Fermi National Accelerator Laboratory

FNAL–PUB -93/034-A
February 1993

On the Correspondence between Theory and Observations in the Inflationary Cosmology

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Abstract

We study the dynamical correspondence between the scalar fluctuation spectrum and the inflationary potential $V(\phi)$ and consider a power law spectrum with index β as an example. We show that the range of β relevant to the interpretation of *COBE* observations, this correspondence is fragile, in the sense that small changes in β could produce dramatic changes in the corresponding $V(\phi)$. This is of potential significance for the reconstruction of $V(\phi)$ from such observations and also in the wider context of inflationary models in which $V(\phi)$ is specially chosen to produce a desired spectrum and dynamics.

Submitted to *Physics Letters B*

Operated by Universities Research Association Inc. under contract with the United States Department of Energy

1 Introduction

The inflationary scenario is commonly invoked to resolve a number of outstanding problems in the standard big bang cosmology [1]. An appealing feature of all inflationary models is that they predict a primordial fluctuation spectrum in the early Universe, which may lead to the observed large-scale galactic structure and anisotropies in the cosmic microwave background radiation (CMBR) [2][3]. The details of these fluctuations, such as their amplitude, scale-dependence and statistics are strongly model dependent and provide one of the strongest constraints on the scenario. The usual approach in inflationary cosmology is to obtain the appropriate observational predictions by specifying the particle physics of the model in the form of an inflationary potential. With this policy, though, there still remain a high number of plausible 'working' models. In light of recent advances in observational results, however, an alternative view is to adopt the reverse procedure and reconstruct the model from the set of observations in the hope that such a program would significantly reduce the number of theoretical options available [4]. Clearly in both approaches it is important to consider the structure and stability of the correspondence which relates theory with observation. The aim of this work is to investigate such a correspondence from a *dynamical* point of view by considering the relationship between the primordial fluctuations and the model.

In the chaotic scenario based on a minimally coupled scalar *inflaton* field propagating in the spatially flat Friedmann universe with self-interaction potential $V(\phi)$, the 'energy' and 'momentum' equations are

$$H^2 = \frac{\kappa^2}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) \tag{1}$$

$$2\dot{H} = -\kappa^2\dot{\phi}^2 \tag{2}$$

where a dot denotes d/dt, $\kappa^2 \equiv 8\pi m_{Pl}^{-2}$, m_{Pl} is the Planck mass, $H \equiv \dot{a}/a$, $a(t)$ is the scale factor and units are chosen such that $\hbar = c = 1$ [5]. Short wavelength quantum fluctuations in the inflaton and graviton fields are redshifted beyond the Hubble scale during inflation and re-enter during the radiation- or matter-dominated phases. These produce primordial scalar and tensor perturbation spectra which depend strongly on the form of $V(\phi)$. Scales relevant for large-scale galactic structure lie in the range 1 Mpc to 6000 Mpc and represent only $\ln(6000) \approx 9$ of the total number of inflationary e–foldings. Fluctuations associated with these scales first crossed the horizon approximately 60 e–foldings before the end of inflation when ϕ had some value ϕ_l [2]. (First horizon crossing occurs when the comoving length scale $\lambda = 2\pi(Ha)^{-1} = 2\pi k^{-1}$). It is therefore reasonable to assume that $\dot{\phi}$ does not pass through zero in this region and it is self–consistent to divide Eq. (2) by $\dot{\phi}$ and view the scalar field as the effective dynamical variable [6]. This enables the field equations to be rewritten in the Hamilton–Jacobi form thus:

$$\kappa^4 V(\phi) = 3\kappa^2 H^2(\phi) - 2(H')^2 \tag{3}$$

$$2H' = -\kappa^2 \dot{\phi}, \tag{4}$$

where $' \equiv d/d\phi$. The form of the potential follows immediately from Eq. (3) once $H(\phi)$ is known.

Recently, *COBE* observed anisotropies in the CMBR temperature distribution on angular scales exceeding 10^o [3]. Since the horizon size at decoupling is $\approx 1^o$, the primordial fluctuations responsible for these anisotropies were super–horizon sized at decoupling and arose through the Sachs–Wolfe effect [7]. They therefore entered the Hubble sphere during matter–domination at some time $t_{\rm HC}$ and the amplitudes of the scalar (A_S) and tensor (A_G) modes at $t_{\rm HC}$ are

$$A_S(k(\phi)) = \frac{\kappa^2}{20\pi^{3/2}} \frac{H^2}{|H'|} \tag{5}$$

$$A_G(k(\phi)) = \frac{\kappa}{4\pi^{3/2}} H \tag{6}$$

respectively, where the quantities on the right–hand side are evaluated at $\phi = \phi_l$ [2].

