor field (related to a shear-free null congruence, see Section 6) while the components $A_\mu(x)$ of the 2×2 matrix $A = A^\mu(x)\sigma_\mu$ can be considered as $\mathbb{C}$-valued electromagnetic (EM) potentials.

Properties and interpretation of eqs. (1) are examined throughout the article. Eqs. (1) originate from $\mathbb{B}$-*generalized Cauchy-Riemann equations* (Section 2), appear to be Lorentz and gauge invariant (Section 3) and impose strict restrictions on both the spinor and the EM fields (Section 4). Indeed, for every solution to eqs. (1) *the components of spinor field satisfy the eikonal and the wave equations (Sections 2,5), while EM field strengths obey Maxwell equations for free space.* Moreover, close connections exist between the solutions to eqs.(1) and the solutions to vacuum Yang-Mills and Einstein-Maxwell equations (Sections 4 and 6 respectively).

In view of the above relations between wave-like, gauge and GTR equations (we'll call them *conventional equations* (CE) for brevity) and eqs.(1) on the other hand, the latters have been called *generating system of equations* (GSE) [14]. Since CE are all of vacuum type, in the approach developed *particles are regarded as (bounded in 3-space) singularities of the fields.* We'll see (Sections 5,8) that the structure of singularities of CE (including even that of linear Maxwell equations) is surprisingly rich, complicated (point-, string- or even membrane-like) and presumably unknown up to now.

On the other hand, the characteristics and time evolution of these particles- singularities are completely governed by the *overdetermined* nonlinear structure of GSE (1) since CE serve only as necessary yet not sufficient *compatibility conditions* with respect to the primary GSE. In particular, the Coulomb-type Ansatz (Section 5) corresponds to some solution of GSE iff *the value of electric charge of the source is fixed to be unit*, in spite of linearity of Maxwell equations themselves. Thus, the charge quantization property holds here just on the classical level of consideration and due again to rigid *overdetermined* structure of GSE (1).

From other results presented in the article, close relation between GSE and (well-known in the framework of GTR) equations defining *shear-free geodesic null congruences* could be distinguished (Section 6). In this account, complete integration of GSE can be performed using its *twistor structure* and Kerr theorem for shear-free congruences (Section 7). On the other hand, this makes it possible to define an effective *Riemannian metric of Kerr-Schild type* for every solution to GSE (Section 6). In stationary axisymmetrical case such metric satisfies Einstein-Maxwell electrovacuum system of equations and is just of Reisner-Nördstrem or Kerr-Newman type.

On the caustics of shear-free conguences the curvature of metric field becomes singular as well as the strength of EM field related to GSE. For the latter a remarkable representation via twistor variables is also presented in Section 7. In Section 8 we discuss general interpretation of particles as bounded singularities and point out the links between this concept and the catastrophe theory. Multisingular solutions to GSE are presented and their properties - briefly discussed.

2. Quaternionic differentiability. Algebraic origination and 2-spinor structure of universal equations

Let $\mathbb{A}$ be a finite-dimensional *associative* and *commutative* algebra over $\mathbb{R}$ or $\mathbb{C}$. Natural definition of $\mathbb{A}$-differentiability has been proposed by G.Sheffers as far as in 1893 and has the form (see [2, chapter 5] for details):

$$dF = D(Z) * dZ, \tag{2}$$

$(*)$ being multiplication in $\mathbb{A}$, $F(Z)$ being an $\mathbb{A}$-valued function of $\mathbb{A}$-variable $Z \in \mathbb{A}$, and $D(Z) \in \mathbb{A}$ — some other $\mathbb{A}$-valued function related to $F(Z)$ ("derivative" from $F(Z)$).

Eqs. (2) can be considered as *the condition of* $\mathbb{A}$*-valued differential* 1*-form to be exact*[1] . For a particular case of complex algebra $\mathbb{A} \equiv \mathbb{C}$ eqs. (2), after elimination of the components of $D(Z)$, lead to the Cauchy-Riemann (CR) equations of ordinary form. It should be mentioned that successful generalization of commutative analysis to the case of *supercommutative* algebras has been developed in the works of V.S.Vladimirov and I.V.Volovich [3] (see also [4]).

To succeed in the formulation of differentiability conditions in the case of associative *noncommutative* algebra $\mathbb{G}$ one notices that the most general component-free form of infinitesimally small increment of a $\mathbb{G}$-function is[2]

[1]Usual conditions of smoothness of the components of $F(Z)$ and of existence of positive norm in $\mathbb{A}$-space are assumed to be fulfilled.

[2]For example, in the simplest case of the quadratic function $F(Z) = Z * Z$ one has

$$dF = Z * dZ * E + E * dZ * Z,$$

E being the unit element in $\mathbb{G}$.

$$dF = L_1(Z) * dZ * R_1(Z) + L_2(Z) * dZ * R_2(Z) + ..., \tag{3}$$

where the set of pairs $\{L_i(Z), R_i(Z)\}$ replace the "derivative" $D(Z)$ of the commutative case. Notice that just the representation (3) serves infact as the basis of noncommutative analysis in the version proposed by A.Yu.Khrennikov [4, chapter 7].

