# Differential equations in automorphic forms and an application to particle physics

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*I would like to dedicate this work to Terry George Klinger and Terry Allen Logan.*

*"Life can only be understood backwards; but it must be lived forwards."*

*– Søren Kierkegaard*

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## Abstract

Physicists such as Green, Vanhove, et al show that differential equations involving automorphic forms govern the behavior of gravitons. One particular point of interest is solutions to $(\Delta - \lambda)u = E_\alpha E_\beta$ on an arithmetic quotient of the exceptional group $E_8$. We establish that the existence of a solution to $(\Delta - \lambda)u = E_\alpha E_\beta$ on the simpler space $SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})$ for certain values of $\alpha$ and $\beta$ depends on nontrivial zeros of the Riemann zeta function $\zeta(s)$. Further, when such a solution exists, we use spectral theory to solve $(\Delta - \lambda)u = E_\alpha E_\beta$ on $SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})$ and provide proof of the meromorphic continuation of the solution. The construction of such a solution uses Arthur truncation, the Maass-Selberg formula, and automorphic Sobolev spaces.


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# Chapter 1

# Introduction

$L$-functions can be derived from another class of functions on moduli spaces called *automorphic forms*. Recently, physicists have discovered the behavior of gravitons (hypothetical particles of gravity represented by massless string states) is closely related to properties of automorphic forms [**?**, **?**]. Kyoto Prize recipient E. Witten recently praised this discovery in his Commemorative Lecture. A natural starting point for studying the behavior of gravitons is examining what happens when two gravitons collide and then go in different directions (see Figure **??**). This is investigated through the *4-graviton scattering amplitude* (the likelihood of a certain interaction between four gravitons occurring). In fact, the 4-graviton scattering amplitude gives rise to a quantum correction that could account for inconsistencies between general relativity and experiment. The full string theory for the 4-graviton scattering amplitude is not known. Green, S. Miller, Vanhove, et al. [**?**, **?**] computed the expansions for the scattering amplitude and found that coefficients in these expansions arise as solutions to PDEs involving automorphic forms.

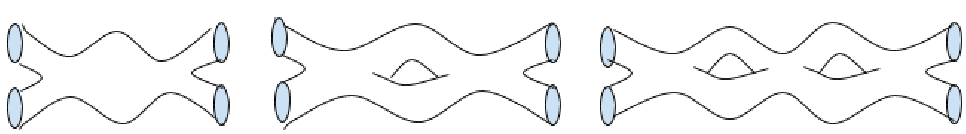

Figure 1.1: Three string world-sheets (with genera $h = 0, 1, 2$ respectively) as they appear in the scattering of four closed strings.

In [?], Green, Miller, Russo and Vanhove study the low energy expansions of string theory amplitudes that generalize the amplitudes of classical supergravity. In doing so they derive differential equations that model the behavior of the 4-loop supergraviton. Such differential equations govern the amplitudes of closed type II superstring theory. These differential equations involve combinations of Eisenstein series in their expressions and have the form [?]:

$$\begin{aligned}
(\Delta - \lambda_w)u_w &= 0 \\
(\Delta - \lambda_w)u_w &= c \\
(\Delta - \lambda_w)u_w &= E_\alpha \\
(\Delta - \lambda_w)u_w &= E_\alpha \cdot E_\beta
\end{aligned}$$

on the exceptional group $E_8$ where $c$ is a constant and $E_\alpha$ and $E_\beta$ are maximal-parabolic Eisenstein series. Solutions for the first three such equations are known. Furthermore, spectral solutions to similar equations are understood (see the work of P. Garrett [?], [?]). The last equation, however, is more challenging to solve. In [?], [?] and [?], the form of this last equation was given where $\alpha = \beta$. The Fourier expansion of an infinite class of solutions has been worked out explicitly in the recent work of D'Hoker and Duke [?].

As a precedent for solving such an equation, we will solve $(\Delta - \lambda_w)u_w = E_\alpha \cdot E_\beta$ on $\Gamma\backslash\mathfrak{H}$ where $\Gamma = SL_2(\mathbb{Z})$ and $\mathfrak{H}$ is the upper half plane, $\Delta = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ is the invariant Laplacian and $\lambda_w = w(w-1)$. There are of course many differences in these domains but examining the simpler domain will illuminate some of the necessary techniques for analyzing solutions elsewhere. Furthermore, this technique allows us to compute the solution to the differential equation in many cases at once. In [?] Green, Miller and Vanhove present a solution on $\Gamma\backslash\mathfrak{H}$ where $\alpha = \beta = 3/2$ and $\lambda_w = 12$ and D'Hoker, Green, Gürdoğan and Vanhove give a solution for integer values of $\alpha$ and $\beta$ in [?]. Our solution will subsume these.

We will solve $(\Delta - \lambda_w)u_w = E_\alpha \cdot E_\beta$ on $\Gamma\backslash\mathfrak{H}$ using spectral theory. This involves finding a spectral expansion for $E_\alpha \cdot E_\beta$; however, given that $E_\alpha \cdot E_\beta \notin L^2(\Gamma\backslash\mathfrak{H})$ no such expansion can be directly computed as methods for computing $L^2$-spectral expansions do not directly apply. Thus, in order to guarantee convergence of the spectral integrals, we will subtract a linear combination of Eisenstein series from $E_\alpha \cdot E_\beta$ and compute the spectral expansion of this new function. We will then be able to solve the differential equation in the usual way using global automorphic Sobolev spaces. The computation of the spectral expansion for this new function implements ideas developed by Zagier [?] and Casselman [?] related to the extending the Rankin-Selberg method for functions not of rapid decay (explanation of this phenomenon can also be found in [?]). This method makes use of Arthur truncation and the Maass-Selberg formula.

In Section **??** and **??**, we will state our main results and prove the existence and uniqueness of solutions to $(\Delta - \lambda_w)u_w = E_\alpha \cdot E_\beta$ on $\Gamma\backslash\mathfrak{H}$ for almost all values of $\alpha$ and $\beta$. In Sections **??**, **??** and **??**, we will compute the spectral expansion of this solution. After computing an explicit form of the solution, we will meromorphically continue the solution in $w$ to the left-half plane in section **??**. This proof relies upon the constructions involving vector-valued integrals as presented by Gelfand, Pettis, and Grothendieck. A brief summary of these constructions is provided in the appendix (Section **??**).

## 1.1 Physics

Quantum field theory does a good job of describing the interactions of elementary particles. However, it does not account for gravity. String theory is an attempt to solve this problem. Instead of representing elementary particles by points and modeling their interactions with Feynman diagrams, elementary particles are represented by vibrational modes of a string. Not only does this representation consistently contain quantum field theory, but it gives us a theory for gravity as well.

There are of course a few problems with the application of string theory to our world. The first being that string theory is consistent only in (9+1) dimensional space-time instead of the (3+1) dimensional space-time in which we live. The second being that there are in fact, five different consistent sting theories in (9+1) dimensions. The first problem is resolved via compactification (where we take six of these dimensions to be small and compact) through work in mirror symmetry. There are of course many ways to consistently compactify each of the five distinct theories to (3+1) dimensional space-time. The second problem is then partially solved by the existence of duality symmetries which remains conjectural. (For more information on duality conjectures see [?].)

Given the description above, string theory is the theory of one-dimensional extended objects propagating in a (9+1) dimensional space-times $M$. During this propagation, strings sweep out to a two-dimensional world-sheet $\Sigma$. Thus string theory can be thought of as the dynamics of the embedding maps $X : \Sigma \to M$ where both $\Sigma$ and $M$ are endowed with an additional structure (like a metric) that enter into the definition. In superstring theory, this additional structure includes world-sheet supersymmetry[1] and thus the space of allowed world-sheets $\Sigma$ is the space of all closed, orientable Riemann surfaces. For the purposes of this investigation, we will considers type IIB superstrings; otherwise we would have to include boundaries and non-orientable surfaces. Riemann surfaces are first classified by their genus $0 \le h \in \mathbb{Z}$ and for fixed genus there is a moduli space.

### 1.1.1 Scattering Amplitudes

Strings interact by various joining and splitting processes. Figure **??** gives examples with few splittings, corresponding to low genus world-sheets. Quantities we would like to compute are *scattering amplitudes*. They are the likelihood of a certain scattering process to occur.

The scattering amplitude depends on the data of the scattering states such as string coupling (the measure of the strength of string-string interaction), the string scale (which separates the masses), and other so-called *moduli fields*. These are realized as aspects of the target space $M$ in the form of additional scalar fields living on them and this new space is called the *moduli space* $\mathcal{M}$.

Much is known for flat target spaces of the type $M = \mathbb{R}^{1,9}$ (flat Minkowski space) and $M = \mathbb{R}^{1,9-d} \times T^d$ (toroidal compactification). In both cases, retaining maximal supersymmetry constrains to moduli spaces. The classical low energy moduli space is a symmetric space of the form

$$\mathcal{M}_{\text{class}} = G(\mathbb{R})/K(\mathbb{R})$$

[1] Supersymmetry is a principle that proposes a relationship between two basic classes of elementary particles: bosons and fermions.

| $D$ | $G_d(\mathbb{R})$ | $K$ | $G_d(\mathbb{Z})$ |
|---|---|---|---|
| 10 | $SL_2(\mathbb{R})$ | $SO(2)$ | $SL_2(\mathbb{Z})$ |
| 9 | $GL_2(\mathbb{R})$ | $SO(2)$ | $SL_2(\mathbb{Z})$ |
| 8 | $SL_3(\mathbb{R}) \times SL_2(\mathbb{R})$ | $SO(3) \times SO(2)$ | $SL_3(\mathbb{Z}) \times SL_2(\mathbb{Z})$ |
| 7 | $SL_5(\mathbb{R})$ | $SO(5)$ | $SL_5(\mathbb{Z})$ |
| 6 | $SO(5,5,\mathbb{R})$ | $(SO(5) \times SO(5))/\mathbb{Z}_2$ | $SO(5,5,\mathbb{Z})$ |
| 5 | $E_6(\mathbb{R})$ | $USp(8)/\mathbb{Z}_2$ | $E_6(\mathbb{Z})$ |
| 4 | $E_7(\mathbb{R})$ | $SU(8)/\mathbb{Z}_2$ | $E_7(\mathbb{Z})$ |
| 3 | $E_8(\mathbb{R})$ | $SO(16)/\mathbb{Z}_2$ | $E_8(\mathbb{Z})$ |

Table 1.2: The duality groups of maximal supersymmetry in $D = 10-d \leq 10$ dimensions for type IIB string theory on a $d$-dimensional torus.

where $G$ and $R$ are the lie groups listed in Table **??**. When passing to quantum theory, the classical symmetries are generally broken and take values in some integral lattice $\Gamma$. The quantum symmetry is defined as the subgroup of $G(\mathbb{R})$ that preserves the lattice $\{g \in G(\mathbb{R}) \mid g\Gamma = \Gamma\}$. The correct moduli space of quantum string theory is not the classical symmetric space but

$$\mathcal{M} = G(\mathbb{Z})\backslash G(\mathbb{R})/K(\mathbb{R}).$$

All observables, including the scattering amplitudes are functions of this space.

The low energy expansion of the scattering amplitude in $D$-dimensional space time as the form:

$$A_D(s,t,u) = A_D^{\text{analytic}}(s,t,u) + A_D^{\text{nonanalytic}}(s,t,u)$$

where we have separated the amplitude into analytic and nonanalytic functions of the Mandelstam invariants, $s$, $t$ and $u$ (with $s = -(k_1+k_2)^2$, $t = -(k_1+k_4)^2$, $u = -(k_1+k_3)^2$ and $s+t+u = 0$ for $k_1, k_2, k_3, k_4$ the momentum of the incoming and outgoing gravitons).

The analytic part of the amplitude has the expansion

$$A_D^{\text{analytic}} = \mathcal{E}^{(D)}_{(0,-1)}(g)\frac{\mathcal{R}^4}{\sigma_3} + \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\mathcal{E}^{(D)}_{(p,q)}(g)\,\sigma_2^p\sigma_3^q\mathcal{R}^4$$

where $g \in \mathcal{M}$ and the Mandelstam invariants above are in dimensionless combinations of the form $\sigma_n = (s^n + t^n + u^n)\frac{\ell_D^{2n}}{4^n}$ for $\ell_D$ the Planck length in $D$ dimensions. The factor $\mathcal{R}^4$ indicated the contraction of four powers of the Riemann curvature tensors linearized around flat space and contracted with a standard sixteen-index tension [?].

The interesting objects in this expansion are the coefficient functions $\mathcal{E}^{(D)}_{(p,q)}(g)$ which are functions on the moduli space $\mathcal{M}$. *These functions* $\mathcal{E}^{(D)}_{(p,q)}(g)$ *are in fact automorphic forms.* Not only do they satisfy the automophy condition, but they also satisfy a growth condition due to the fact that string coupling $g_s$ (and other limits in the moduli space) $g_s$ goes to 0 at the cusps in $G/K$, and they satisfy appropriate differential equations under the action of $G$-invariant operators.

The supersymmetry in string theory imposes such differential conditions on these coefficient functions $\mathcal{E}^{(D)}_{(p,q)}(g)$. Green and Sethi first analyzed this in the case of ten-dimensional (D=10) type IIB string theory for $p = q = 0$ [?]. In this case, they found that $\mathcal{E}^{(10)}_{(0,0)}(g)$ is the non-holomorphic Eisenstein series on $SL_2(\mathbb{R})$ (see Section 2.4 of [?] for a detailed explanation of this deduction).

The other coefficients $\mathcal{E}^{(D)}_{(p,q)}(g)$ appears as solutions $u_w$ to various differential equations. Such differential equations, as presented in [?] and [?], are of the forms:

$$\begin{aligned}(\Delta - \lambda_s)u_w &= 0\\ (\Delta - \lambda_w)u_w &= c\\ (\Delta - \lambda_w)u_w &= E_\alpha\\ (\Delta - \lambda_w)u_w &= E_\alpha \cdot E_\beta\end{aligned}$$

on the exceptional group $E_8$ where $c$ is a constant and $E_\alpha$ and $E_\beta$ are Eisenstein series. Solutions for the first three equations are known at least on $SL_2(\mathbb{R})$. Specifically, the second term $\mathcal{E}^{(10)}_{(0,1)}(g)$ satisfies the inhomogeneous Laplace eigenvalue equation

$$(\Delta - 12)\mathcal{E}^{(10)}_{(0,1)}(g) = -\left(\mathcal{E}^{(10)}_{(0,0)}(g)\right)^2$$

where $\mathcal{E}^{(10)}_{(0,0)}(g) = E_{3/2}(g)$ on $SL_2(\mathbb{R})$. This solution is found $\mathcal{E}^{(10)}_{(0,1)}(g)$ is found in [?]. In our work we will find a general spectral solution for

$$(\Delta - \lambda_w)u_w = E_\alpha \cdot E_\beta$$

on $\Gamma\backslash\mathfrak{H}$, show uniqueness of that solution and meromorphically continue it in the variable $w$ to the entire complex plane. Our solution provides rigor of via Sobolev spaces, and provides a mechanism which one may hope to find a solution on higher rank groups corresponding to other dimensions $D$.

## 1.2 Number Theory

In number theory, automorphic forms give rise to $L$-functions. The canonical example being Riemann's theta function, which he used to prove the meromorphic continuation of the Riemann zeta function. My first exposure to differential equations involving automorphic data was not through physics but through the following story.

### 1.2.1 The Story

The foundations for this work are provided by a project begun by Bombieri and Garrett, originating in work of ColinDeVerdière and Hejhal in the 1980s. Their project was inspired by a story that begins in 1977 when Haas [?] miscomputed eigenvalues of the invariant Laplacian $\Delta = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ on $\Gamma\backslash\mathfrak{H}$ for $\Gamma = SL_2(\mathbb{Z})$. In his Master's thesis, Haas attempted to compute eigenvalues $\lambda_s = s(s-1)$ by numerical solution of the differential equation $(\Delta - \lambda_s)u = 0$. Since the invariant Laplacian $\Delta$ descends from the Casimir element of the universal enveloping algebra $U\mathfrak{sl}_2(\mathbb{R})$ and, furthermore, automorphic forms are eigenfunctions for such operators, solutions to such differential equations had a natural significance to number theorists. Haas' advisor Neunhoffer mailed the list of parameters $s$ to A. Terras in San Diego, but without detailed explanation of their derivation. In this list, Stark observed some zeros of $\zeta(s)$ and Hejhal noticed zeros of $L(s, \chi_{-3})$.[2]

[2] This list only went up to height 20.45578 and was quite short due to the restricted computing power of the time. Only one zero of $\zeta$ was initially observed; however, there is only one nontrivial zero of $\zeta$ below height 21. See [?] or [?].

Though it was not surprising that the values had interest to number theorists, it was unexpected that zeros of $L$-functions appeared as spectral parameters $s$. Given Hilbert and Pólya's remarks about the possibility of proving the Riemann Hypothesis by producing self-adjoint operators with eigenvalues $s(s-1)$ for zeros $s$ of $\zeta(s)$, this finding was quite exciting (since $\Delta$ is a non-positive self-adjoint operator, this would mean that $\lambda_s \in \mathbb{R}$ and $\lambda_s \leq 0$ and hence either $s \in [0,1]$ or $\mathrm{Re}(s) = 1/2$). It follows then that *if* all zeros of $\zeta(s)$ are on this list of parameters of $\lambda_s$ we have the Riemann Hypothesis. The mere numerical artifact does not suggest a *proof* and, in particular, there is no visible guarantee that *all* zeros of $\zeta$ are on this list of spectral parameters. However, given any strong correlation between zeros of $\zeta(s)$ spectral parameters for eigenvalues of a self-adjoint operator one might hope to retrieve a definite percentage using such a differential equation.

For instance, one hopeful route might be to show that all spectral parameters $s$ are zeros of $\zeta(s)$. In this case, using the fact that the number $N(T)$ of zeros of $\zeta$ in the critical strip below height $T$ is

$$N(T) = \frac{1}{2\pi} T \cdot \log\left(\frac{T}{2\pi e}\right) + O(\log T)$$

one might check what percentage of zeros of $\zeta$ are found on the critical line below any given height $T$. In the best-case scenario, asymptotically 100% of zeros would appear this way. Indeed, any fraction over 40% would be progress, and, furthermore, any correlation between spectral parameters for a self-adjoint operator and zeros of $\zeta(s)$ would give motivation for continued research [?].

Such an exciting prospect should naturally be met with skepticism. In 1979-1981, Hejhal [?] recomputed the eigenvalues and found that all of these interesting parameters, the zeros, were *missing*. He realized that Haas had inadvertently allowed for some non-smoothness, misapplyng of Henrici collocation method at the corners of the fundamental domain, and had found $s$-values that were solutions to the inhomogeneous equation

$$(\Delta - \lambda_s)u = \delta_\omega^{\mathrm{afc}} \tag{1.1}$$

where $\delta_\omega^{\mathrm{afc}} := \sum_{\gamma\in\Gamma} \delta_\omega^{\mathfrak{H}} \circ \gamma$ is the automorphic Dirac delta at the corner $\omega = e^{2\pi i/3}$ of the fundamental domain of $\Gamma\backslash\mathfrak{H}$, as opposed to the homogeneous equation $(\Delta - \lambda_s)u = 0$.

More relevant to the story of Haas and Hejhal, the implications of this new differential equation (**??**) toward RH are not as initially hoped since values $\lambda_s = s(s-1)$ admitting non-trivial solutions $u$ are not genuine eigenfunctions for $\Delta$. However, not all is lost. Bombieri and Garrett, building on work of Hejhal and ColinDeVerdière, have clarified and made precise ideas from [**?**] and proved some basic results showing the relevance of operator theory to location of zeros and other periods of Eisenstein series. More explicitly, Bombieri and Garrett make (*necessarily*) subtler operators related to $\Delta$ to better exploit the that fact the constant term of $u_s$ is essentially $\theta E_s$ .

The work of Bombieri and Garrett clarifies two promising observations initially made by ColinDeVerdière in 1982-3 [**?**]. The first was Lax and Phillips (1976) result [**?**] that for $a > 1$, if we define

$$L^2_a(\Gamma\backslash\mathfrak{H}) = \left\{ f \in L^2(\Gamma\backslash\mathfrak{H}) \;\middle|\; c_P f(x) = 0 \text{ for } y > a \right\}$$

for $c_P f(x) := \int_0^1 f(x+iy)\ dx$ then the Friedrichs' extension $\widetilde{\Delta}_a$ of $\Delta$ restricted to $C_c^\infty(\Gamma\backslash\mathfrak{H}) \cap L^2_a(\Gamma\backslash\mathfrak{H})$ has purely discrete spectrum. Furthermore, $\widetilde{\Delta}_a$ is self-adjoint so that the eigenvalues are real and for a distribution $\eta_a$ at $a$ defined by $\eta_a f = c_P f(ia)$

$$(\widetilde{\Delta}_a - \lambda_s)u = 0 \iff (\Delta - \lambda_s)u = c \cdot \eta_a \quad \text{and} \quad \eta_a u = 0$$

for some constant $c$. If

$$(\widetilde{\Delta}_{z_0} - \lambda_s)u = 0 \quad \iff \quad (\Delta - \lambda_s)u = c \cdot \delta^{\text{afc}}_{z_0} \;\; \text{and} \;\; \delta^{\text{afc}}_{z_0} u = 0 \quad \text{(for some c)}$$

a pseudo-Laplacians $\widetilde{\Delta}_{z_0}$ attached to the automorphic Dirac delta $\delta^{\text{afc}}_{z_0}$ may contain spectral parameters relating to zeros of $\zeta$. The issue then becomes that, in order to have a Friedrichs extension attached to a distribution, that distribution must be contained in $H^{-1}(\Gamma\backslash\mathfrak{H})$. However, $\delta^{\text{afc}}_\omega \in H^{-1-\epsilon}$.

The second relevant observation made by ColinDeVerdière (1983) in [**?**] was (approximately) that projecting $\delta^{\text{afc}}_\omega$ to the non-cuspidal spectrum would allow this new distribution $\theta$ be in $H^{-3/4-\epsilon}(\Gamma\backslash\mathfrak{H}) \subseteq H^{-1}(\Gamma\backslash\mathfrak{H})$ by the moment bound of Hardy and Littlewood [**?**]. In 2011, Bombieri and Garrett made ColinDeVerdière's speculation precise and proved that the discrete spectrum $\lambda_s = s(s-1)$ (*if any*) of $\widetilde{\Delta}_\theta$ has parameters contained in the on-line zeros of $\zeta(s)L(s,\chi_{-3})$. There is no guarantee that the spectrum is non-empty.

In fact, the first purely new result of Bombieri and Garrett is their limitation of the fraction of zeros which could occur as $s$-values for discrete spectrum $\lambda_s$. They achieve this by showing a connection to the relatively regular behavior of $\zeta(s)$ on the edge of the critical strip, leading to conflict with Montgomery's pair correlation conjecture [**?**]. This provides a strong reason to believe that the most optimistic version of ColinDeVerdière's simplest formulation of a conjecture in the style of Hilbert-Pólya is false (barring significant failure of RH!). Further, the influence of the spectral theory of self-adjoint operators on spaces of automorphic forms is more complicated than a literal manifestation of Hilbert-Pólya. However, given that there is an overall lack of candidate operators that fit Hilbert and Pólya's suggestion, any promising suggestion in this direction progress. Furthermore, not all is lost – though this 'simple' case yields a negative result, the hope is that given this jumping-off point, we can recover the lost spectral parameters using more complicated boundary conditions. One virtue of this approach is that the same set-up and conclusions can apply to much broader contexts.

## 1.3 Relevant background on Eisenstein series on $GL(2)$

Let $E_\alpha$ and $E_\beta$ be two Eisenstein series on $\Gamma\backslash SL_2(\mathbb{R})$ for $\Gamma = SL_2(\mathbb{Z})$. Each $E_s$ can then be described as

$$E_s(z) = \sum_{\gamma\in P\backslash\Gamma} \mathrm{Im}(\gamma z)^s$$

which converges absolutely and uniformly for $\mathrm{Re}(s) > 1$ and $z$ in compacts (where $P$ is the standard parabolic of $SL_2(\mathbb{R})$ restricted to $\Gamma$). The following result of the analytic continuation and functional equation is well-known and its proof can be found many places including (but not limited to) Epstein's [**?**] and Garrett's [**?**] explication of Godement's [**?**] 1966 work.

**Theorem 1.** *For each $z \in \mathfrak{H}$, $s(s-1)\xi(s)\cdot E_s(z)$ has an analytic continuation to an entire function of $s$ and functional equation given by*

$$\xi(2s)E_s = \xi(2-2s)E_{1-s}$$

*where* $\xi(s) = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$ *is the completed Riemann zeta function.*

Note that we will employ the notation $c_s = \dfrac{\xi(2-2s)}{\xi(2s)}$ so that the function equation for $E_s$ is given by $E_s = c_s \cdot E_{1-s}$.

Furthermore, it is known (proof in [?]) that the Fourier-Whittaker expansion for $E_s$ (for $s \neq 1$) is given by

$$E_s(x+iy) = y^s + c_s y^{1-s} + \frac{1}{\pi^{-s}\Gamma(s)\zeta(2s)} \sum_{n\neq 0} \frac{\sigma_{2s-1}(|n|)}{|n|^{s-\frac{1}{2}}} \cdot \sqrt{y} \int_0^\infty t^{s-1/2} e^{-(t+\frac{1}{t})\pi|n|y} \frac{dt}{t} \cdot e^{2\pi inx}$$

$$= y^s + c_s y^{1-s} + \sum_{n\neq 0} \varphi(n,s) \cdot W_s(|n|y) \cdot e^{2\pi inx}$$

where

$$W_s(|n|y) = \sqrt{y} \int_0^\infty t^{s-1/2} e^{-(t+\frac{1}{t})\pi|n|y} \frac{dt}{t}$$

is the Whittaker function – the unique (up to scalars) moderate-growth solution $u$ of $u'' - \left(\frac{\lambda_s}{y^2} + 4\pi^2 n^2\right) \cdot u = 0$ for $\lambda_s = s(s-1)$ – and $\varphi(n,s) = \dfrac{1}{\pi\Gamma(s)\zeta(2s)} \dfrac{\sigma_{2s-1}(|n|)}{|n|^{s-\frac{1}{2}}}$ and $\sigma_{2s-1}(|n|)$ is the sum of the $(2s-1)^{th}$ powers of positive divisors of $n$. We will use the notation $c_P E_s$ to refer to the constant term of the Eisenstein series at $s$.

Since $E_s$ has a simple pole at $s = 1$, the constant term $c_P E_1^*$ for the $a_{-1}$ coefficient of the Laurent expansion $E_s$ at $s = 1$ will not have the form $y^s + c_s y^{1-s}$. Instead, the Fourier-Whittaker expansion for $E_1^*$ is

$$E_1^*(x+iy) = y + C - \frac{3}{\pi}\log y + \sum_{n\neq 0} \varphi(n,1) \cdot W_1(|n|y) \cdot e^{2\pi inx}$$

where $C = \frac{d}{ds}\left((s-1)c_s\right)\Big|_{s=1}$ and $W_s$ is as above.

We will later also need the Fourier-Whittaker expansion of cuspforms. Indeed the archimedean parts of that of the Fourier-Whittaker functions for a cuspform with $\Delta$-eigenvalue $\lambda_s = s(s-1)$ are the same as the Eisenstein series (see [?]). We then have for $f$ a cuspform on $\Gamma\backslash\mathfrak{H}$ that

$$f(x+iy) = \sum_{n\neq 0} c_n \cdot W_s(|n|y) \cdot e^{2\pi inx}$$

for some constants $c_n$ with $W_s(|n|y)$ as above.

## 1.4 Approach to the solution

In what follows, we solve

$$(\Delta - \lambda_w)u_w = E_\alpha \cdot E_\beta$$

on $\Gamma\backslash\mathfrak{H}$ where $\Gamma = SL_2(\mathbb{Z})$ and $\mathfrak{H}$ is the upper half plane, $\Delta = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ is the invariant Laplacian and $\lambda_w = w(w-1)$. First we write out a spectral expansion for $E_\alpha \cdot E_\beta$.

For $0 \le k \in \mathbb{Z}$, the $k^{th}$-Sobolev norm on $C_c^\infty(\Gamma\backslash\mathfrak{H})$ is given by

$$|f|_k^2 := \langle (1-\Delta)^k f, f\rangle_{L^2(\Gamma\backslash\mathfrak{H})}$$

and we define the global automorphic Sobolev space $H^k(\Gamma\backslash\mathfrak{H})$ to be the completion of $C_c^\infty(\Gamma\backslash\mathfrak{H})$ with respect to $|\cdot|_k$. Ordinarily, for $S$ in some Sobolev space $H^k(\Gamma\backslash\mathfrak{H})$ we can write

$$S = \sum_{f \text{ cfm}} \langle S, f\rangle \cdot f + \frac{\langle S, 1\rangle \cdot 1}{\langle 1, 1\rangle} + \frac{1}{4\pi i}\int_{(1/2)} \langle S, E_s\rangle \cdot E_s \, ds$$

(see Section **??**, [**?**] or [**?**] for further explanation of global automorphic Sobolev spaces and [**?**] for the spectral expansion). The problem is that $E_\alpha \cdot E_\beta$ is not in such a Sobolev space so we cannot properly write this spectral decomposition for $S = E_\alpha \cdot E_\beta$. The device we use is subtraction of a finite linear combination of $E_\alpha$ and $E_\beta$ so that

$$S = E_\alpha \cdot E_\beta - \sum_i c_i E_{s_i}$$

which will be in $L^2$ or even possibly in $H^\infty$ and we can give a decomposition for $\Delta$.

# Chapter 2

# Results

We will use the spectral relation in Section **??** to solve $(\Delta - \lambda_w)u = E_\alpha \cdot E_\beta$ on $\Gamma\backslash SL_2(\mathbb{R})$. The automorphic Sobolev space $H^k$ in which this solution exists is also defined in Section **??**. Furthermore, we will show that the solution we have found is unique.

## 2.1 Main Results

Consider the set

$$\mathcal{C} := \{(\alpha,\beta) \in (\mathbb{C}-\{1\})^2 \mid \mathrm{Re}(\alpha) \geq 1/2, \mathrm{Re}(\beta) \geq 1/2, \mathrm{Re}(\alpha+\beta) \neq 3/2, \mathrm{Re}(\beta) \neq \pm 1/2 + \mathrm{Re}(\alpha)\}.$$

The following guarantees the existence of a unique solution to $(\Delta - \lambda)u = E_\alpha \cdot E_\beta$ on $\Gamma\backslash\mathfrak{H}$ for all $(\alpha,\beta) \in \mathcal{C}$. There are a few complex values eliminated from the set $\mathcal{C}$. We will address what happens with the solution when $\mathrm{Re}(\alpha+\beta) = 3/2$ and $\mathrm{Re}(\beta) = \pm 1/2 + \mathrm{Re}(\alpha)$ in Section **??**. However, it should be noted that the reason for the exclusion of the value 1 is that $E_s$ has a pole at $s = 1$.

Let $\mathcal{E}$ be the vector space consisting of finite linear combinations of Eisenstein series so that

$$\mathcal{E}(\Gamma\backslash\mathfrak{H}) := \left\{ \sum_i a_i F_{s_i}(z) \;\middle|\; a_i \in \mathbb{C} \text{ and } F_{s_i}(z) \in \{\mathbb{C}, E_1^*(z), E_{s_i}(z) \text{ for } s_i \in \mathbb{C}\backslash\{1\}\} \right\}.$$

This space has an LF-space structure as locally convex colimit of finite-dimensional spaces.

**Theorem 2.** *In Re(w) > 1/2, for* $(\alpha, \beta) \in \mathcal{C}$, $(\Delta - \lambda)u = E_\alpha \cdot E_\beta$ *on* $\Gamma\backslash\mathfrak{H}$ *has a unique solution in* $H^{-\infty}(\Gamma\backslash\mathfrak{H}) \oplus \mathcal{E}(\Gamma\backslash\mathfrak{H})$ *with spectral expansion which lies in* $H^2(\Gamma\backslash\mathfrak{H}) \oplus \mathcal{E}(\Gamma\backslash\mathfrak{H})$.

*Proof.* The existence of the solution can be seen in the computation of the spectral expansion. First, we will subtract a finite linear combination of Eisenstein series $E_{s_i}$ so that

$$S = E_\alpha \cdot E_\beta - \sum_i c_i E_{s_i}$$

which will be in $L^2(\Gamma\backslash\mathfrak{H})$.

