Relativistic invariance of a many body system with a Dirac oscillator interaction

M. Moshinsky*, **G. Loyola**, **C. Villegas**†
Instituto de Física, UNAM Apdo. Postal 20-364, México, D.F., 01000 México

Abstract. In a recent publication Moshinsky and Szczepaniak considered a one particle Dirac equation linear in *both* momenta and coordinates. As in the non–relativistic limit it gave a Hamiltonian containing an ordinary oscillator, the equation was referred to as that of a Dirac oscillator. In this paper we extend the concept of Dirac oscillator interactions to a system of n particles showing that it can be derived in a relativistically invariant way. Thus the eigenstates for the mass operators of Dirac oscillators with 2 or 3 particles, that were discussed in previous publications, are basis for irreducible representations of the Poincaré group, and they can be used to derive a relativistic mass formula for baryons.

In a recent publication [1], Moshinsky and Szczepaniak discussed a single particle Dirac equation that was linear in *both* momenta and coordinates. As the large component in this equation turned out to be an eigenfunction of an ordinary oscillator plus a strong spin orbit force, they gave to the problem the name of Dirac oscillator.

The authors mentioned [1] extended their analysis to many body systems with Dirac oscillator type of interactions, but their discussion of the relativistic invariance of the problem was not considered complete. Nevertheless they obtained [2] in an exact and analytic form the eigenstates and eigenvalues for the two particle Dirac oscillator.

For three particles the problem could still be solved exactly, but required the determination of roots of secular equations associated with *finite* matrices, for which numerical computation were needed. This problem was discussed [3] and further extended to determine a relativistic mass formula for baryons [4].

In this note our wish though is to prove that the Dirac oscillator problem for a many body system can be obtained from a relativistically invariant equation, *i.e.* one that commutes with the generators of the Poincaré group. Thus the eigenfunctions mentioned in the previous paragraphes, that were obtained in the frame of reference where the center of mass was at rest, are basis for irreducible representations of the Poincaré group.

---

* Member of El Colegio Nacional

† Present address: IMAS, UNAM

We shall start by briefly reviewing the mass operator for an n–body system with Dirac oscillator interaction and later proceed to rewrite it in a relativistically invariant form.

For a single free particle the Dirac Hamiltonian takes the form [5]

$$H_o = \boldsymbol{\alpha} \cdot \mathbf{p} + m\beta, \tag{1}$$

where

$$\mathbf{p} = -i\nabla, \boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{2a, b, c}$$

with $\boldsymbol{\sigma}$ being the vector of Pauli spin matrices and in units where $\hbar = c = 1$.

For n–free particle of the same mass m, the Hamiltonian becomes

$$H_o = \sum_{s=1}^{n} (\boldsymbol{\alpha}_s \cdot \mathbf{p}_s + m\beta_s), \tag{3}$$

where the matrices are direct products such as

$$\beta_s = I \otimes \cdots I \otimes \beta \otimes I \cdots \otimes I, \tag{4}$$

with β in the position s, and similarly for the $\boldsymbol{\alpha}_s$.

Introducing the total momentum

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 + \cdots + \mathbf{p}_n, \tag{5}$$

the Hamiltonian can be rewritten as

$$H_o = n^{-1}(\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 + \cdots + \boldsymbol{\alpha}_n) \cdot \mathbf{P} + H_o', \tag{6}$$

where

$$H_o' = \sum_{s=1}^{n} (\boldsymbol{\alpha}_s \cdot \mathbf{p}_s' + m\beta_s) \quad ; \mathbf{p}_s' = \mathbf{p}_s - n^{-1}\mathbf{P}, \tag{7a, b}$$

In the frame of reference where the center of mass of this system of n–particles is at rest *i.e.* when $\mathbf{P} = 0$, the Hamiltonian becomes the H_o' of (7a) and can be interpreted as the mass of the n–free particle system with relative motions between them.

The Dirac oscillator n–body mass operator, which we shall designate by $\mathcal{M}$ was obtained [1,3] when in (7a) we made the replacement

$$\mathbf{p}_s' \longrightarrow \mathbf{p}_s' - i\mathbf{x}_s' B, \tag{8}$$

where

$$\mathbf{x}_s' = \mathbf{x}_s - \mathbf{X}, \mathbf{X} = n^{-1}(\mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_n), \tag{9a, b}$$

$$B = \beta \otimes \beta \otimes \beta ..... \otimes \beta \tag{9c}$$

Thus we have

$$\mathcal{M} = \sum_{s=1}^{n} \left[\boldsymbol{\alpha}_s \cdot (\mathbf{p}_s' - i\mathbf{x}_s' B) + m\beta_s \right] \tag{10}$$

whose spectra and eigenfunctions for $n = 2$ and 3 were determined in reference [2] and [3].

Clearly the previous analysis does not seem invariant under transformations of the Poincaré group, but we shall show below the $\mathcal{M}$ can also be derived from an equation that is explicitly relativistically invariant.