2 The dynamical correspondence and potential fragility of models

An important feature of Eqs. (1) and (2) is that they may be viewed as setting up a nonlinear mathematical correspondence between the 'dynamics' as determined by the scale factor a and the inflationary potential $V(\phi)$ [8]. Observationally one can also think of a table of the correspondences between, for example, the primordial scalar fluctuation spectrum A_S as a function of the comoving wave number k and the inflationary potential (or equivalently $H(\phi)$). In this way one may consider two function spaces: one consisting of all the possible functional forms of $A_S(k)$ say, labelled as a 'space of observations', and the other containing all the functional forms of $H(\phi)$ (or $V(\phi)$), labelled as the 'space of models'.

Now observations are always imprecise and the exact nature of the inflationary potential $V(\phi)$ is not well known. As a result, in practice one must always consider mappings between *neighbourhoods* in these spaces. The question then arises as to the nature of such 'approximate correspondences', i.e. whether they are stable or not in the sense of neighbourhoods corresponding to neighbourhoods. This would clearly be the case if such correspondences turn out to be structurally stable (i.e. one–to–one), where small errors or ignorance on one side of the correspondence would not lead to *qualitatively* important changes on the other. But within the context of our discussion here, this only makes sense if errors involved in observations do not in turn lead to qualitatively different functional forms for $V(\phi)$. Now an important outcome of the developments in the nonlinear dynamical systems theory is that structural stability as a common property of dynamical systems cannot be assumed a priori; in fact the appropriate theoretical framework in a variety of settings in cosmology may turn out to be that of *fragility*[1] [9]. As a result the nature of the approximate correspondences between the imprecise observations of such spectra and the inflationary potential need to be studied concretely.

Towards this aim, we proceed to set up the mathematical correspondence between the scalar and tensor fluctuation spectra and the potential [4]. A fluctuation has physical size H^{-1} when it first crosses the horizon at ϕ_l and its present day length scale, λ, is related to H^{-1} by the total number of e–foldings, $N \equiv \int H\mathrm{d}t$, that occur between ϕ_l and the current epoch. We write $N = N_{\rm inf} + N_*$, where $N_{\rm inf}$ represents the e–foldings from ϕ_l to the end of inflation at ϕ_f and N_* the number from ϕ_f to the current epoch. It follows that

$$\lambda = 2\pi H^{-1}(\phi) \exp\left[N_* + \frac{\kappa^2}{2}\int_{\phi_f}^{\phi_l} \mathrm{d}\phi \frac{\mathrm{H}}{\mathrm{H}'}\right]. \qquad (7)$$

[1]In other words, fragile behaviour arises when the correspondence between neighbourhoods in the two function spaces is one-to-many.

A differential equation relating A_S to A_G can now be derived by considering the ratio of Eqs. (5) and (6) in the form

$$\frac{\kappa}{5}\frac{A_G}{A_S} = \left|\frac{\mathrm{d}\,\lambda}{\mathrm{d}\,\phi}\frac{\mathrm{d}\ln \mathrm{A_G}}{\mathrm{d}\lambda}\right| \tag{8}$$

and differentiating Eq. (7) with respect to ϕ, i.e.

$$\frac{\mathrm{d}\,\lambda}{\mathrm{d}\,\phi} = \frac{5\kappa}{2}\left(\frac{A_S}{A_G} - \frac{2}{25}\frac{A_G}{A_S}\right)\lambda. \tag{9}$$

Combining Eqs. (8) and (9) yields the result:

$$\left(\frac{A_S}{A_G}\right)^2 = \frac{2}{25}\left(1 + \frac{\mathrm{d}\ln\lambda}{\mathrm{d}\ln \mathrm{A_G}}\right). \tag{10}$$

Hence, given the spectrum of the tensor modes, one may derive the scalar spectrum and vice–versa. By substituting Eq. (9) into Eq. (3) we find

$$\kappa^4 V(\lambda) = 16\pi^3 A_G^2(\lambda)\left[3 - \frac{2}{25}\frac{A_G^2}{A_S^2}\right] \tag{11}$$

and the solution of Eq. (8) is

$$\phi(\lambda) - \phi_0 = \sqrt{\frac{2}{\kappa^2}}\int_{\lambda_0}^{\lambda}\mathrm{d}(\ln \mathrm{A_G})\left[1 + \frac{\mathrm{d}\,\ln\lambda}{\mathrm{d}\,\ln \mathrm{A_G}}\right]^{1/2}. \tag{12}$$

The form of $V(\phi)$ follows by substituting the inverse of this solution into Eq. (11). The set of equations (10)–(12) defines the nonlinear map that carries the function space of the primordial perturbation spectra over to the space of possible inflationary potentials, and vice-versa. It generalizes the formalism of ref. [12] to include the gravitational wave contribution. One may equally well specify the scalar or tensor spectrum and then derive the corresponding potential rather than beginning with $V(\phi)$.