If $S \in L^2(\Gamma\backslash\mathfrak{H})$, we can write a convergent spectral expansion

$$S = \sum_{f \text{ cfm}} \langle S, f\rangle \cdot f + \frac{\langle S, 1\rangle \cdot 1}{\langle 1, 1\rangle} + \frac{1}{4\pi i}\int_{(1/2)} \langle S, E_s\rangle \cdot E_s \, ds$$

where this convergence occurs in $L^2$. Furthermore, this expansion can be extended by isometry to all of $H^{-\infty}$. It the follows that we can write

$$E_\alpha \cdot E_\beta = \sum_i c_i E_{s_i} + \sum_{f \text{ cfm}} \langle S, f\rangle \cdot f + \frac{\langle S, 1\rangle \cdot 1}{\langle 1, 1\rangle} + \frac{1}{4\pi i}\int_{(1/2)} \langle S, E_s\rangle \cdot E_s \, ds$$

which also converges in $L^2$. Then, given that the spectral data in the expansions above is given by eigenfunctions for $\Delta$, the solution to $(\Delta - \lambda_w)u = E_\alpha \cdot E_\beta$ is given by division by the corresponding eigenvalues.

It can be found in many sources such as [**?**] that the theory of the constant term implies that $E_\alpha = y^\alpha + c_\alpha y^{1-\alpha} + R_\alpha$ where $R_\alpha$ is rapidly decreasing. Thus

$$E_\alpha E_\beta = (y^\alpha + c_\alpha y^{1-\alpha} + R_\alpha)(y^\beta + c_\beta y^{1-\beta} + R_\beta)$$

$$= y^{\alpha+\beta} + c_\beta y^{1+\alpha-\beta} + c_\alpha y^{1-\alpha+\beta} + c_\alpha c_\beta y^{2-\alpha-\beta} + R$$

where $R$ is rapidly decreasing since $y^\alpha + c_\alpha y^{1-\alpha}$ and $y^\beta + c_\beta y^{1-\beta}$ are of moderate growth (and rapidly decreasing times moderate growth is rapidly decreasing). Notice that different values of $\alpha$ and $\beta$ will imply different vanishing for terms of $E_\alpha \cdot E_\beta$. Thus in different regimes, we will be required to subtract different linear combinations of Eisenstein series as follows.

Assume that $\alpha \neq 1$ and $\beta \neq 1$ since $E_s$ has a pole at $s = 1$. Also, without loss of generality, assume that $\mathrm{Re}(\alpha) \leq \mathrm{Re}(\beta)$.

**(I):** Suppose that $1/2 \leq \mathrm{Re}(\alpha) < \mathrm{Re}(\alpha) + 1/2 < \mathrm{Re}(\beta)$. Then

$$\sum_i c_i E_{s_i} = E_{\alpha+\beta} + c_\alpha \cdot E_{1-\alpha+\beta}$$

$$= y^{\alpha+\beta} + c_{\alpha+\beta} y^{1-\alpha-\beta} + c_\alpha y^{1-\alpha+\beta} + c_\alpha c_{1-\alpha+\beta} y^{\alpha-\beta} + R_{\alpha+\beta} + c_\alpha R_{1-\alpha+\beta}$$

Thus

$$S = E_\alpha \cdot E_\beta - \sum_i c_i E_{s_i}$$

$$= c_\beta y^{1+\alpha-\beta} + c_\alpha c_\beta y^{2-\alpha-\beta} - c_{\alpha+\beta} y^{1-\alpha-\beta} - c_\alpha c_{1-\alpha+\beta} y^{\alpha-\beta} + R - R_{\alpha+\beta} - c_\alpha R_{1-\alpha+\beta}$$

Since $1/2 \leq \mathrm{Re}(\alpha) < \mathrm{Re}(\alpha) + 1/2 < \mathrm{Re}(\beta)$,

$$c_\beta y^{1+\alpha-\beta} + c_\alpha c_\beta y^{2-\alpha-\beta} - c_{\alpha+\beta} y^{1-\alpha-\beta} - c_\alpha c_{1-\alpha+\beta} y^{\alpha-\beta} \in L^2(\Gamma\backslash\mathfrak{H}).$$

**(II):** Suppose $1/2 \leq \mathrm{Re}(\alpha) \leq \mathrm{Re}(\beta) < \mathrm{Re}(\alpha) + 1/2$ but that $\alpha \neq \beta$. This case yields two subcases depending on $\mathrm{Re}(\alpha + \beta)$:

**(IIa)** Suppose also that $\mathrm{Re}(\alpha + \beta) > 3/2$. Then

$$\sum_i c_i E_{s_i} = E_{\alpha+\beta} + c_\beta \cdot E_{1+\alpha-\beta} + c_\alpha \cdot E_{1-\alpha+\beta}$$

$$= y^{\alpha+\beta} + c_{\alpha+\beta} y^{1-\alpha-\beta} + R_{\alpha+\beta} + c_\beta y^{1+\alpha-\beta} + c_\beta c_{1+\alpha-\beta} y^{-\alpha+\beta} + c_\beta R_{1+\alpha-\beta}$$

$$+ c_\alpha y^{1-\alpha+\beta} + c_\alpha c_{1-\alpha+\beta} y^{\alpha-\beta} + c_\alpha R_{1-\alpha+\beta}$$

Thus

$$S = E_\alpha \cdot E_\beta - \sum_i c_i E_{s_i}$$

$$= c_\alpha c_\beta y^{2-\alpha-\beta} - c_{\alpha+\beta} y^{1-\alpha-\beta} - c_\beta c_{1+\alpha-\beta} y^{-\alpha+\beta} - c_\alpha c_{1-\alpha+\beta} y^{\alpha-\beta}$$

$$+ R - R_{\alpha+\beta} - c_\beta R_{1+\alpha-\beta} - c_\alpha R_{1-\alpha+\beta}$$

Since $1/2 \leq \mathrm{Re}(\alpha) \leq \mathrm{Re}(\beta) < \mathrm{Re}(\alpha) + 1/2$ and $\mathrm{Re}(\alpha + \beta) > 3/2$,

$$c_\alpha c_\beta y^{2-\alpha-\beta} - c_{\alpha+\beta} y^{1-\alpha-\beta} - c_\beta c_{1+\alpha-\beta} y^{-\alpha+\beta} - c_\alpha c_{1-\alpha+\beta} y^{\alpha-\beta} \in L^2(\Gamma\backslash\mathfrak{H}).$$

**(IIb)** Now suppose instead that $\mathrm{Re}(\alpha+\beta) < 3/2$. Then

$$\sum_i c_i E_{s_i} = E_{\alpha+\beta} + c_\beta \cdot E_{1+\alpha-\beta} + c_\alpha \cdot E_{1-\alpha+\beta} + c_\alpha c_\beta \cdot E_{2-\alpha-\beta}$$

$$= y^{\alpha+\beta} + c_{\alpha+\beta} y^{1-\alpha-\beta} + R_{\alpha+\beta} + c_\beta y^{1+\alpha-\beta} + c_\beta c_{1+\alpha-\beta} y^{-\alpha+\beta} + c_\beta R_{1+\alpha-\beta+c_\alpha y^{1-\alpha+\beta}}$$

$$+ c_\alpha c_{1-\alpha+\beta} y^{\alpha-\beta} + c_\alpha R_{1-\alpha+\beta} + c_\alpha c_\beta y^{2-\alpha-\beta} + c_\alpha c_\beta c_{2-\alpha-\beta} y^{\alpha+\beta-1} + c_\alpha c_\beta R_{2-\alpha-\beta}$$

Thus

$$S = E_\alpha \cdot E_\beta - \sum_i c_i E_{s_i}$$

$$= -c_{\alpha+\beta} y^{1-\alpha-\beta} - c_\beta c_{1+\alpha-\beta} y^{-\alpha+\beta} - c_\alpha c_{1-\alpha+\beta} y^{\alpha-\beta} - c_\alpha c_\beta c_{2-\alpha-\beta} y^{\alpha+\beta-1}$$

$$+ R - R_{\alpha+\beta} - c_\beta R_{1+\alpha-\beta} - c_\alpha R_{1-\alpha+\beta} - c_\alpha c_\beta R_{2-\alpha-\beta}$$

Since $1/2 \le \mathrm{Re}(\alpha) \le \mathrm{Re}(\beta) < \mathrm{Re}(\alpha) + 1/2$ and $\mathrm{Re}(\alpha+\beta) < 3/2$,

$$-c_{\alpha+\beta} y^{1-\alpha-\beta} - c_\beta c_{1+\alpha-\beta} y^{-\alpha+\beta} - c_\alpha c_{1-\alpha+\beta} y^{\alpha-\beta} - c_\alpha c_\beta c_{2-\alpha-\beta} y^{\alpha+\beta-1} \in L^2(\Gamma\backslash\mathfrak{H}).$$

**(III):** Suppose that $\alpha = \beta$. This will again yield two different cases based on $\mathrm{Re}(\alpha)$:

**(IIIa)** Suppose also that $\mathrm{Re}(\alpha) > 3/4$. Then

$$\sum_i c_i E_{s_i} = E_{2\alpha} + 2c_\alpha E_1^* - \frac{\pi}{3} C_\alpha = y^{2\alpha} + c_{2\alpha} y^{1-2\alpha} + R_{2\alpha} + 2c_\alpha\left(y - \frac{3}{\pi}\log y + C - \frac{\pi}{3} C_\alpha + R_1\right)$$

where $C_\alpha = \frac{d}{ds} c_s \Big|_{s=\alpha}$. Thus

$$S = (E_\alpha)^2 - \sum_i c_i E_{s_i} = (E_\alpha)^2 - E_{2\alpha} - 2c_\alpha E_1^* + \frac{\pi}{3} C_\alpha$$

$$= c_\alpha^2 y^{2-2\alpha} - c_{2\alpha} y^{1-2\alpha} + 2c_\alpha \frac{3}{\pi}\log y + C + \frac{\pi}{3}C_\alpha + R$$

and so $(E_\alpha)^2 - E_{2\alpha} - 2c_\alpha E_1 \in L^2(\Gamma\backslash\mathfrak{H})$ for $\mathrm{Re}(\alpha) > 3/4$.

Note that adding the constant $\frac{\pi}{3}C_\alpha$ does not affect whether $S$ is in $L^2$; however, this regime will aid computation later in the paper and arrises when taking the limit as $\beta \to \alpha$ as seen in Lemma **??** below.

**(IIIb)** Instead suppose that $1/2 \leq \mathrm{Re}(\alpha) < 3/4$. Then

$$\sum_i c_i E_{s_i} = E_{2\alpha} + 2c_\alpha E_1^* + c_\alpha^2 E_{2-2\alpha} - \frac{\pi}{3}C_\alpha$$

$$= y^{2\alpha} + c_{2\alpha} y^{1-2\alpha} + R_{2\alpha} + 2c_\alpha(y - \frac{3}{\pi}\log y + C + R_1) + c_\alpha^2(y^{2-2\alpha} + c_{2-2\alpha} y^{1-(2-2\alpha)} + R - \frac{\pi}{3}C_\alpha$$

Thus

$$S = (E_\alpha)^2 - \sum_i c_i E_{s_i} = (E_\alpha)^2 - E_{2\alpha} - 2c_\alpha E_1^* - c_\alpha^2 E_{2-2\alpha} + \frac{\pi}{3}C_\alpha$$

$$= -c_{2\alpha} y^{1-2\alpha} + 2c_\alpha \frac{3}{\pi}\log y - c_\alpha^2 c_{2-2\alpha} y^{2\alpha-1} + C + \frac{\pi}{3}C_\alpha + R$$

so $(E_\alpha)^2 - E_{2\alpha} - 2c_\alpha E_1 - c_\alpha^2 E_{2-2\alpha} \in L^2(\Gamma\backslash\mathfrak{H})$ for $1/2 \leq \mathrm{Re}(\alpha) < 3/4$.

We have shown that for each $\alpha$ and $\beta$ there is a linear combination of Eisenstein series $\sum_i c_i E_{s_i}$ so that $S = E_\alpha \cdot E_\beta - \sum_i c_i E_{s_i} \in L^2$. We can thus write a spectral expansion for each case for $S$ and get

$$E_\alpha \cdot E_\beta = \sum_i c_i E_{s_i} + \sum_{f \text{ cfm}} \langle S, f\rangle \cdot f + \frac{\langle S, 1\rangle \cdot 1}{\langle 1, 1\rangle} + \frac{1}{4\pi i}\int_{(1/2)} \langle S, E_s\rangle \cdot E_s \, ds$$

in $L^2(\Gamma\backslash\mathfrak{H}) \oplus \mathcal{E}(\Gamma\backslash\mathfrak{H})$.

To establish uniqueness, suppose that there are two solutions $u$ and $v$ to $(\Delta - \lambda_w)u = E_\alpha \cdot E_\beta$ in $H^2(\Gamma\backslash\mathfrak{H}) \oplus \mathcal{E}(\Gamma\backslash\mathfrak{H})$. Then $(\Delta - \lambda_w)(u - v) = E_\alpha \cdot E_\beta - E_\alpha \cdot E_\beta = 0$. Thus $u - v$ is a solution to the homogeneous equation $(\Delta - \lambda_w)(u - v) = 0$ and $\lambda_w \in \mathbb{R}$ but this cannot be the case if $\mathrm{Re}(w) > 1/2$ and $\mathrm{Im}(w) > 0$. □

Observe that for $\mathrm{Re}(s) < 1/2$, the functional equation gives $E_s = c_s \cdot E_{1-s}$. Thus it is sufficient to consider the case where $\mathrm{Re}(\alpha) \geq 1/2$ and $\mathrm{Re}(\beta) \geq 1/2$. Many of the other values excluded from $\mathcal{C}$ are in fact problematic as will will see in Section **??**. However, before we consider what is happening at these values, we will give a spectral expansion for the solution $u_w$.

**Theorem 3.** *In Re(w) > 1/2, for* $\alpha, \beta \in \mathcal{C}$, $(\Delta - \lambda)u = E_\alpha \cdot E_\beta$ *on* $\Gamma\backslash\mathfrak{H}$ *has a unique solution in* $H^{-\infty}(\Gamma\backslash\mathfrak{H})\oplus\mathcal{E}(\Gamma\backslash\mathfrak{H})$ *with spectral expansion which lies in* $H^2(\Gamma\backslash\mathfrak{H})\oplus\mathcal{E}(\Gamma\backslash\mathfrak{H})$ *and is given by*

$$u_w = \sum_i \frac{c_i E_{s_i}}{\lambda_{s_i} - \lambda_w} - \mathbb{1}_{\alpha=\beta} \cdot \frac{\frac{\pi}{3} C_\alpha}{\lambda_1 - \lambda_w} + \sum_{f\ cfm} \frac{\Lambda(\alpha, \overline{f} \times E_\beta) \cdot f}{\lambda_{s_f} - \lambda_w}$$

$$+ \frac{1}{4\pi i} \int_{(1/2)} \Lambda(\overline{s}, E_\alpha \times E_\beta) \cdot \frac{E_s}{\lambda_s - \lambda_w}\, ds$$

*where* $\mathbb{1}_{\alpha=\beta} = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta \end{cases}$ *and* $C_\alpha = \frac{d}{ds} c_s \Big|_{s=\alpha}$.

The proof of this result will be given in Sections 2, 3, 4, and 5 where we will construct the solution. Theorem **??** in Section **??** calculates the cuspidal spectrum, Theorem **??** in Section **??** calculates the continuous spectrum and Theorem **??** in Section **??** calculates the residual spectrum. In these sections, we will follow the regime presented in the proof of Theorem **??** and the final solution will be obtained by division in Section **??**. Finally, at the end of Section 5, we will prove that the solution can be meromorphically continued in $w$ to Re$(w) < 1/2$.

Before we turn to the derivation of the solution, we will address what appear to be oddities at some of the borderline cases in $\mathcal{C}$.

## 2.2 Limits in $\alpha$ and $\beta$

One would expect that the equality regimes ($\alpha = \beta$) presented in Theorem **??** can be recognized as a limits if those of Re$(\alpha)$ = Re$(\beta)$ and this is in fact the case due to the addiction of the constant $\frac{\pi}{3} C_\alpha$.

**Lemma 4.** $\lim_{\beta \to \alpha} c_\alpha E_{1-\alpha+\beta} + c_\beta \cdot E_{1+\alpha-\beta} = -\frac{\pi}{3} C_\alpha + 2c_\alpha E_1^*$ *where* $C_\alpha = \frac{d}{ds} c_s \Big|_{s=\alpha}$.

*Proof.* Recall that $E_s$ has a simple pole at $s = 1$ and thus the Laurent expansion for $E_s$

is given by

$$E_s = \frac{a_{-1}}{s-1} + a_o + a_1(s-1) + a_2(s-1)^2 + \dots$$

Using this we have

$$\lim_{\beta\to\alpha} c_\alpha E_{1-\alpha+\beta} + c_\beta \cdot E_{1+\alpha-\beta}$$

$$= \lim_{\beta\to\alpha} c_\alpha \cdot \left(\frac{a_{-1}}{\beta-\alpha} + a_o + a_1(\beta-\alpha) + \dots\right) + c_\beta\left(\frac{a_{-1}}{\alpha-\beta} + a_o + a_1(\alpha-\beta) + \dots\right)$$

$$= \lim_{\beta\to\alpha} c_\alpha \cdot \left(\frac{a_{-1}}{\beta-\alpha} + a_o + a_1(\beta-\alpha) + \dots\right) + c_\beta\left(-\frac{a_{-1}}{\beta-\alpha} + a_o - a_1(\beta-\alpha) - \dots\right)$$

$$= \lim_{\beta\to\alpha} \frac{a_{-1}(c_\alpha - c_\beta)}{\beta-\alpha} + 2c_\alpha a_o = -\frac{\pi}{3}\frac{d}{ds}c_s\Big|_{s=\alpha} + 2c_\alpha a_o = -\frac{\pi}{3}C_\alpha + 2c_\alpha E_1^*$$

where $C_\alpha = \frac{d}{ds}c_s\Big|_{s=\alpha}$ since $\lim_{\beta\to\alpha}\frac{a_{-1}(c_\alpha - c_\beta)}{\beta-\alpha} = -a_{-1}\frac{d}{ds}c_s\Big|_{s=\alpha} = -\frac{\pi}{3}\frac{d}{ds}c_s\Big|_{s=\alpha}.$ □

We can now express the equality case of **(III)** a limit of case **(II)**. Suppose as in case **(II)**, $1/2 \geq \mathrm{Re}(\alpha) \leq \mathrm{Re}(\beta) < \mathrm{Re}(\alpha) + 1/2$ where $\alpha \neq \beta$.

If we also suppose as in **(IIa)** that $\mathrm{Re}(\alpha+\beta) > 3/2$ then

$$S = E_\alpha \cdot E_\beta - (E_{\alpha+\beta} + c_\alpha \cdot E_{1-\alpha+\beta} + c_\beta \cdot E_{1+\alpha-\beta}).$$

As $\beta \to \alpha$ the first two terms become $E_{\alpha^2} - E_{2\alpha}$. Thus $\beta \to \alpha$, when $\mathrm{Re}(\alpha) > 3/4$, we get that

$$S \to (E_\alpha)^2 - E_{2\alpha} - 2c_\alpha E_1^* + \frac{\pi}{3}C_\alpha.$$

Similarly, if we suppose as in **(IIb)** that $\mathrm{Re}(\alpha+\beta) < 3/2$ then

$$S = E_\alpha \cdot E_\beta - (E_{\alpha+\beta} + c_\alpha \cdot E_{1-\alpha+\beta} + c_\beta \cdot E_{1+\alpha-\beta} + c_\alpha c_\beta E_{2-\alpha-\beta}).$$

Thus $\beta \to \alpha$, when $\mathrm{Re}(\alpha) > 3/4$, we get that

$$S \to (E_\alpha)^2 - E_{2\alpha} - 2c_\alpha E_1^* - c_\alpha^2 E_{2-2\alpha} + \frac{\pi}{3}C_\alpha.$$

However, despite this nice continuity where near where $\alpha = \beta$, one can see that we are not guaranteed the existence of the solution when $\alpha = \beta$ and $\mathrm{Re}(\alpha) = 3/4$. In fact, the strategy presented in Theorem **??** breaks down. When $\mathrm{Re}(\alpha) = 3/4$ we have

$E_\alpha^2 = y^{3/2+2\mathrm{Im}(\alpha)i} + 2c_\alpha y + c_\alpha^2 y^{1/2-2\mathrm{Im}(\alpha)i} + R$ and subtracting $E_{2-2\alpha}$ for example will cause the first term to vanish but will also introduce a new non-$L^2$ term $y^{1/2-2\mathrm{Im}(\alpha)i}$ to appear. In fact, we have the following results which only guarantee the existence of a solution under certain conditions.

**Theorem 5.**

*(i) In $Re(w) > 1/2$, for $\alpha \neq \beta$, $1/2 \leq Re(\alpha) \leq Re(\beta) < Re(\alpha) + 1/2$ and $Re(\alpha+\beta) = 3/2$, $(\Delta - \lambda)u = E_\alpha E_\beta$ on $\Gamma\backslash\mathfrak{H}$ has a unique solution in $H^{-\infty}(\Gamma\backslash\mathfrak{H}) \oplus \mathcal{E}(\Gamma\backslash\mathfrak{H})$ when $2\alpha - 1$ or $2\beta - 1$ is a nontrivial zero of $\zeta(s)$. If it is also the case that $Re(\beta) = Re(\alpha) + 1/2$, $(\Delta - \lambda)u = E_\alpha E_\beta$ on $\Gamma\backslash\mathfrak{H}$ has a unique solution in $H^{-\infty}(\Gamma\backslash\mathfrak{H}) \oplus \mathcal{E}(\Gamma\backslash\mathfrak{H})$ when $2\alpha - 1$ is a nontrivial zero of $\zeta(s)$.*

*(ii) In $Re(w) > 1/2$, for $Re(\alpha) = 3/4$, $(\Delta - \lambda)u = E_\alpha^2$ on $\Gamma\backslash\mathfrak{H}$ has a unique solution in $H^{-\infty}(\Gamma\backslash\mathfrak{H}) \oplus \mathcal{E}(\Gamma\backslash\mathfrak{H})$ when $2\alpha - 1$ is a nontrivial zero of $\zeta(s)$.*

Before we proceed with the proof, it should be noted that in the theorem above (ii) is a special instance of (i). However, we will provided a proof of both for a few reasons. One reason being that we will be using limits from the left and right of the solutions previously found and the solutions appear to be slightly different for $\alpha = \beta$ versus $\alpha \neq \beta$ (since the $S$'s constructed are differently) even though there limits are equal. However, the main reason is that it is easier to follow the argument in the $\alpha = \beta$ case and then see how it extends to the inequality case.

*Proof.* As shown in Theorem **??**, to demonstrate the existence and uniqueness of the solution, it suffices to construct appropriate $S$ in $L^2(\Gamma\backslash\mathfrak{H})$.

We will begin with a proof of (ii) since it is a simplified case of (i) and exemplifies the same general phenomenon. Observe that in the regime where $\alpha = \beta$, the $S$ given by **(IIIa)** and **(IIIb)** differ only by one term $c_\alpha^2 E_{2-2\alpha}$. This implies that we cannot have a simultaneous solution corresponding to both

$$S = (E_\alpha)^2 - E_{2\alpha} - 2c_\alpha E_1^* + \frac{\pi}{3} C_\alpha$$

and

$$S = (E_\alpha)^2 - E_{2\alpha} - 2c_\alpha E_1^* - c_\alpha^2 E_{2-2\alpha} + \frac{\pi}{3} C_\alpha$$

at $\mathrm{Re}(\alpha) = 3/4$ since their difference $\dfrac{c_\alpha^2 E_{2-2\alpha}}{\lambda_{2-2\alpha} - \lambda_w}$ is not in $L^2(\Gamma\backslash\mathfrak{H})$. In fact, in general, neither of these contrived $S$'s will be in $L^2(\Gamma\backslash\mathfrak{H})$ in general since:

In regime **(IIIa)**,

$$S = (E_\alpha)^2 - E_{2\alpha} - 2c_\alpha E_1^* + \frac{\pi}{3}C_\alpha$$

$$= c_\alpha^2 y^{2-2\alpha} - c_{2\alpha} y^{1-2\alpha} + 2c_\alpha \frac{3}{\pi}\log y + C + \frac{\pi}{3}C_\alpha + R$$

and $y^{2-2\alpha} \notin L^2(\Gamma\backslash\mathfrak{H})$ for $\mathrm{Re}(\alpha) = 3/4$. Thus we would need $c_\alpha^2 = 0$ in order for $S$ to be in $L^2$. Recall that $c_\alpha = \frac{\xi(2-2\alpha)}{\xi(2\alpha)} = \frac{\xi(2\alpha-1)}{\xi(2\alpha)}$. For $\alpha = 3/4 + it$, this yields $\xi(2\alpha - 1) = \xi(1/2 + 2it)$. Then $S \in L^2(\Gamma\backslash\mathfrak{H})$ when $2\alpha - 1$ is a nontrivial zero of $\xi$. Thus a solution to $(\Delta - \lambda)u = E_\alpha^2$ on $\Gamma\backslash\mathfrak{H}$ exists when $2\alpha - 1$ is a nontrivial zero of $\zeta$.

In regime **(IIIb)**,

$$S = (E_\alpha)^2 - E_{2\alpha} - 2c_\alpha E_1^* - c_\alpha^2 E_{2-2\alpha} + \frac{\pi}{3}C_\alpha$$

$$= -c_{2\alpha} y^{1-2\alpha} + 2c_\alpha \frac{3}{\pi}\log y - c_\alpha^2 c_{2-2\alpha} y^{2\alpha-1} + C + \frac{\pi}{3}C_\alpha + R$$

and $y^{2\alpha-1} \notin L^2(\Gamma\backslash\mathfrak{H})$ for $\mathrm{Re}(\alpha) = 3/4$. Thus we would need either $c_\alpha^2 = 0$ or $c_{2-2\alpha} = 0$. Observe that $c_{2-2\alpha} = 0$ when $\xi(4ti) = 0$ for $\alpha = 3/4 + it$. Since $\xi$ has no zeros on the imaginary axis, we need only consider where $c_\alpha^2 = 0$. As above, a solution to $(\Delta - \lambda)u = E_\alpha^2$ on $\Gamma\backslash\mathfrak{H}$ exists when $2\alpha - 1$ is a nontrivial zero of $\zeta$.

Since, as previously stated, these solutions, given by regime **(IIIa)** and **(IIIb)** may not be distinct. In fact, for a solution to exist, we need $c_\alpha^2 E_{2-2\alpha} \to 0$ as $\mathrm{Re}(\alpha) \to 3/4^-$. This will happen when $c_\alpha^2 = 0$ or when $E_{2-2\alpha} = 0$.[1] When $c_\alpha^2 = 0$ we see that the limit of the solution from each side of $\mathrm{Re}(\alpha) = 3/4$ will approach the above solution at $\mathrm{Re}(\alpha) = 3/4$.

Now let's turn to case (i). Observe that in the regime where $\alpha \neq \beta$ and $1/2 \leq \mathrm{Re}(\alpha) \leq \mathrm{Re}(\beta) < \mathrm{Re}(\alpha) + 1/2$ , the $S$ given by **(IIa)** and **(IIb)** differ only by one term $c_\alpha c_\beta \cdot E_{2-\alpha-\beta}$. This implies that we cannot have a simultaneous solution corresponding

[1] Note that when $E_{2-2\alpha} = 0$, the limits of $S$ from the left and right of $\mathrm{Re}(\alpha) = 3/4$ will be equal but that neither $S$ will be in $L^2(\Gamma\backslash\mathfrak{H})$.

to both

$$S = E_\alpha \cdot E_\beta - E_{\alpha+\beta} - c_\beta \cdot E_{1+\alpha-\beta} - c_\alpha \cdot E_{1-\alpha+\beta}$$

and

$$S = E_\alpha \cdot E_\beta - E_{\alpha+\beta} - c_\beta \cdot E_{1+\alpha-\beta} - c_\alpha \cdot E_{1-\alpha+\beta} - c_\alpha c_\beta \cdot E_{2-\alpha-\beta}$$

at $\mathrm{Re}(\alpha+\beta) = 3/2$ since their difference $\dfrac{c_\alpha c_\beta \cdot E_{2-\alpha-\beta}}{\lambda_{2-\alpha-\beta} - \lambda_w}$ is not in $L^2(\Gamma\backslash\mathfrak{H})$. In fact, in general, neither of these contrived $S$'s will be in $L^2(\Gamma\backslash\mathfrak{H})$ in general since:

In regime **(IIa)**,

$$S = E_\alpha \cdot E_\beta - E_{\alpha+\beta} - c_\beta \cdot E_{1+\alpha-\beta} - c_\alpha \cdot E_{1-\alpha+\beta}$$

$$= c_\alpha c_\beta y^{2-\alpha-\beta} - c_{\alpha+\beta} y^{1-\alpha-\beta} - c_\beta c_{1+\alpha-\beta} y^{-\alpha+\beta} - c_\alpha c_{1-\alpha+\beta} y^{\alpha-\beta} + R$$

and $y^{2-\alpha-\beta} \notin L^2(\Gamma\backslash\mathfrak{H})$ for $\mathrm{Re}(\alpha+\beta) = 3/2$. Thus we would need $c_\alpha = 0$ or $c_\beta = 0$ in order for $S$ to be in $L^2$.

In regime **(IIb)**,

$$S = E_\alpha \cdot E_\beta - E_{\alpha+\beta} - c_\beta \cdot E_{1+\alpha-\beta} - c_\alpha \cdot E_{1-\alpha+\beta}$$

$$= -c_{\alpha+\beta} y^{1-\alpha-\beta} - c_\beta c_{1+\alpha-\beta} y^{-\alpha+\beta} - c_\alpha c_{1-\alpha+\beta} y^{\alpha-\beta} - c_\alpha c_\beta c_{2-\alpha-\beta} y^{\alpha+\beta-1} + R$$

and $y^{\alpha+\beta-1} \notin L^2(\Gamma\backslash\mathfrak{H})$ for $\mathrm{Re}(\alpha+\beta) = 3/2$. Thus we would need either $c_\alpha = 0$, $c_\beta = 0$ or $c_{1-\alpha+\beta} = 0$.

Thus solution to $(\Delta - \lambda)u = E_\alpha \cdot E_\beta$ on $\Gamma\backslash\mathfrak{H}$ exists when $2\alpha - 1$ or $2\beta - 1$ is a nontrivial zero of $\zeta$. Furthermore, when $\mathrm{Re}(\beta) = \mathrm{Re}(\alpha) + 1/2$, we also have $y^{\alpha-\beta} \notin L^2(\Gamma\backslash\mathfrak{H})$. We will then need either $c_\alpha = 0$ or $c_{1-\alpha+\beta} = 0$. However, in this case, we also have $c_{1-\alpha+\beta} = c_{3/2+it} \neq 0$.

Since, as previously stated, these solutions, given by regime **(IIa)** and **(IIb)** may not be distinct. In fact, for a solution to exist, we need $c_\alpha c_\beta \cdot E_{2-\alpha-\beta} \to 0$ as $\mathrm{Re}(\alpha+\beta) \to 3/2^-$. This will happen when $c_\alpha = 0$, $c_\beta = 0$ or when $E_{2-\alpha-\beta} = 0$. [2]

When $c_\alpha = 0$ or when $c_\beta = 0$ we see that the limit of the solution from each side of $\mathrm{Re}(\alpha+\beta) = 3/2$ will approach the above solution at $\mathrm{Re}(\alpha+\beta) = 3/2$. □

[2] Note that when $E_{2-\alpha-\beta} = 0$, the limits of $S$ from the left and right of $\mathrm{Re}(\alpha+\beta) = 3/2$ will be equal but that neither $S$ will be in $L^2(\Gamma\backslash\mathfrak{H})$.