We shall start this part of our discussion by returning to the one body problem and defining the γ^μ matrices, where $\mu = 0, 1, 2, 3$, as

$$\gamma^o = \beta, \gamma^i = \beta\alpha_i, i = 1, 2, 3, \tag{11a, b}$$

where our metric will be

$$g_{\mu\nu} = 0, \mu \neq \nu; g_{11} = g_{22} = g_{33} = -g_{00} = 1, \tag{12}$$

and the γ^μ matrices satisfy the anticommutation relation

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = -2g^{\mu\nu}. \tag{13}$$

As usual [6] the spin part of the generators of the Lorentz group is given by

$$S^{\mu\nu} = (i/4)(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu), \tag{14}$$

so that from (11) and (12) we obtain

$$[S^{\mu\nu}, \gamma^\tau] = i(g^{\mu\tau}\gamma^\nu - g^{\nu\tau}\gamma^\mu). \tag{15}$$

On the other hand the orbital part of the generators of the Lorentz group is given by [5]

$$L^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu, \tag{16}$$

so that from $[x^\mu, p^\nu] = ig^{\mu\nu}$ we obtain

$$[L^{\mu\nu}, x^\tau] = i(g^{\mu\tau}x^\nu - g^{\nu\tau}x^\mu), \tag{17}$$

and similarly for the momentum p^τ.

The full generator of the Lorentz group for the one body problem is then [5]

$$J^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu}, \tag{18}$$

and from (15) and (17) we see that $\gamma^\tau, x^\tau, p^\tau$, transform in the same way under the $J^{\mu\nu}$ so we can interpret all three of them as four vectors.

For an n–body problem $\gamma_s^\mu, s = 1, 2, \cdots, n$ are direct products as for example

$$\gamma_s^o = I \otimes I \cdots I \otimes \beta \otimes I \cdots \otimes I, \tag{19}$$

were we made use of (11a) and the β is in the s^{th} position. The $\gamma_s^\mu, x_s^\mu, p_s^\mu$ remain though four vectors as their commutation relation with

$$J^{\mu\nu} = \sum_{s=1}^{n} J_s^{\mu\nu}, \tag{20}$$

continue to be of the type (15) or (17), where $J_s^{\mu\nu}$ are defined by (14), (16), (18), adding an index s to all the variables involved.

The total momentum four vector P_μ, defined as in (5), will play an important role in the following discussion, as well as does a unit time like four vector which we shall designate by u_μ. As a final point in notation we define the scalars

$$\Gamma_s = (\gamma_s^\mu u_\mu)^{-1}\Gamma, \quad \Gamma = \prod_{r=1}^{n} \gamma_r^\mu u_\mu \tag{21a, b}$$

where repeated indices are summed over $\mu = 0, 1, 2, 3$. Note that $(\gamma_s^\mu u_\mu)^{-1}$ in (21a) just eliminates the corresponding term in the Γ of (21b) so Γ_s is still in product form.

We now turn our attention to papers of Barut and Komy [7] and Barut and Strobel [8]. In them they derive, from an appropriate variational procedure applied to a field theoretical action, a *single* covariant equation for an n–body system. For non–interacting particles, and in the notation introduced above, this equation takes the form [7,8]

$$\left[\sum_{s=1}^{n} \Gamma_s \left(\gamma_s^\mu p_{\mu s} + m\right)\right] \psi = 0, \tag{22}$$

where so far the u_μ appearing in the Γ_s of (21) is arbitrary, except for the requirement that it should be time like, of unit length, and transform as a four vector.

We first show that (22) is the covariant form of the equation obtained when we apply the operator (3) to ψ, *i.e.* for the system of n non–interacting particles. For this purpose we choose the frame of reference in which $(u_\mu) = (1000)$ where (22) takes the form

$$\left[\Gamma^o \sum_{s=1}^{n} p_{os} + \sum_{s=1}^{n} \Gamma_s^o(\gamma_s^\mu p_{\mu s} + m)\right] \psi = 0, \tag{23}$$

with

$$\Gamma^o \equiv \prod_{r=1}^{n} \gamma_r^o = B, \Gamma_s^o \equiv (\gamma_s^o)^{-1}\Gamma^o. \tag{24a, b}$$

Multiplying (23) by Γ^o and making use of (2b,c) and (11) we obtain

$$\left[-P^o + \sum_{s=1}^{n} (\boldsymbol{\alpha}_s \cdot \mathbf{p}_s + m\beta_s)\right] \psi = 0, \tag{25}$$

where we put the time like component of the four momentum in its covariant form using the metric (12). Comparing (25) with (3) we see that we achieve the objective indicated at the beginning of the paragraph if we interpret P^o as H_o, as is usually done.

Equation (22) can also be written in the form

$$\left[n^{-1}\sum_{s=1}^{n}\Gamma_s(\gamma_s^{\mu}P_{\mu})+\sum_{s=1}^{n}\Gamma_s(\gamma_s^{\mu}p'_{\mu s}+m)\right]\psi=0, \tag{26}$$

where P_μ and $p'_{\mu s}$ are the total and relative four momenta defined respectively by (5) and (7b) when we put an index $\mu = 0,1,2,3$ on all the variables.