Unfortunately, no constraints are known to exist, generally, between the components of a "good" $\mathbb{G}$-function, i.e. of a function which differential can be presented in the form (3) (for details, see e.g. the paper of A.Sudbery [5]). The situation is quite contrary to that in the commutative case, in $\mathbb{C}$-case with respactive CR-equations in particular. Besides, from geometrical point of view, functions satisfying eqs.(3) show no analogy with conformal mappings in the complex case. For these reasons the version of noncommutative analysis presented in [4] cannot be recognized as fully satisfactory.

Direct account of noncommutativity in the very definition of $\mathbb{G}$-differentiability seems, however, quite natural and promising. In 1980 just this way towards the construction of noncommutative analysis has been proposed by one of the authors in [6] (see [1] and the references therein). However, in order to impose some restrictions on the components of $F(Z)$ (*generalized CR-equations*) it was proposed to regard as "true" $\mathbb{G}$-differentiable only such $\mathbb{G}$-functions for which *representation (3) is reduced to one "elementary" $\mathbb{G}$-valued differential 1-form only*, i.e. for which it holds

$$dF = L(Z) * dZ * R(Z), \tag{4}$$

where $L(Z), R(Z) \in \mathbb{G}$ had been called *semi-derivatives* of $F(Z)$ (they are defined up to an element from the *centre* of $\mathbb{G}$, see [1, 9, 14]).

Definition of $\mathbb{G}$-differentiability (4) can be considered as the requirement on an elementary $\mathbb{G}$-valued 1-form to be exact[3]. For $\mathbb{G}$ being commutative again, conditions (4) evidently reduce to the old ones (2) (and, therefore, to CR-equations in $\mathbb{C}$-case).

Definition (4) appreciably narrows down the class of "good" $\mathbb{G}$-functions, cutting off, say, all of polynomials (exept trivial linear ones). The situation looks like rather unexpected from the point of view of customary complex analysis. Nevertheless, condition (4) singles out just the class of $\mathbb{G}$-functions which is natural from algebraic considerations, extremely interesting in geometrical properties and which admits a natural field-theoretical interpretation.

In the exclusive case of real Hamilton quaternions $\mathbb{G} \equiv \mathbb{H}$ eqs.(4) appear to be just (necessary and sufficient) *algebraic conditions for mapping $F : Z \to F(Z)$ to be conformal in E^4* (see [1, 9] for details). However, since the conformal group of E^4 is known to be finite (15-) parametrical, $\mathbb{H}$-valued functions satisfying eqs.(4) are too trivial to be treated, say, as field variables. Fortunately, the situation becomes quite different when one turns to consider *the complex extention of $\mathbb{H}$, i.e. the algebra of biquaternions $\mathbb{B}$* which only we are going to deal with below[4].

For $\mathbb{B}$-algebra the 2×2 complex matrix representation is suitable. To realize the latter, for every $Z \in \mathbb{B}$ we take

$$Z \Leftrightarrow \begin{pmatrix} z^0 + z^3 & z^1 - iz^2 \\ z^1 + iz^2 & z^0 - z^3 \end{pmatrix} \equiv z^\mu \sigma_\mu, \tag{5a}$$

where $z^\mu \in \mathbb{C}$, $\sigma_\mu = \{E, \ \sigma_a\}$ are unit and three Pauli matrices respectively (as usual, $\mu, \nu, ... = 0, 1, 2, 3$ and $a, b, ... = 1, 2, 3$).

Applying now the column- or the full row-column splitting to eqs. (4) we obtain the following two conditions:

$$d\xi = L(Z) * dZ * \eta(Z), \tag{6a}$$

$$df = \phi(Z) * dZ * \psi(Z), \tag{6b}$$

[3]Note that *elementary* $\mathbb{G}$-form (4) can be defined as *the most general $\mathbb{G}$-valued 1-form* which can be constructed by means of operation of multiplication in $\mathbb{G}$ only.

[4]Some considerations about differentiability in Dirac-Clifford and even in non-associative octonion algebras have been presented in [1, chapter 2].

where $\xi(Z), \eta(Z), \psi(Z) \in \mathbb{C}^2$ are 2-columns and $\phi(Z) \in \mathbb{C}^2$ is a 2-row, functionally independent in general, while $f(Z) \in \mathbb{C}$ is some any *matrix* component of $F(Z)$. According to symmetry properties of eqs.(6a,b) the quantities ξ, η, ϕ, ψ manifest themselves as 2-spinors, whereas $f(Z)$ - as a scalar (see Section 3 and article [10] for details).

From condition (6b) in account of well-known *Fiertz identities*, complexified *eikonal equation* for every (matrix) component $f(Z)$ of a $\mathbb{B}$-differentiable function $F(Z)$ immegiately follows [1, 15]

$$\eta^{\nu\lambda}\partial_\nu f \partial_\lambda f = 0, \tag{7}$$

$\partial_\nu \equiv \partial/\partial z^\nu$ being partial derivatives and $\eta^{\nu\lambda}$ being metric tensor of Minkowsky space, of the form $\eta^{\nu\lambda} = diag(+1, -1, -1, -1)$ (with respect to representation (5)).