The solution on these regions where say $\mathrm{Re}(\alpha+\beta) = 3/2$ will present itself as a limit and will thus be identified with the corresponding limit of the solution in Theorem **??**. Explicitly, when $\alpha \neq \beta$, $1/2 \leq \mathrm{Re}(\alpha) \leq \mathrm{Re}(\beta) < \mathrm{Re}(\alpha) + 1/2$ and $\mathrm{Re}(\alpha+\beta) = 3/2$ and $2\alpha - 1$ is a zero of $\zeta(s)$ (i.e. $c_\alpha = 0$),

$$S = E_\alpha \cdot E_\beta - E_{\alpha+\beta} - c_\beta \cdot E_{1+\alpha-\beta}$$

is in $L^2$ for $\mathrm{Re}(\alpha+\beta) = 3/2$. In this case, the equation $(\Delta - \lambda)u = E_\alpha \cdot E_\beta$ on $\Gamma\backslash\mathfrak{H}$ has a unique solution in $H^{-\infty}(\Gamma\backslash\mathfrak{H}) \oplus \mathcal{E}(\Gamma\backslash\mathfrak{H})$ with spectral expansion which lies in $H^2(\Gamma\backslash\mathfrak{H}) \oplus \mathcal{E}(\Gamma\backslash\mathfrak{H})$ and is given by

$$u_w = \frac{E_{\alpha+\beta}}{\lambda_{\alpha+\beta} - \lambda_w} + \frac{c_\beta \cdot E_{1+\alpha-\beta}}{\lambda_{1+\alpha-\beta} - \lambda_w} + \sum_{f \text{ cfm}} \frac{\Lambda(\alpha, \overline{f} \times E_\beta) \cdot f}{\lambda_{s_f} - \lambda_w}$$

$$+ \frac{1}{4\pi i} \int_{(1/2)} \Lambda(\overline{s}, E_\alpha \times E_\beta) \cdot \frac{E_s}{\lambda_s - \lambda_w}\, ds.$$

We will conclude this section by showing that there are no solutions in $H^2(\Gamma\backslash\mathfrak{H}) \oplus \mathcal{E}(\Gamma\backslash\mathfrak{H})$ on the lines $\mathrm{Re}(\alpha+\beta) = 3/2$ or $\mathrm{Re}(\alpha) = 3/4$ where neither $c_\alpha$ nor $c_\beta$ are zero. We will need the following preliminary results in order to establish the other direction of the implied biconditional.

**Lemma 6.** *Let $\xi_i, \ldots, \xi_n$ be distinct real numbers and $\sigma_1, \ldots, \sigma_n$ real. For non-zero complex $c_1, \ldots, c_n$, the function $f(y) = \sum_j c_j y^{\sigma_j + i\xi_j}$ is in $L^2([1,\infty), \frac{dy}{y^2})$ for if and only if $\sigma_j < 1/2$ for all $j$.*

*Proof.* If $\mu := \max_j \sigma_j < 1/2$, then $f(y) \in L^2([1,\infty), \frac{dy}{y^2})$.

On the other hand, suppose that $\mu = 1/2$. Observe that

$$\left| \sum_j c_j y^{\sigma_j + i\xi_j} \right|^2_{L^2([1,\infty), \frac{dy}{y^2})} = \lim_{b\to\infty} \int_1^b \left| \sum_j c_j y^{\sigma_j + i\xi_j} \right|^2 \frac{dy}{y^2}$$

$$= \lim_{b\to\infty} \int_1^b \sum_{j,k} c_j \overline{c}_k\, y^{i(\xi_j - \xi_k)}\, y^{\sigma_j + \sigma_k}\, \frac{dy}{y^2}.$$

If $\mu = 1/2$ then all the terms $c_j \overline{c}_k\, y^{i(\xi_j - \xi_k)}\, y^{\sigma_j + \sigma_k}$ are in $L^1$ except for possibly the sum over $j, k$ with $\sigma_j = 1/2 = \sigma_k$.

Suppose now that $\mu = 1/2$ and $\sigma_j = 1/2 = \sigma_k$. Among the tails for the improper integral for the $L^2$-norm-squared integrals are

$$\sum_{j,k} c_j \bar{c}_k \int_T^{T^2} y^{i(\xi_j - \xi_k)} \, \frac{dy}{y}.$$

For $j = k$, the term is $|c_j|^2 \cdot \log T$. For $j \neq k$, the term is $c_j \bar{c}_k \frac{(T^2)^{i(\xi_j - \xi_k)} - T^{i(\xi_j - \xi_k)}}{i(\xi_j - \xi_k)}$. The sum of the $j \neq k$ is uniformly bounded in $T$. The sum of the $j = k$ term is a strictly positive real multiple of $\log T$ and goes to $\infty$ as $T \to \infty$. Thus an expression of the form $\sum_j c_j y^{1/2 + i\xi_j}$ will be in $L^2([1, \infty), \frac{dy}{y^2})$ only when $c_j = 0$ for each $j$. Furthermore, if $\mu = 1/2$ then $f(y)$ cannot be in $L^2([1, \infty), \frac{dy}{y^2})$.

Finally, in the case of $\mu > 1/2$, $y^{1/2 - \mu} \cdot f(y)$ is in $L^2([1, \infty), \frac{dy}{y^2})$ if $f(y)$ is and this reduces to the case where $\mu = 1/2$ just treated.

□

**Lemma 7.**

*(i) For $\alpha \neq \beta$, $1/2 \leq Re(\alpha) \leq Re(\beta) < Re(\alpha) + 1/2$ and $Re(\alpha + \beta) = 3/2$, then $E_\alpha E_\beta \notin L^2(\Gamma \backslash \mathfrak{H}) \oplus \mathcal{E}(\Gamma \backslash \mathfrak{H})$ unless $2\alpha - 1$ or $2\beta - 1$ is a zero of $\zeta(s)$.*

*(ii) For $Re(\alpha) = 3/4$, then $E_\alpha^2 \notin L^2(\Gamma \backslash \mathfrak{H}) \oplus \mathcal{E}(\Gamma \backslash \mathfrak{H})$ unless $2\alpha - 1$ is a zero of $\zeta(s)$.*

*Proof.* We will again first establish the result of (ii) first. Assume $\alpha = \beta$ and $\mathrm{Re}(\alpha) = 3/4$. We have $E_\alpha^2 = y^{3/2 + 2\mathrm{Im}(\alpha)i} + 2c_\alpha y + c_\alpha^2 y^{1/2 - 2\mathrm{Im}(\alpha)i} + R$. Subtracting $E_{2\alpha}$ and $2c_\alpha E_1^*$ will eliminate the first term terms and what remains will be in $L^2$ with the exception of the term $c_\alpha^2 y^{1/2 - 2\mathrm{Im}(\alpha)i}$. Subtracting $c_\alpha^2 E_{2-2\alpha}$ will cause the last term to vanish but will also introduce a new non-$L^2$ term $c_\alpha^2 c_{2-2\alpha} y^{1/2 + 2\mathrm{Im}(\alpha)i}$ to appear. Furthermore, observe that $c_{2-2\alpha}$ cannot be zero for $\mathrm{Re}(\alpha) = 3/4$ since $\zeta(s)$ has no zeros on the line $\mathrm{Re}(s) = 1$. More formally, the non-rapidly decreasing terms of $\mathcal{E}(\Gamma \backslash \mathfrak{H})$ can be written as linear combinations of the form $\sum_j c_j y^{\sigma_j + i\xi_j}$ and so by Lemma **??** $c_\alpha^2 y^{1/2 - 2\mathrm{Im}(\alpha)i} + \sum_j c_j y^{\sigma_j + i\xi_j}$ is not in $L^2$ except when $c_\alpha = 0$. Thus the only way for $E_\alpha^2$ to be in $L^2(\Gamma \backslash \mathfrak{H}) \oplus \mathcal{E}(\Gamma \backslash \mathfrak{H})$ is by $c_\alpha$ being 0 and thus it is necessary that $\zeta(2\alpha - 1) = 0$.

For (i), assume $\alpha \neq \beta$, $1/2 \leq \text{Re}(\alpha) \leq \text{Re}(\beta) < \text{Re}(\alpha)+1/2$ and $\text{Re}(\alpha+\beta) = 3/2$. Then $E_\alpha E_\beta = y^{3/2+\text{Im}(\alpha+\beta)i} + c_\beta y^{1+\alpha-\beta} + c_\alpha y^{1-\alpha+\beta} + c_\alpha c_\beta y^{1/2-\text{Im}(\alpha+\beta)i} + R$. Again, the first three terms can be eliminated putting what remains in $L^2$ with the exception of the term $c_\alpha c_\beta y^{1/2-\text{Im}(\alpha+\beta)i}$. Subtracting $c_\alpha c_\beta E_{2-\alpha-\beta}$ will cause the first term to vanish but will also introduce a new non-$L^2$ term $c_\alpha c_\beta c_{2-\alpha-\beta} y^{1/2+\text{Im}(\alpha+\beta)i}$ to appear. Again $c_{2-\alpha-\beta}$ cannot be zero for $\text{Re}(\alpha+\beta) = 3/4$ since $\zeta(s)$ has no zeros on the line $\text{Re}(s) = 1$. Furthermore, by Lemma **??** $c_\alpha c_\beta c_{2-\alpha-\beta} y^{1/2+\text{Im}(\alpha+\beta)i} + \sum_j c_j y^{\sigma_j + i\xi_j}$ is not in $L^2$ except when $c_\alpha = 0$ or $c_\beta = 0$. Thus the only way for $E_\alpha E_\beta$ to be in $L^2(\Gamma\backslash\mathfrak{H}) \oplus \mathcal{E}(\Gamma\backslash\mathfrak{H})$ is by $c_\alpha$ or $c_\beta$ being 0 and thus it is necessary that $\zeta(2\alpha - 1) = 0$ or $\zeta(2\beta - 1) = 0$. □

**Lemma 8.** *If there exists a solution $u$ to $(\Delta - \lambda)u = E_\alpha \cdot E_\beta$ on $\Gamma\backslash\mathfrak{H}$ in $H^2(\Gamma\backslash\mathfrak{H}) \oplus \mathcal{E}(\Gamma\backslash\mathfrak{H})$ then $E_\alpha E_\beta \in L^2(\Gamma\backslash\mathfrak{H}) \oplus \mathcal{E}(\Gamma\backslash\mathfrak{H})$.*

*Proof.* Suppose $u$ is a solution to $(\Delta-\lambda)u = E_\alpha \cdot E_\beta$ on $\Gamma\backslash\mathfrak{H}$ and $u \in H^2(\Gamma\backslash\mathfrak{H}) \oplus \mathcal{E}(\Gamma\backslash\mathfrak{H})$. Say $u = f + \sum_k a_k F_{s_k}$ for $f \in H^2(\Gamma\backslash\mathfrak{H})$ and $\sum_k a_k F_{s_k} \in \mathcal{E}(\Gamma\backslash\mathfrak{H})$. Then

$$E_\alpha \cdot E_\beta = (\Delta - \lambda_w)u = (\Delta - \lambda_w)\left(f + \sum_k a_k F_{s_k}\right)$$

$$= (\Delta - \lambda_w)f + \sum_k a_k(\lambda_{s_k} - \lambda_w)F_{s_k}$$

Thus

$$E_\alpha \cdot E_\beta - \sum_k a_k(\lambda_{s_k} - \lambda_w)F_{s_k} = (\Delta - \lambda_w)f \in H^0(\Gamma\backslash\mathfrak{H}) = L^2(\Gamma\backslash\mathfrak{H})$$

since $f \in H^2(\Gamma\backslash\mathfrak{H})$.

□

Combining the last two results, we see that if there were a solution $u$ in $H^2(\Gamma\backslash\mathfrak{H}) \oplus \mathcal{E}(\Gamma\backslash\mathfrak{H})$ it would be contrived as above and thus there is no such solutions on $\text{Re}(\alpha+\beta) = 3/2$ where neither $2\alpha - 1$ nor $2\beta - 1$ is a zero of $\zeta$.

**Theorem 9.**

*(i) In Re$(w) > 1/2$, for $\alpha \neq \beta$, $1/2 \leq$ Re$(\alpha) \leq$ Re$(\beta) <$ Re$(\alpha) + 1/2$ and Re$(\alpha + \beta) = 3/2$, $(\Delta - \lambda)u = E_\alpha E_\beta$ on $\Gamma\backslash\mathfrak{H}$ has a unique solution in $H^2(\Gamma\backslash\mathfrak{H}) \oplus \mathcal{E}(\Gamma\backslash\mathfrak{H})$ if and only if $2\alpha - 1$ or $2\beta - 1$ is a nontrivial zero of $\zeta(s)$.*

*(ii) In Re*$(w) > 1/2$*, for Re*$(\alpha) = 3/4$*,* $(\Delta - \lambda)u = E^2_\alpha$ *on* $\Gamma\backslash\mathfrak{H}$ *has a unique solution in* $H^2(\Gamma\backslash\mathfrak{H}) \oplus \mathcal{E}(\Gamma\backslash\mathfrak{H})$ *if and only if* $2\alpha - 1$ *is a nontrivial zero of* $\zeta(s)$*.*

The proof of this result follows directly from Theorem **??** in conjunction with Lemma **??** and Lemma **??**.

# Chapter 3

# The Cuspidal Spectrum

We will now compute the cuspidal spectrum for the expansion of the solution. Let $f$ be a cuspform with Fourier expansion

$$f(z) = \sum_{n \neq 0} c_n \cdot W_s(|n|y) \cdot e^{2\pi i n x}$$

**Theorem 10.** *For $f$ a cuspform and $(\alpha, \beta) \in \mathcal{C}$,*

$$\begin{aligned} \langle S, f \rangle_{L^2} &= L(\alpha, \overline{f} \times E_\beta) \cdot \frac{\pi^{\beta+\overline{s}-\alpha}}{2\,\Gamma(\beta)\Gamma(\overline{s})} \cdot \frac{\Gamma(\frac{\alpha+\beta-\overline{s}}{2})\Gamma(\frac{\alpha-\beta+\overline{s}}{2})\Gamma(\frac{\alpha+1-\beta-\overline{s}}{2})\Gamma(\frac{\alpha-1+\beta+\overline{s}}{2})}{\Gamma(\alpha)} \\ &= \Lambda(\alpha, \overline{f} \times E_\beta) \end{aligned}$$

*for each $S$ proposed in Theorem* **??**.

The proof of this result is given in the what remains of this section. Before we investigate each case for each different $S$, we will first perform two useful computations. Many examples of the following computations can be found in relevant literature – for example, in [**?**] or [**?**].

**Lemma 11.** *For each $\alpha$ and $\beta$,*

$$\int_{\Gamma\backslash\mathfrak{H}} E_\alpha \cdot E_\beta \cdot \overline{f} \, \frac{dx\, dy}{y^2}$$

$$= L(\alpha, \overline{f} \times E_\beta) \cdot \frac{\pi^{\beta+\overline{s}-\alpha}}{2\,\Gamma(\beta)\Gamma(\overline{s})} \cdot \frac{\Gamma(\frac{\alpha+\beta-\overline{s}}{2})\Gamma(\frac{\alpha-\beta+\overline{s}}{2})\Gamma(\frac{\alpha+1-\beta-\overline{s}}{2})\Gamma(\frac{\alpha-1+\beta+\overline{s}}{2})}{\Gamma(\alpha)}$$

$$= \Lambda(\alpha, \overline{f} \times E_\beta)$$

*Proof.* The computation that follows we can will begin by examining $1/2 < \mathrm{Re}(\alpha) < \mathrm{Re}(\alpha) + 1/2 < \mathrm{Re}(\beta)$ since $\int_{\Gamma\backslash\mathfrak{H}} E_\alpha \cdot E_\beta \cdot \overline{f}\, \frac{dx\,dy}{y^2}$ is holomorphic on this region. Since it extends to meromorphic function of $\alpha$ and $\beta$ (since $f$ is cuspform), we can evaluate it via identity principle by moving $\alpha$ to $\mathrm{Re}(\alpha) > 1$ so that we can then unwind $E_\alpha$.

Thus, by unwinding, we have

$$\int_{\Gamma\backslash\mathfrak{H}} E_\alpha \cdot E_\beta \cdot \overline{f}\, \frac{dx\,dy}{y^2} = \int_{\Gamma\backslash\mathfrak{H}} \sum_{\gamma \in P\backslash\Gamma} \mathrm{Im}(\gamma z)^\alpha \cdot E_\beta \cdot \overline{f}\, \frac{dx\,dy}{y^2} = \int_{P\backslash\mathfrak{H}} y^\alpha \cdot E_\beta \cdot \overline{f}\, \frac{dx\,dy}{y^2}$$

$$= \int_0^\infty \int_0^1 y^\alpha \cdot E_\beta \cdot \overline{f}\, \frac{dx\,dy}{y^2}$$

since the fundamental domain of $P\backslash\mathfrak{H}$ is $\{z = x+iy \in \mathfrak{H} \mid 0 \le x \le 1\}$. Now, writing out the Fourier-Whittaker expansions for $E_\beta$ and $f$, we have

$$\int_0^\infty \int_0^1 y^\alpha \cdot \left( c_P E_\beta + \sum_{n\neq 0} \varphi(n,\beta) \cdot W_\beta(|n|y) \cdot e^{2\pi i n x} \right) \cdot \left( \sum_{m\neq 0} \overline{c}_m \cdot \overline{W_s}(|m|y) \cdot e^{-2\pi i m x} \right) \frac{dx\,dy}{y^2}$$

where $\varphi$, $W_s$ and $c_m$ are defined in Section **??**

$$= \int_0^\infty \int_0^1 y^\alpha \Big[ c_P E_\beta \cdot \sum_{m\neq 0} \overline{c}_m \cdot \overline{W_s}(|m|y) \cdot e^{-2\pi i m x} + \left( \sum_{n\neq 0} \varphi(n,\beta) \cdot W_\beta(|n|y) \cdot e^{2\pi i n x} \right) \cdot \left( \sum_{m\neq 0} \overline{c}_m \cdot \overline{W_s}(|m|y) \cdot e^{-2\pi i m x} \right) \Big] \frac{dx\,dy}{y^2}$$

$$= \int_0^\infty y^\alpha \Big[ c_P E_\beta \cdot \sum_{m\neq 0} \overline{c}_m \cdot \overline{W_s}(|m|y) \cdot \int_0^1 e^{-2\pi i m x}\, dx$$

$$+ \sum_{m,n\neq 0} \varphi(n,\beta) W_\beta(|n|y) \cdot \overline{c}_m \overline{W_s}(|m|y) \int_0^1 e^{2\pi i (n-m)x}\, dx \Big] \frac{dy}{y^2}$$

$$= \int_0^\infty y^\alpha \Big[ c_P E_\beta \cdot \sum_{m\neq 0} \overline{c}_m \cdot \overline{W_s}(|m|y) \cdot \delta_{0,m} + \sum_{m,n\neq 0} \varphi(n,\beta) W_\beta(|n|y) \cdot \overline{c}_m \overline{W_s}(|m|y) \cdot \delta_{n,m} \Big] \frac{dy}{y^2}$$

We see that the sum is zero when $n \neq m$ (furthermore, since $f$ is a cuspform the $n = 0$ term vanishes) and we get

$$\int_0^\infty y^\alpha \sum_{n\neq 0} \varphi(n,\beta) W_\beta(|n|y) \cdot \overline{c}_m \overline{W_s}(|m|y) \,\frac{dy}{y^2}$$

$$= \sum_{n\neq 0} \varphi(n,\beta) \cdot \overline{c}_n \cdot \int_0^\infty y^\alpha \cdot W_\beta(|n|y)\, \overline{W_s}(|n|y) \,\frac{dy}{y^2}$$

Replacing $y$ by $y/n$, we have

$$\sum_{n\neq 0} \frac{\varphi(n,\beta)\cdot \overline{c}_n}{n^{\alpha-1}} \cdot \int_0^\infty y^\alpha \cdot W_\beta(y)\, \overline{W_s}(y) \,\frac{dy}{y^2} = L(\alpha, \overline{f} \times E_\beta) \cdot \int_0^\infty y^\alpha \cdot W_\beta(y)\, \overline{W_s}(y) \,\frac{dy}{y^2}$$

$$= L(\alpha, \overline{f} \times E_\beta) \cdot \frac{\pi^{\beta+\overline{s}-\alpha}}{2\,\Gamma(\beta)\Gamma(\overline{s})} \cdot \frac{\Gamma(\frac{\alpha+\beta-\overline{s}}{2})\Gamma(\frac{\alpha-\beta+\overline{s}}{2})\Gamma(\frac{\alpha+1-\beta-\overline{s}}{2})\Gamma(\frac{\alpha-1+\beta+\overline{s}}{2})}{\Gamma(\alpha)} = \Lambda(\alpha, \overline{f} \times E_\beta)$$

□

**Lemma 12.** *For any $r \neq 1$, $\int_{\Gamma\backslash\mathfrak{H}} E_r \cdot \overline{f}\ \frac{dx\,dy}{y^2} = 0$ and $\int_{\Gamma\backslash\mathfrak{H}} E_1^* \cdot \overline{f}\ \frac{dx\,dy}{y^2} = 0$.*

*Proof.* Since the integrals extend to meromorphic function of $r$ (since $f$ is cuspform), we can evaluate it via identity principle by moving $r$ to $\mathrm{Re}(r) > 1$ so that we can then unwind $E_r$. Thus

$$\int_{\Gamma\backslash\mathfrak{H}} E_r \cdot \overline{f}\ \frac{dx\,dy}{y^2} = \int_{\Gamma\backslash\mathfrak{H}} \sum_{\gamma\in P\backslash\Gamma} \mathrm{Im}(\gamma z)^r\, \overline{f}\ \frac{dx\,dy}{y^2} = \int_{P\backslash\mathfrak{H}} \mathrm{Im}(z)^r\, \overline{f}\ \frac{dx\,dy}{y^2}$$

where the fundamental domain of $P\backslash\mathfrak{H}$ is $\{z = x + iy \in \mathfrak{H} \mid 0 \le x \le 1\}$

$$= \int_{P\backslash\mathfrak{H}} y^r \overline{f}(z)\,\frac{dx\,dy}{y^2} = \sum_{n>0} \overline{c}_n \int_{y>0} y^r \cdot \overline{W}_s(|n|y) \left(\int_{0\le x\le 1} e^{2\pi inx}\,dx\right) \frac{dy}{y^2}$$

$$= \sum_{n>0} \overline{c_n} \int_{y>0} y^r \overline{W}_s(|n|y)\delta_{n,0}\,\frac{dy}{y^2} = 0$$

□

Finally, we should note that constants (such as $\frac{\pi}{3}C_\alpha$) are orthogonal to cuspforms in $L^2(\Gamma\backslash\mathfrak{H})$ so

$$\int_{\Gamma\backslash\mathfrak{H}} \frac{\pi}{3} C_\alpha \cdot \overline{f}\,\frac{dx\,dy}{y^2} = 0$$

We can now quickly evaluate each case of $S$ for each $\alpha$ and $\beta$ presented above.

## 3.1 Regimes

Recall the regimes set up in the proof of Theorem **??**. Again, suppose that $\alpha \ne 1$ and $\beta \ne 1$.

**(I):** When $1/2 \le \mathrm{Re}(\alpha) < \mathrm{Re}(\alpha) + 1/2 < \mathrm{Re}(\beta)$,

$$\langle S, f\rangle_{L^2} = \int_{\Gamma\backslash\mathfrak{H}} (E_\alpha \cdot E_\beta - E_{\alpha+\beta} - c_\alpha \cdot E_{1-\alpha+\beta}) \cdot \overline{f}\,\frac{dx\,dy}{y^2}$$

$$= \int_{\Gamma\backslash\mathfrak{H}} E_\alpha \cdot E_\beta \cdot \overline{f} - E_{\alpha+\beta} \cdot \overline{f} - c_\alpha \cdot E_{1-\alpha+\beta} \cdot \overline{f}\,\frac{dx\,dy}{y^2}$$

**(II):** Suppose $1/2 \le \mathrm{Re}(\alpha) \le \mathrm{Re}(\beta) < \mathrm{Re}(\alpha) + 1/2$ but that $\alpha \ne \beta$.

**(IIa)** If $\mathrm{Re}(\alpha+\beta) > 3/2$ then

$$\langle S, f\rangle_{L^2} = \int_{\Gamma\backslash\mathfrak{H}} (E_\alpha \cdot E_\beta - E_{\alpha+\beta} - c_\beta \cdot E_{1+\alpha-\beta} - c_\alpha \cdot E_{1-\alpha+\beta}) \cdot \overline{f}\,\frac{dx\,dy}{y^2}$$

$$= \int_{\Gamma\backslash\mathfrak{H}} E_\alpha \cdot E_\beta \cdot \overline{f} - E_{\alpha+\beta} \cdot \overline{f} - c_\beta \cdot E_{1+\alpha-\beta} \cdot \overline{f} - c_\alpha \cdot E_{1-\alpha+\beta} \cdot \overline{f}\,\frac{dx\,dy}{y^2}$$

**(IIb)** If $\mathrm{Re}(\alpha+\beta) < 3/2$ then $\langle S, f\rangle_{L^2}$

$$= \int_{\Gamma\backslash\mathfrak{H}} (E_\alpha \cdot E_\beta - E_{\alpha+\beta} - c_\beta \cdot E_{1+\alpha-\beta} - c_\alpha \cdot E_{1-\alpha+\beta} - c_\alpha c_\beta \cdot E_{2-\alpha-\beta}) \cdot \overline{f}\,\frac{dx\,dy}{y^2}$$

$$= \int_{\Gamma\backslash\mathfrak{H}} E_\alpha \cdot E_\beta \cdot \overline{f} - E_{\alpha+\beta} \cdot \overline{f} - c_\beta \cdot E_{1+\alpha-\beta} \cdot \overline{f} - c_\alpha \cdot E_{1-\alpha+\beta} \cdot \overline{f}$$

$$- c_\alpha c_\beta \cdot E_{2-\alpha-\beta} \cdot \overline{f}\,\frac{dx\,dy}{y^2}$$

**(III):** Suppose $\alpha = \beta$.

**(IIIa)** Suppose also that $\mathrm{Re}(\alpha) > 3/4$ then

$$\langle S, f\rangle_{L^2} = \int_{\Gamma\backslash\mathfrak{H}} \Big((E_\alpha)^2 - E_{2\alpha} - 2c_\alpha E_1^* + \frac{\pi}{3}C_\alpha\Big) \cdot \overline{f}\,\frac{dx\,dy}{y^2}$$

$$= \int_{\Gamma\backslash\mathfrak{H}} (E_\alpha)^2 \cdot \overline{f} - E_{2\alpha} \cdot \overline{f} - 2c_\alpha E_1^* \cdot \overline{f} + \frac{\pi}{3}C_\alpha \cdot \overline{f}\,\frac{dx\,dy}{y^2}$$

**(IIIb)** Now suppose $1/2 \le \mathrm{Re}(\alpha) < 3/4$ then

$$\langle S, f\rangle_{L^2} = \int_{\Gamma\backslash\mathfrak{H}} \Big((E_\alpha)^2 - E_{2\alpha} - 2c_\alpha E_1^* - c_\alpha^2 E_{2-2\alpha} + \frac{\pi}{3}C_\alpha\Big) \cdot \overline{f}\,\frac{dx\,dy}{y^2}$$

$$= \int_{\Gamma\backslash\mathfrak{H}} (E_\alpha)^2 \cdot \overline{f} - E_{2\alpha} \cdot \overline{f} - 2c_\alpha E_1^* \cdot \overline{f} - c_\alpha^2 E_{2-2\alpha} \cdot \overline{f} + \frac{\pi}{3}C_\alpha \cdot \overline{f}\,\frac{dx\,dy}{y^2}$$

In each of the above cases, we can use Lemma **??** to evaluate the integral of the first term and see that each of the remaining terms will integrate to be zero using Lemma **??**. Thus for each $S$, we get

$$\langle S, f\rangle_{L^2} = L(\alpha, \overline{f} \times E_\beta) \cdot \frac{\pi^{\beta+\overline{s}-\alpha}}{2\,\Gamma(\beta)\Gamma(\overline{s})} \cdot \frac{\Gamma(\frac{\alpha+\beta-\overline{s}}{2})\Gamma(\frac{\alpha-\beta+\overline{s}}{2})\Gamma(\frac{\alpha+1-\beta-\overline{s}}{2})\Gamma(\frac{\alpha-1+\beta+\overline{s}}{2})}{\Gamma(\alpha)}$$

$$= \Lambda(\alpha, \overline{f} \times E_\beta)$$

# Chapter 4

# The Continuous Spectrum

We want to compute

$$\int_{(1/2)} \langle S, E_s \rangle \cdot E_s \, ds$$

for each case of $S$. Though we have designed $S$ so that $S \in L^2(\Gamma\backslash\mathfrak{H})$, there is no guarantee that $S \cdot \overline{E_s}$ is in $L^1(\Gamma\backslash\mathfrak{H})$. However, observe that on $\mathrm{Re}(s) = 1/2$, $\langle S, E_s \rangle$ exists as a literal integral since $S$ is $\mathcal{O}(y^{\frac{1}{2}-\epsilon})$ for some $\epsilon > 0$. This can be seen by observing that $E_s = y^s + c_s y^{1-s} + R_s$ where $R_s$ is rapidly decreasing and so $\overline{E}_s \cdot S$ is $\mathcal{O}(y^{1-\epsilon})$. Thus

$$\int_{\Gamma\backslash\mathfrak{H}} \overline{E}_s \cdot S \, \frac{dy \, dx}{y^2} < \infty.$$

Futhermore, in what follows we will show

**Theorem 13.** *For each $\alpha, \beta \in \mathcal{C}$,*

$$\langle S, E_s \rangle_{L^2} = L(\overline{s}, E_\alpha \times E_\alpha) \cdot \int_0^\infty y^{\overline{s}} \cdot W_\alpha(y) W_\alpha(y) \, \frac{dy}{y^2} = \Lambda(\overline{s}, E_\alpha \times E_\alpha)$$

*for each $S$ given in Theorem* **??**.

Knowing that these integrals converge and computing them directly are two different things. In the style of Zagier [**?**] and Casselman [**?**], we will use Arthur truncation to compute these spectral integrals. To make proper use of the truncated Eisenstein series, we will also need that the limit of these truncated Eisenstein series converges to the Eisenstein series itself.

## 4.1 Convergence of Truncated Eisenstein Series

Recall that Arthur truncation is defined as

$$\wedge^T E_s := E_s - \sum_{\gamma \in P\backslash\Gamma} \tau_s(\gamma z) \qquad \text{where} \qquad \tau_s(z) = \begin{cases} y^s + c_s y^{1-s} & y \geq T \\ 0 & y < T \end{cases}.$$

For convenience we will label the sum $\Theta_s^T(z) := \sum_{\gamma \in P\backslash\Gamma} \tau_s(\gamma z)$ so that $\wedge^T E_s := E_s - \Theta_s^T(z)$.

Let

$$\Psi_\epsilon(z) := \sum_{\gamma \in P\backslash\Gamma} \varphi_\epsilon\left(\mathrm{Im}(\gamma z)\right)$$

be the Eisenstein series where $\varphi_\epsilon(y) = \begin{cases} y^\epsilon & y > 1 \\ 0 & y < 1 \end{cases}$ and define

$$\mathcal{B}_\epsilon^k := \{f \in L^2(\Gamma\backslash\mathfrak{H}) \mid \langle(1 + |\Psi_\epsilon|)^k f, f\rangle_{L^2} < \infty\}$$

for $k \in \mathbb{Z}$ with norm $|f|^2_{\mathcal{B}_\epsilon^k} = \langle(1 + |\Psi_\epsilon|)^k f, f\rangle$. Let $\mathcal{B}_\epsilon^{-k}$ be the dual to $\mathcal{B}_\epsilon^k$ for each $k$.

**Lemma 14.** *For some $\epsilon > 0$, $S \in \mathcal{B}_\epsilon^1$.*

*Proof.* Recall that $S$ is in $L^2(\Gamma\backslash\mathfrak{H})$ by design and in fact by examining the construction of each $S$ we see that $S$ is $\mathcal{O}(y^{\frac{1}{2}-\epsilon})$. By design, $|\Psi_\epsilon| \cdot S \cdot \overline{S}$ is $\mathcal{O}(y^{1-\epsilon})$ and $\langle(1+|\Psi_\epsilon|)S, S\rangle_{L^2} < \infty$ as desired. □

**Lemma 15.** *Given $\epsilon > 0$ and $s$ with Re$(s) = 1/2$, both $E_s$ and $\wedge^T E_s$ are in $\mathcal{B}_\epsilon^{-1}$.*

*Proof.* Let $s$ be such that $\mathrm{Re}(s) = 1/2$. In the cases of $\langle(1 + |\Psi_{1+\epsilon}(z)|)^{-1} \cdot E_s, E_s\rangle_{L^2}$ and $\langle(1 + |\Psi_{1+\epsilon}(z)|)^{-1} \cdot \wedge^T E_s, \wedge^T E_s\rangle_{L^2}$, both integrands are of order $\mathcal{O}(y^{1-\epsilon})$ since $E_s$ and $\wedge^T E_s$ are $\mathcal{O}(y^{1/2})$. When integrated against the measure $\frac{dy}{y^2}$, these integrals will converge. □

Now we must show that the limit of the truncated Eisenstein series approaches the original Eisenstein series in this topology.