We now give to the u_μ appearing in the Γ_s of (26) a dynamical meaning by writing

$$u_\mu = (P_\mu/P), P = (-P_\mu P^\mu)^{1/2}, \tag{27a, b}$$

which implies that the unit time like four vector u_μ takes its form (1000) in the frame of reference in which the center of mass is at rest. By an analysis entirely similar to the one leading from (22) to (25) we see that in this frame (26) becomes

$$\left[-P^o+\sum_{s=1}^{n}(\boldsymbol{\alpha}_s\cdot\mathbf{p}'_s+m\beta_s)\right]\psi=0, \tag{28}$$

which is exactly what we obtain if we apply (7a) to ψ and identify P^o with H'_o.

Thus we see that, with the choice (27) for u_μ, we get from the covariant equation (26) the total energy for a system of n non–interacting particles in the frame of reference in which their center of mass is at rest.

We can immediately generalize this analysis to the case when there is a Dirac oscillator interaction between the particles if we make the replacement

$$p'_{\mu s}\longrightarrow p'_{\mu s}-ix'_{\mu s}\Gamma \tag{29}$$

in (26), where $p'_{\mu s}, x'_{\mu s}$ are given respectively by (7a), (9a) when we put an index $\mu = 0,1,2,3$ on all the variables and Γ was defined in (21b). The covariant equation for the n–body Dirac oscillator becomes then

$$\left\{n^{-1}\sum_{s=1}^{n}\Gamma_s(\gamma_s^{\mu}P_{\mu})+\sum_{s=1}^{n}\Gamma_s\left[\gamma_s^{\mu}(p'_{\mu s}-ix'_{\mu s}\Gamma)+m\right]\right\}\psi=0, \tag{30}$$

and it reduces to the equation that is obtained when we apply (10) to ψ when we pass to the frame of reference in which the center of mass is at rest, where now we identify P^o with $\mathcal{M}$. Note that u_μ appearing in the Γ, Γ_s of (30) is now given by (27) and is *not* the unit time like four vector required by Barut [7,8] in his discussion of the *single* covariant n body equation (22) in an *arbitrary* frame of reference.

Equation (30) commutes with P_μ, $J^{\mu\nu}$ and thus is an invariant of the Poincaré group whose Casimir operators [6] are

$$P^2 = -P_\mu P^\mu, W^2 = W_\mu W^\mu, W_\mu = \frac{1}{2}\epsilon_{\mu\nu\sigma\tau} P^\nu J^{\sigma\tau}, \qquad (31a, b, c)$$

which in the center of mass frame of reference reduce to [6]

$$P^2 = (P^o)^2, W^2 = (P^o)^2 J^2, \qquad (32a, b)$$

where P^o is now the total mass $\mathcal{M}$ and J^2 the total angular momentum of the system of n particles.

In the solution of the equation

$$\mathcal{M}\psi = \mu\psi, \qquad (33)$$

where $\mathcal{M}$ is given by (10), for systems of 2 and 3 particles, [2,3] we not only considered the eigenstates corresponding to eigenvalues μ of the operator $\mathcal{M}$, but required also that they should be eigenstates of the total angular momentum J^2 with eigenvalues $j(j+1)$. Thus our states [2,3] for two and three particles are basis for irreducible representations of the Poincaré group characterized by μ^2 and $\mu^2 j(j+1)$. Fur thermore these states, and their corresponding eigenvalues, were used in the determination of a relativistic mass formula [4], whose predictions were compared with experiment.

We wish to acknowledge fruitful discussions on this subject with C. Quesne.

References

1. Moshinsky, M. and Szczepaniak, A. *J. Phys. A: Math. Gen.* **22**, L817 (1989).

2. Moshinsky, M., Loyola, G., Szczepaniak, A. *The two body Dirac oscillator* (Anniversary Volume in Honor of J.J. Giambiagi, World Scientific Press, Singapore, 1990).

3. Moshinsky, M., Loyola, G., Szczepaniak, A., Villegas, C., Aquino, N.,*The Dirac oscillator and its contribution to the baryon mass formula* (Proceedings of the Rio de Janeiro International Workshop on Relativistic Aspects of Nuclear Physics, 1989, World Scientific Press, Singapore, 1990) pp. 271-307.

4. M. Moshinsky, G. Loyola y C. Villegas, *Relativistic Mass Formula for Baryons*, Proceedings of the 13^{th} Oaxtepec Symposium on Nuclear Physics, Notas de Física, Vol **13**, No. 1 pp. 187-195, (1990).

5. Schiff, L.I. *Quantum Mechanics* (New York, McGraw–Hill) Third Edition, Chapter 13, (1968).

6. Kim, Y.S. and Noz, E.M., 1986 *Theory and Applications of the Poincaré Group* (Dordrecht, Reidel Publishing Co, 1986) Chapter III.

7. Barut, A.O. and Komy, S.,*Fortsch. Phys.* **33**, 6, (1985).

8. Barut, A.O. and Strobel, G.L., *Few-Body systems,* **1**, 167, (1986).