The eikonal eq. (7) plays in $\mathbb{B}$-analysis the role similar to that of the Laplace equation in two-dimensional complex case. Thus, definition (4) of $\mathbb{G}$-differentiability links together the *non-commutativity* of $\mathbb{G}$-algebra (entering directly into eqs. (4)) with the *nonlinearity* of generalized CR-equations resulted. There is nothing surprising in this correlation from the usual standpoint of gauge theories where non-Abelian groups result in nonlinearity of Yang-Mills strengths). However, within the framework of noncommutative analisis similar interrelation was demonstrated, perhaps, for the first time (all of the previous works on (bi)quaternionic analysis dealt with trivial linear generalizations of the CR-equations, see for example [7, 8] and excellent review in [2, chapter 5]).

Noticing that eqs.(6b) follow directly from eqs. (6a) and the latters make it possible to reconstruct an arbitrary solution of the full system (4), we come to *fundamental 2-spinor structure* of the primary system (4). Together with its nonlinear character this property allows to formulate a field theory on the base of eqs. (6a) only. This program was partially realized in [15] where general analytical solution to the eikonal equation (7) has been obtained. However, in this article we restrict ourselves with a particular case for which $\xi(Z) \equiv \eta(Z)$ in eqs. (6a). In accord with the results obtained in [15], this case exhausts one of two classes of solutions to eikonal equation and, perhaps, appears to be the most interesting both from physical and geometrical standpoints.

The only *ad hoc* conjecture we are obliged to accept here is the requirement for coordinates z^μ in (5a) *to be real*, $z^\mu \equiv x^\mu \in \mathbb{R}$, i.e. to belong to *Minkowsky space* which is here only a subspace of full complex vector space of $\mathbb{B}$-algebra. Notice that this demands from the coordinate-representing matrix in (5a) to be Hermitian

$$Z \Leftrightarrow X = X^+ == \begin{pmatrix} u & w \\ \bar{w} & v \end{pmatrix} \equiv x^\mu \sigma_\mu \tag{5b}$$

$u, v = x^0 \pm x^3;\ w, \bar{w} = x^1 \pm ix^2$ being *spinor (null) coordinates* which are permanently used below)

In account of two above-presented limitations, from which the latter is evidently necessary to ensure *relativistic invariance* of theory, conditions of $\mathbb{B}$-differentiability (4) reduce to GSE (1) announced at the beginning of the paper. The latter is considered as the basic system of equations of *algebraic nonlinear field theory* which deals with a spinor field as well as with a gauge field represented by matrix $A(x)$ (see the next Section). Obviously, however, such a theory will be quite exotic due to the *overdetermined, nonLagrangian structure* of its dynamical background - the GSE, to detailed examination of which we now proceed.

3. Geometrodynamical interpretation and "weak" gauge structure of GSE

In 4-index notation GSE (1) takes the form

$$\partial_\nu \xi = A(x) * \sigma_\nu * \xi(x), \tag{8}$$

where $A(x) = A^\mu(x)\sigma_\mu$. According to (1) or (8) GSE can be considered from geometrodynamical point of view as *condition for fundamental spinor field $\xi(x)$ to be covariantly constant* with respect to the effective affine connection

$$\Gamma = A(X) * dX, \qquad \Gamma_\nu = A(x) * \sigma_\nu \tag{9}$$

which may be called *left* $\mathbb{B}$*-connection.* It form is completely determined by the structure of $\mathbb{B}$-algebra and induces a specific affine geometry of *Weyl-Cartan type* on the complex vector space of $\mathbb{B}$-algebra. To see this, one should return back in (8) from the spinor $\xi(x)$ to the full 2×2 matrix $R(x) = F(x) = F^\mu(x)\sigma_\mu$ representing the "right semi-derivative" in conitions of $\mathbb{B}$-differentiability (4); then one gets for the components

$$\partial_\nu F^\mu = \Gamma^\mu_{\nu\rho} F^\rho(x), \tag{10}$$

where the connection coefficients get the form

$$\Gamma^\mu_{\nu\rho} \equiv \Gamma^\mu_{\nu\rho}(A(x)) = \delta^\mu_\nu A_\rho + \delta^\mu_\rho A_\nu - \eta_{\nu\rho} A^\mu - i\varepsilon^\mu_{.\nu\rho\lambda} A^\lambda, \tag{11}$$

which includes the *Weyl nonmetricity* and the *totally skew-symmetric torsion* terms related to each other (with complex Weyl vector $A_\mu(x)$ being proportional to the pseudo-trace $iA_\mu(x)$ of the torsion tensor).

Note that $\mathbb{B}$-induced complex Weyl-Cartan connection (11) has been proposed firstly in [1, 10] and recently used by V.G.Kretchet in his search for geometric theory of electroweak interactions [17] (based on *the break of* P*-invariance* by the torsion term in (11)).

GSE (1) is evidently form-invariant under the global transformations of coordinates and field variables

$$X \Rightarrow X' = M^+ * X * M, \tag{12a}$$

$$\xi \Rightarrow \xi' = M^{-1}\xi, \quad A \Rightarrow A' = M^{-1} * A * (M^+)^{-1}, \tag{12b}$$

$M \in SL(2,\mathbb{C})$ being an arbitrary unimodular 2×2 complex matrix.