**Lemma 16.** *Given $\epsilon > 0$ and $s$ with Re$(s) = 1/2$, we have $\mathcal{B}_\epsilon^{-1}\text{–}\lim_T \wedge^T E_s = E_s$.*

*Proof.* Consider $E_s$ where $\mathrm{Re}(s) = 1/2$ and $\sigma_o > \mathrm{Re}(s)$. We have

$$|\wedge^T E_s|^2_{\mathcal{B}_\epsilon^{-1}} = \left\langle \frac{1}{1+|\Psi_{1+\epsilon}|} \cdot \wedge^T E_s, \wedge^T E_s \right\rangle_{L^2} = \int_{\Gamma\backslash\mathfrak{H}} \frac{1}{1+|\Psi_{1+\epsilon}|} \wedge^T E_s \cdot \wedge^T E_{\overline{s}} \frac{dy\,dx}{y^2}$$

$$= \int_{\Gamma\backslash\mathfrak{H}} \frac{1}{1+|\Psi_{1+\epsilon}|} \left(E_s - \Theta_s^T(z)\right) \cdot \left(E_{\overline{s}} - \Theta_{\overline{s}}^T(z)\right) \frac{dy\,dx}{y^2}$$

$$= \int_0^\infty \int_{|x|\leq 1/2 x^2 \geq 1-y^2} \frac{1}{1+|\Psi_{1+\epsilon}|} \left(E_s - \Theta_s^T(z)\right) \cdot \left(E_{\overline{s}} - \Theta_{\overline{s}}^T(z)\right) \frac{dy\,dx}{y^2}$$

$$= \int_0^T \int_{|x|\leq 1/2 x^2 \geq 1-y^2} \frac{1}{1+|\Psi_{1+\epsilon}|} \cdot E_s \cdot E_{\overline{s}} \frac{dx\,dy}{y^2}$$

$$+ \int_T^\infty \int_{|x|\leq 1/2 x^2 \geq 1-y^2} \frac{1}{1+|\Psi_{1+\epsilon}|} \left(E_s - (y^s + c_s y^{1-s})\right) \cdot \left(E_{\overline{s}} - (y^{\overline{s}} + c_{\overline{s}} y^{1-\overline{s}})\right) \frac{dy\,dx}{y^2}$$

Now $\frac{1}{1+|\Psi_{1+\epsilon}|} \left(E_s - y^s - c_s y^{1-s}\right) \cdot \left(E_{\overline{s}} - y^{\overline{s}} - c_{\overline{s}} y^{1-\overline{s}}\right) < \frac{1}{E_{\sigma_o}} E_s E_{\overline{s}} \in L^2$ thus by Lebesgue's Convergence Theorem as $T \to \infty$, the second integral disappears and this becomes

$$\int_0^\infty \int_{|x|\leq 1/2 x^2 \geq 1-y^2} \frac{1}{1+|\Psi_{1+\epsilon}|} \cdot E_s \cdot E_{\overline{s}} \frac{dx\,dy}{y^2} = \left\langle \frac{1}{1+|\Psi_{1+\epsilon}|} \cdot E_s, E_s \right\rangle_{L^2} = |E_s|^2_{\mathcal{B}_\epsilon^{-1}}$$

□

## 4.2 Integrals of Truncated Eisenstein series

It remains to compute

$$\int_{\Gamma\backslash\mathfrak{H}} \wedge^T \overline{E}_s \cdot S \frac{dy\,dx}{y^2}$$

for each $S$.

We have that each $S \in L^2$. However, we will need something a bit stronger to actually compute $\langle S, \wedge^T E_s \rangle_{L^2}$. For instance, we may know that $\int_{\Gamma\backslash\mathfrak{H}} \wedge^T \overline{E_s} \cdot S \frac{dy\,dx}{y^2} < \infty$ but since each $S$ involved many terms that we would like to be able to separate and compute, we need that each integral exists term-wise. We will need a few results to address each of the terms for each $S$.

In order to compute term-wise, truncated pairings we will need the following three results. Lemma **??** will give us the first for the part of $S$ which consists of $E_\alpha E_\beta$ paired against $\wedge^T \overline{E}_s$. Lemma **??** and Theorem **??** will allow us to compute the integrals corresponding to the part of $S$ which consists of linear combinations of Eisenstein series.

**Lemma 17.** *For $\alpha, \beta \neq 1$,*

$$\int_{\Gamma\backslash\mathfrak{H}} \wedge^T \overline{E}_s \cdot E_\alpha E_\beta \frac{dy\,dx}{y^2}$$

$$= \frac{1}{\overline{s}+\alpha+\beta-1}T^{\overline{s}+\alpha+\beta-1} + \frac{c_\alpha}{\overline{s}-\alpha+\beta}T^{\overline{s}-\alpha+\beta} + \frac{c_\beta}{\overline{s}+\alpha-\beta}T^{\overline{s}+\alpha-\beta}$$
$$+\frac{c_\alpha c_\beta}{\overline{s}-\alpha-\beta+1}T^{\overline{s}-\alpha-\beta+1} + L(\overline{s}, E_\alpha \times E_\beta) \cdot \int_{y \leq T} y^{\overline{s}} \cdot W_\alpha(y) W_\beta(y) \frac{dy}{y^2}$$
$$+\frac{c_{\overline{s}}}{-\overline{s}+\alpha+\beta}T^{-\overline{s}+\alpha+\beta} + \frac{c_\alpha c_{\overline{s}}}{1-\overline{s}-\alpha+\beta}T^{1-\overline{s}-\alpha+\beta} + \frac{c_\beta c_{\overline{s}}}{1-\overline{s}+\alpha-\beta}T^{1-\overline{s}+\alpha-\beta}$$
$$+\frac{c_\alpha c_\beta c_{\overline{s}}}{2-\overline{s}-\alpha-\beta}T^{2-\overline{s}-\alpha-\beta} - c_{\overline{s}} \sum_{n\neq 0} \varphi(n,\alpha)\varphi(n,\beta) n^{\overline{s}} \int_{y \geq T} y^{-1-\overline{s}} \cdot W_\alpha(y) W_\beta(y)\, dy$$

*Proof.* In the first term, we will compute $\int_{\Gamma\backslash\mathfrak{H}} \wedge^T \overline{E}_s \cdot E_\alpha E_\beta \frac{dy\,dx}{y^2}$:

$$\int_{\Gamma\backslash\mathfrak{H}} \wedge^T \overline{E}_s \cdot E_\alpha E_\beta \frac{dy\,dx}{y^2} = \int_{\Gamma\backslash\mathfrak{H}} \left( \sum_{\gamma \in P\backslash\Gamma} \mathrm{Im}(\gamma z)^{\overline{s}} - \sum_{\gamma \in P\backslash\Gamma} \tau_{\overline{s}}(\gamma z) \right) \cdot E_\alpha E_\beta \frac{dy\,dx}{y^2}$$

$$= \int_{\Gamma\backslash\mathfrak{H}} \sum_{\gamma \in P\backslash\Gamma} \left(\mathrm{Im}(\gamma z)^{\overline{s}} - \tau_{\overline{s}}(\gamma z)\right) \cdot E_\alpha E_\beta \frac{dy\,dx}{y^2} = \int_{P\backslash\mathfrak{H}} \left(y^{\overline{s}} - \tau_{\overline{s}}(z)\right) \cdot E_\alpha E_\beta \frac{dy\,dx}{y^2}$$

by unwinding

$$= \int_{P\backslash\mathfrak{H} y \leq T} \left(y^{\overline{s}} - \tau_{\overline{s}}(z)\right) \cdot E_\alpha E_\beta \frac{dy\,dx}{y^2} + \int_{P\backslash\mathfrak{H} y > T} \left(y^{\overline{s}} - \tau_{\overline{s}}(z)\right) \cdot E_\alpha E_\beta \frac{dy\,dx}{y^2}$$

$$= \int_{P\backslash\mathfrak{H} y \leq T} y^{\overline{s}} \cdot E_\alpha E_\beta \frac{dy\,dx}{y^2} - \int_{P\backslash\mathfrak{H} y > T} c_{\overline{s}} y^{1-\overline{s}} \cdot E_\alpha E_\beta \frac{dy\,dx}{y^2}$$

(A) Examining $\displaystyle\int_{P\backslash\mathfrak{H} y\le T} y^{\overline{s}}\cdot E_\alpha E_\beta \,\frac{dy\,dx}{y^2}$:

Recall that the fundamental domain of $P\backslash\mathfrak{H}$ is $\{z=x+iy\in\mathfrak{H}\mid 0\le x\le 1\}$ so we have

$$\int_{P\backslash\mathfrak{H} y\le T} y^{\overline{s}}\cdot E_\alpha E_\beta \,\frac{dy\,dx}{y^2} = \int_0^1\int_{y\le T} y^{\overline{s}-2}\cdot E_\alpha E_\beta\,dy\,dx$$

$$= \int_0^1\int_{y\le T} y^{\overline{s}-2}[y^\alpha + c_\alpha y^{1-\alpha} + \sum_{n\neq 0}\varphi(n,\alpha)W_\alpha(|n|y)e^{2\pi i n x}][y^\beta + c_\beta y^{1-\beta}$$

$$+\sum_{m\neq 0}\varphi(m,\beta)W_\beta(|m|y)e^{2\pi i m x}]\,dy\,dx$$

$$= \int_{y\le T} y^{\overline{s}-2}(y^\alpha + c_\alpha y^{1-\alpha})(y^\beta + c_\beta y^{1-\beta}) + y^{\overline{s}-2}\sum_{n\neq 0}\varphi(n,\alpha)\varphi(n,\beta)\cdot W_\alpha(|n|y)W_\beta(|n|y)\,dy$$

since the product vanishes off the diagonal.

(1) Examining the first term $\displaystyle\int_{y\le T} y^{\overline{s}-2}(y^\alpha + c_\alpha y^{1-\alpha})(y^\beta + c_\beta y^{1-\beta})\,dy$:

$$\int_{y\le T} y^{\overline{s}-2}(y^\alpha + c_\alpha y^{1-\alpha})(y^\beta + c_\beta y^{1-\beta})\,dy$$

$$= \int_{y\le T} y^{\overline{s}-2}(y^{\alpha+\beta} + c_\alpha y^{1-\alpha+\beta} + c_\beta y^{1+\alpha-\beta} + c_\alpha c_\beta y^{2-\alpha-\beta})\,dy$$

$$= \int_{y\le T} y^{\overline{s}+\alpha+\beta-2} + c_\alpha y^{\overline{s}-\alpha+\beta-1} + c_\beta y^{\overline{s}+\alpha-\beta-1} + c_\alpha c_\beta y^{\overline{s}-\alpha-\beta}\,dy$$

$$= \frac{1}{\overline{s}+\alpha+\beta-1}y^{\overline{s}+\alpha+\beta-1} + \frac{c_\alpha}{\overline{s}-\alpha+\beta}y^{\overline{s}-\alpha+\beta} + \frac{c_\beta}{\overline{s}+\alpha-\beta}y^{\overline{s}+\alpha-\beta}$$

$$+ \frac{c_\alpha c_\beta}{\overline{s}-\alpha-\beta+1}y^{\overline{s}-\alpha-\beta+1}\Big|_{y=0}^{T}$$

$$= \frac{1}{\overline{s}+\alpha+\beta-1}T^{\overline{s}+\alpha+\beta-1} + \frac{c_\alpha}{\overline{s}-\alpha+\beta}T^{\overline{s}-\alpha+\beta} + \frac{c_\beta}{\overline{s}+\alpha-\beta}T^{\overline{s}+\alpha-\beta}$$

$$+\frac{c_\alpha c_\beta}{\overline{s}-\alpha-\beta+1}T^{\overline{s}-\alpha-\beta+1} - \lim_{t\to 0^+}\Big(\frac{1}{\overline{s}+\alpha+\beta-1}t^{\overline{s}+\alpha+\beta-1} + \frac{c_\alpha}{\overline{s}-\alpha+\beta}t^{\overline{s}-\alpha+\beta}$$

$$+ \frac{c_\beta}{\overline{s}+\alpha-\beta}t^{\overline{s}+\alpha-\beta} + \frac{c_\alpha c_\beta}{\overline{s}-\alpha-\beta+1}t^{\overline{s}-\alpha-\beta+1}\Big)$$

For $\mathrm{Re}(\overline{s}) > \mathrm{Re}(\alpha+\beta) > 1$ where $\mathrm{Re}(\alpha) > 0$ and $\mathrm{Re}(\beta) > 0$, this last term

$$\lim_{t\to 0^+}\Big(\frac{1}{\overline{s}+\alpha+\beta-1}t^{\overline{s}+\alpha+\beta-1} + \frac{c_\alpha}{\overline{s}-\alpha+\beta}t^{\overline{s}-\alpha+\beta} + \frac{c_\beta}{\overline{s}+\alpha-\beta}t^{\overline{s}+\alpha-\beta} + \frac{c_\alpha c_\beta}{\overline{s}-\alpha-\beta+1}t^{\overline{s}-\alpha-\beta+1}\Big) = 0$$

Thus, by the Identity Principle, we can meromorphically continue to get that

$$\int_{y\le T} y^{\overline{s}-2}(y^\alpha + c_\alpha y^{1-\alpha})(y^\beta + c_\beta y^{1-\beta})\,dy$$

$$= \frac{1}{\overline{s}+\alpha+\beta-1}T^{\overline{s}+\alpha+\beta-1} + \frac{c_\alpha}{\overline{s}-\alpha+\beta}T^{\overline{s}-\alpha+\beta} + \frac{c_\beta}{\overline{s}+\alpha-\beta}T^{\overline{s}+\alpha-\beta} + \frac{c_\alpha c_\beta}{\overline{s}-\alpha-\beta+1}T^{\overline{s}-\alpha-\beta+1}$$

(2) Examining the second term $\displaystyle\int_{y\le T} y^{\overline{s}-2}\sum_{n\ne 0}\varphi(n,\alpha)\varphi(n,\beta)\cdot W_\alpha(|n|y)W_\beta(|n|y)\,dy$:

$$\int_{y\le T} y^{\overline{s}-2}\sum_{n\ne 0}\varphi(n,\alpha)\varphi(n,\beta)\cdot W_\alpha(|n|y)W_\beta(|n|y)\,dy$$

$$= \sum_{n\ne 0}\varphi(n,\alpha)\varphi(n,\beta)\int_{y\le T} y^{\overline{s}}\cdot W_\alpha(|n|y)W_\beta(|n|y)\,\frac{dy}{y^2}$$

replacing $y$ by $y/n$ we have

$$= \sum_{n\ne 0}\frac{\varphi(n,\alpha)\varphi(n,\beta)}{n^{\overline{s}-1}}\int_{y\le T} y^{\overline{s}}\cdot W_\alpha(y)W_\beta(y)\,\frac{dy}{y^2} = L(\overline{s}, E_\alpha\times E_\beta)\cdot\int_{y\le T} y^{\overline{s}}\cdot W_\alpha(y)W_\beta(y)\,\frac{dy}{y^2}$$

(B) Examining $\displaystyle\int_{P\backslash\mathfrak{H} y>T} c_{\overline{s}}y^{1-\overline{s}}\cdot E_\alpha E_\beta\,\frac{dy\,dx}{y^2}$:

Recall that the fundamental domain of $P\backslash\mathfrak{H}$ is $\{z = x+iy\in\mathfrak{H} \mid 0\le x\le 1\}$ so we have

$$\int_{P\backslash\mathfrak{H} y>T} c_{\overline{s}}y^{1-\overline{s}}\cdot E_\alpha E_\beta\,\frac{dy\,dx}{y^2} = \int_0^1\int_{y\ge T} c_{\overline{s}}y^{1-\overline{s}}\cdot E_\alpha E_\beta\,\frac{dy}{y^2}\,dx$$

$$= \int_0^1 \int_{y \geq T} c_{\overline{s}} y^{1-\overline{s}} \left( y^\alpha + c_\alpha y^{1-\alpha} + \sum_{n \neq 0} \varphi(n,\alpha) W_\alpha(|n|y) e^{2\pi i n x} \right)$$

$$\cdot \left( y^\beta + c_\beta y^{1-\beta} + \sum_{m \neq 0} \varphi(m,\beta) W_\beta(|m|y) e^{2\pi i m x} \right) \frac{dy\, dx}{y^2}$$

$$= \int_{y \geq T} c_{\overline{s}} y^{-1-\overline{s}} (y^\alpha + c_\alpha y^{1-\alpha})(y^\beta + c_\beta y^{1-\beta})$$

$$+ c_{\overline{s}} y^{-1-\overline{s}} \sum_{n \neq 0} \varphi(n,\alpha)\varphi(n,\beta) \cdot W_\alpha(|n|y) W_\beta(|n|y)\, dy$$

since the product vanishes off the diagonal.

(1) Examining the first term $\int_{y \geq T} c_{\overline{s}} y^{-1-\overline{s}} (y^\alpha + c_\alpha y^{1-\alpha})(y^\beta + c_\beta y^{1-\beta})\, dy$:

$$\int_{y \geq T} c_{\overline{s}} y^{-1-\overline{s}} (y^\alpha + c_\alpha y^{1-\alpha})(y^\beta + c_\beta y^{1-\beta})\, dy$$

$$= \int_{y \geq T} c_{\overline{s}}\, y^{-1-\overline{s}+\alpha+\beta} + c_\alpha c_{\overline{s}}\, y^{-\overline{s}-\alpha+\beta} + c_\beta c_{\overline{s}}\, y^{-\overline{s}+\alpha-\beta} + c_\alpha c_\beta c_{\overline{s}}\, y^{1-\overline{s}-\alpha-\beta}\, dy$$

$$= \Big( \frac{c_{\overline{s}}}{-\overline{s}+\alpha+\beta}\, y^{-\overline{s}+\alpha+\beta} + \frac{c_\alpha c_{\overline{s}}}{1-\overline{s}-\alpha+\beta}\, y^{1-\overline{s}-\alpha+\beta} + \frac{c_\beta c_{\overline{s}}}{1-\overline{s}+\alpha-\beta}\, y^{1-\overline{s}+\alpha-\beta}$$

$$+ \frac{c_\alpha c_\beta c_{\overline{s}}}{2-\overline{s}-\alpha-\beta}\, y^{2-\overline{s}-\alpha-\beta} \Big) \Big|_{y=T}^{\infty}$$

$$= \lim_{t \to \infty} \Big[ \Big( \frac{c_{\overline{s}}}{-\overline{s}+\alpha+\beta}\, t^{-\overline{s}+\alpha+\beta} + \frac{c_\alpha c_{\overline{s}}}{1-\overline{s}-\alpha+\beta}\, t^{1-\overline{s}-\alpha+\beta} + \frac{c_\beta c_{\overline{s}}}{1-\overline{s}+\alpha-\beta}\, t^{1-\overline{s}+\alpha-\beta}$$

$$+ \frac{c_\alpha c_\beta c_{\overline{s}}}{2-\overline{s}-\alpha-\beta}\, t^{2-\overline{s}-\alpha-\beta} \Big)$$

$$- \Big( \frac{c_{\overline{s}}}{-\overline{s}+\alpha+\beta}\, T^{-\overline{s}+\alpha+\beta} + \frac{c_\alpha c_{\overline{s}}}{1-\overline{s}-\alpha+\beta}\, T^{1-\overline{s}-\alpha+\beta}$$

$$+ \frac{c_\beta c_{\overline{s}}}{1-\overline{s}+\alpha-\beta}\, T^{1-\overline{s}+\alpha-\beta} + \frac{c_\alpha c_\beta c_{\overline{s}}}{2-\overline{s}-\alpha-\beta}\, T^{2-\overline{s}-\alpha-\beta} \Big) \Big]$$

For $\mathrm{Re}(\overline{s}) > \mathrm{Re}(\alpha+\beta) > 1$ where $\mathrm{Re}(\alpha) > 1/2$ and $\mathrm{Re}(\beta) > 1/2$ the first term

$$\lim_{t\to\infty} \Big(\frac{c_{\overline{s}}}{-\overline{s}+\alpha+\beta}\, t^{-\overline{s}+\alpha+\beta} + \frac{c_\alpha c_{\overline{s}}}{1-\overline{s}-\alpha+\beta}\, t^{1-\overline{s}-\alpha+\beta} + \frac{c_\beta c_{\overline{s}}}{1-\overline{s}+\alpha-\beta}\, t^{1-\overline{s}+\alpha-\beta} + \frac{c_\alpha c_\beta c_{\overline{s}}}{2-\overline{s}-\alpha-\beta}\, t^{2-\overline{s}-\alpha-\beta}\Big) = 0$$

Thus, by the Identity Principle, we can meromorphically continue to get that

$$\int_{y\geq T} c_{\overline{s}} y^{-1-\overline{s}} (y^\alpha + c_\alpha y^{1-\alpha})(y^\beta + c_\beta y^{1-\beta})\, dy$$

$$= -\Big(\frac{c_{\overline{s}}}{-\overline{s}+\alpha+\beta}\, T^{-\overline{s}+\alpha+\beta} + \frac{c_\alpha c_{\overline{s}}}{1-\overline{s}-\alpha+\beta}\, T^{1-\overline{s}-\alpha+\beta} + \frac{c_\beta c_{\overline{s}}}{1-\overline{s}+\alpha-\beta}\, T^{1-\overline{s}+\alpha-\beta} + \frac{c_\alpha c_\beta c_{\overline{s}}}{2-\overline{s}-\alpha-\beta}\, T^{2-\overline{s}-\alpha-\beta}\Big)$$

(2) Examining the second term $\int_{y\geq T} c_{\overline{s}} y^{-1-\overline{s}} \sum_{n\neq 0} \varphi(n,\alpha)\varphi(n,\beta)\cdot W_\alpha(|n|y)W_\beta(|n|y)\, dy$:

$$\int_{y\geq T} c_{\overline{s}} y^{-1-\overline{s}} \sum_{n\neq 0} \varphi(n,\alpha)\varphi(n,\beta)\cdot W_\alpha(|n|y)W_\beta(|n|y)\, dy$$

$$= \int_{y\geq T} c_{\overline{s}} y^{-1-\overline{s}} \sum_{n\neq 0} \varphi(n,\alpha)\varphi(n,\beta)\cdot W_\alpha(|n|y)W_\beta(|n|y)\, dy$$

$$= \sum_{n\neq 0} \varphi(n,\alpha)\varphi(n,\beta) \int_{y\geq T} c_{\overline{s}} y^{-1-\overline{s}} \cdot W_\alpha(|n|y)W_\beta(|n|y)\, dy$$

replacing $y$ by $y/n$ we have

$$= c_{\overline{s}} \sum_{n\neq 0} \varphi(n,\alpha)\varphi(n,\beta) n^{\overline{s}} \int_{y\geq T} y^{-1-\overline{s}} \cdot W_\alpha(y)W_\beta(y)\, dy$$

Putting (A) and (B) together, we get:

$$\int_{\Gamma\backslash\mathfrak{H}} \wedge^T\, \overline{E}_s \cdot E_\alpha E_\beta\, \frac{dy\, dx}{y^2}$$

$$= \frac{1}{\overline{s}+\alpha+\beta-1}T^{\overline{s}+\alpha+\beta-1} + \frac{c_\alpha}{\overline{s}-\alpha+\beta}T^{\overline{s}-\alpha+\beta} + \frac{c_\beta}{\overline{s}+\alpha-\beta}T^{\overline{s}+\alpha-\beta}$$

$$+\frac{c_\alpha c_\beta}{\overline{s}-\alpha-\beta+1}T^{\overline{s}-\alpha-\beta+1} + L(\overline{s}, E_\alpha \times E_\beta) \cdot \int_{y\leq T} y^{\overline{s}} \cdot W_\alpha(y)W_\beta(y)\,\frac{dy}{y^2}$$

$$+\frac{c_{\overline{s}}}{-\overline{s}+\alpha+\beta}\,T^{-\overline{s}+\alpha+\beta} + \frac{c_\alpha c_{\overline{s}}}{1-\overline{s}-\alpha+\beta}\,T^{1-\overline{s}-\alpha+\beta} + \frac{c_\beta c_{\overline{s}}}{1-\overline{s}+\alpha-\beta}\,T^{1-\overline{s}+\alpha-\beta}$$

$$+\frac{c_\alpha c_\beta c_{\overline{s}}}{2-\overline{s}-\alpha-\beta}\,T^{2-\overline{s}-\alpha-\beta} - c_{\overline{s}} \sum_{n\neq 0} \varphi(n,\alpha)\varphi(n,\beta)n^{\overline{s}} \int_{y\geq T} y^{-1-\overline{s}} \cdot W_\alpha(y)W_\beta(y)\,dy$$

□

We will also need the following two results for the parts of $S$ which consist of linear combinations of Eisenstein series.

**Lemma 18.** $\displaystyle\int_{\Gamma\backslash\mathfrak{H}} \wedge^T\overline{E}_s \cdot \wedge^T E_r\,\frac{dy\,dx}{y^2} = \int_{\Gamma\backslash\mathfrak{H}} \wedge^T\overline{E}_s \cdot E_r\,\frac{dy\,dx}{y^2}$

*Proof.* Recall that the fundamental domain for $\Gamma\backslash\mathfrak{H}$ is $F = \{z \in \mathfrak{H} \mid |z| \geq 1 \;\&\; |\mathrm{Re}(z)| \leq 1/2\}$ so rewriting our integral we have

$$\int_{\Gamma\backslash\mathfrak{H}} \wedge^T\,\overline{E}_s \cdot \wedge^T E_r\,\frac{dx\,dy}{y^2} = \int_{0\leq y\leq\infty}\int_{|x|\leq 1/2 x^2\geq 1-y^2} \wedge^T\,\overline{E}_s \cdot \wedge^T E_r\,\frac{dx\,dy}{y^2}$$

$$= \int_{0\leq y\leq T}\int_{|x|\leq 1/2 x^2\geq 1-y^2} \wedge^T\,\overline{E}_s\cdot\wedge^T E_r\,\frac{dx\,dy}{y^2} + \int_{T\leq y\leq\infty}\int_{|x|\leq 1/2 x^2\geq 1-y^2} \wedge^T\,\overline{E}_s\cdot\wedge^T E_r\,\frac{dx\,dy}{y^2}$$

Notice that since the first integral is only defined for $y \leq T$ and on this region, $\wedge^T E_r = E_r$ by definition,

$$\int_{0\leq y\leq T}\int_{|x|\leq 1/2 x^2\geq 1-y^2} \wedge^T\,\overline{E}_s \cdot \wedge^T E_r\,\frac{dx\,dy}{y^2} = \int_{0\leq y\leq T}\int_{|x|\leq 1/2 x^2\geq 1-y^2} \wedge^T\,\overline{E}_s \cdot E_r\,\frac{dx\,dy}{y^2}$$

Thus it remains to show this result for the second integral $\displaystyle\int_{T\leq y\leq\infty}\int_{|x|\leq 1/2 x^2\geq 1-y^2} \wedge^T\,\overline{E}_s\cdot \wedge^T E_r\,\frac{dx\,dy}{y^2}$.

For $T > 1$, this domain of integration is a cylinder so that

$$\int_{T\leq y\leq\infty}\int_{|x|\leq 1/2 x^2\geq 1-y^2} \wedge^T\,\overline{E}_s \cdot \wedge^T E_r\,\frac{dx\,dy}{y^2} = \int_{T\leq y\leq\infty}\int_{|x|\leq 1/2} \wedge^T\,\overline{E}_s \cdot \wedge^T E_r\,\frac{dx\,dy}{y^2}$$

Writing this in terms of Fourier expansions, we have

$$= \int_{T\le y\le\infty}\int_{|x|\le 1/2}\left(\sum_{n\ne 0}\varphi(n,s)\overline{W}_s(|n|y)e^{-2\pi inx}\right)\cdot\left(\sum_{m\ne 0}\varphi(m,s)W_s(|m|y)e^{2\pi imx}\right)\frac{dx\,dy}{y^2}$$

$$= \int_{T\le y\le\infty}\int_{|x|\le 1/2}\left(\sum_{n\ne 0}\varphi(n,s)\overline{W}_s(|n|y)e^{-2\pi inx}\right)\cdot\left(\sum_{m\in\mathbb{Z}}\varphi(m,s)W_s(|m|y)e^{2\pi imx}\right)\frac{dx\,dy}{y^2}$$

since the integral will be zero when $n\ne m$ i.e. when $m=0$ (this computation was seen previously as $\int_0^1 e^{2\pi i(-m)x}dx=\delta_{0,m}$ and the $0^{th}$ coefficient of the first Eisenstein series has been truncated to be made 0)

$$= \int_{T\le y\le\infty}\int_{|x|\le 1/2}\wedge^T\,\overline{E}_s\cdot E_r\,\frac{dx\,dy}{y^2}$$

as desired. Combining the domains as originally stated, we have

$$\int_{\Gamma\backslash\mathfrak{H}}\wedge^T\,\overline{E}_s\cdot\wedge^T E_r\,\frac{dy\,dx}{y^2}=\int_{\Gamma\backslash\mathfrak{H}}\wedge^T\,\overline{E}_s\cdot E_r\,\frac{dy\,dx}{y^2}$$

□

We will use this to compute the pairing for the linear combination terms in $S$ with the truncated Eisenstein series. Lemma **??** allows for each of the terms in the linear combination to become

$$\int_{\Gamma\backslash\mathfrak{H}}\wedge^T\,\overline{E}_s\cdot E_r\,\frac{dy\,dx}{y^2}=\int_{\Gamma\backslash\mathfrak{H}}\wedge^T\,\overline{E}_s\cdot\wedge^T E_r\,\frac{dy\,dx}{y^2}$$

and then we will use Maass-Selberg and unwinding of $\wedge^T\,\overline{E}_s$.

Recall the following the Maass-Selberg relation (see Casselman [**?**] or Garrett [**?**] for proof) states that

**Theorem 19.** *For two complex numbers $r,s\ne 1$ with $r(r-1)\ne s(s-1)$,*

$$\int_{\Gamma\backslash\mathfrak{H}}\wedge^T\,E_s\cdot\wedge^T E_r\,\frac{dy\,dx}{y^2}$$

$$= \frac{T^{r+s-1}}{r+s-1} + c_r \frac{T^{(1-r)+s-1}}{(1-r)+s-1} + c_s \frac{T^{r+(1-s)-1}}{r+(1-s)-1} + c_r c_s \frac{T^{(1-r)+(1-s)-1}}{(1-r)+(1-s)-1}.$$

Observe that when we are computing $\int_{\Gamma\backslash\mathfrak{H}} \wedge^T \overline{E}_s \cdot S \frac{dy\,dx}{y^2}$, the last few terms of $S$ will appear as $\int_{\Gamma\backslash\mathfrak{H}} \wedge^T \overline{E}_s \cdot E_r \frac{dy\,dx}{y^2}$. Using the previous two results, for each $r$ in our linear combination $S$, we will have something of the form

$$\int_{\Gamma\backslash\mathfrak{H}} \wedge^T \overline{E}_s \cdot E_r \frac{dy\,dx}{y^2}$$

$$= \frac{T^{r+\overline{s}-1}}{r+\overline{s}-1} + c_r \cdot \frac{T^{(1-r)+\overline{s}-1}}{(1-r)+\overline{s}-1} + c_{\overline{s}} \cdot \frac{T^{r+(1-\overline{s})-1}}{r+(1-\overline{s})-1} + c_r c_{\overline{s}} \cdot \frac{T^{(1-r)+(1-\overline{s})-1}}{(1-r)+(1-\overline{s})-1}$$

The following is a version of the Maass-Selberg relation for when $r = 1$. We follow the style of argument for the original Maass-Selberg relation, thus we will label it as a corollary.