3. Power law scalar fluctuations

Having set up the correspondence we now demonstrate its fragility by means of a specific example. We consider potentials which lead to a power law scalar fluctuation spectrum,

$$A_S(\lambda) \propto \lambda^{-\beta} \propto k^{\beta}, \tag{13}$$

to see whether small changes in the power index β, which are bound to be present due to inevitable errors in observations, result in significant changes in $H(\phi)$. This is important because recent results from *COBE* suggest that such a spectrum is *consistent* with the data if β lies in the range $-0.2 < \beta < 0.3$ at the 1–sigma level independent of the dark matter content [10]. The case of $\beta = 0$ is the scale–invariant Harrison–Zeldovich spectrum often employed as an initial condition for dark matter models of galaxy formation [11]. Although *COBE* allows for such a possibility it does not currently restrict the sign of β. (Indeed it is hard for any observation to do so).

It is possible to derive the functional forms of $H(\phi)$ that lead precisely to this power law form for A_S by equating the logarithms of Eqs. (5) and (13) and differentiating with respect to ϕ [13]. This removes all arbitrary constants and substituting for Eq. (9) yields the second–order equation

$$(2-\beta)\frac{(H')^2}{H} - H'' = -\frac{\beta\kappa^2}{2}H \tag{14}$$

By using the identity $2H'' \equiv \mathrm{d(H')^2/dH}$, which follows from Eq. (4), we may write this as the first–order equation

$$\frac{\mathrm{d\,(H')^2}}{\mathrm{d\,H}} - 2(2-\beta)\frac{H'^2}{H} = \beta\kappa^2 H \tag{15}$$

which has the exact integral

$$(H')^2 = \left[\frac{\beta\kappa^2}{2(\beta-1)}\right] H^2 + CH^{2(2-\beta)}, \qquad (\beta \neq 1) \tag{16}$$

where C is an arbitrary integration constant. This equation gives the correspondence between the power index β for a polynomial scalar fluctuations spectrum and $H(\phi)$. It has a number of families of solutions, depending upon the signs of β and C, which are conveniently summarized in table 1.

C/β	$\beta < 0$	$\beta = 0$	$\beta > 0$
$C < 0$	$\lambda \text{sech}^n(\omega\phi)$	NS	NS
$C = 0$	$\exp(\pm\sqrt{\frac{\beta}{2(\beta-1)}}\kappa\phi)$	Const.	NS
$C > 0$	$\lambda \text{cosech}^n(\omega\phi)$	$\frac{1}{\sqrt{C}\phi}$	$\lambda \sec^n(\omega\phi)$

Table 1

Without loss of generality the second integration constant in these solutions has been removed by means of a linear translation in the scalar field. As can be seen, there are regions of $C-\beta$ space in which small changes in C or β (either singly or simultaneously) produce qualitative changes in the corresponding $H(\phi)$. For example, for fixed values of C, small changes in the physically motivated $\beta = 0$ neighbourhood produce qualitatively different changes in $H(\phi)$. The neighbourhood around the Harrison-Zel'dovich is therefore carried to three separate neighbourhoods in the space of models. In this way the $\beta - H(\phi)$ correspondence may be said to be fragile around $\beta = 0$. Although no real solution exists in certain regions, the correspondence is still fragile because the form of $H(\phi)$ changes.

The origin of the fragile behaviour around $C = 0$ for fixed β can be traced to the nonlinear nature of Eq. (10). By comparing Eqs. (10) and (11) we see that the potential is essentially determined by the tensor spectrum, but the scalar spectrum only determines the tensor modes up to an integration constant. Different values of this constant result in different forms for $V(\phi)$.

To get a better feel for these results, we have plotted the various forms of $H(\phi)$ for $C > 0$ in figure 1.