The 6-parametric group of transformations (12a) $2:1$ corresponds to the continious transformations of the coordinates $\{x^\mu\}$ from Lorentz group. Thus, *GSE is relativistic invariant* and, according to the laws of transformations (12b), the quantities $A_\mu(x)$ and $\xi_B(x), B = 0,1$ behave themselves as the components of 4-vector and 2-spinor respectively. As to local symmetries of GSE, system (8) can be shown to preserve its form under the so called "*weak gauge transformations*" [10, 13, 14]

$$\xi_B \Rightarrow \xi'_B = \lambda \xi_B, \qquad A_\mu \Rightarrow A'_\mu = A_\mu + \frac{1}{2}\partial_\mu \ln\lambda, \tag{13}$$

where the gauge parameter $\lambda \equiv \lambda(\xi^1, \xi^2, \tau_1, \tau_2) \in \mathbb{C}$ is a smooth scalar function *dependent on two spinor components of the original solution and their twistor counterparts* $\tau = X\xi$ *only* (instead of its direct dependence on the 4-coordinates $\{x^\mu\}$ themselves in generally accepted gauge approach). For detailed discussion of this new concept which is based on twistor structure of GSE (see below, Section 7) we refer the reader to our papers [14, 13].

Besides, GSE is invariant under the gauge transformations of Weyl type, related to conformal transformation of the original Minkowsky metric; discussion of *double gauge group* so arising can be found in [11].

In account of the gauge nature of the 4-vector $A_\mu(x)$ and of its close relation to the Weyl nonmetricity vector, it seems quite natural *to identify* $A_\mu(x)$ *(up to a dimensional factor) with the 4-vector of potentials of (complexified) electromagnetic field.* Leaving for the next Section the discussion of complex structure of EM field, we recall only that both the spinor and the EM fields can be found from the only system (8) in a self-consistent way due to overdetermined structure of the latter . Then the question arises what sort of restrictions on EM strengths are imposed by GSE, and in which way are they related to Maxwell equations?

4. Self-duality, Maxwell & Yang-Mills equations as the compatibility conditions of GSE

Since the set of original equations (1) or (8) is overdetermined (8 equations for 2 spinor plus 4 potential components), some *compatibility conditions* should be satisfied "on shell". In particular, commutators of partial derivatives $\partial_{[\mu}\partial_{\nu]}\xi = 0$ in (8) should turn to zero, this being in correspondence with the *closeness* of $\mathbb{B}$-valued 1-form in (1) accord with the Poincaré lemma. After trivial calculations we get then

$$0 = R_{[\mu\nu]}\xi, \tag{14}$$

where the quantities

$$R_{[\mu\nu]} = \partial_{[\mu}A\sigma_{\nu]} - [A\sigma_\mu,\ A\sigma_\nu] \tag{15}$$

represent $\mathbb{B}$-*curvature tensor* of left $\mathbb{B}$-connection (9). From (14) it doesn't follow $R_{[\mu\nu]} \equiv 0$, since the spinor $\xi(x)$ is, certainly, not at all arbitrary. However, it can be shown (see [1, 10] or [14] where 2-spinor formalism has been used) that *self-dual part* $R^+_{[\mu\nu]}$ of (15)

$$R^+_{[\mu\nu]} \equiv R_{[\mu\nu]} + \frac{i}{2}\varepsilon_{\mu\nu}^{\cdot\cdot\rho\lambda}R_{[\rho\lambda]} = 0 \tag{16}$$

should turn to zero by virtue of eqs.(14). Being written in components, expressions (15),(16) result in the following 3+1 set of equations:

$$\mathcal{F}_{[\mu\nu]} + \frac{i}{2}\varepsilon_{\mu\nu}^{\cdot\cdot\rho\lambda}\mathcal{F}_{[\rho\lambda]} = 0, \tag{17}$$

$$\partial_\mu A^\mu + 2A_\mu A^\mu = 0, \tag{18}$$

where the tensor

$$\mathcal{F}_{[\mu\nu]} = \partial_{[\mu}A_{\nu]} \tag{19}$$

is a usual tensor of EM field strength. 3-vector form of eqs.(17)

$$\vec{\mathcal{E}} + i\vec{\mathcal{H}} = 0 \tag{20}$$

relates the (C-valued) *electric* $\vec{\mathcal{E}}$ and *magnetic* $\vec{\mathcal{H}}$ vectors of field strength

$$\mathcal{E}_a = \mathcal{F}_{[oa]} = \partial_o A_a - \partial_a A_o, \quad \mathcal{H}_a = \frac{1}{2}\varepsilon_{abc}\mathcal{F}_{[bc]} = \varepsilon_{abc}\partial_b A_c. \tag{21}$$

Thus, we have found that *self-duality conditions (17) and "inhomogeneous Lorentz condition" (18)*[5] *are just the integrability conditions of GSE.*

According to definitions of field strengths via the potentials (21) and to self-duality conditions (20) we conclude then that *free-space Maxwell equations are satisfied identically for every solution to GSE.*

Complex variable' nature of field strengths (21), however, doesn't result in the doubling of the number of degrees of freedom of EM field just because of the self-duality constraints (20). Indeed, from the latters we get only

$$\vec{B} = \vec{E}, \qquad \vec{D} = -\vec{H}, \tag{22}$$

where $\{\vec{E},\vec{H}\}$ and $\{\vec{D},\vec{B}\}$ represent respectively the real ($\Re$) and imaginary ($\Im$) parts of the initial complex fields $\{\vec{\mathcal{E}},\vec{\mathcal{H}}\}$. The real-part fields $\vec{E}$ and $\vec{H}$ are therewith mutually independent

[5] Geometrically the latter corresponds to the condition for scalar 4-curvature invariant R of the Weyl tensor to be null, see [10].

algebraically (i.e. in a space-time point) and satisfy themselves free Maxwell equations owing to linearity of the latters[6] .