**Corollary 20.** *For all complex $s$ with $0 \neq s(s-1)$,*

$$\int_{\Gamma\backslash\mathfrak{H}} \wedge^T \overline{E}_s \cdot E_1^* \frac{dy\,dx}{y^2}$$

$$= \frac{T^{\overline{s}}}{\overline{s}} + C\frac{T^{\overline{s}-1}}{\overline{s}-1} - \frac{3}{\pi}\frac{T^{\overline{s}-1}}{\overline{s}-1}\log T + \frac{3}{\pi}\frac{T^{\overline{s}-1}}{(\overline{s}-1)^2} + c_{\overline{s}}\left(\frac{T^{1-\overline{s}}}{1-\overline{s}} - C\frac{T^{-\overline{s}}}{\overline{s}} + \frac{3}{\pi}\frac{T^{-\overline{s}}}{\overline{s}}\log T + \frac{3}{\pi}\frac{T^{-\overline{s}}}{\overline{s}^2}\right)$$

*Proof.*

$$\int_{\Gamma\backslash\mathfrak{H}} \wedge^T \overline{E}_s \cdot E_1^* \frac{dy\,dx}{y^2} = \int_{\Gamma\backslash\mathfrak{H}} \left(\sum_{\gamma\in P\backslash\Gamma} \mathrm{Im}(\gamma z)^{\overline{s}} - \sum_{\gamma\in P\backslash\Gamma} \tau_{\overline{s}}(\gamma z)\right) \cdot E_1^* \frac{dy\,dx}{y^2}$$

$$= \int_{\Gamma\backslash\mathfrak{H}} \sum_{\gamma\in P\backslash\Gamma} \left(\mathrm{Im}(\gamma z)^{\overline{s}} - \tau_{\overline{s}}(\gamma z)\right) \cdot E_1^* \frac{dy\,dx}{y^2} = \int_{P\backslash\mathfrak{H}} \left(y^{\overline{s}} - \tau_{\overline{s}}(z)\right) \cdot E_1^* \frac{dy\,dx}{y^2}$$

by unwinding

$$= \int_{P\backslash\mathfrak{H}y\leq T} y^{\overline{s}} \cdot E_1^* \frac{dy\,dx}{y^2} - \int_{P\backslash\mathfrak{H}y>T} c_{\overline{s}} y^{1-\overline{s}} \cdot E_1^* \frac{dy\,dx}{y^2}$$

from the definitions of $\tau_s$

(A) Examining $\displaystyle\int_{P\backslash\mathfrak{H} y\leq T} y^{\overline{s}} \cdot E_1^* \frac{dy\,dx}{y^2}$:

Recall that the fundamental domain of $P\backslash\mathfrak{H}$ is $\{z = x+iy \in \mathfrak{H} \mid 0 \leq x \leq 1\}$ so we have

$$\int_{P\backslash\mathfrak{H} y\leq T} y^{\overline{s}} \cdot E_1^* \frac{dy\,dx}{y^2} = \int_0^1 \int_{y\leq T} y^{\overline{s}-2} \cdot E_1^*\,dy\,dx$$

$$= \int_0^1 \int_{y\leq T} y^{\overline{s}-2} \left( c_P E_1^* + \sum_{n\neq 0} \varphi(n,1) W_1(|n|y) e^{2\pi i n x} \right) dy\,dx$$

$$= \int_0^1 \int_{y\leq T} y^{\overline{s}-2} \cdot c_P E_1^* + y^{\overline{s}-2} \cdot \sum_{n\neq 0} \varphi(n,1) W_1(|n|y) e^{2\pi i n x}\,dy\,dx$$

(1) Examining the first term $\displaystyle\int_0^1 \int_{y\leq T} y^{\overline{s}-2} \cdot c_P E_1^*\,dy\,dx$:

$$\int_{y\leq T} y^{\overline{s}-2} \cdot c_P E_1^*\,dy = \int_{y\leq T} y^{\overline{s}-2} \cdot \left( y + C - \frac{3}{\pi}\log y \right) dy$$

$$= \int_{y\leq T} y^{\overline{s}-1} + C y^{\overline{s}-2} - \frac{3}{\pi} y^{\overline{s}-2} \log y\,dy$$

$$= \frac{y^{\overline{s}}}{\overline{s}} + C\frac{y^{\overline{s}-1}}{\overline{s}-1} - \frac{3}{\pi}\frac{y^{\overline{s}-1}}{\overline{s}-1}\log y + \frac{3}{\pi}\frac{y^{\overline{s}-1}}{(\overline{s}-1)^2}\Big|_0^T$$

$$= \frac{T^{\overline{s}}}{\overline{s}} + C\frac{T^{\overline{s}-1}}{\overline{s}-1} - \frac{3}{\pi}\frac{T^{\overline{s}-1}}{\overline{s}-1}\log T + \frac{3}{\pi}\frac{T^{\overline{s}-1}}{(\overline{s}-1)^2}$$
$$- \lim_{t\to 0^+} \left( \frac{t^{\overline{s}}}{\overline{s}} + C\frac{t^{\overline{s}-1}}{\overline{s}-1} - \frac{3}{\pi}\frac{t^{\overline{s}-1}}{\overline{s}-1}\log t + \frac{3}{\pi}\frac{t^{\overline{s}-1}}{(\overline{s}-1)^2} \right)$$

For $\mathrm{Re}(\overline{s}) > 1$, the second term

$$\lim_{t\to 0^+} \left( \frac{t^{\overline{s}}}{\overline{s}} + C\frac{t^{\overline{s}-1}}{\overline{s}-1} - \frac{3}{\pi}\frac{t^{\overline{s}-1}}{\overline{s}-1}\log t + \frac{3}{\pi}\frac{t^{\overline{s}-1}}{(\overline{s}-1)^2} \right) = 0$$

Thus, by the Identity Principle, we can meromorphically continue to get that

$$\int_0^1 \int_{y\leq T} y^{\overline{s}-2} \cdot c_P E_1^*\,dy\,dx = \frac{T^{\overline{s}}}{\overline{s}} + C\frac{T^{\overline{s}-1}}{\overline{s}-1} - \frac{3}{\pi}\frac{T^{\overline{s}-1}}{\overline{s}-1}\log T + \frac{3}{\pi}\frac{T^{\overline{s}-1}}{(\overline{s}-1)^2}$$

(2) Examining the second term $\int_0^1 \int_{y \le T} y^{\overline{s}-2} \cdot \sum_{n \ne 0} \varphi(n,1) W_1(|n|y) e^{2\pi i n x} \, dy \, dx$:

$$\int_0^1 \int_{y \le T} y^{\overline{s}-2} \cdot \sum_{n \ne 0} \varphi(n,1) W_1(|n|y) e^{2\pi i n x} \, dy \, dx$$

$$= \int_{y \le T} y^{\overline{s}-2} \cdot \sum_{n \ne 0} \varphi(n,1) W_1(|n|y) \, dy \cdot \int_0^1 e^{2\pi i n x} \, dx$$

$$= \int_{y \le T} y^{\overline{s}-2} \cdot \sum_{n \ne 0} \varphi(n,1) W_1(|n|y) \, dy \cdot \delta_{0,n} = 0$$

(B) Examining $\int_{P \backslash \mathfrak{H} y > T} c_{\overline{s}} y^{1-\overline{s}} \cdot E_1^* \frac{dy \, dx}{y^2}$:

Recall that the fundamental domain of $P \backslash \mathfrak{H}$ is $\{z = x + iy \in \mathfrak{H} \mid 0 \le x \le 1\}$ so we have

$$\int_{P \backslash \mathfrak{H} y > T} c_{\overline{s}} y^{1-\overline{s}} \cdot E_1^* \frac{dy \, dx}{y^2} = \int_0^1 \int_{y \ge T} c_{\overline{s}} y^{1-\overline{s}} \cdot E_1^* \frac{dy}{y^2} \, dx$$

$$= \int_0^1 \int_{y \ge T} c_{\overline{s}} y^{1-\overline{s}} \left( c_P E_1^* + \sum_{n \ne 0} \varphi(n,1) W_1(|n|y) e^{2\pi i n x} \right) \frac{dy \, dx}{y^2}$$

$$= \int_0^1 \int_{y \ge T} c_{\overline{s}} y^{-1-\overline{s}} \cdot c_P E_1^* + c_{\overline{s}} y^{-1-\overline{s}} \cdot \sum_{n \ne 0} \varphi(n,1) W_1(|n|y) e^{2\pi i n x} \, dy \, dx$$

since the product vanishes off the diagonal.

(1) Examining the first term $\int_0^1 \int_{y \ge T} c_{\overline{s}} y^{-1-\overline{s}} \cdot c_P E_1^* \, dy \, dx$:

$$\int_{y \ge T} c_{\overline{s}} y^{-1-\overline{s}} \cdot c_P E_1^* \, dy = c_{\overline{s}} \int_{y \ge T} y^{-1-\overline{s}} \cdot \left( y + C - \frac{3}{\pi} \log y \right) \, dy$$

$$= c_{\overline{s}} \int_{y \ge T} y^{-\overline{s}} + C y^{-1-\overline{s}} - \frac{3}{\pi} y^{-1-\overline{s}} \log y \, dy$$

$$= c_{\overline{s}} \left( \frac{y^{-\overline{s}+1}}{-\overline{s}+1} + C \frac{y^{-\overline{s}}}{-\overline{s}} - \frac{3}{\pi} \frac{y^{-\overline{s}}}{-\overline{s}} \log y + \frac{3}{\pi} \frac{y^{-\overline{s}}}{\overline{s}^2} \right) \Big|_T^{\infty}$$

$$= c_{\overline{s}} \lim_{t \to \infty} \left( \frac{t^{-\overline{s}+1}}{-\overline{s}+1} + C \frac{t^{-\overline{s}}}{-\overline{s}} - \frac{3}{\pi} \frac{t^{-\overline{s}}}{-\overline{s}} \log t + \frac{3}{\pi} \frac{t^{-\overline{s}}}{\overline{s}^2} \right)$$

$$- c_{\overline{s}} \left( \frac{T^{-\overline{s}+1}}{-\overline{s}+1} + C\frac{T^{-\overline{s}}}{-\overline{s}} - \frac{3}{\pi}\frac{T^{-\overline{s}}}{-\overline{s}} \log T + \frac{3}{\pi}\frac{T^{-\overline{s}}}{\overline{s}^2} \right)$$

For $\mathrm{Re}(\overline{s}) > 1$, the first term

$$c_{\overline{s}} \lim_{t\to\infty} \left( \frac{t^{-\overline{s}+1}}{-\overline{s}+1} + C\frac{t^{-\overline{s}}}{-\overline{s}} - \frac{3}{\pi}\frac{t^{-\overline{s}}}{-\overline{s}} \log t + \frac{3}{\pi}\frac{t^{-\overline{s}}}{\overline{s}^2} \right) = 0$$

Thus, by the Identity Principle, we can meromorphically continue to get that

$$\int_0^1 \int_{y\geq T} c_{\overline{s}} y^{-1-\overline{s}} \cdot c_P E_1^* \, dy\, dx = -c_{\overline{s}} \left( \frac{T^{1-\overline{s}}}{1-\overline{s}} - C\frac{T^{-\overline{s}}}{\overline{s}} + \frac{3}{\pi}\frac{T^{-\overline{s}}}{\overline{s}} \log T + \frac{3}{\pi}\frac{T^{-\overline{s}}}{\overline{s}^2} \right)$$

(2) Examining the second term $\int_0^1 \int_{y\geq T} c_{\overline{s}} y^{-1-\overline{s}} \cdot \sum_{n\neq 0} \varphi(n,1) W_1(|n|y) e^{2\pi i n x}\, dy\, dx$:

$$\int_0^1 \int_{y\geq T} c_{\overline{s}} y^{-1-\overline{s}} \cdot \sum_{n\neq 0} \varphi(n,1) W_1(|n|y) e^{2\pi i n x}\, dy\, dx$$

$$= \int_{y\geq T} c_{\overline{s}} y^{-1-\overline{s}} \cdot \sum_{n\neq 0} \varphi(n,1) W_1(|n|y)\, dy \cdot \int_0^1 e^{2\pi i n x}\, dx$$

$$= \int_{y\geq T} c_{\overline{s}} y^{-1-\overline{s}} \cdot \sum_{n\neq 0} \varphi(n,1) W_1(|n|y)\, dy \cdot \delta_{0,n} = 0$$

Thus

$$\int_{P\backslash\mathfrak{H} y>T} c_{\overline{s}} y^{1-\overline{s}} \cdot E_1^* \frac{dy\, dx}{y^2} = -c_{\overline{s}} \left( \frac{T^{1-\overline{s}}}{1-\overline{s}} - C\frac{T^{-\overline{s}}}{\overline{s}} + \frac{3}{\pi}\frac{T^{-\overline{s}}}{\overline{s}} \log T + \frac{3}{\pi}\frac{T^{-\overline{s}}}{\overline{s}^2} \right)$$

Putting (A) and (B) together, we get:

$$\int_{\Gamma\backslash\mathfrak{H}} \wedge^T \overline{E}_s \cdot E_1^* \frac{dy\, dx}{y^2}$$

$$= \frac{T^{\overline{s}}}{\overline{s}} + C\frac{T^{\overline{s}-1}}{\overline{s}-1} - \frac{3}{\pi}\frac{T^{\overline{s}-1}}{\overline{s}-1} \log T + \frac{3}{\pi}\frac{T^{\overline{s}-1}}{(\overline{s}-1)^2} + c_{\overline{s}} \left( \frac{T^{1-\overline{s}}}{1-\overline{s}} - C\frac{T^{-\overline{s}}}{\overline{s}} + \frac{3}{\pi}\frac{T^{-\overline{s}}}{\overline{s}} \log T + \frac{3}{\pi}\frac{T^{-\overline{s}}}{\overline{s}^2} \right)$$

□

Lastly, for when $\alpha = \beta$, we will need to compute $\int_{\Gamma\backslash\mathfrak{H}} E_s \cdot \frac{\pi}{3} C_\alpha \frac{dy\, dx}{y^2}$.

**Lemma 21.** *For each $s$,*

$$\int_{\Gamma\backslash\mathfrak{H}} \wedge^T \overline{E}_s \cdot \frac{\pi}{3} C_\alpha \frac{dy\,dx}{y^2} = \frac{\pi}{3} C_\alpha \cdot \frac{T^{\overline{s}-1}}{\overline{s}-1} + \frac{\pi}{3} c_{\overline{s}}\, C_\alpha \cdot \frac{T^{-\overline{s}}}{-\overline{s}}$$

*Proof.*

$$\int_{\Gamma\backslash\mathfrak{H}} \wedge^T \overline{E}_s \cdot \frac{\pi}{3} C_\alpha \frac{dy\,dx}{y^2} = \frac{\pi}{3} C_\alpha \int_{\Gamma\backslash\mathfrak{H}} \sum_{\gamma\in P\backslash\Gamma} \mathrm{Im}(\gamma z)^{\overline{s}} - \sum_{\gamma\in P\backslash\Gamma} \tau_{\overline{s}}(\gamma z) \frac{dy\,dx}{y^2}$$

$$= \frac{\pi}{3} C_\alpha \int_{\Gamma\backslash\mathfrak{H}} \sum_{\gamma\in P\backslash\Gamma} \left(\mathrm{Im}(\gamma z)^{\overline{s}} - \ \tau_{\overline{s}}(\gamma z)\right) \frac{dy\,dx}{y^2} = \frac{\pi}{3} C_\alpha \int_{P\backslash\mathfrak{H}} \left(y^{\overline{s}} - \ \tau_{\overline{s}}(z)\right) \frac{dy\,dx}{y^2}$$

by unwinding

$$= \frac{\pi}{3} C_\alpha \int_{P\backslash\mathfrak{H} y\leq T} y^{\overline{s}} \frac{dy\,dx}{y^2} - \frac{\pi}{3} C_\alpha \int_{P\backslash\mathfrak{H} y>T} c_{\overline{s}} y^{1-\overline{s}} \frac{dy\,dx}{y^2}$$

from the definitions of $\tau_s$

(A) Examining $\frac{\pi}{3} C_\alpha \int_{P\backslash\mathfrak{H} y\leq T} y^{\overline{s}} \frac{dy\,dx}{y^2}$:

Recall that the fundamental domain of $P\backslash\mathfrak{H}$ is $\{z = x + iy \in \mathfrak{H} \mid 0 \leq x \leq 1\}$ so we have

$$\frac{\pi}{3} C_\alpha \int_{P\backslash\mathfrak{H} y\leq T} y^{\overline{s}} \frac{dy\,dx}{y^2} = \frac{\pi}{3} C_\alpha \int_{y\leq T} y^{\overline{s}-2}\, dy = \frac{\pi}{3} C_\alpha \cdot \left.\frac{y^{\overline{s}-1}}{\overline{s}-1}\right|_0^T$$

$$= \frac{\pi}{3} C_\alpha \cdot \frac{T^{\overline{s}-1}}{\overline{s}-1} - \frac{\pi}{3} C_\alpha \cdot \lim_{t\to 0^+} \frac{t^{\overline{s}-1}}{\overline{s}-1}$$

For $\mathrm{Re}(\overline{s}) > 1$, the second term

$$\lim_{t\to 0^+} \frac{t^{\overline{s}-1}}{\overline{s}-1} = 0$$

Thus, by the Identity Principle, we can meromorphically continue to get that

$$\frac{\pi}{3} C_\alpha \int_{P\backslash\mathfrak{H} y\leq T} y^{\overline{s}} \frac{dy\,dx}{y^2} = \frac{\pi}{3} C_\alpha \cdot \frac{T^{\overline{s}-1}}{\overline{s}-1}$$

(B) Examining $\frac{\pi}{3} C_\alpha \int_{P\backslash\mathfrak{H} y>T} c_{\overline{s}} y^{1-\overline{s}} \frac{dy\,dx}{y^2}$:

Recall that the fundamental domain of $P\backslash\mathfrak{H}$ is $\{z = x+iy \in \mathfrak{H} \mid 0 \le x \le 1\}$ so we have

$$\frac{\pi}{3}C_\alpha \int_{P\backslash\mathfrak{H} y>T} c_{\overline{s}} y^{1-\overline{s}} \frac{dy\,dx}{y^2} = \frac{\pi}{3}C_\alpha \int_{y\ge T} c_{\overline{s}} y^{-1-\overline{s}}\,dy = \frac{\pi}{3}c_{\overline{s}}C_\alpha \cdot \frac{y^{-\overline{s}}}{-\overline{s}}\Big|_T^{infty}$$

$$= \frac{\pi}{3}c_{\overline{s}}\,C_\alpha \cdot \lim_{t\to\infty}\frac{t^{-\overline{s}}}{-\overline{s}} - \frac{\pi}{3}c_{\overline{s}}\,C_\alpha \cdot \frac{T^{-\overline{s}}}{-\overline{s}}$$

For $\mathrm{Re}(\overline{s}) > 0$, the first term

$$c_{\overline{s}} \lim_{t\to\infty}\frac{t^{-\overline{s}}}{-\overline{s}} = 0$$

Thus, by the Identity Principle, we can meromorphically continue to get that

$$\frac{\pi}{3}C_\alpha \int_{P\backslash\mathfrak{H} y>T} c_{\overline{s}} y^{1-\overline{s}} \frac{dy\,dx}{y^2} = -\frac{\pi}{3}c_{\overline{s}}\,C_\alpha \cdot \frac{T^{-\overline{s}}}{-\overline{s}}$$

Putting (A) and (B) together, we get:

$$\int_{\Gamma\backslash\mathfrak{H}} \wedge^T \overline{E}_s \cdot \frac{\pi}{3}C_\alpha \frac{dy\,dx}{y^2} = \frac{\pi}{3}C_\alpha \cdot \frac{T^{\overline{s}-1}}{\overline{s}-1} + \frac{\pi}{3}c_{\overline{s}}\,C_\alpha \cdot \frac{T^{-\overline{s}}}{-\overline{s}}$$

□

Finally we can apply these results to compute each $\int_{\Gamma\backslash\mathfrak{H}} \wedge^T S \cdot E_s \frac{dy\,dx}{y^2}$. We will now address each of the regimes presented in Theorem **??**.

## 4.3 Regimes

Recall the regimes set up in the proof of Theorem **??**. Again, suppose that $\alpha \ne 1$ and $\beta \ne 1$.

**(I):** Assume $1/2 \le \mathrm{Re}(\alpha) < \mathrm{Re}(\alpha)+1/2 < \mathrm{Re}(\beta)$ so $S = E_\alpha \cdot E_\beta - (E_{\alpha+\beta} + c_\alpha \cdot E_{1-\alpha+\beta})$.

First assume that $\alpha \ne 1$. Using the above Lemma **??**, Lemma **??** and Theorem **??** above, after canceling terms, we have

$$\int_{\Gamma\backslash\mathfrak{H}} \wedge^T S \cdot E_s \, \frac{dy\,dx}{y^2} = \langle E_\alpha E_\beta, \wedge^T E_s\rangle_{L^2} - \langle E_{\alpha+\beta}, \wedge^T E_s\rangle_{L^2} - \langle c_\alpha \cdot E_{1-\alpha+\beta}, \wedge^T E_s\rangle_{L^2}$$

$$= \frac{c_\beta}{\overline{s}+\alpha-\beta} T^{\overline{s}+\alpha-\beta} + \frac{c_\alpha c_\beta}{\overline{s}-\alpha-\beta+1} T^{\overline{s}-\alpha-\beta+1} + L(\overline{s}, E_\alpha \times E_\beta) \cdot \int_{y\le T} y^{\overline{s}} \cdot W_\alpha(y) W_\beta(y) \, \frac{dy}{y^2}$$

$$+ \frac{c_\beta c_{\overline{s}}}{1-\overline{s}+\alpha-\beta} \, T^{1-\overline{s}+\alpha-\beta} + \frac{c_\alpha c_\beta c_{\overline{s}}}{2-\overline{s}-\alpha-\beta} \, T^{2-\overline{s}-\alpha-\beta}$$

$$-c_{\alpha+\beta} \cdot \frac{T^{-\alpha-\beta+\overline{s}}}{-\alpha-\beta+\overline{s}} - c_{\alpha+\beta} c_{\overline{s}} \cdot \frac{T^{1-\alpha-\beta-\overline{s}}}{1-\alpha-\beta-\overline{s}} - c_\alpha c_{1-\alpha+\beta} \cdot \frac{T^{\alpha-\beta+\overline{s}-1}}{\alpha-\beta+\overline{s}-1}$$

$$-c_\alpha c_{1-\alpha+\beta} c_{\overline{s}} \cdot \frac{T^{\alpha-\beta-\overline{s}}}{\alpha-\beta-\overline{s}} - c_{\overline{s}} \sum_{n\neq 0} \varphi(n,\alpha)\varphi(n,\beta) n^{\overline{s}} \int_{y\ge T} y^{-1-\overline{s}} \cdot W_\alpha(y) W_\beta(y) \, dy$$

As $T \to \infty$, the polynomials will vanish on $1/2 < \mathrm{Re}(\alpha) < \mathrm{Re}(\alpha) + 1/2 < \mathrm{Re}(\beta)$ since $\mathrm{Re}(s) = 1/2$. Furthermore, since

$$c_{\overline{s}} \sum_{n\neq 0} \varphi(n,\alpha)\varphi(n,\beta) n^{\overline{s}} \int_{y\ge T} y^{-1-\overline{s}} \cdot W_\alpha(y) W_\beta(y) \, dy \to 0$$

as $T \to \infty$, we have that

$$\langle S, E_s\rangle_{L^2} = \mathcal{B}^{-1} - \lim_T \langle S, \wedge^T E_s\rangle_{L^2} = L(\overline{s}, E_\alpha \times E_\beta) \cdot \int_0^\infty y^{\overline{s}} \cdot W_\alpha(y) W_\beta(y) \, \frac{dy}{y^2}$$

$$= L(\overline{s}, E_\alpha \times E_\beta) \cdot \frac{\pi^{\alpha+\beta-\overline{s}}}{2\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{\Gamma(\frac{\overline{s}+\alpha-\beta}{2})\Gamma(\frac{\overline{s}-\alpha+\beta}{2})\Gamma(\frac{\overline{s}+1-\alpha-\beta}{2})\Gamma(\frac{\overline{s}-1+\alpha+\beta}{2})}{\Gamma(\overline{s})} = \Lambda(\overline{s}, E_\alpha \times E_\beta)$$

**(II):** Assume $1/2 \le \mathrm{Re}(\alpha) \le \mathrm{Re}(\beta) < \mathrm{Re}(\alpha) + 1/2$ but that $\alpha \neq \beta$.

**(IIa)** Suppose also that $\mathrm{Re}(\alpha+\beta) > 3/2$ so that
$S = E_\alpha \cdot E_\beta - (E_{\alpha+\beta} + c_\beta \cdot E_{1+\alpha-\beta} + c_\alpha \cdot E_{1-\alpha+\beta})$.

Using the above Lemma **??**, Lemma **??** and Theorem **??** above, after canceling terms, we have

$$\int_{\Gamma\backslash\mathfrak{H}} \wedge^T S \cdot E_s \, \frac{dy\, dx}{y^2}$$

$$= \langle E_\alpha E_\beta, \wedge^T E_s \rangle_{L^2} - \langle E_{\alpha+\beta}, \wedge^T E_s \rangle_{L^2} - \langle c_\beta \cdot E_{1+\alpha-\beta}, \wedge^T E_s \rangle_{L^2} - \langle c_\alpha \cdot E_{1-\alpha+\beta}, \wedge^T E_s \rangle_{L^2}$$

$$= \frac{c_\alpha c_\beta}{-\alpha-\beta+\overline{s}+1} T^{-\alpha-\beta+\overline{s}+1} + L(\overline{s}, E_\alpha \times E_\beta) \cdot \int_{y \le T} y^{\overline{s}} \cdot W_\alpha(y) W_\beta(y) \, \frac{dy}{y^2}$$

$$+ \frac{c_\alpha c_\beta c_{\overline{s}}}{-\alpha-\beta-\overline{s}+2} T^{-\alpha-\beta-\overline{s}+2} + c_{\overline{s}} \sum_{n\neq 0} \varphi(n,\alpha)\varphi(n,\beta) n^{\overline{s}} \int_{y\ge T} y^{-1-\overline{s}} \cdot W_\alpha(y) W_\beta(y) \, dy$$

$$-c_{\alpha+\beta} \cdot \frac{T^{-\alpha-\beta+\overline{s}}}{-\alpha-\beta+\overline{s}} - c_{\alpha+\beta} c_{\overline{s}} \cdot \frac{T^{-\alpha-\beta-\overline{s}+1}}{-\alpha-\beta-\overline{s}+1} - c_\beta c_{1+\alpha-\beta} \cdot \frac{T^{-\alpha+\beta+\overline{s}-1}}{-\alpha+\beta+\overline{s}-1}$$

$$-c_\beta c_{1+\alpha-\beta} c_{\overline{s}} \cdot \frac{T^{-\alpha+\beta-\overline{s}}}{-\alpha+\beta-\overline{s}} - c_\alpha c_{1-\alpha+\beta} \cdot \frac{T^{\alpha-\beta+\overline{s}-1}}{\alpha-\beta+\overline{s}-1} - c_\alpha c_{1-\alpha+\beta} c_{\overline{s}} \cdot \frac{T^{\alpha-\beta-\overline{s}}}{\alpha-\beta-\overline{s}}$$

As $T \to \infty$, the polynomials will vanish on $1/2 \le \mathrm{Re}(\alpha) \le \mathrm{Re}(\beta) < \mathrm{Re}(\alpha) + 1/2$ where $\mathrm{Re}(\alpha+\beta) > 3/2$ since $\mathrm{Re}(s) = 1/2$. Furthermore, since

$$c_{\overline{s}} \sum_{n\neq 0} \varphi(n,\alpha)\varphi(n,\beta) n^{\overline{s}} \int_{y\ge T} y^{-1-\overline{s}} \cdot W_\alpha(y) W_\beta(y) \, dy \to 0$$

as $T \to \infty$, we have that

$$\langle S, E_s \rangle_{L^2} = \mathcal{B}^{-1} - \lim_T \langle S, \wedge^T E_s \rangle_{L^2} = L(\overline{s}, E_\alpha \times E_\beta) \cdot \int_0^\infty y^{\overline{s}} \cdot W_\alpha(y) W_\beta(y) \, \frac{dy}{y^2}$$

$$= L(\overline{s}, E_\alpha \times E_\beta) \cdot \frac{\pi^{\alpha+\beta-\overline{s}}}{2\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{\Gamma(\frac{\overline{s}+\alpha-\beta}{2})\Gamma(\frac{\overline{s}-\alpha+\beta}{2})\Gamma(\frac{\overline{s}+1-\alpha-\beta}{2})\Gamma(\frac{\overline{s}-1+\alpha+\beta}{2})}{\Gamma(\overline{s})} = \Lambda(\overline{s}, E_\alpha \times E_\beta)$$

**(IIb)** Now suppose $\mathrm{Re}(\alpha+\beta) < 3/2$ so

$$S = E_\alpha \cdot E_\beta - (E_{\alpha+\beta} + c_\beta \cdot E_{1+\alpha-\beta} + c_\alpha \cdot E_{1-\alpha+\beta} + c_\alpha c_\beta \cdot E_{2-\alpha-\beta}) .$$

Using the above Lemma **??**, Lemma **??** and Theorem **??** above, after canceling terms, we have

$$\int_{\Gamma\backslash\mathfrak{H}} \wedge^T S \cdot E_s \, \frac{dy\, dx}{y^2}$$

$$= \langle E_\alpha E_\beta, \wedge^T E_s\rangle_{L^2} - \langle E_{\alpha+\beta}, \wedge^T E_s\rangle_{L^2} - \langle c_\beta \cdot E_{1+\alpha-\beta}, \wedge^T E_s\rangle_{L^2} - \langle c_\alpha \cdot E_{1-\alpha+\beta}, \wedge^T E_s\rangle_{L^2}$$

$$- \langle c_\alpha c_\beta \cdot E_{2-\alpha-\beta}, \wedge^T E_s\rangle_{L^2}$$

$$= L(\overline{s}, E_\alpha \times E_\beta) \cdot \int_{y\le T} y^{\overline{s}} \cdot W_\alpha(y) W_\beta(y) \, \frac{dy}{y^2}$$

$$-c_{\overline{s}} \sum_{n\neq 0} \varphi(n,\alpha)\varphi(n,\beta) n^{\overline{s}} \int_{y\ge T} y^{-1-\overline{s}} \cdot W_\alpha(y) W_\beta(y)\, dy$$

$$-c_{\alpha+\beta} \cdot \frac{T^{-\alpha-\beta+\overline{s}}}{-\alpha-\beta+\overline{s}} - c_{\alpha+\beta} c_{\overline{s}} \cdot \frac{T^{-\alpha-\beta-\overline{s}+1}}{-\alpha-\beta-\overline{s}+1} - c_\beta c_{\overline{s}} \cdot \frac{T^{\alpha-\beta-\overline{s}+1}}{\alpha-\beta-\overline{s}+1}$$

$$-c_\beta c_{1+\alpha-\beta} c_{\overline{s}} \cdot \frac{T^{-\alpha+\beta-\overline{s}}}{-\alpha+\beta-\overline{s}} - c_\alpha c_{1-\alpha+\beta} \cdot \frac{T^{\alpha-\beta+\overline{s}-1}}{\alpha-\beta+\overline{s}-1} - c_\alpha c_{1-\alpha+\beta} c_{\overline{s}} \cdot \frac{T^{\alpha-\beta-\overline{s}}}{\alpha-\beta-\overline{s}}$$

$$-c_\alpha c_\beta c_{2-\alpha-\beta} \cdot \frac{T^{\alpha+\beta+\overline{s}-2}}{\alpha+\beta+\overline{s}-2} - c_\alpha c_\beta c_{2-\alpha-\beta} c_{\overline{s}} \cdot \frac{T^{\alpha+\beta-\overline{s}-1}}{\alpha+\beta-\overline{s}-1}$$