Figure 1

4 Conclusion

An assumption that is often made, at least implicitly, in the interpretation of observational data and construction of mathematical models for physical phenomena is that of 'stability'; in the sense that the uncertainties due to errors involved in observations would not qualitatively change the nature of the corresponding models. In this paper, we have argued that this may not *necessarily* be a valid assumption to make in the inflationary cosmology. We have derived the mathematical correspondence between the scalar and tensor fluctuation spectra and the potential, which will allow the stability of the correspondence to be studied in more detail. We have shown that the correspondence between the index β of a polynomial scalar fluctuation spectrum and $H(\phi)$ may be fragile in the sense that small uncertainties in fixing β in certain neighbourhoods, including the physically motivated $\beta = 0$, can give rise to qualitatively different types of models (i.e. $H(\phi)$). This could be of significance in the interpretation of observational data, such as those obtained by *COBE*, especially in view of the fact that the relevant range of β is currently $-0.2 < \beta < 0.3$ [10].

It should be emphasized that, in view of the error bars present in observations and uncertainties regarding the nature of the potential $V(\phi)$, it is the correspondence between *neighbourhoods* in the two function spaces that has been studied here. As shown by table (1) there does exist a one-to-one correspondence between points once C is fixed, but this breaks down when one considers local neighbourhoods around these points.

It is perhaps important to note that even though inflation 'washes out' dependence on initial conditions by exponentially blowing up small regions, the (non-linear) dynamics of the inflationary correspondence defined here, between say $H(\phi)$ (or $V(\phi)$) and the spectrum of fluctuations, may be fragile under small changes. The crucial difference to note is the difference between the space of initial conditions on which the dynamics acts and the space of dynamical correspondences (systems). There is an analogy here with non-linear dynamics where for example a given chaotic dynamical system may act to wash out any reference to initial conditions, but the dynamical system itself may be fragile with respect to small perturbations.

Finally, given that observations are always imprecise and the exact form of the inflationary potential is not known, our result could be of potential importance for those applications of the 'designer' inflationary models in which special forms of $V(\phi)$ are adopted in order to produce a desired type of spectrum or dynamics [14]. Such correspondences could be fragile in the sense that different potentials lead to identical scalar fluctuation spectra, as is shown in table (1) by fixing $\beta < 0$ and varying C around zero.

Acknowledgments JEL is supported by a Science and Engineering Research Council (SERC) UK postdoctoral research fellowship and is supported at Fermilab by the DOE and NASA under Grant NAGW-2381. RKT is supported by SERC UK Grant $\#H09454$. We acknowledge J. D. Barrow, B. J. Carr, E. J. Copeland, G. F. R. Ellis, E. W. Kolb and A. R. Liddle for useful discussions.

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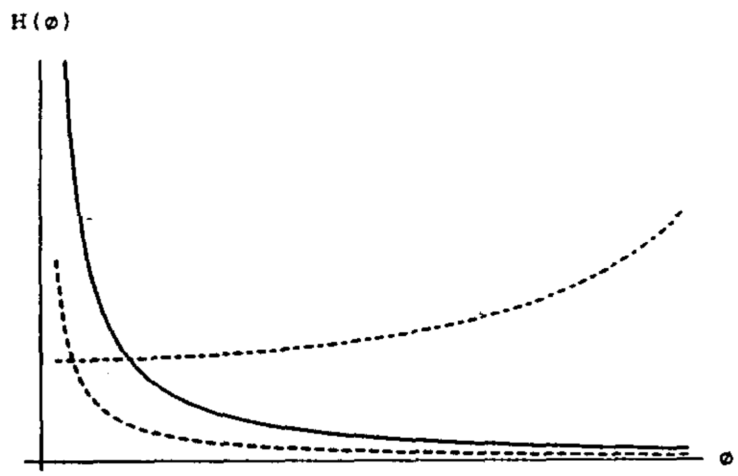
Table 1

Behaviour of $H(\phi)$ for positive, vanishing or negative C and β. The parameter λ is a positive–definite constant determined by C and 'NS' implies no real solution for $H(\phi)$ exists in the given region. $n = 1/(1-\beta)$, $\omega^2 = [\beta(\beta-1)]\kappa^2/2$ if $\beta < 0$ and $\omega^2 = [\beta(1-\beta)]\kappa^2/2$ if $\beta > 0$. For a given value of C or β, the correspondence is fragile around $\beta = 0$ or $C = 0$ respectively. Fragile behaviour is also seen as C and β are simultaneously allowed to vary around zero.

Figure 1

A schematic plot illustrating the fragile behaviuor of $H(\phi)$ around $\beta = 0$ for $C > 0$. The dashed curve represents $\beta = 0$, the dot-dashed curve represents $\beta > 0$ and the solid curve $\beta < 0$.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