Physical meaning of decomposition of unique complex field into its real and imaginary parts is the following [1, Appendix]. The density of *energy-momentum tensor* can be defined via the latters in a usual way while for original complex fields respective densities (as well as the density of angular momentum)

$$w \propto \vec{\mathcal{E}}^2 + \vec{\mathcal{H}}^2, \qquad \vec{p} \propto \vec{\mathcal{E}} \times \vec{\mathcal{H}} \tag{23}$$

vanish in account of self-duality conditions (20). Moreover, some preferance of $\Re$-part fields $\vec{E}, \vec{H}$ may be therewith justified from geometrical and physical considerations (see Section 5).

In addition to all this, it can be shown [10, 14] that the structure of GSE, and of $\mathbb{B}$-connection (9) in particular, makes it possible to define also the $\mathbb{C}$-valued *Yang-Mills field.* Infact, connection (9) can be rewritten in the form

$$\Gamma_\nu = A(X) * \sigma_\nu = A^\mu(x)\sigma_\mu * \sigma_\nu = A^\mu(x) B^\rho_{\mu\nu}\sigma_\rho \equiv A_\nu(x) + N^a_\nu(x)\sigma_a, \tag{24}$$

where $B^\rho_{\mu\nu}$ are *the structure constants* of $\mathbb{B}$- algebra, and the trace part of connection corresponds to EM 4-potentials $A_\mu(x)$. As to the *trace-free-part* variables $N^a_\nu(x)$,

$$N^a_o = A_a(x); \qquad N^a_b = \delta_{ab} A_o(x) + i\varepsilon_{abc} A_c(x) \tag{25}$$

they can be identified with the potentials of a Yang-Mills (YM) field of a special type. The trace-free part of $\mathbb{B}$-curvature tensor (15) gives then for the *strengths* of YM potentials (25) usual expression

$$\mathcal{L}^a_{[\mu\nu]} = \partial_{[\mu} N^a_{\nu]} - \frac{i}{2}\varepsilon_{abc} N^b_\mu \, N^c_\nu. \tag{26}$$

For a nonzero solution $\xi(x)$ it follows then from eqs.(14) *for every* $[\mu\nu]$ *component of curvature and strengths*

$$\det \| R_{[\mu\nu]} \| \equiv \mathcal{F}^2_{[\mu\nu]} - \mathcal{L}^a_{[\mu\nu]}\mathcal{L}^a_{[\mu\nu]} = 0, \tag{27}$$

In view of (27) *EM field (21) should be regarded as a modulus of isotopic vector of YM-triplet field.* Both fields are described via unique left $\mathbb{B}$-connection (9): EM field is related to the trace part of correspondent curvature while YM field - to the trace-free part of it.

Such an interrelation of EM and YM fields which was proposed firstly in [10] is gauge invariant and requires no participation of auxiliary chiral field as it occures in generally accepted gauge approach. However, the subclass of YM fields (25) can't be *pure*, being always accompanied by EM field due to positive definite norm of isotopic field 3-space (see (27)).

The above speculations would be significant if only the YM equations would be satisfied by the fields (25),(26). Fortunately, it is just the case, since the trace-free part of self dual $\mathbb{B}$-curvature (15) includes only corresponding self-dual configuration of Maxwell strength tensor and Lorentz inhomogeneous form [10], both being null in account of the integrability conditions (17), (18). Thus, for every solution to GSE *field strengths (26) are self-dual and satisfy therefore YM equations for free space.*

It may be noted in conclusion that, contrary to EM case, the $\Re$ and $\Im$ parts of $\mathbb{C}$-valued strengths (26) won't satisfy nonlinear YM equations separately. Thus, YM fields arising here are essentially complex-valued. On the other hand, it can be proved directly that non-Abelian (commutator) part of YM strengths (26) does not vanish for the potentials (25) neither identically nor on the solutions to GSE, so that they can't be reduced generally to EM fields[7] .

[6]The same is true, of course, for the $\Im$-part fields $\vec{D}, \vec{B}$ as well providing, in account of (22), a dual solution to Maxwell equations.

[7]Possible nonAbelian nature of EM field itself was discussed recently in [18], also in the framework of Weyl geometry.

5. Fundamental unisingular solution, quantization of electric charge and ring-like model of electron

Vacuum Maxwell equations hold identically for every solution of GSE. Hence, no regular "soliton-like" field distribution can exist for the model considered. Nevertheless, *particles can be brought into correspondence with singular points (or strings, membranes etc.) of the field functions* in which $\mathbb{B}$-differentiability conditions are violated and which manifest themselves as point-like or extended source of physical fields.