As $T \to \infty$, the polynomials will vanish on $1/2 \le \mathrm{Re}(\alpha) \le \mathrm{Re}(\beta) < \mathrm{Re}(\alpha) + 1/2$ where $\mathrm{Re}(\alpha+\beta) < 3/2$ since $\mathrm{Re}(s) = 1/2$. Furthermore, since

$$c_{\overline{s}} \sum_{n\neq 0} \varphi(n,\alpha)\varphi(n,\beta) n^{\overline{s}} \int_{y\ge T} y^{-1-\overline{s}} \cdot W_\alpha(y) W_\beta(y)\, dy \to 0$$

as $T \to \infty$, we have that

$$\langle S, E_s\rangle_{L^2} = \mathcal{B}^{-1} - \lim_T \langle S, \wedge^T E_s\rangle_{L^2} = L(\overline{s}, E_\alpha \times E_\beta) \cdot \int_0^\infty y^{\overline{s}} \cdot W_\alpha(y) W_\beta(y)\, \frac{dy}{y^2}$$

$$= L(\overline{s}, E_\alpha \times E_\beta) \cdot \frac{\pi^{\alpha+\beta-\overline{s}}}{2\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{\Gamma(\frac{\overline{s}+\alpha-\beta}{2})\Gamma(\frac{\overline{s}-\alpha+\beta}{2})\Gamma(\frac{\overline{s}+1-\alpha-\beta}{2})\Gamma(\frac{\overline{s}-1+\alpha+\beta}{2})}{\Gamma(\overline{s})}$$

$$= \Lambda(\overline{s}, E_\alpha \times E_\beta)$$

**(III):** Suppose that $\alpha = \beta$.

**(IIIa)** Also assume $\mathrm{Re}(\alpha) > 3/4$ so $S = (E_\alpha)^2 - E_{2\alpha} - 2c_\alpha E_1^* + \frac{\pi}{3} C_\alpha$. Using the above Lemma **??**, Lemma **??**, Theorem **??**, Corollary **??** and Lemma **??** above, after canceling terms, we have

$$\int_{\Gamma\backslash\mathfrak{H}} \wedge^T S \cdot E_s \,\frac{dy\,dx}{y^2}$$

$$= \langle (E_\alpha)^2, \wedge^T E_s\rangle_{L^2} - \langle E_{2\alpha}, \wedge^T E_s\rangle_{L^2} - \langle 2c_\alpha E_1^*, \wedge^T E_s\rangle_{L^2} + \langle \frac{\pi}{3} C_\alpha, \wedge^T E_s\rangle_{L^2}$$

$$= \frac{c_\alpha^2}{\overline{s} - 2\alpha + 1} T^{\overline{s} - 2\alpha + 1} + L(\overline{s}, E_\alpha \times E_\alpha) \cdot \int_{y \le T} y^{\overline{s}} \cdot W_\alpha(y) W_\alpha(y)\,\frac{dy}{y^2}$$

$$+ \frac{c_\alpha^2 c_{\overline{s}}}{2 - \overline{s} - 2\alpha} T^{2 - \overline{s} - 2\alpha} - c_{\overline{s}} \sum_{n \neq 0} \varphi(n,\alpha)\varphi(n,\alpha) n^{\overline{s}} \int_{y \ge T} y^{-1-\overline{s}} \cdot W_\alpha(y) W_\alpha(y)\,dy$$

$$-c_{2\alpha} \cdot \frac{T^{(1-2\alpha)+\overline{s}-1}}{(1-2\alpha)+\overline{s}-1} - c_{2\alpha} c_{\overline{s}} \cdot \frac{T^{(1-2\alpha)+(1-\overline{s})-1}}{(1-2\alpha)+(1-\overline{s})-1} - 2c_\alpha C \frac{T^{\overline{s}-1}}{\overline{s}-1} - 2c_\alpha \frac{3}{\pi} \frac{T^{\overline{s}-1}}{\overline{s}-1} \log T$$

$$+2c_\alpha \frac{3}{\pi} \frac{T^{\overline{s}-1}}{(\overline{s}-1)^2} - 2c_\alpha c_{\overline{s}} \left( 2c_\alpha C \frac{T^{-\overline{s}}}{-\overline{s}} - \frac{3}{\pi} \frac{T^{-\overline{s}}}{-\overline{s}} \log T + \frac{3}{\pi} \frac{T^{-\overline{s}}}{\overline{s}^2} \right) + \frac{\pi}{3} C_\alpha \cdot \frac{T^{\overline{s}-1}}{\overline{s}-1} + \frac{\pi}{3} c_{\overline{s}} C_\alpha \cdot \frac{T^{-\overline{s}}}{-\overline{s}}$$

As $T \to \infty$, the polynomials will vanish on $\mathrm{Re}(\alpha) > 3/4$ since $\mathrm{Re}(s) = 1/2$. Furthermore, since

$$c_{\overline{s}} \sum_{n \neq 0} \varphi(n,\alpha)\varphi(n,\alpha) n^{\overline{s}} \int_{y \ge T} y^{-1-\overline{s}} \cdot W_\alpha(y) W_\alpha(y)\,dy \to 0$$

as $T \to \infty$ we have that

$$\langle S, E_s\rangle_{L^2} = \mathcal{B}^{-1} - \lim_T \langle S, \wedge^T E_s\rangle_{L^2} = L(\overline{s}, E_\alpha \times E_\alpha) \cdot \int_0^\infty y^{\overline{s}} \cdot W_\alpha(y) W_\alpha(y)\,\frac{dy}{y^2}$$

**(IIIb)** Suppose that $\alpha = \beta$ and $\mathrm{Re}(\alpha) < 3/4$ so $S = (E_\alpha)^2 - E_{2\alpha} - 2c_\alpha E_1^* - c_\alpha^2 E_{2-2\alpha} + \frac{\pi}{3} C_\alpha$. Using the above Lemma **??**, Lemma **??**, Theorem **??**, Corollary **??** and Lemma **??** above, after canceling terms, we have

$$\int_{\Gamma\backslash\mathfrak{H}} \wedge^T S \cdot E_s \,\frac{dy\,dx}{y^2}$$

$$= \langle (E_\alpha)^2, \wedge^T E_s\rangle_{L^2} - \langle E_{2\alpha}, \wedge^T E_s\rangle_{L^2} - \langle 2c_\alpha E_1^*, \wedge^T E_s\rangle_{L^2} - \langle c_\alpha^2 E_{2-2\alpha}, \wedge^T E_s\rangle_{L^2}$$

$$+ \langle \frac{\pi}{3} C_\alpha, \wedge^T E_s\rangle_{L^2}$$

$$= L(\overline{s}, E_\alpha \times E_\alpha) \cdot \int_{y \le T} y^{\overline{s}} \cdot W_\alpha(y) W_\alpha(y) \, \frac{dy}{y^2}$$

$$-c_{\overline{s}} \sum_{n \neq 0} \varphi(n,\alpha)\varphi(n,\alpha) n^{\overline{s}} \int_{y \ge T} y^{-1-\overline{s}} \cdot W_\alpha(y) W_\alpha(y) \, dy$$

$$-c_{2\alpha} \cdot \frac{T^{(1-2\alpha)+\overline{s}-1}}{(1-2\alpha)+\overline{s}-1} - c_{2\alpha} c_{\overline{s}} \cdot \frac{T^{(1-2\alpha)+(1-\overline{s})-1}}{(1-2\alpha)+(1-\overline{s})-1}$$

$$-2c_\alpha C \frac{T^{\overline{s}-1}}{\overline{s}-1} - 2c_\alpha \frac{3}{\pi} \frac{T^{\overline{s}-1}}{\overline{s}-1} \log T + 2c_\alpha \frac{3}{\pi} \frac{T^{\overline{s}-1}}{(\overline{s}-1)^2} - 2c_\alpha c_{\overline{s}} \left( C \frac{T^{-\overline{s}}}{-\overline{s}} - \frac{3}{\pi} \frac{T^{-\overline{s}}}{-\overline{s}} \log T + \frac{3}{\pi} \frac{T^{-\overline{s}}}{\overline{s}^2} \right)$$

$$-c_\alpha^2 c_{2-2\alpha} \cdot \frac{T^{(1-(2-2\alpha))+\overline{s}-1}}{(1-(2-2\alpha))+\overline{s}-1} - c_\alpha^2 c_{2-2\alpha} c_{\overline{s}} \cdot \frac{T^{(1-(2-2\alpha))+(1-\overline{s})-1}}{(1-(2-2\alpha))+(1-\overline{s})-1}$$

$$+\frac{\pi}{3} C_\alpha \cdot \frac{T^{\overline{s}-1}}{\overline{s}-1} + \frac{\pi}{3} c_{\overline{s}} C_\alpha \cdot \frac{T^{-\overline{s}}}{-\overline{s}}$$

As $T \to \infty$, the polynomials will vanish on $1/2 \le \mathrm{Re}(\alpha) < 3/4$ since $\mathrm{Re}(s) = 1/2$. Furthermore, since

$$c_{\overline{s}} \sum_{n \neq 0} \varphi(n,\alpha)\varphi(n,\alpha) n^{\overline{s}} \int_{y \ge T} y^{-1-\overline{s}} \cdot W_\alpha(y) W_\alpha(y) \, dyy \to 0$$

as $T \to \infty$ we have that

$$\langle S, E_s \rangle_{L^2} = \mathcal{B}^{-1} - \lim_T \langle S, \wedge^T E_s \rangle_{L^2} = L(\overline{s}, E_\alpha \times E_\alpha) \cdot \int_0^\infty y^{\overline{s}} \cdot W_\alpha(y) W_\alpha(y) \, \frac{dy}{y^2}$$

Finally, for each $\alpha, \beta \in \mathcal{C}$, we have that

$$\langle S, E_s \rangle_{L^2} = L(\overline{s}, E_\alpha \times E_\alpha) \cdot \int_0^\infty y^{\overline{s}} \cdot W_\alpha(y) W_\alpha(y) \, \frac{dy}{y^2} = \Lambda(\overline{s}, E_\alpha \times E_\alpha).$$

# Chapter 5

# The Residual Spectrum

We will compute the residual spectrum $\langle S, 1\rangle_{L^2}$ for each $S$.

**Theorem 22.** $\langle S, 1\rangle_{L^2} = 0$ *for each $S$ presented in Theorem* **??**.

We will prove this in what follows with the following Lemma and the use of truncated Eisenstein series.

**Lemma 23.** *For each $\beta$ and $\alpha \neq 1$,* $\displaystyle\int_{\Gamma\backslash\mathfrak{H}} E_\alpha \cdot E_\beta - E_{\alpha+\beta} - c_\alpha \cdot E_{1-\alpha+\beta} \frac{dx\,dy}{y^2} = 0$

*Proof.* $\displaystyle\int_{\Gamma\backslash\mathfrak{H}} E_\alpha \cdot E_\beta - E_{\alpha+\beta} - c_\alpha \cdot E_{1-\alpha+\beta} \frac{dx\,dy}{y^2}$

$$= \int_{\Gamma\backslash\mathfrak{H}} E_\alpha \cdot \sum_{\gamma_1 \in P\backslash\Gamma} \mathrm{Im}(\gamma_1 z)^\beta - \sum_{\gamma_2 \in P\backslash\Gamma} \mathrm{Im}(\gamma_2 z)^{\alpha+\beta} - c_\alpha \cdot \sum_{\gamma_3 \in P\backslash\Gamma} \mathrm{Im}(\gamma_3 z)^{1-\alpha+\beta} \frac{dx\,dy}{y^2}$$

$$= \int_{\Gamma\backslash\mathfrak{H}} \sum_{\gamma \in P\backslash\Gamma} \left( (\gamma y)^\beta \cdot E_\alpha - (\gamma y)^{\alpha+\beta} - c_\alpha \cdot (\gamma y)^{1-\alpha+\beta} \right) \frac{dx\,dy}{y^2}$$

by unwinding

$$= \int_{P\backslash\mathfrak{H}} \left( y^\beta \cdot E_\alpha - y^{\alpha+\beta} - c_\alpha \cdot y^{1-\alpha+\beta} \right) \frac{dx\,dy}{y^2}$$

Now, writing out the Fourier-Whittaker expansions for $E_\alpha$, we have

$$= \int_{P\backslash\mathfrak{H}} \left( y^\beta \cdot \left( y^\alpha + c_\alpha y^{1-\alpha} + \sum_{n\neq 0} \varphi(n,\alpha) \cdot W_\alpha(|n|y) e^{2\pi i n x} \right) \right.$$

$$\left. - y^{\alpha+\beta} - c_\alpha \cdot y^{1-\alpha+\beta} \right) \frac{dx\,dy}{y^2}$$

$$= \int_{P\backslash\mathfrak{H}} \left( y^{\alpha+\beta} + c_\alpha y^{1-\alpha+\beta} + y^\beta \cdot \sum_{n\neq 0} \varphi(n,\alpha) \cdot W_\alpha(|n|y) e^{2\pi i n x} - y^{\alpha+\beta} \right.$$

$$\left. - c_\alpha \cdot y^{1-\alpha+\beta} \right) \frac{dx\,dy}{y^2}$$

$$= \int_{P\backslash\mathfrak{H}} y^\beta \cdot \sum_{n\neq 0} \varphi(n,\alpha) \cdot W_\alpha(|n|zy) e^{2\pi i n x} \frac{dx\,dy}{y^2}$$

$$= \sum_{n\neq 0} \varphi(n,\alpha) \int_0^\infty \int_0^1 y^\beta \cdot W_\alpha(|n|y) e^{2\pi i n x} \frac{dx\,dy}{y^2} = 0.$$

□

## 5.1 Regimes

Recall the regimes set up in the proof of Theorem **??**. Again, suppose that $\alpha \neq 1$ and $\beta \neq 1$.

**(I):** Suppose that $1/2 \leq \mathrm{Re}(\alpha) \leq \mathrm{Re}(\alpha) + 1/2 < \mathrm{Re}(\beta)$ then

$$\langle S, 1\rangle_{L^2} = \int_{\Gamma\backslash\mathfrak{H}} E_\alpha \cdot E_\beta - E_{\alpha+\beta} - c_\alpha \cdot E_{1-\alpha+\beta} \frac{dx\,dy}{y^2} = 0$$

by Lemma **??**.

**(II):** Suppose $1/2 \leq \mathrm{Re}(\alpha) \leq \mathrm{Re}(\beta) < \mathrm{Re}(\alpha) + 1/2$.

**(IIa)** Suppose also that $\mathrm{Re}(\alpha+\beta) > 3/2$. Then

$$S = E_\alpha \cdot E_\beta - (E_{\alpha+\beta} + c_\alpha \cdot E_{1-\alpha+\beta} + c_\beta \cdot E_{1+\alpha-\beta})$$

which gives

$$\langle S, 1\rangle_{L^2} = \int_{\Gamma\backslash\mathfrak{H}} E_\alpha \cdot E_\beta - (E_{\alpha+\beta} + c_\alpha \cdot E_{1-\alpha+\beta} + c_\beta \cdot E_{1+\alpha-\beta})\ \frac{dx\,dy}{y^2}$$

$$= -c_\beta \cdot \int_{\Gamma\backslash\mathfrak{H}} E_{1+\alpha-\beta}\ \frac{dx\,dy}{y^2}$$

by Lemma **??**.

We will again use Arthur truncation to compute this integral as well as a trick which involves passing the computation of a residue outside of an integral. Given that $E_r$ is a vector-valued holomorphic function, vector-valued Cauchy (-Goursat) theory, as well as Gelfand-Pettis, implies that we can pass the linear functional outside the integral (see [**?**] or [**?**]).

Since $\mathrm{Res}_{r=1}(E_r) = \frac{3}{\pi}$,

$$\langle \wedge^T E_{1+\alpha-\beta}, 1\rangle_{L^2} = \int_{\Gamma\backslash\mathfrak{H}} \wedge^T E_{1+\alpha-\beta}\ \frac{dy\,dx}{y^2} = \int_{\Gamma\backslash\mathfrak{H}} \wedge^T E_{1+\alpha-\beta} \cdot \mathrm{Res}_{r=1}(E_r) \cdot \frac{\pi}{3}\ \frac{dy\,dx}{y^2}$$

$$= \frac{\pi}{3} \cdot \mathrm{Res}_{r=1}\left(\int_{\Gamma\backslash\mathfrak{H}} \wedge^T E_{1+\alpha-\beta} \cdot E_r\ \frac{dy\,dx}{y^2}\right) = \frac{\pi}{3} \cdot \mathrm{Res}_{r=1}\left(\int_{\Gamma\backslash\mathfrak{H}} \wedge^T E_{1+\alpha-\beta} \cdot \wedge^T E_r\ \frac{dy\,dx}{y^2}\right)$$

by Lemma **??**

$$= \frac{\pi}{3} \cdot \mathrm{Res}_{r=1}\Big(\frac{T^{r+\alpha-\beta}}{r+\alpha-\beta} + c_r \frac{T^{1-r+\alpha-\beta}}{1-r+\alpha-\beta} + c_{1+\alpha-\beta} \frac{T^{r-1-\alpha+\beta}}{r-1-\alpha+\beta}$$

$$+ c_r c_{1+\alpha-\beta} \frac{T^{-r-\alpha+\beta}}{-r-\alpha+\beta}\Big) = 0$$

**(IIb)** Now suppose also that $\mathrm{Re}(\alpha+\beta) < 3/2$. Then

$$S = E_\alpha \cdot E_\beta - (E_{\alpha+\beta} + c_\alpha \cdot E_{1-\alpha+\beta} + c_\beta \cdot E_{1+\alpha-\beta} + c_\alpha c_\beta \cdot E_{2-\alpha-\beta})$$

which gives

$$\langle S, 1\rangle_{L^2} = \int_{\Gamma\backslash\mathfrak{H}} E_\alpha \cdot E_\beta - E_{\alpha+\beta} - c_\alpha \cdot E_{1-\alpha+\beta} - c_\beta \cdot E_{1+\alpha-\beta} - c_\alpha c_\beta \cdot E_{2-\alpha-\beta}\ \frac{dx\,dy}{y^2}$$

$$= -c_\beta \cdot \int_{\Gamma\backslash\mathfrak{H}} E_{1+\alpha-\beta} + c_\alpha \cdot E_{2-\alpha-\beta} \, \frac{dx\,dy}{y^2}$$

by Lemma **??**.

As **(IIa)**, we again have that $\langle \wedge^T E_{1+\alpha-\beta}, 1\rangle_{L^2} = 0$. We will again use Arthur truncation to compute the last integral

$$\langle \wedge^T E_{2-\alpha-\beta}, 1\rangle_{L^2} = \int_{\Gamma\backslash\mathfrak{H}} \wedge^T E_{2-\alpha-\beta} \, \frac{dy\,dx}{y^2} = \int_{\Gamma\backslash\mathfrak{H}} \wedge^T E_{2-\alpha-\beta} \cdot \mathrm{Res}_{r=1}(E_r) \cdot \frac{\pi}{3} \, \frac{dy\,dx}{y^2}$$

since $\mathrm{Res}_{r=1}(E_r) = \frac{3}{\pi}$

$$= \frac{\pi}{3} \cdot \mathrm{Res}_{r=1} \left( \int_{\Gamma\backslash\mathfrak{H}} \wedge^T E_{1+\alpha-\beta} \cdot E_r \, \frac{dy\,dx}{y^2} \right) = \frac{\pi}{3} \cdot \mathrm{Res}_{r=1} \left( \int_{\Gamma\backslash\mathfrak{H}} \wedge^T E_{2-\alpha-\beta} \cdot \wedge^T E_r \, \frac{dy\,dx}{y^2} \right)$$

by Lemma **??**

$$= \frac{\pi}{3} \cdot \mathrm{Res}_{r=1} \Big( \frac{T^{r+1-\alpha-\beta}}{r+1-\alpha-\beta} + c_r \frac{T^{-r+2-\alpha-\beta}}{-r+2-\alpha-\beta} + c_{2-\alpha-\beta} \frac{T^{r-2+\alpha+\beta}}{r-2+\alpha+\beta}$$

$$+ c_r c_{2-\alpha-\beta} \frac{T^{-r-1+\alpha+\beta}}{-r-1+\alpha+\beta} \Big) = 0$$

**(III):** Suppose that $\alpha = \beta$. Unlike the other spectral integrals, we will consider this case as a limit of case **(II)**. Since both the limit and the integrals converge nicely (as already proven in Section 1), we can interchange the limit and the integral to get the following.

**(IIIa):** Also suppose $\mathrm{Re}(\alpha) > 3/4$ so $S = (E_\alpha)^2 - E_{2\alpha} - 2c_\alpha E_1^* - c_\alpha^2 E_{2-2\alpha} + \frac{\pi}{3} C_\alpha$ which gives

$$\langle S, 1\rangle_{L^2} = \int_{\Gamma\backslash\mathfrak{H}} (E_\alpha)^2 - E_{2\alpha} - 2c_\alpha E_1^* + \frac{\pi}{3} C_\alpha \, \frac{dx\,dy}{y^2}$$

$$= \int_{\Gamma\backslash\mathfrak{H}} \lim_{\beta\to\alpha} \left( E_\alpha \cdot E_\beta - E_{\alpha+\beta} - c_\alpha \cdot E_{1-\alpha+\beta} - c_\beta \cdot E_{1+\alpha-\beta} - c_\alpha c_\beta \cdot E_{2-\alpha-\beta} \right) \, \frac{dx\,dy}{y^2}$$

by Lemma **??**

$$= \lim_{\beta\to\alpha} \int_{\Gamma\backslash\mathfrak{H}} E_\alpha \cdot E_\beta - E_{\alpha+\beta} - c_\alpha \cdot E_{1-\alpha+\beta} - c_\beta \cdot E_{1+\alpha-\beta} - c_\alpha c_\beta \cdot E_{2-\alpha-\beta} \, \frac{dx\,dy}{y^2} = 0$$

by part **(IIa)**.

**(IIIb):** Now suppose $1/2 \le \mathrm{Re}(\alpha) < 3/4$ so $S = (E_\alpha)^2 - E_{2\alpha} - 2c_\alpha E_1^* - c_\alpha^2 E_{2-2\alpha} + \frac{\pi}{3} C_\alpha$ which gives

$$\langle S, 1\rangle_{L^2} = \int_{\Gamma\backslash\mathfrak{H}} (E_\alpha)^2 - E_{2\alpha} - 2c_\alpha E_1^* - c_\alpha^2 E_{2-2\alpha} + \frac{\pi}{3} C_\alpha \, \frac{dx\, dy}{y^2}$$

$$= \int_{\Gamma\backslash\mathfrak{H}} \lim_{\beta\to\alpha} \left(E_\alpha \cdot E_\beta - E_{\alpha+\beta} - c_\alpha \cdot E_{1-\alpha+\beta} - c_\beta \cdot E_{1+\alpha-\beta}\right) \frac{dx\, dy}{y^2}$$

by Lemma **??**

$$= \lim_{\beta\to\alpha} \int_{\Gamma\backslash\mathfrak{H}} E_\alpha \cdot E_\beta - E_{\alpha+\beta} - c_\alpha \cdot E_{1-\alpha+\beta} - c_\beta \cdot E_{1+\alpha-\beta} \, \frac{dx\, dy}{y^2} = 0$$

by part **(IIb)**.

Putting these cases together we see that the residual spectrum $\langle S, 1\rangle_{L^2} = 0$ for each $S$.

# Chapter 6

# Spectral Decomposition of the Solution

Finally, putting everything together we have

$$\begin{aligned} S &= \sum_{f \text{ cfm}} \langle S, f\rangle_{L^2} \cdot f + \frac{\langle S, 1\rangle_{L^2} \cdot 1}{\langle 1, 1\rangle_{L^2}} + \frac{1}{4\pi i}\int_{(1/2)} \langle S, E_s\rangle_{L^2} \cdot E_s \, ds \\ &= \sum_{f \text{ cfm}} \Lambda(\alpha, \overline{f} \times E_\beta) \cdot f + \frac{1}{4\pi i}\int_{(1/2)} \Lambda(\overline{s}, E_\alpha \times E_\beta) \cdot E_s \, ds \end{aligned}$$

where this $S \in L^2(\Gamma\backslash\mathfrak{H})$.

Recall that we have found the spectral decomposition for $S = E_\alpha \cdot E_\beta - \sum_i c_i E_i + \mathbb{1}_{\alpha=\beta} \cdot \frac{\pi}{3} C_\alpha$ where $\mathbb{1}_{\alpha=\beta} = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta \end{cases}$ but we want to use this to solve $(\Delta - \lambda_s)u_w = E_\alpha \cdot E_\beta$ on $\Gamma\backslash SL_2(\mathbb{R})$.

$$E_\alpha \cdot E_\beta = \sum_i c_i E_{s_i} - \mathbb{1}_{\alpha=\beta} \cdot \frac{\pi}{3} C_\alpha + \sum_{f \text{ cfm}} \Lambda(\alpha, \overline{f} \times E_\beta) \cdot f + \frac{1}{4\pi i}\int_{(1/2)} \Lambda(\overline{s}, E_\alpha \times E_\beta) \cdot E_s \, ds$$

where

$$\sum_i c_i E_{s_i} = E_{\alpha+\beta} + c_\alpha E_{1-\alpha+\beta}$$

on $1/2 \leq \text{Re}(\alpha) < \text{Re}(\alpha) + 1/2 < \text{Re}(\beta)$,

$$\sum_i c_i E_{s_i} = E_{\alpha+\beta} + c_\beta E_{1+\alpha-\beta} + c_\alpha \cdot E_{1-\alpha+\beta}$$

on $1/2 \le \mathrm{Re}(\alpha) \le \mathrm{Re}(\beta) < \mathrm{Re}(\alpha) + 1/2$ and $\mathrm{Re}(\alpha+\beta) > 3/2$ but $\alpha \neq \beta$,

$$\sum_i c_i E_{s_i} = E_{\alpha+\beta} + c_\beta E_{1+\alpha-\beta} + c_\alpha E_{1-\alpha+\beta} + c_\alpha c_\beta E_{2-\alpha-\beta}$$

on $1/2 \le \mathrm{Re}(\alpha) \le \mathrm{Re}(\beta) < \mathrm{Re}(\alpha) + 1/2$ and $\mathrm{Re}(\alpha+\beta) < 3/2$ but $\alpha \neq \beta$,

$$\sum_i c_i E_{s_i} = E_\alpha^2 + 2c_\alpha E_1^*$$

when $\alpha = \beta$ and $\mathrm{Re}(\alpha) > 3/4$ , and

$$\sum_i c_i E_{s_i} = E_\alpha^2 + 2c_\alpha E_1^* + c_\alpha^2 E_{2-2\alpha}$$

when $\alpha = \beta$ and $1/2 \le \mathrm{Re}(\alpha) < 3/4$.

Now we can use this as well as the spectral relation in Section **??** to solve

$$(\Delta - \lambda_w) u_w = E_\alpha \cdot E_\beta$$

on $\Gamma \backslash SL_2(\mathbb{R})$. In $\mathrm{Re}(w) > 1/2$, for $(\alpha, \beta) \in C$, the solution is given by

$$u_w = \sum_i \frac{c_i E_{s_i}}{\lambda_{s_i} - \lambda_w} - \mathbb{1}_{\alpha=\beta} \cdot \frac{\frac{\pi}{3} C_\alpha}{\lambda_1 - \lambda_w} + \sum_{f \text{ cfm}} \frac{\Lambda(\alpha, \overline{f} \times E_\beta) \cdot f}{\lambda_{s_f} - \lambda_w}$$

$$+ \frac{1}{4\pi i} \int_{(1/2)} \Lambda(\overline{s}, E_\alpha \times E_\beta) \cdot \frac{E_s}{\lambda_s - \lambda_w} \, ds$$

and lies in $H^2(\Gamma \backslash \mathfrak{H}) \oplus \mathcal{E}(\Gamma \backslash \mathfrak{H})$. Also, note that the automorphic Sobolev space $H^k$ in which this solution exists is also defined in Section **??**. This concludes our proof of Theorem **??**.

## 6.1 Meromorphic Continuation of the Solution

We will now meromorphically continue the solution

$$u_w = \sum_i \frac{c_i E_{s_i}}{\lambda_{s_i} - \lambda_w} - \mathbb{1}_{\alpha=\beta} \cdot \frac{\frac{\pi}{3} C_\alpha}{\lambda_1 - \lambda_w} + \sum_{f \text{ cfm}} \frac{\Lambda(\alpha, \overline{f} \times E_\beta) \cdot f}{\lambda_{s_f} - \lambda_w}$$

$$+ \frac{1}{4\pi i} \int_{(1/2)} \Lambda(\overline{s}, E_\alpha \times E_\beta) \cdot \frac{E_s}{\lambda_s - \lambda_w} \, ds$$

in $V := H^2(\Gamma\backslash\mathfrak{H}) \oplus \mathcal{E}(\Gamma\backslash\mathfrak{H})$ which is initially defined on $\mathrm{Re}(w) > 1/2$.

Observe that the first three terms of $u_w$ will have meromorphic continuation. Since Eisenstein series (and also constants) are constant in $w$ and we are only dividing by at most a simple pole given by these discrete combinations of $\alpha$ and $\beta$, the first two terms have meromorphic continuation. In the third term of $u_w$, again the $L$-function and cuspform will be constant in $w$. Furthermore, we can see that the eigenvalues attached to cuspforms are also discrete by examining the pre-trace formula:

$$\sum_{F: \, : \, |\lambda_F| \leq T} |F(z_o)|^2 + \frac{|\langle F, 1 \rangle|^2}{\langle 1, 1 \rangle} + \frac{1}{4\pi i} \int_{(1/2)} |E_s(z_o)|^2 \, ds \ll_C T^2$$

For the fourth term, it is important to note that the visual symmetry on the continuous spectrum in misleading. More work must be done to meromorphically continue this piece for the spectral expansion of $u$. These meromorphic continuations do not exist in $V$ but in a larger space $M$ of moderate-growth functions that includes Eisenstein series. For this reason meromorphic continuation is best described in terms of vector-valued integrals. This will require a bit of topological set-up.

Define

$$M := \left\{ f \in C^o(\Gamma\backslash\mathfrak{H}) \;\Big|\; \sup_{\mathrm{Im}(z) \geq \sqrt{3}/2} y^r \cdot |f(x+iy)| < \infty \text{ for some } r \in \mathbb{R} \right\}$$

The topology on $M$ is a an inductive limit of Banach spaces

$$M_o^r = \left\{ f \in C^o(\Gamma\backslash\mathfrak{H}) \;\Big|\; \sup_{\mathrm{Im}(z) \geq \sqrt{3}/2} y^r \cdot |f(x+iy)| < \infty \text{ for } r \in \mathbb{R} \right\}$$

obtained by the completion of $C^o(\Gamma\backslash\mathfrak{H})$ with respect to norms

$$|f|_{M_o^r} := \sup_{\mathrm{Im}(z)\geq\sqrt{3}/2} y^r \cdot |f(x+iy)|$$

for $f \in M_o^r$. Thus $M$ is a strict colimit in the locally convex category of Banach spaces so is quasi-complete and locally convex.

Let $\Phi : M \to N$ be a continuous linear map to a quasi-complete locally convex topological vector space $N$ and consider the $N$-valued integrals

$$u_{w,\Phi} = \sum_i \frac{c_i \Phi E_{s_i}}{\lambda_{s_i} - \lambda_w} - \mathbb{1}_{\alpha=\beta} \cdot \frac{\frac{\pi}{3} C_\alpha \cdot \Phi(1)}{\lambda_1 - \lambda_w} + \sum_{f \text{ cfm}} \frac{\Lambda(\alpha, \overline{f} \times E_\beta) \cdot \Phi f}{\lambda_{s_f} - \lambda_w} + \frac{1}{4\pi i} \int_{(1/2)} \Lambda(\overline{s}, E_\alpha \times E_\beta) \cdot \frac{\Phi E_s}{\lambda_s - \lambda_w} \, ds$$

Of course, for $\Phi$ the identity map $M \to M$ gives $u_w$ itself and we anticipate that $\Phi(u_w) = u_{w,\Phi}$.