In account of complex variable and self-dual nature of gauge fields established above, *charged singular solutions*, if exist, should be *dions*, i.e. should carry both electric and magnetic charges of equal (up to a factor "i") magnitudes. Indeed, elementary unisingular dion-like solution has been found in [1, 10]. To obtain it here, we'll fix the gauge so that the only component $G(x)$ of fundamental 2-spinor $\xi(x)$ will remain in GSE[8]:

$$\xi^T(x) = (\ 1,\ G\ ); \tag{28}$$

then for complex EM potentials $A(x)$ one gets

$$A_w = \partial_u G, \quad A_v = \partial_{\bar{w}} G, \quad A_u = A_{\bar{w}} \equiv 0, \tag{29}$$

and GSE reduces to a couple of nonlinear differential equations for *a unique unknown function* $G(x)$

$$\partial_w G = G \partial_u G, \qquad \partial_v G = G \partial_{\bar{w}} G, \tag{30}$$

where the spinor space-time coordinates $\{u, v, w, \bar{w}\}$ defined previously by eq.(5b) have been used.

By mutual multiplication of eqs.(31) we'll come then to the constraint

$$(\partial_u G)(\partial_v G) - (\partial_w G)(\partial_{\bar{w}} G) = 0, \tag{31}$$

which is nothing but the eikonal eq.(7) in spinor coordinates. If we'll then write out the commutator of derivatives in the l.h.p. of eqs. (30) we get with respect to eq. (31)

$$\partial_u \partial_v G - \partial_w \partial_{\bar{w}} G = 0, \tag{32}$$

where the latter equation is just the *wave (d'Alembert) equation* $\bigtriangledown^2 G = 0$. It can be shown that the last result is gauge invariant in the following sense: *the ratio of two components of the spinor field $\{\xi_B(x)\}$ obeys "on shell" both the eikonal and d'Alembert equations.*.

Fundamental static axisymmetric solution to GSE (which also satisfy eqs. (31), (32) has been found in [1, 10]. It corresponds to *stereographic mapping* $S^2 \mapsto \mathbb{C}$ of the Riemannian 2-sphere onto the complex plane:

$$G = \frac{x^1 + ix^2}{x^3 \pm r} \equiv \frac{\bar{w}}{z \pm r} \equiv \tan^{\pm 1} \frac{\theta}{2} \exp^{i\varphi}, \tag{33}$$

$\{r, \theta, \varphi\}$ being usual spherical coordinates. From the solution (33), which satisfy the couple of eqs. (30) under consideration, complex EM potentials (29) A_w, A_v can be found; then for the scalar (A_o) and spherical components $\{A_r, A_\theta, A_\varphi\}$ of 4-potential we'll have

$$A_o = \pm \frac{1}{4r}, \quad A_r = -\frac{1}{4r}, \quad A_\varphi = \pm i A_\theta = \frac{i}{4r} \tan^{\pm 1} \frac{\theta}{2} \tag{34}$$

Now for nonzero components of $\mathbb{C}$-valued EM field strengths (21) we get (the electric field appears to be pure real, while the magnetic - pure imaginary)

$$\mathcal{E}_r = \pm \frac{1}{4r^2}, \quad \mathcal{H}_r = \pm \frac{i}{4r^2}, \tag{35}$$

[8] This gauge is possible for every "physically nontrivial" solution to GSE, see [14] for details.

(note that the components $A_r,\ A_\theta$ don't contribute into the magnitude of field strengths, being a "pure gauge"). We see that fundamental solution (33) corresponds to *a point source of real electric (and imaginary magnetic) field and with fixed value of electric charge* $q = \pm 1/4$ *(and equal value of imaginary magnetic charge* $m = \pm i/4$*)*.

At this stage of consideration, factor (1/4) is unessential since *physical* EM potentials are determined up to an arbitrary dimensional factor only. What is really significant is that in this model 1) all values of electric charge except *the only possible one* are not allowed for a point singular source to possess, and 2) its Coulomb field is always accompanied by the magnetic monopole field with the charge equal in modulus to the electric one! For the proof of general theorem on charge quantization which is based on self-duality condition (20) and gauge invariance (13) of GSE we refer the reader to the paper [16]. For similar models where the property of charge quantization arises see [11].

Let us consider now an interesting modification of solution (33)-(35), which can be obtained via *complex translation* $z \mapsto z + ia,\ \ a \in R$, the latter being obviously a symmetry of GSE. In this way we come to a new solution which electric field structure instead of Coulomb form (35) corresponds to the known *Appel solution* (see e.g. [29]). Singular locus of EM field would be then defined by the condition

$$r^* \equiv \sqrt{(z+ia)^2 + x^2 + y^2} = 0, \quad \Rightarrow \quad \{x^2 + y^2 = a^2, \ \ z = 0\}, \tag{36}$$

and is a *ring* of radius $|a|$. For real-part fields ($\Re$-fields) $\{\vec{E}, \vec{H}\}$ the magnetic component appears, and the following asypmtotic behaviour at distances $r >> |a|$ is true:

$$E_r \simeq \frac{q}{r^2}(1 - \frac{3a^2}{2r^2}(3\cos^2\theta - 1)),\ E_\theta \simeq -\frac{qa^2}{r^4} 3\cos\theta \sin\theta,\ H_r \simeq \frac{2qa}{r^3}\cos\theta,\ H_\theta \simeq \frac{qa}{r^3}\sin\theta. \tag{37}$$