**Lemma 24.** *$\Phi(u_w) = u_{w,\Phi}$ in the region Re$(w) > 1/2$.*

*Proof.* Observe that

$$\Phi(u_w) = \sum_i \frac{c_i \Phi E_{s_i}}{\lambda_{s_i} - \lambda_w} - \mathbb{1}_{\alpha=\beta} \cdot \frac{\frac{\pi}{3} C_\alpha \cdot \Phi(1)}{\lambda_1 - \lambda_w} + \sum_{f \text{ cfm}} \frac{\Lambda(\alpha, \overline{f} \times E_\beta) \cdot \Phi f}{\lambda_{s_f} - \lambda_w} + \frac{1}{4\pi i} \Phi \left( \int_{(1/2)} \Lambda(\overline{s}, E_\alpha \times E_\beta) \cdot \frac{E_s}{\lambda_s - \lambda_w} \, ds \right)$$

In $\mathrm{Re}(w) > 1/2$, the integral for $u_w$ is a $v$-valued holomorphic function in $w$. We have In that region, due to the properties of compactly supported continuous-integrand Gelfand-Pettis integrals [?],

$$\Phi \left( \int_{(1/2)} \Lambda(\overline{s}, E_\alpha \times E_\beta) \cdot \frac{E_s}{\lambda_s - \lambda_w} \, ds \right) = \Phi \left( \lim_{T\to\infty} \int_{|\mathrm{Im}(s)|\leq T} \Lambda(\overline{s}, E_\alpha \times E_\beta) \cdot \frac{E_s}{\lambda_s - \lambda_w} \, ds \right)$$

$$= \lim_{T\to\infty} \Phi \left( \int_{|\mathrm{Im}(s)|\leq T} \Lambda(\overline{s}, E_\alpha \times E_\beta) \cdot \frac{E_s}{\lambda_s - \lambda_w} \, ds \right)$$

$$= \lim_{T\to\infty} \int_{|\mathrm{Im}(s)|\le T} \Lambda(\overline{s}, E_\alpha \times E_\beta) \cdot \frac{\Phi E_s}{\lambda_s - \lambda_w}\, ds = \int_{(1/2)} \Lambda(\overline{s}, E_\alpha \times E_\beta) \cdot \frac{\Phi E_s}{\lambda_s - \lambda_w}\, ds$$

since the limit is approached in $V \subset M$. □

**Theorem 25.** *With continuous linear $\Phi : M \to N$ with $N$ quasi-complete and locally convex, the $\Phi M$-valued function $w \mapsto u_{w,\Phi}$ has meromorphic continuation as an $N$-valued function of $w$. Explicitly, the function*

$$J_{w,\Phi} = \sum_i \frac{c_i \Phi E_{s_i}}{\lambda_{s_i} - \lambda_w} - \mathbb{1}_{\alpha=\beta} \cdot \frac{\frac{\pi}{3} C_\alpha \cdot \Phi(1)}{\lambda_1 - \lambda_w} + \sum_{f\ cfm} \frac{\Lambda(\alpha, \overline{f} \times E_\beta) \cdot \Phi f}{\lambda_{s_f} - \lambda_w}$$

$$+ \frac{1}{4\pi i} \int_{(1/2)} \frac{\Lambda(1-s, E_\alpha \times E_\beta) \cdot \Phi E_s - \Lambda(1-w, E_\alpha \times E_\beta) \cdot \Phi E_w}{\lambda_s - \lambda_w}\, ds$$

*has a meromorphic continuation to an $N$-valued function with the functional equation $J_{1-w,\Phi} = J_{w,\Phi}$ and*

$$u_{w,\Phi} = J_{w,\Phi} + \frac{\Lambda(1-w, E_\alpha \times E_\beta) \cdot \Phi E_w}{2(1-2w)}$$

*Proof.* From Lemma **??**, in $\mathrm{Re}(w) > 1/2$ the expression for $u_{w,\Phi}$ converges as an $N$-valued integral. The meromorphic continuation of $u_{w,\Phi}$ will be obtained through rearranging the integral.

First, in $\mathrm{Re}(w) > 1/2$ we add and subtract to obtain

$$u_{w,\Phi} = \sum_i \frac{c_i \Phi E_{s_i}}{\lambda_{s_i} - \lambda_w} - \mathbb{1}_{\alpha=\beta} \cdot \frac{\frac{\pi}{3} C_\alpha \cdot \Phi(1)}{\lambda_1 - \lambda_w} + \sum_{f \text{ cfm}} \frac{\Lambda(\alpha, \overline{f} \times E_\beta) \cdot \Phi f}{\lambda_{s_f} - \lambda_w}$$

$$+ \frac{1}{4\pi i} \int_{(1/2)} \frac{\Lambda(1-s, E_\alpha \times E_\beta) \cdot \Phi E_s}{\lambda_s - \lambda_w}\, ds$$

$$= \sum_i \frac{c_i \Phi E_{s_i}}{\lambda_{s_i} - \lambda_w} - \mathbb{1}_{\alpha=\beta} \cdot \frac{\frac{\pi}{3} C_\alpha \cdot \Phi(1)}{\lambda_1 - \lambda_w} + \sum_{f \text{ cfm}} \frac{\Lambda(\alpha, \overline{f} \times E_\beta) \cdot \Phi f}{\lambda_{s_f} - \lambda_w}$$

$$+ \frac{1}{4\pi i} \int_{(1/2)} \frac{\Lambda(1-s, E_\alpha \times E_\beta) \cdot \Phi E_s - \Lambda(1-w, E_\alpha \times E_\beta) \cdot \Phi E_w}{\lambda_s - \lambda_w}\, ds$$

$$+ \Lambda(1-w, E_\alpha \times E_\beta) \cdot \Phi E_w \frac{1}{4\pi i} \int_{(1/2)} \frac{1}{\lambda_s - \lambda_w}\, ds$$

$$= J_{w,\Phi} + \Lambda(1-w, E_\alpha \times E_\beta) \cdot \Phi E_w \cdot \frac{1}{4\pi i} \int_{(1/2)} \frac{1}{\lambda_s - \lambda_w} \, ds$$

By residues,

$$\Lambda(1-w, E_\alpha \times E_\beta) \cdot \Phi E_w \cdot \frac{1}{4\pi i} \int_{(1/2)} \frac{1}{\lambda_s - \lambda_w} \, ds$$

$$= \Lambda(1-w, E_\alpha \times E_\beta) \cdot \Phi E_w \cdot \left( -\frac{1}{2} \cdot \mathrm{Res}_{s=w} \frac{1}{\lambda_s - \lambda_w} \right) = \frac{\Lambda(1-w, E_\alpha \times E_\beta) \cdot \Phi E_w}{2(1-2w)}$$

Since $\Lambda(1-w, E_\alpha \times E_\beta)$ is a meromorphic $\mathbb{C}$-valued function and $w \mapsto \Phi E_w$ is a meromorphic $N$-valued function, $\Lambda(1-w, E_\alpha \times E_\beta) \cdot \Phi E_w$ is a meromorphic $N$-valued function with a meromorphic continuation from the meromorphic continuation of Eisenstein series and $\Lambda(w, E_\alpha \times E_\beta)$. Observe that although the Eisenstein series is invariant under $w \mapsto 1-w$, the denominator is skew-symmetric.

We will now meromorphically continue the integral $J_{w,\Phi}$. First constrain $w$ so that is lies in a fixed compact set $C$ and take $T$ large enough so that $T \geq 2|w|$ for all $w \in C$. First, for $\mathrm{Re}(w) > 1/2$ and $s = \frac{1}{2} + it$, we make an attempt to cancel the vanishing denominator when $s$ is close to $w$ by rearranging

$$J_{w,\Phi} - \left( \sum_i \frac{c_i \Phi E_{s_i}}{\lambda_{s_i} - \lambda_w} - \mathbb{1}_{\alpha=\beta} \cdot \frac{\frac{\pi}{3} C_\alpha \cdot \Phi(1)}{\lambda_1 - \lambda_w} + \sum_{f \text{ cfm}} \frac{\Lambda(\alpha, \overline{f} \times E_\beta) \cdot \Phi f}{\lambda_{s_f} - \lambda_w} \right)$$

$$= \frac{1}{4\pi i} \int_{(1/2)} \frac{\Lambda(1-s, E_\alpha \times E_\beta) \cdot \Phi E_s - \Lambda(1-w, E_\alpha \times E_\beta) \cdot \Phi E_w}{\lambda_s - \lambda_w} \, ds$$

$$= \frac{1}{4\pi i} \int_{|t| \geq T} \frac{\Lambda(1-s, E_\alpha \times E_\beta) \cdot \Phi E_s}{\lambda_s - \lambda_w} \, ds - \Lambda(1-w, E_\alpha \times E_\beta) \cdot \Phi E_w \cdot \frac{1}{4\pi i} \int_{|t| \geq T} \frac{1}{\lambda_s - \lambda_w} \, ds$$

$$+ \frac{1}{4\pi i} \int_{|t| \leq T} \frac{\Lambda(1-s, E_\alpha \times E_\beta) \cdot \Phi E_s - \Lambda(1-w, E_\alpha \times E_\beta) \cdot \Phi E_w}{\lambda_s - \lambda_w} \, ds$$

The meromorphy of the leading integral is understood via the Plancherel Theorem on the continuous automorphic spectrum. Up to constants, the Plancherel Theorem for $L^2$ states that $A(t) \in L^2(\mathbb{R})$ the spectral synthesis integral

$$B = \frac{1}{4\pi} \int_{-\infty}^{\infty} A(t) \cdot E_s \, dt$$

for $z \in \mathfrak{H}$ produces a function in $H^0$ and the map the $A \mapsto B$ gives an isometry.

Observe that $\Lambda(1-s, E_\alpha \times E_\beta) \in L^2(\frac{1}{2}+i\mathbb{R})$ since $S \in L^2(\Gamma\backslash\mathfrak{H})$ and $\Lambda(\overline{s}, E_\alpha \times E_\beta) = \langle S, E_s\rangle_{L^2(\Gamma\backslash\mathfrak{H})}$. Hence for $w$ in a fixed compact, $\dfrac{\Lambda(1-s,E_\alpha\times E_\beta)}{\lambda_s-\lambda_w} \in L^2\left(\dfrac{1}{2}+i\mathbb{R}\right)$. Composition with Plancherel isometry shows that

$$w \mapsto \frac{1}{4\pi i}\int_{|t|\geq T} \frac{\Lambda(1-s,E_\alpha\times E_\beta)\cdot E_s}{\lambda_s-\lambda_w}\,ds$$

is a meromorphic $L^2(\frac{1}{2}+i\mathbb{R})$-valued function in $w$ in the fixed compact. Now, since $|w| \ll T$ the meromophic continuation is given by the same integral, the invariance of the integrand under $w \mapsto 1-w$ remains.

In the second summand,

$$\Lambda(1-w,E_\alpha\times E_\beta)\cdot \Phi E_w \cdot \frac{1}{4\pi i}\int_{|t|\geq T}\frac{1}{\lambda_s-\lambda_w}\,ds$$

the leading coefficient $\Lambda(1-w,E_\alpha\times E_\beta)\cdot\Phi E_w$ has meromorphic continuation and is invariant under $w \mapsto 1-w$. Since $|w| \ll T$ the meromorphic continuation of the integrand is given by the same integral and the invariance under $w \mapsto 1-w$ remains.

Finally, in the remaining summand,

$$\frac{1}{4\pi i}\int_{|t|\leq T}\frac{\Lambda(1-s,E_\alpha\times E_\beta)\cdot\Phi E_s - \Lambda(1-w,E_\alpha\times E_\beta)\cdot\Phi E_w}{\lambda_s-\lambda_w}\,ds$$

is a compactly-supported vector-valued integral. In order to show that the integral is a meromorphic $N$-valued function of $w$, we will use the Gelfand-Pettis criterion for existence of a weak integral.

Let $\mathrm{Hol}(\Omega, N)$ be the topological vector space of holomorphic $N$-valued functions on a fixed open $\Omega$ which avoids the poles if $E_w$ and has compact closure $C$. It suffices to show that the integrand extends to a continuous $\mathrm{Hol}(\Omega,N)$-valued function of $s$ where $\mathrm{Hol}(\Omega,N)$ has the natural quasi-complete locally convex topology from Corollary **??**. To show that the integral extends to a holomorphic (and hence continuous) $\mathrm{Hol}(\Omega,N)$-valued function of $s$, it suffices to show that the integral extends to a holomorphic $N$-valued function of two complex variables $s$ and $w$.

By Cauchy-Goursat theory for vector-valued holomorphic functions (see Appendix), near a point $s_o$, the $N$-valued function $s \mapsto \Phi E_s$ has a convergent power series expansion

$$\Phi E_s = A_0 + A_1(s - s_o) + A_2(s - s_o)^2 + \dots$$

with $A_i \in N$ and so $\Lambda(1-s, E_\alpha \times E_\beta) \cdot \Phi E_s$ has power series expansion

$$\Lambda(1-s, E_\alpha \times E_\beta) \cdot \Phi E_s = B_0 + B_1(s - s_o) + B_2(s - s_o)^2 + \dots$$

for some $B_n \in N$. Then we have

$$\Lambda(1-s, E_\alpha \times E_\beta) \cdot \Phi E_s - \Lambda(1-w, E_\alpha \times E_\beta) \cdot \Phi E_w$$

$$= B_1((s - s_o) - (w - s_o)) + B_2((s - s_o)^2 - (w - s_o)^2) + \dots$$

$$= ((s - s_o) - (w - s_o)) \cdot (B_1 + B_2((s - s_o) - (w - s_o)) + \dots)$$

$$= (s - w) \cdot (B_1 + B_2((s - s_o) - (w - s_o)) + \dots)$$

where $(B_1 + B_2((s - s_o) - (w - s_o)) + \dots)$ is a convergent power series in $s - s_o$ and $w - s_o$. Thus the integrand, initially defined only for $s \neq w$ extends to a holomorphic $N$-valued function $F(s, w)$ including the diagonal $s = w = \frac{1}{2} + it$ with $|t| \leq T$. Thus the $\mathrm{Hol}(\Omega, N)$-valued function $f(s)$ given by $f(s)(w) = F(s, w)$ is holomorphic in $w$. Thus there is a Gelfand-Pettis integral $\int_{|t| \leq T} f(\frac{1}{2} + it)\, dt$ in $\mathrm{Hol}(\Omega, N)$ as desired. Thus we have shown the meromorphic continuation. The $w \mapsto 1 - w$ symmetry is retained by the extension of the integral to the diagonal.

□

# Chapter 7

# Future Directions

I will expand upon these reults by computing the coefficients of the scattering amplitude of supergravitons on a broader class of domains. Different target spacetimes present varying domains on which to solve the differential equation (**??**). For a target spacetime $M = \mathbb{R}^{1,9-d} \times T^d$, where $T$ is a torus, we get moduli spaces by attaching an additional $\mathbb{C}$-scalar field to $M$. These moduli spaces are of the form $G(\mathbb{Z})\backslash G(\mathbb{R})/K$ for the duality groups of maximal supergravity in $D = 10 - d \leq 10$ dimensions as in Table **??** [**?**]. My prior work covers the cases of $D = 10$ and $D = 9$, but it remains to prove even the existence of a solution for smaller $D$.

To approach the subtleties of this domain I will first strengthen the machinery so that it can be used to solve differential equations on higher rank groups beginning with $GL(3)$ and $GL(4)$. Luckily, the application to the scattering amplitude that we are considering is really only concerned with solutions to (**??**) where $E_\alpha$ and $E_\beta$ are *maximal* parabolic Eisenstein series with degenerate data. As the rank of the group $G$ increases, this will cut down drastically on the amount of cases that need to be considered.

## 7.1 $GL(2)$ over number fields and congruence conditions

Before moving on to the more complicated cases of higher rank groups, I should note that there are other cases involving $GL(2)$ that we could examine in more detail. Specifically, we could examine the case of $G = SL_2(k)$ where $k$ is a number field or choosing the discrete subgroup of $G = SL_2(\mathbb{R})$ to be a congruence subgroup of $\Gamma = SL_2(\mathbb{Z})$ (as opposed to Γ itself). At present, there is no immediate tie to the physics application in either of these two cases.

Furthermore, the computation and results found in this work will vary little in either of these cases. In the case of $G = SL_2(k)$, this will introduce a summation over the central characters. When looking at different congruence subgroups of $SL_2(\mathbb{Z})$, the spectral solution will only change by the need to sum over the different cusps that are introduced. The scattering matrix for the Eisenstein series will also have a higher dimension. The subtleties introduced by these two cases are something we could address in more detail; however, the essential analytic issues have already been addressed in the case of $SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})/SO(2)$.

## 7.2 $GL(3)$ case

In the $GL(2)$ case there is just one parabolic subgroup and there are no Eisenstein series made from cuspidal data because $GL(2)$ is too small. The first thing to note in the case of $GL(3)$ is that there are now many types of Eisenstein series associated to the different parabolic subgroups.

While functions on $GL(3)$ have a spectral expansion similar to those on $GL(2)$, this case will inform our approach for $E_8$ in that there are three types (instead of one) of Eisenstein series that occur depending on the parabolic subgroup chosen. Furthermore, examining $GL(3)$ will give us a solution in the case of $D = 8$ above. The main issue to address is that the root space for $SL(3)$ contains two distinct positive roots and thus, the positive Weyl chamber will not become compact when truncating with respect to only one root. S. Miller specified the proper form of Arthur truncation on $SL(3)$ in his PhD thesis so he will be very helpful in this regard [**?**]. The higher rank analogues of the Maaß-Selberg relations will provide helpful insight into which type of truncation may leave computation of the spectral integrals manageable.

The application to physics which we are are considering is mainly concerned with *maximal* parabolic Eisenstein series with maximally degenerate data. In this case there are only two maximal parabolic Eisenstein series and their parabolics are associate to one another, $P^{2,1}$ and $P^{1,2}$. For $f_1$ and $f_2$ cuspforms on $GL(1)$ and $GL(2)$ respectively (of course there are no cuspforms on $GL(1)$ so this condition is vacuous), their corresponding

Eisenstein series will be of the form

$$E^{2,1}_{s,f_2\otimes f_1}(znmk) = \sum_{\gamma\in P^{2,1}\backslash G} \left|\frac{\det(m_2)^2}{\det(m_1)^1}\right|^s \cdot f_2(m_2)$$

and

$$E^{1,2}_{s,f_1\otimes f_2}(znmk) = \sum_{\gamma\in P^{1,2}\backslash G} \left|\frac{\det(m_1)^1}{\det(m_2)^2}\right|^s \cdot f_2(m_2).$$

Recall from Theorem **??** in the Appendix that $c_Q E^P_{s,f} = 0$ unless $Q = P$ or $Q$ is the associate of $P$. From Theorem **??**, we have

$$c_{2,1}E^{2,1}_{s,f_2\otimes f_1} = \varphi^{2,1}_{s,f_2\otimes f_1} = \left|\tfrac{\det(m_2)^2}{\det(m_1)^1}\right|^s \cdot f_2(m_2)f_1(m_1)$$

$$c_{1,2}E^{2,1}_{s,f_2\otimes f_1} = c^{1,2}_{s,f_2\otimes f_1}\varphi^{1,2}_{1-s,f_1\otimes f_2} = c^{1,2}_{s,f_2\otimes f_1}\left|\tfrac{\det(m_1)^1}{\det(m_2)^2}\right|^{1-s} \cdot f_1(m_1)f_2(m_2)$$

$$c_{1,2}E^{1,2}_{s,f_1\otimes f_2} = \varphi^{1,2}_{s,f_1\otimes f_2} = \left|\tfrac{\det(m_1)^1}{\det(m_2)^2}\right|^s \cdot f_1(m_1)f_2(m_2)$$

$$c_{2,1}E^{1,2}_{s,f_1\otimes f_2} = c^{2,1}_{s,f_1\otimes f_2}\varphi^{2,1}_{1-s,f_2\otimes f_1} = c^{2,1}_{s,f_1\otimes f_2}\left|\tfrac{\det(m_2)^2}{\det(m_1)^1}\right|^{1-s} \cdot f_2(m_2)f_1(m_1)$$

Observe that again the constant terms of these Eisenstein series do not allow them to be in $L^2$. We will employ the same trick of subtracting a linear combination of Eisenstein series (depending on the value of the real part parameter $s$) to construct some $\widetilde{S} \in L^2$ in order to decompose it spectrally.

As in the previous case of $SL(2)$, in order to compute the spectral integrals of this new $\widetilde{S}$, we will need to use truncate these Eisenstein series. However, there is no longer one notion of height that we need to consider. S. Miller's choice of truncation in [**?**] will not only serve well on $SL(3)$ but will also work well as a generalization for other higher rank groups.

### 7.2.1 Truncation

In order to compute the spectral integrals we will need the appropriate notion of truncation of $SL(3)$. For the parabolic subgroups of $SL(3)$ we have the following truncations for $T \in \mathbb{R}$.

Recall that for $\delta^P$ the modular function of $P_{\mathbb{A}}$

$$\delta\left(\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\right) = \left|\frac{(\det m_1)^{r_1}}{(\det m_2)^{r_2}}\right|$$

we can extend this to a *height function* aligned with $P$ by making it right $K_{\mathbb{A}}$-invariant

$$h^P(nmk) = \delta^P(nm) = \delta^P(m)$$

for $n \in N_{\mathbb{A}}^P$, $m \in M_{\mathbb{A}}^P$ and $k \in K_{\mathbb{A}}$. Furthermore, for fixed large real $T$, the $T$-tail of the $P$-constant term of an automorphic form $F$ is

$$c_P^T F(g) = \begin{cases} c_P F(g) & \text{for } h^P(g) \geq T \\ 0 & \text{for } h^P(g) < T \end{cases}$$

Then for $P = P^{2,1}$,

$$c_{2,1}^T F(g) = \begin{cases} c_{2,1} F(g) & \text{for } h^{2,1}(g) = \left|\frac{(\det m_1)^2}{(\det m_2)^1}\right| \geq T \\ 0 & \text{for } h^{2,1}(g) = \left|\frac{(\det m_1)^2}{(\det m_2)^1}\right| < T \end{cases}$$

and for $P = P^{1,2}$,

$$c_{1,2}^T F(g) = \begin{cases} c_{2,1} F(g) & \text{for } h^{1,2}(g) = \left|\frac{(\det m_1)^1}{(\det m_2)^2}\right| \geq T \\ 0 & \text{for } h^{1,2}(g) = \left|\frac{(\det m_1)^1}{(\det m_2)^2}\right| < T \end{cases}$$

Thus in the case the $SL(3)$ we have the following truncated Eisenstein series for $T \in \mathbb{R}$:

$$\wedge^T E_{s,f}^{2,1} = E_{s,f}^{2,1} - \Psi^{2,1}(c_{2,1}^T E_{s,f}^{2,1}) - \Psi^{1,2}(c_{1,2}^T E_{s,f}^{2,1})$$

$$\wedge^T E_{s,f}^{1,2} = E_{s,f}^{1,2} - \Psi^{1,2}(c_{1,2}^T E_{s,f}^{1,2}) - \Psi^{2,1}(c_{2,1}^T E_{s,f}^{1,2})$$

where $\Psi^P(\varphi) = \Psi_\varphi^P$ for the pseudo-Eisenstein series attached to the data $\varpi$ so that these truncations are automorphic. Then for $\left|\frac{(\det m_1)^2}{(\det m_2)^1}\right| \geq T$ and $\left|\frac{(\det m_1)^1}{(\det m_2)^2}\right| \geq T$

$$\begin{aligned}
\wedge^T E^{2,1}_{s,f} &= E^{2,1}_{s,f} - \Psi^{2,1}(c_{2,1}E^{2,1}_{s,f}) - \Psi^{1,2}(c_{1,2}E^{2,1}_{s,f}) \\
&= E^{2,1}_{s,f} - \Psi^{2,1}\left(\left|\frac{\det(m_2)^2}{\det(m_1)^1}\right|^s \cdot f_2(m_2)f_1(m_1)\right) \\
&\qquad - \Psi^{1,2}\left(c^{1,2}_{s,f_2\otimes f_1}\left|\frac{\det(m_1)^1}{\det(m_2)^2}\right|^{1-s} \cdot f_1(m_1)f_2(m_2)\right) \\
= E^{2,1}_{s,f} &- \sum_{\gamma\in P^{2,1}_k\backslash G_k}\left(\left|\frac{\det(\gamma m_2)^2}{\det(\gamma m_1)^1}\right|^s \cdot f_2(\gamma m_2)f_1(\gamma m_1)\right) \\
&- \sum_{\gamma\in P^{1,2}_k\backslash G_k}\left(c^{1,2}_{s,f_2\otimes f_1}\left|\frac{\det(\gamma m_1)^1}{\det(\gamma m_2)^2}\right|^{1-s} \cdot f_1(\gamma m_1)f_2(\gamma m_2)\right)
\end{aligned}$$

and

$$\begin{aligned}
\wedge^T E^{1,2}_{s,f} &= E^{1,2}_{s,f} - \Psi^{1,2}(c_{1,2}E^{1,2}_{s,f}) - \Psi^{2,1}(c_{2,1}E^{1,2}_{s,f}) \\
&= E^{1,2}_{s,f} - \Psi^{1,2}\left(\left|\frac{\det(m_1)^1}{\det(m_2)^2}\right|^s \cdot f_1(m_1)f_2(m_2)\right) \\
&\qquad - \Psi^{2,1}\left(c^{2,1}_{s,f_1\otimes f_2}\left|\frac{\det(m_2)^2}{\det(m_1)^1}\right|^{1-s} \cdot f_2(m_2)f_1(m_1)\right) \\
= E^{1,2}_{s,f} &- \sum_{\gamma\in P^{1,2}_k\backslash G_k}\left(\left|\frac{\det(\gamma m_1)^1}{\det(\gamma m_2)^2}\right|^s \cdot f_1(\gamma m_1)f_2(\gamma m_2)\right) \\
&- \sum_{\gamma\in P^{2,1}_k\backslash G_k}\left(c^{2,1}_{s,f_1\otimes f_2}\left|\frac{\det(\gamma m_2)^2}{\det(\gamma m_1)^1}\right|^{1-s} \cdot f_2(\gamma m_2)f_1(\gamma m_1)\right).
\end{aligned}$$

For the purposes of the higher rank Lie groups with no simple matrix model (i.e. $E_8$) it will be helpful to have a characterization of truncation in terms of the roots. The following is adapted from S. Miller's dissertation [?]. Let $A$ be the connected center of $M$ for $P = NM$ and $\mathfrak{a}$ be its Lie algebra. Let $\mathfrak{a}_0$ be the $\mathfrak{a}$ associated with the minimal parabolic $P^{min}$. In the case of the rank 2 group $SL(3)$, there are two positive simple roots $\sigma = \{\alpha_1, \alpha_2\} \subset \mathfrak{a}^*$,

$$\alpha_1\left(\begin{bmatrix} h_1 & 0 & 0\\ 0 & h_1 & 0\\ 0 & 0 & h_3\end{bmatrix}\right) = h_1 - h_2 \qquad \text{and} \qquad \alpha_2\left(\begin{bmatrix} h_1 & 0 & 0\\ 0 & h_1 & 0\\ 0 & 0 & h_3\end{bmatrix}\right) = h_2 - h_3.$$

There is a third positive root $\alpha_3 = \alpha_1 + \alpha_2$. Let $c > 0$ be large and set

$$C := c\,(1, 0, -1) \in \mathfrak{a}_0.$$

define

$$\mathcal{F}_C = \{x \in \mathcal{F} \mid (2\alpha_1 + \alpha_2)(H(x) - C), (\alpha_1 + 2\alpha_2)(H(x) - C) \leq 0\}$$

for $H(x) = (\log|\det(m_1)|, \log|\det(m_2)|, \log|\det(m_3)|)$.

Recall that the coroots $\{\alpha_1^\vee, \alpha_2^\vee\}$ form a basis of $\mathfrak{a}_0$. Let $\hat{\tau}_P(x)$ be the characteristic function of $\{x = c_s\alpha_1^\vee + c_2\alpha_2^\vee \in \mathfrak{a}_0 \mid c_i > 0, \forall \alpha_i \in \Delta_P\}$. Thus

$$\hat{\tau}_{min}(x) = \{x = c_s\alpha_1^\vee + c_2\alpha_2^\vee \in \mathfrak{a}_0 \mid c_, c_2 > 0\}$$

$$\hat{\tau}_{2,1}(x) = \{x = c_s\alpha_1^\vee + c_2\alpha_2^\vee \in \mathfrak{a}_0 \mid c_2 > 0\}$$

$$\hat{\tau}_{1,2}(x) = \{x = c_s\alpha_1^\vee + c_2\alpha_2^\vee \in \mathfrak{a}_0 \mid c_1 > 0\}$$

The truncation of an automorphic from $\psi$ is a sum over all parabolic subgroups

$$\wedge^C\psi(x) := \sum_P (-1)^{\dim A} \sum_{\gamma\in\Gamma\cap P\backslash\Gamma} \hat{\tau}_P\left(H(\gamma x) - C\right) \int_{\Gamma\cap N\backslash N} \psi(n\gamma x)\, dn$$

$$= \sum_P (-1)^{\dim A} \sum_{\gamma\in\Gamma\cap P\backslash\Gamma} \hat{\tau}_P\left(H(\gamma x) - C\right) c_P\psi(x)$$

### 7.2.2 Conjecture

**Conjecture 26.** *In $Re(w) > 1/2$, for $P, Q$ parabolic subgroups of $SL_3(\mathbb{R})$,*

*$(\Delta-\lambda)u = E^P_\alpha \cdot E^Q_\beta$ on $SL_3(\mathbb{Z})\backslash SL_3(\mathbb{R})/SO(3)$ for $E^P_\alpha, E^Q_\beta$ are cuspidal data Eisenstein series has a unique solution in $H^{-\infty} \oplus \mathcal{E}$ of the form*

$$u_w = \sum_{i,P} \frac{c_i E^P_{s_i}}{\lambda_{s_i} - \lambda_w} + \frac{C}{\lambda_1 - \lambda_w} + \sum_{f\ cfm} \frac{\Lambda(\alpha, \overline{f} \times E^Q_\beta)\cdot f}{\lambda_{s_f} - \lambda_w} + \frac{1}{4\pi i}\int_{\rho + i\mathfrak{a}^*} \Lambda(\overline{s}, E^P_\alpha \times E^Q_\beta)\cdot \frac{E^{min}_s}{\lambda_s - \lambda_w}\, ds$$

$$+\frac{1}{4\pi i}\sum_{F\ cfm\ on\ M}\int_{(1/2)} \Lambda(\overline{s}, E^P_\alpha \times E^Q_\beta)\cdot \frac{E^{2,1}_{s,F}}{\lambda_s - \lambda_w}\, ds$$

## 7.3 $GL(4)$ case

When considering the case of $GL(4)$, in addition to addressing the issue of truncation, new features with the spectral expansion appear since in the spectral expansion for $GL(4)$, Speh forms are present. However, recent work of Lapid and Mao [**?**] on Rankin-Selberg integrals for Speh representations will provide insight on these computations.

## 7.4 $E_8$ case

Ultimately, $E_8$ will provide its own obstacles. A Chevalley basis for $E_8$ contains matrices of size $248 \times 248$. This means finding an explicit solution will be computationally-intensive. I will use spectral methods to demonstrate the existence of a solution. This involves finding an appropriate truncation to compute spectral integrals. Examining the cases of GL(3) and GL(4) will provide insight on the form of these truncations. Pioline's theta functions constructions for Spin$(5, 5)$ may also provide insight on our approach [**?**]. Given the level of complexity of the exceptional groups, it will be important to develop an efficient notation and description of what is needed to express the existence of a solution in this case.

# Chapter 8

# Appendix

## 8.1 Parabolic subgroups of $GL(r)$

Parabolic subgroups play an important role in the definition of Eisenstein series. For $k$ an arbitrary field $G = GL_r(k)$ acts on $k^r$ by matrix multiplication. A *flag* in $k^r$ is a proper nested sequence of one or more non-zero $k$-subspaces $V_1 \subset \cdots \subset V_\ell \subset k^r$. The corresponding parabolic subgroup $P$ is the stabilizer of the flag $F$. The whole group $G$ stabilizes the improper flag $k^r$ so is itself a parabolic subgroup. The *proper* parabolics are the stabilizers of flags $V^1 \subset \dot{V}_\ell \subset k^r$ with $\ell \geq 1$.

The *maximal* parabolic subgroups are the stabilizers of $P^{V \subset k^r}$ of flags consisting of single proper subspaces $V \subset k^r$. Every proper parabolic subgroup $P^F$ for a flag $F = (V_1 \subset \cdots \subset V_\ell \subset k^r)$ is the itersecection of the maximal proper parabolics $P^{V_i \subset k^r}$. A *minimal* parabolic (stabilizing a maximal flag) is a *Borel* subgroup.