In view of eqs. (36)-(37), $\Re$-field solution is related to electrically charged singular ring equipped with a quantized "elementary" value of electric charge $q = \pm 1/4$, *dipole magnetic moment* $\mu = qa$ and *quadrupole electric moment* $\vartheta = -2qa^2$. If we'll choose dimensional physical units so that to have $q = e$, e being elementary charge, and accept for the radius of the ring the value

$$|a| = \frac{\hbar}{2Mc}, \qquad 38$$

M being the mass of the source, then for dipole magnetic moment μ we'll get the known Dirac value $\mu = e\hbar/2Mc$. Moreover, according to (37) we conclude that *fundamental charged fermion should necessarily possess quadrupole electric moment* ϑ equal in magnitude to

$$\vartheta = \frac{e\hbar^2}{2M^2c^2} \tag{39}$$

It should be marked that prediction of quadrupole moment (39) for elementary Dirac-like particles, to our knowledge, has been made firstly by C.A. Lopes [25] in the framework of GTR and on the base of Kerr-Newman metric. We'll see further that our fundamental solution is also deeply related to Kerr-Newman metric. At present, the statement about necessary existence of quadrupole electric moment looks rather speculative; nontheless, possibility of its experimental proof may be discussed.

However, much more fundamental seems to us the fact that for $\Re$-part fields their asymptotic structure (37) is in complete agreement with that observed for elementary particles, whereas the $\Im$-fields contain only "phantom" terms proportional, say, to magnetic charge or to dipole electric moment! This property is based just on complex self-dual structure of EM fields and, as well as the property of charge quantization, *is peculiar only for our model!*

Geometrically, phantom-like $\Im$-fields representing, in particular, magnetic monopole, contribute only into the *torsion* term[9] of real projection of original complex $\mathbb{B}$-connection (9) onto Minkowsky space [10]. Therefore, owing to specific (totally skew symmetric) structure of torsion figurating in

[9]Relation between the magnetic monopole field and the geometries with torsion has been advocated, in particular, by G.Lochak [26].

(9), $\Im$-fields won't enter into equations of *geodesics*, thus resulting in *non-observability of magnetic charges and electric dipole moments of elementary particles* (see [10] for details). Other considerations about the possible role of magnetic monopoles and, generally, of "phantom" $\Im$-fields in GSE-based model can be found in [16].

6. Shear-free null congruences and effective metrics related to GSE

In special gauge (28),(29) GSE was shown to reduce to the couple of eqs.(30). The latter is well-known in GTR as the system defining a field of null vector $l_\mu(x)$ tangent to a *shear-free (geodesic null) congruence* (SFC). For this vector we can take, say, the spinor representation

$$l_\mu \Leftrightarrow L = \begin{pmatrix} 1 & \bar{G} \\ G & \bar{G}G \end{pmatrix}, \quad \det \|L\| \equiv 0.$$

Via SFC, effective Riemannian metric of well-known *Kerr-Schild type* can be defined as

$$ds^2 = ds_o^2 + H(l_\mu dx^\mu)^2 \equiv (dudv - dwd\bar{w}) + H(du + Gd\bar{w} + \bar{G}dw + G\bar{G}dv)^2, \tag{40}$$

where ds_o is Minkowsky part of the interval, and function $H(x)$ can be determined from the requirement for metric (40) to satisfy either vacuum Einstein or electrovacuum Einstein-Maxwell equations. It's well known [20] that both opportunities can be realized for SFC generated by point-source solution (33) or its ring-like deformation, for the latter metrics (40) being Kerr or Kerr-Newman (in electrovacuum case) solutions. For particular case of solution (33) with $|a| = 0$ one has Schwarzchild or Reisner-Nördstrem metrics respectively.

From the viewpoint of the considered GSE-model correspondence with electrovacuum system seems, of course, more consistent. Remarkably, EM fields related to fundamental solution (with asymptotics (37)) belong to the so called class of *invariant fields* [30] and obey free Maxwell equations *both in flat and Kerr-Newman space.* Thus, efective Kerr-Newman (or Reisner-Nördstrem in particular) solution can be defined for fundamental stationary Ansatz of GSE *which EM fields, contrary to classical case of GTR, possess unit and only unit electric charge.* If one reminds now well-known expression $J = Mc|a|$ for the angular momentum (spin) J of Kerr-Newman source, then from previous assumption (39) on the value of Kerr parameter a one immegiately has $J = \hbar/2$, so that *all quantum numbers for ring-like singular source represented by fundamental solution to GSE coincide with those of Dirac-like fermion!* However, for *arbitrary* solution to GSE its relationship to solutions of Einstein-Maxwell system is not yet clear.

In conclusion, it should be mentioned that Riemannian metrics (40) defined for solutions of GSE may be regarded only as *effective* and strongly resumble those used in A.A. Logunov's *relativistic theory of gravitation* [27] in which physical space-time preserves its flat Minkowsky geometry. Genuine significance of these metrics is completely dynamical and in the framework of GSE model can be displayed only via consideration of interactions of "particle-like" singular sources (see Section 8 below).

7. Twistor variables and general solution to GSE. Twistor representation of electromagnetic field strengths and caustic condition for singularities

We pass now to the demonstration of complete integrability of SFC system of eqs. (30) (and, therefore, of GSE itself) which is based on a famous *Kerr theorem* (see [21, chapter 7]). According to the latter, general (analytical) solution to SFC equations has the form of implicit functional dependence

$$\Pi(G, \tau_0, \tau_1) = 0 \tag{41}$$

of the projective spinor component $G(x)$ on related *twistor variables*

$$\tau = X\xi \qquad (\tau_0 = u + wG, \quad \tau_1 = \bar{w} + vG). \tag{42}$$

where fundamental spinor field $\xi(x)$ is taken in the gauge (28) and X, as before (5b), is the Hermitian matrix of space-time coordinates.

In eq. (41) function Π may arbitrarily yet smoothly depend on its three complex arguments, and for every analytical solution to SFC system some generating function Π can be found. For a given Π, resolving eq. (41) with respect to unknown $G(x)$ one obtains a solution to SFC eqs. (30).

However, resolving procedure is unique everywhere exept in the *branching points* of (generally multi-valued) function $G(x)$. These points correspond to the *caustic* condition

$$Q = \frac{d\Pi}{dG} = 0, \tag{43}$$

where infinitesimal rays of SFC intersect each other. It's well known [20, 22] that just on the caustics defined by eq. (43) *curvature of the Kerr-Schild metric (40) becomes singular.* On the other hand, *the same condition (43) can be verified to fix the singularities of EM-field* which can be constructed from $G(x)$-function via 4-potentials (29). Indeed, in our paper [14] for spintensor $\{F_{(AB)}\}$, $A, B = 0, 1$ of EM field strengths the following remarkable expression involving only twistor variables has been obtained:

$$F_{(AB)} = \frac{1}{Q}\left(\Pi_{AB} - \frac{d}{dG}\left(\frac{\Pi_A \Pi_B}{Q}\right)\right), \tag{44}$$

where by $\Pi_A, \Pi_{AB}, ...$ first and second order derivatives of generating function Π (41) with respect to its twistor arguments (42) are denoted. On the caustics (43) EM fields (44) evidently become singular.

8. Bounded field singularities as the model of particles in GSE

Structure of GSE (1), purely abstract, algebraic in origin and compact in form, was found to be unexpectedly rich, being related to spinors, twistors, gauge fields and effective geometries. Every solution to GSE identically satisfy both linear Maxwell and nonlinear Yang-Mills equations. Singular structure of EM fields[10] coincides with that of curvature of related Riemannian metrics both being defined by the caustic condition (43). If this common singular locus is bounded in 3-space (at every finite moment of time), it seems quite natural *to identify it with elementary particle.* Note that for stationary case bounded singularities of SFC and correspondent curvature are known [23, 24] to be exhausted by the singular ring of Kerr-Newman solution.

On the other hand, arbitrary solution to linear Maxwell equations doesn't necessarily correspond to some any solution of overdetermined nonlinear GSE. From such "hidden nonlinearity" of GSE-electrodynamics both quantization of electric charge of singular sources and nontrivial time evolution of particles-singularities do follow.

From purely mathematical point of view, the shape and dynamics of singularities bear direct relationship with rapidly progressing *catastrophe theory* in which framework classification of singularities of differentiable mappings and of their bifurcations has been obtained (see, e.g. [28, 29]). In our approach, time evolution and bifurcations of (bounded) singular sources can be interpreted as the *process of interactions and transmutations of elementary particles!*

Confirmation for such a conjecture has been found recently in our works [13, 14] in which, in particular, exact *bisingular* solution to GSE (and, hence, to Maxwell and Yang-Mills equations) has been presented. Its structure describes axisymmetric "interaction" of two point-like and oppositely charged singularities for which the magnitudes of charges are equal to the charge of unisingular

[10] YM fields possess additional singularities correspondent to the poles of G-function itself.

solution (35). Under imaginary value of integration constant, the solution undergoes bifurcation strongly resumbling the *"creation - annihilation" process* of particles - singularities, and manifests also an *intermediate resonance structure of toroidal-like geometry* [14]. In [12] exact solutions with 8-figure- and helix-like singularities have been obtained while in [16] more complicated multisingular solutions have been announced. We hope to examine them in detail elsewhere, as well as the general problem of interactions and bifurcations of singularities in the framework of unified algebraic model based on GSE (1).

To conclude, we'll enumerate some features of the considered model which can be of interest from general (orthodox) viewpoint. In the first place, powerful and simple algebraic method of generation of complicated solutions to Maxwell and (complexified) Yang-Mills vacuum equations based on eqs. (41), (42) and (44) should be marked. The most intriguing feature of this method is that it selects just those solutions to gauge equations for which *the charge of every bounded singularity is equal to unit (elementary) one.*

Secondly, we want to attract reader's attention to, rather exotic, "weak" gauge invariance (13) of GSE model which is deeply related to projective transformations in twistor space and to special subgroups of the gauge $SL(2,\mathbb{C})$ group (for details, see [14]). Discovered "weak" invariance can be used to construct a new class of gauge field models in the framework of generally accepted Lagrangian approach.

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