With $e_1, e_2, \ldots, e_r$ the standard basis for $k^r$ identify $k^d = ke_e + \ldots ke_d$. Because $G$ is transitive on ordered bases of $k^r$, every orbit in the action of $G$ on flags has a unique representative among standard flags. In other words, for some ordered partition $d_1 + g_2 + \cdots + d_\ell = r$ with $0 < d_j \in \mathbb{Z}$, the corresponding flag is

$$F^{d_1,\ldots,d_\ell} = (k^{d_1} \subset k^{d_1+d_2} \subset k^{d_1+d_2+d_3} \subset \cdots \subset k^{d_1+\cdots+d_\ell})$$

The stabilizer of $F^{d_1,\ldots,d_\ell}$ is the standard (proper) parabolic subgroup $P^{d_1,\ldots,d_\ell}$ of $G$ and is the intersection of the maximal proper parabolics containing it, i.e.

$$P^{d_1,\ldots,d_\ell} = \bigcap_{1\leq i\leq \ell-1} P^{(d_1+\cdots+d_i),(d_{i+1}+\cdots+d_\ell)}.$$

Distinguishing types of Eisenstein series is, in part, done by the type of parabolic subgroup with which they are associated. We say two standard parabolics $P^{d_1,\ldots,d_\ell}$ and $P^{d'_1,\ldots,d'_{\ell'}}$ are *associate* when $\ell = \ell'$ and $d_1,\ldots,d_\ell$ and $d'_1,\ldots,d_{\ell'}$ differ only by permutation. A parabolic $P^{d_1,\ldots,d_\ell}$ is *self-associate* if $d_i = d_j$ for some $i \neq j$.

The standard maximal parabolic subgroups are block upper diagonal of the form

$$P^{r',r-r'} = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \ : \ s \in GL(r'), b \in M_{r'\times(r-r'))}, d \in GL(r-r') \right\}.$$

The next-to-maximal proper parabolics have the form

$$P^{r_1,r_2,r_3} = \left\{ \begin{bmatrix} m_1 & * & * \\ 0 & m_2 & * \\ 0 & 0 & m_3 \end{bmatrix} \ : \ m_1 \in GL(r_1), m_2 \in GL(r_2), m_3 \in GL(r_3) \right\}$$

for $r_1 + r_2 + r_3 = r$. The standard proper parabolic $P^{d-1,\ldots,d_\ell}$ consists of block-upper-triangular matrices with diagonal blocks of sizes $d_1\times d_1, d_2\times d_2, \ldots, d_\ell\times d_\ell$. The standard *Borel subgroup* (i.e. minimal parabolic) is the subgroup of upper triangular matrices.

The *unipotent radical* $N^P$ of a parabolic subgroup $P$ stabilizing a flag $F = (V_1 \subset \cdots \subset V_\ell \subset k^r)$ is the subgroup tat fixes quotients $V_\ell/V_{\ell-1}$ pointwise. The characterization shows that $N^P$ is a normal subgroup of $P$. For the standard maximal parabolic, $P = P^{r',r-r'}$ the unipotent radical is

$$N = N^P = N^{r',r-r'} = \left\{ \begin{bmatrix} 1_{r'} & b \\ 0 & 1_{r-r'} \end{bmatrix} \ : \ b = r' \times (r-r') \right\}.$$

Note that containment of the parabolics reverses the containment of the unipotent radicals (i.e. $P \subset Q$ implies $N^P \supset N^Q$). Explicitly the next-to-maximal standard parabolic

$P = P^{r_1,r_2,r_3}$, the unipotent radical is

$$N = N^P = N^{r_1,r_2,r_3} = \left\{ \begin{bmatrix} 1_{r_1} & * & * \\ 0 & 1_{r_2} & * \\ 0 & 0 & 1_{r_3} \end{bmatrix} \right\}.$$

The *standard Levi component* (or Levi-Malcev component) $M = M^P = M^{d_1,\ldots,d_\ell}$ of the standard parabolic $P = P^{d_1,\ldots,d_\ell}$ is the subgroup of $P = P^{d_1,\ldots,d_\ell}$ with the blocks *above* the diagonal 0, ie.

$$M = M^P + M^{d_1,\ldots,d_\ell} = \left\{ \begin{bmatrix} m_1 & 0 & 0 & \ldots & 0 \\ 0 & m_2 & 0 & \ldots & 0 \\ \vdots & 0 & \ddots & & \vdots \\ & & & \ddots & 0 \\ 0 & \ldots & & 0 & m_\ell \end{bmatrix} : m_j \in GLd_j \right\}.$$

The Levi component is *not* normal in the parabolic. However, we do have the Levi-Malcev decomposition

$$P = N^P \cdot M^P.$$

The standard *Weyl group* $W$ can be identified with permutation matrices in $G$, i.e. matrices with exactly one non-zero entry in each row and column where that nonzero entry is 1. The Weyl group normalizes $P^{\min}$ and we have the following *Bruhat decompositions*:

**Theorem 27.** *With $P^{min}$ the standard minimal parabolic and $N^{min}$ its unpotent radical, we have a disjoint union*

$$GL_r(k) = \bigsqcup_{w \in W} P^{min} w P^{min} = \bigsqcup_{w \in W} P^{min} w N^{min}.$$

See Section 3.1 of [?] for proof.

**Corollary 28.** $G = \bigcup_{w\in W} PwQ$ *for any standard parabolics $P$ and $Q$.*

## 8.2 Siegel Sets

Recall that any semisimple Lie group $G$ can be written via the Iwasawa decomposition as

$$G = KAN$$

where $K$, $A$ and $N$ are the Lie subgroups of $G$ generated by $\mathfrak{k}_0$ (for Cartan decomposition $\mathfrak{g}_0 = \mathfrak{p}_0 \oplus \mathfrak{k}_0$), $\mathfrak{a}_0$ (maximal abelian subalgebra), and $\mathfrak{n}_0$ (the nilpotent Lie algebra given as the sum of root spaces of positive roots $\Sigma^+$).

On $SL(2)$, matrices $a \in A$ can be written as $a = \begin{bmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{bmatrix}$ and for $C \subset N$ a *standard Siegel set* is a subset of $G$ of the form

$$\mathfrak{S}_{t,C} := \{na_y k \mid n \in C, k \in K, y \geq t\}.$$

The notion of a standard Siegel set becomes more complicated on $GL(r)$. There is no longer a notion of a single numerical height but rather a family. The standard positive simple roots are characters on $M = M^P$ for $P = P^{\min} = P^{1,\ldots,1}$ the standard minimal parabolic by the action

$$\alpha_i\left(\begin{bmatrix} m_1 & & \\ & \ddots & \\ & & m_r \end{bmatrix}\right) = \frac{m_i}{m_{i+1}}$$

for $1 \leq i < r$. Then the standard *Siegel set aligned with* $P = P^{min}$ is a set of the form

$$\mathfrak{S}^P = \mathfrak{S}^P_{t,C} := \{nmk \mid n \in C, m \in M_{\mathbb{A}}, k \in K_{\mathbb{A}}, \text{ and } |a_i(m)| \geq t \text{ for all } 1 \leq i < r\}$$

with idele norm $|\cdot|$, for $0 < t \in \mathbb{R}$ and compact $C \subset N^P_{\mathbb{A}}$.

**Theorem 29.** *For given $k$, there is a $y > 0$ and a compact group $C \subset N^{min}_{\mathbb{A}}$ such that $G_k \cdot \mathfrak{S}^P_{t,C} = G_{\mathbb{A}}$. That is $G_k \backslash G_{\mathbb{A}}$ is covered by a single, sufficiently large Siegel set.*

See Section 3.3 of [?] for proof.

## 8.3 Eisenstein series on $GL(r)$

### 8.3.1 Minimal-parabolic Eisenstein series on $GL(r)$

Let $P = NM$ be the minimal parabolic subgroup of $G_{\mathbb{A}} = SL_r(\mathbb{A})$ with $N$ the unipotent radical and $M$ the standard Levi-Malcev component. Let

$$M^1 = \left\{ \begin{bmatrix} m_1 & & \\ & \ddots & \\ & & m_r \end{bmatrix} \;:\; m_1, \ldots, m_r \in \mathbb{J}, |m_1| = 1, \ldots, |m_r| = 1 \right\}.$$

For a pseudo-Eisenstein series $\Psi^P_\varphi(g) = \sum_{\gamma \in P_k \backslash G_k} \varphi(\gamma \cdot g)$ for $\varphi \in C^\infty_c(Z^+ N_{\mathbb{A}} M_k \backslash G_{\mathbb{A}})$, such $\varphi$ admit decompositions in $L^2(Z^+ N_{\mathbb{A}} M_k \backslash G_{\mathbb{A}})$ by characters $\chi$ of the compact abelian group $M_k \backslash M^1$ acting on the left.

Let $\delta$ map $(0, \infty)$ to the archimedean factors of $\mathbb{J}$ so that $|\delta(t)| = t$ and describe *Hecke characters* $\widetilde{\chi}$ as

$$\widetilde{\chi}(\delta(t) \cdot t_1) = t^s \chi(t_1)$$

for $t > 0$, $t_1 \in \mathbb{J}$ and $s \in$ c. Given an $r$-tuple of Hecke characters $\widetilde{\chi}_1, \ldots, \widetilde{\chi}_r$ with relation $s_1 + \cdots + s_r = 0$, the *minimal-parabolic Eisenstein series* on $GL(r)$ is

$$E^{min}_{s,\chi}(g) = \sum_{\gamma \in P_k \backslash G_k} \varphi^o_{s,\chi}(\gamma \cdot g)$$

where $\varphi_{s,\chi}(nmk) = \widetilde{\chi}_1(m_1) \cdot \cdots \cdot \widetilde{\chi}_r(m_r)$ for $n \in N^{min}_{\mathbb{A}}, m = \begin{bmatrix} m_1 & & \\ & \ddots & \\ & & m_r \end{bmatrix}$ and $k \in K_{\mathbb{A}}$.

**Theorem 30.** *The minimal parabolic Eisenstein series $E^{min}_{s,\chi}(g)$ on $GL(r)$ converges (absolutely and uniformly for $g$ in compacts) for $\frac{\sigma_j - \sigma_{j+1}}{2} > 1$ for $j = 1, \ldots, r-1$ where $s = (s_1, \ldots, s_r) \in$ c$^r$ and $\sigma = (Re(s_1), \ldots, Re(s_r))$.*

(For a proof of this result see Chapter 3 of [**?**].)

For Lie groups, it is most natural to be able to express Eisenstein series in terms of root spaces. this is done in the following manner. For Hecke characters of the simplest form $\widetilde{\chi}_i(\delta(t) \cdot t_1) = t^{s_i}$, we have

$$\varphi_s^o(nmk) = \varphi_{s,1}^o(nmk) = |m_1|^{s_1}|m_2|^{s_2} \dots |m_r|^{s_r}.$$

In order words, in terms of the parameter $s$, $E_{s,\chi}^{min}$ is a function-valued function of $r-1$ complex variables, but the parameter space is the complex hyperplane $s_1 + \cdots + s_r = 0$ in $c^r$, rather than $c^{r-1}$. In terms of simple roots $\alpha_i(m) = \frac{m_i}{m_{i+1}}$, so using the fact that $s_1 + \cdots + s_r = 0$, we have

$$\varphi_{s,\chi}^o(nmk) = |\alpha_1(m)|^{s_1}|\alpha_2(m)|^{s_1+s_2} \dots |\alpha_r(m)|^{s_1+\cdots+s_{r-1}-1}.$$

Let $\mathfrak{gl}_r(\mathbb{R})$ be the lie algebra of $GL_r(\mathbb{R})$, that is, all $r \times r$ real matrices. Let $\mathfrak{a}$ be the Lie algebra of diagonal matrices in $GL_r(\mathbb{R})$. The nonzero eigenvalues (roots) of $\mathfrak{a}$ on $\mathfrak{gl}_r(\mathbb{R})$ are functionals $a \mapsto a_i - a_j$ in the dual space $\mathfrak{a}^*$. For $i \neq j$, the corresponding roots and (rootspaces) are those with $i < j$. Write $\beta > 0$ for a positive root $\beta$ and $\beta < 0$ when $-\beta > 0$. The standard simple positive roots are $a \mapsto a_i - a_{i-1}$. The *half-sum* of positive roots is

$$\rho(a) = \frac{1}{2}\sum_{i<j}(a_i - a_j)$$

for $a \in \mathfrak{a}$. Define a type of logarithm map $M_{\mathbb{A}} \to \mathfrak{a}$ by

$$\log \left| \begin{bmatrix} m_1 & & \\ & \ddots & \\ & & m_r \end{bmatrix} \right| = \begin{bmatrix} \log|m_1| & & \\ & \ddots & \\ & & \log|m_r| \end{bmatrix}$$

and for $m \in M_{\mathbb{A}}$ and $\alpha \in \mathfrak{a}^*$ write

$$m^\alpha = e^{\alpha(\log|m|)}.$$

This context allows interpretation of the parameter $s$ as an element of the complexification $\mathfrak{a}^* \otimes_{\mathbb{R}} c$ of the dual $\mathfrak{a}^*$ of $\mathfrak{a}$. Furthermore, using $\langle x, y \rangle = tr(xy)$ on $\mathfrak{a}$, we can identify $\mathfrak{a}$ with $\mathfrak{a}^*$ and tranport $\mathfrak{a}^*$ to the pairing $\langle , \rangle$.

Thus Theorem **??** can be reframed as the following:

**Corollary 31.** *The minimal parabolic Eisenstein series $E_{s,\chi}^{min}(g)$ on $GL(r)$ converges* *(absolutely and uniformly for $g$ in compacts) for $\langle \alpha, \sigma - 2\rho \rangle > 0$ for all positive simple roots $\alpha$.*

In other words, the Eisenstein series $E^{min}_{s,\chi}(g)$ converges absolutely for $\sigma \in \mathfrak{a}$ in the translate by $2\rho$ of the positive Weyl chamber:

$$\text{positive Weyl chamber } = \{x \in \mathfrak{a}^* \mid \langle x, \alpha \rangle > 0 \text{ for all positive roots } \alpha\} \subset \mathfrak{a}^*$$

**Theorem 32.** *For $P$ the minimal parabolic subgroup of $GL(r)$, in the region of convergence, for suitable holomorphic functions $s \mapsto c_{w,s}$ with $c_{1,s} = 1$ the constant term is*

$$c_P E^P_{\rho+s}(m) = m^{\rho+s} + \sum_{1 \neq w \in W} c_{w,s} m^{\rho + w \cdot s}$$

*for $W$ the Weyl group.*

**Corollary 33.** *For reflections $\tau$, $c_{\tau,s} = \dfrac{\xi\langle s, \alpha\rangle}{\xi(1 + \langle s, \alpha \rangle)}$ and the cocycle relation $c_{w', w\cdot s} \cdot c_{w,s} = c_{ww',s}$ holds for $w, w' \in W$ and $s \in \mathfrak{a}^* \otimes_{\mathbb{R}} \mathrm{c}$. We have*

$$c_{w,s} = \prod_{\beta > 0 \ ; \ w \cdot \beta < 0} \frac{\xi\langle s, \beta \rangle}{\xi(1 + \langle s, \beta \rangle)}.$$

(For proofs of the Theorem and Corollary, see [?] Chapter 3.)

### 8.3.2 Maximal-parabolic Eisenstein series on $GL(r)$

The general case of a cuspidal-data Eisenstein series is a combination of the features of the minimal-parabolic and maximal-parabolic so we will only discuss these two cases in detail.

Let $f_1, f_2$ be cuspforms on $GL_{r_1}(\mathbb{A})$ and $GL_{r_2}(\mathbb{A})$ respectively, right-invariant by the standard maximal compacts, with trivial central characters. We will require that $f_1$ and $f_2$ are cuspforms in the *strong sense* meaning that they are eigenfunctions for all spherical Hecke algebras (including at the archimedean places) in addition to satisfying the Gelfand condition on vanishing constant terms. Note that they will also be eigenfunctions for the invariant Laplacians.

Cuspforms in this strong sense are of rapid decay (see [?] Chapter 7.3). The cuspidal data $f_1 \otimes f_2$ is a function on $GL_{r_1}(\mathbb{A}) \times GL_{r_2}(\mathbb{A}) \cong M_{\mathbb{A}}^P$. In the case where $r_1 = 1$ or $r_2 = 1$ , the situation degenerates somewhat: there is no corresponding cuspform $f_j$ (i.e. it is identically 1).

Let

$$\varphi(znmk) = \varphi_{s,f}(znmk) = \left|\frac{(\det\, m_1)^{r_2}}{(\det\, m_2)^{r_1}}\right|^s \cdot f_1(m_1) \cdot f_2(m_2)$$

where $m = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \in M_{\mathbb{A}}^P$ $z \in Z^+$, $n \in N_{\mathbb{A}}$, and $k \in K_{\mathbb{A}}$. The exponents on the idele norms of determinants make $\varphi$ invariant under $Z_{\mathbb{A}}$. The corresponding *cuspidal-data Eisenstein series* is

$$E_{s,f}(g) = \sum_{\gamma \in P_k \backslash G_k} \varphi_{s,f}(\gamma \cdot g).$$

When $\varphi_{s,f}$ is replaced by $\varphi_s(nmk) = \left|\frac{(\det\, m_1)^{r_2}}{(\det\, m_2)^{r_1}}\right|^s$, the sum $E_s(g) = \sum_{\gamma \in P_k \backslash G_k} \varphi_s(\gamma \cdot g)$ dominates that for $E_{s,f}$. This $E_s$ is a *degenerate* Eisenstein series when $r_1 + r_2 > 2$. it is missing the cuspidal data and does not play a role in the spectral theory despite its relevance to physics.

For proofs of the following Theorems, see Chapter 3.11 of [**?**].

**Theorem 34.** *The cuspidal-data Eisenstein series $E_{s,f}(g)$ converges (absolutely and uniformly for $g$ in compacts) for Re$(s) > 1$.*

Many constant terms for cuspidal-data Eisenstein series vanish for general reasons.

**Theorem 35.** *Let $P = P^{r_1,r_2}$ and $f = f_1 \otimes f_2$ cuspforms on $M = M^P$. Let $Q$ be another parabolic of $GL_r(\mathbb{A})$. Then $c_Q E_{s,f}^P = 0$ unless $Q = P$ or $Q = P^{r_2,r_1}$ (the associate of $P$).*

In the $P$ and $Q$ associate cases, we will need an addition assumption. *Strong multiplicity one* is that the only other cuspforms on $M^P \cong GL_{r_1} \times GL_{r_2}$ with the same spherical Hecke eigenvalues at all finite primes are scalar multiples of $f = f_1 \otimes f_2$. Let $f^w = (f_1 \otimes f_2)^w = f_2 \otimes f_1$.

**Theorem 36.** *In the non-vanishing cases, with maximal proper $P$, and $Q = P$ or its* *associate, with the strong multiplicity one assumption above,*

$$\begin{cases} c_P E^P_{s,f} = \varphi^P_{s,f} & \text{for } r_1 \neq r_2 \text{ (not self-associate)} \\ c_P E^P_{s,f} = \varphi^P_{s,f} + c^P_{s,f}\varphi^P_{1-s,f^w} & \text{for } r_1 = r_2 \text{ (self-associate), meromorphic } c^P_{s,f} \\ c_Q E^P_{s,f} = c^Q_{s,f}\varphi^Q_{1-s,f^w} & \text{for } r_1 \neq r_2, Q = P^{r_2,r_1}, \text{ meromorphic } c^Q_{s,f} \end{cases}$$

The meromorphic functions $c^P_{s,f}$ have Euler products expansions attached to $f_1$ and $f_2$. This was investigated by Langlands in [?] and completed by Shahidi [?, ?].

**Theorem 37.** *With the maximal proper $P$ and $Q = P$ or its associate, with the strong multiplicity one assumption, $E^P_{s,f}$ has meromorphic continuations in $s$ with functional equation*

$$E^P_{1-s,f} = (c^P_{s,f^w})^{-1} E^Q_{s,f^w} \qquad \text{and} \qquad c^q_{1-s,f} \cdot c^P_{s,f^w} = 1.$$

For proof see [?] Chapter 11.

### 8.3.3 Truncation

The computation of the spectral expansion for the solution of the differential equation involving Eisenstein series requires truncation in order to compute the spectral integrals. However, for higher rank groups we must make precise this notion of truncation relative to the choice of parabolic subgroup. For self-associate maximal parabolic $P^{r,r}$ in $GL(2r)$, the computation of the Maaß-Selberg relation is the same as in $GL(2)$.

The simplest non-trivial example of Maaß-Selberg relations are corollaries concerns spherical Eisenstein series on $GL(n)$ associated to cuspidal data on the Levi component of maximal parabolics $P = P^{r_1,r_1}$.

Consider right $K_{\mathbb{A}}$-invariant Eisenstein series $E^P_{s,f_1\otimes f_2}$ where $f_1$ and $f_2$ are cuspforms in the strong sense (are spherical Hecke eigenfunctions) with trivial central characters. When $P$ is not self-associate ($r_1 \neq r_2$), let $Q = P^{r_2,r_1}$ be its associate parabolic. Let $\delta^P$ be the modular function of $P_{\mathbb{A}}$

$$\delta\left(\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\right) = \left|\frac{(\det m_1)^{r_1}}{(\det m_2)^{r_2}}\right|$$

and extend this to a *height function* aligned with $P$ by making it right $K_{\mathbb{A}}$-invariant $h^P(nmk) = \delta^P(nm) = \delta^P(m)$ for $n \in N_{\mathbb{A}}^P$, $m \in M_{\mathbb{A}}^P$ and $k \in K_{\mathbb{A}}$. For fixed large real $T$, the $T$-tail of the $P$-constant term of an automorphic form $F$ is

$$c_P^T F(g) = \begin{cases} c_P F(g) & \text{for } h^P(g) \geq T \\ 0 & \text{for } h^P(g) < T \end{cases}$$

We can define the $T$-tail for the $Q$-constant term mutatis mutandis. When truncating an automorphic from it is important to maintain the automorphic nature of the truncation. Thus it will be important to "wind-up" these constant terms. Write $\Psi^P(\varphi) = \Psi_\varphi^P$ for the pseudo-Eisenstein series attached to the data $\varpi$. Then the *truncation* at height $T$ of the Eisenstein series is

$$\wedge^T E_{s,f}^P = \begin{cases} E_{s,f}^P - \Psi^P(c_P^T E_{s,f}^P) & \text{for } n_1 = n_2 \text{ (P self-associate)} \\ E_{s,f}^P - \Psi^P(c_P^T E_{s,f}^P) - \Psi^Q(c_Q^T E_{s,f}^P) & \text{for } n_1 \neq n_2 \text{ (P not self-associate)} \end{cases}$$

**Theorem 38.** *The truncated Eisenstein series $\wedge^T E_{s,f}^P$ is of rapid decay in Siegel sets.*

## 8.4 Automorphic Spectral Expansions

The general pattern for the spectral expansion of automorphic forms is there is that an orthonormal basis of cuspforms. The orthogonal compliment of cuspforms is spanned by pseudo-Eisenstein series. These pseudo-Eisenstein series are integrals of Eisenstein series, the latter eigenfunctions for the invariant differential operators. The functional equation for Eisenstein series attached to associate parabolics shows that they will produce the same functions on the group. Thus part of the indexing of the $L^2$ expansion is by associate-class of parabolics. The expression of the pseudo-Eisenstein series also involves residues of cuspidal-data Eisenstein series These residues are square-integrable and inherit eigenfunction properties from the associated Eisenstein series. For $GL(2)$, these are essentially constants.

For $GL(3)$, the maximal parabolics $P^(2,1)$ and $P^{1,2}$ have no relevant residues (see Section 3.14 of [**?**]) and the only residues are those from the minimal parabolic $P^{1,1,1}$

which are constants. Thus for $\Phi \in L^2(Z_{\mathbb{A}}GL_3(\mathbb{Q})\backslash GL_3(\mathbb{A})/K_{\mathbb{A}})$, we have

$$\Phi = \sum_{GL(3)\text{ cfm } F} \langle \Phi, F\rangle \cdot F + \sum_{GL(2)\text{ cfm } f} \frac{1}{2\pi i}\int_{(1/2)} \langle \Phi, E^{2,1}_{s,f}\rangle \cdot E^{2,1}_{s,f}\, ds$$

$$+\frac{1}{3!\cdot 2\pi i}\int_{i\mathfrak{q}^*} \langle \Phi, E^{\min}_{\rho+s}\rangle \cdot E^{\min}_{\rho+s}\, ds + \frac{\langle \Phi, 1\rangle \cdot 1}{\langle 1, 1\rangle}$$

where the first sum is over an orthonormal basis of spherical cuspforms for $GL_3(\mathbb{Z})$ with trivial central character and the second sum is over an orthonormal basis of spherical cuspforms for $GL_2(\mathbb{Z})$ with trivial central character. The right hand side converges in an $L^2$ sense and the integrals involving Eisenstein series are isometric extensions of the corresponding literal integrals.

The expansion for $GL(4)$ is the smallest group for which Eisenstein series ($E^{2,2}_{s,f}$ with real-values $f$) produce non-constant residues. The computation of the constant terms for $E^{2,2}_{s,f}$ shows that it is a ratio of Rankin-Selberg $L$-functions attached to $f\times \overline{f}$ which has a poles in $\mathrm{Re}(s) > 1/2$ yielding a square-integrable residue. For $\Phi \in L^2(Z_{\mathbb{A}}GL_4(\mathbb{Q})\backslash GL_4(\mathbb{A})/K_{\mathbb{A}})$, we have

$$\Phi = \sum_{GL(4)\text{ cfm } F} \langle \Phi, F\rangle \cdot F + \sum_{GL(3)\text{ cfm } f} \frac{1}{2\pi i}\int_{(1/2)} \langle \Phi, E^{3,1}_{s,f}\rangle \cdot E^{3,1}_{s,f}\, ds$$

$$+ \sum_{GL(2)\text{ cfms } f_1, f_2;\ f_1\neq \overline{f_2}} \int_{(1/2)} \langle \Phi, E^{2,2}_{s,f_1\otimes f_2}\rangle \cdot E^{2,2}_{s,f_1\otimes f_2}\, ds$$

$$+ \sum_{GL(2)\text{ cfm } f} \frac{1}{4\pi i}\int_{(1/2)} \langle \Phi, E^{2,2}_{s,f\otimes \overline{f}}\rangle \cdot E^{2,2}_{s,f\otimes \overline{f}}\, ds$$

$$+ \sum_{GL(2)\text{ cfm } f} \langle \Phi, F_f\rangle \cdot F_f + \sum_{GL(2)\text{ cfm } f} \frac{1}{4\pi i}\int_{(1/2)} \langle \Phi, E^{2,1,1}_{s,f}\rangle \cdot E^{2,1,1}_{s,f}\, ds$$

$$+\frac{1}{4!\cdot 2\pi i}\int_{i\mathfrak{q}^*} \langle \Phi, E^{\min}_{\rho+s}\rangle \cdot E^{\min}_{\rho+s}\, ds + \frac{\langle \Phi, 1\rangle \cdot 1}{\langle 1, 1\rangle}$$

where $F_f$ are Speh forms. Again right hand side converges in an $L^2$ sense and the integrals involving Eisenstein series are isometric extensions of the corresponding literal integrals.

## 8.5 Spectral relation

The following can be found many places including P. Garrett's [?] and A. DeCelles' [?].

**Theorem 39.** *For $f \in C_c^\infty(\Gamma\backslash\mathfrak{H})$, then $\langle \Delta f, E_s\rangle_{L^2(\Gamma\backslash\mathfrak{H})} = \lambda_s \cdot \langle f, E_s\rangle_{L^2(\Gamma\backslash\mathfrak{H})}$.*

*Proof.* Let $f \in C_c^\infty(\Gamma\backslash\mathfrak{H})$. Note that the symmetry of $\Delta$ and compact support of elements of $D$ allows integration by parts. Then we have the following spectral relation

$$\langle \Delta f, E_s\rangle_{L^2(\Gamma\backslash\mathfrak{H})} = \int_{\Gamma\backslash\mathfrak{H}} \Delta f(z) \cdot E_{1-s}(z)\,\frac{dx\,dy}{y^2} = \int_{\Gamma\backslash\mathfrak{H}} f(z)\cdot \Delta E_{1-s}(z)\,\frac{dx\,dy}{y^2}$$

$$= \int_{\Gamma\backslash\mathfrak{H}} f(z)\cdot \lambda_s E_{1-s}(z)\,\frac{dx\,dy}{y^2} = \lambda_s\langle f, E_s\rangle_{L^2(\Gamma\backslash\mathfrak{H})}$$

□

For $0 \le k \in \mathbb{Z}$, the $k^{th}$-Sobolev norm on $C_c^\infty(\Gamma\backslash\mathfrak{H})$ is given by

$$|f|_k^2 := \langle (1-\Delta)^k f, f\rangle_{L^2(\Gamma\backslash\mathfrak{H})}$$

and $H^k(\Gamma\backslash\mathfrak{H})$ is the completion of $C_c^\infty(\Gamma\backslash\mathfrak{H})$ with respect to $|\cdot|_k$.

**Theorem 40.** *There is a continuous injection $H^k(\Gamma\backslash\mathfrak{H}) \to H^{k+1}(\Gamma\backslash\mathfrak{H})$ with dense image.*

*Proof.* Let $f \in C_c^\infty(\Gamma\backslash\mathfrak{H})$ then $\langle -\Delta f, f\rangle \ge 0$. We would like to show that for a polynomial $p$ with non-negative real coefficients $\langle p(-\Delta)f, f\rangle \ge 0$. It suffices to show that $\langle (-\Delta)^n f, f\rangle \ge 0$.

For $n = 2m$ even,

$$\langle (-\Delta)^n f, f\rangle = \langle (-\Delta)^{2m} f, f\rangle = \langle (-\Delta)^m f, (-\Delta)^m f\rangle \ge 0.$$

For $n = 2m+1$ odd,

$$\langle (-\Delta)^n f, f\rangle = \langle (-\Delta)^{2m+1} f, f\rangle = \langle (-\Delta)((-\Delta)^m f), (-\Delta)^m f\rangle \ge 0.$$

This gives

$$|f|_{k+1}^2 = \langle (1-\Delta)^{k+1} f, f\rangle = \langle (1+(-\Delta))^k f, f\rangle + \langle (1+(-\Delta))^k(-\Delta) f, f\rangle$$

$$\geq \langle (1+(-\Delta))^k f, f\rangle + 0 = |f|_k^2$$

Thus the identity map $C_c^\infty(\Gamma\backslash\mathfrak{H})$ extends to a continuous injection $H^{k+1} \to H^k$ since $C_c^\infty(\Gamma\backslash\mathfrak{H})$ is dense in both. Furthermore, the image is dense.

□

**Theorem 41.** *The differential operator $\Delta : C_c^\infty(\Gamma\backslash\mathfrak{H}) \to C_c^\infty(\Gamma\backslash\mathfrak{H})$ is continuous when the source is given the $H^{k+2}$ topology and the target is given the $H^k$ topology for $0 \geq k \in \mathbb{Z}$.*

*Proof.* Using the latter negativity property of the previous proof, we have

$$|\Delta f|_k^2 = \langle (1-\Delta)^k(\Delta f), (\Delta f)\rangle = \langle (-\Delta)^2(1+(-\Delta))^k f, f\rangle$$

$$\leq \langle (-\Delta)^2(1+(-\Delta))^k f, f\rangle + \langle (2(-\Delta)+1)f, f\rangle = \langle (1+(-\Delta))^{k+2} f, f\rangle = |f|_{k+1}^2$$

□

**Corollary 42.** *$\Delta$ extends by continuity from test functions to a continuous linear map $\Delta : H^{k+2}(\Gamma\backslash\mathfrak{H}) \to H^k(\Gamma\backslash\mathfrak{H})$ for each $0 \leq k \in \mathbb{Z}$.*

*Proof.* For test functions $\{f_n\}$ forming a Cauchy sequence in the $H^{k+1}$ topology, the continuity on the respective topologies on test functions means that the extension-by-continuity definition

$$\Delta(H^{k+2}\text{–}\lim_n f_n) = H^k\text{–}\lim_n \Delta f_n$$

is well-defined and given a continuous map in those topologies. □

**Corollary 43.** *For $f \in H^k(\Gamma\backslash\mathfrak{H})$, then $\langle \Delta f, E_s\rangle_{L^2(\Gamma\backslash\mathfrak{H})} = \lambda_s \cdot \langle f, E_s\rangle_{L^2(\Gamma\backslash\mathfrak{H})}$.*

*Proof.* Because $\langle \cdot, E_s\rangle_{L^2(\Gamma\backslash\mathfrak{H})} : L^2(\Gamma\backslash\mathfrak{H}) \to L^2(1/2+i[0,\infty))$ is an isometric isomorphism obtained by extension by continuity on test functions, the literal spectral integrals in Theorem **??** extend by continuity to give the result. □

The same argument can be given for each function in

$$\Xi = \{\text{orthonormal basis of cuspforms}\} \cup \{1\} \cup 1/2 + i[0,\infty)$$

where the half-line parametrizes the Eisenstein series $E_{1/2+it}$.

## 8.6 Vector-valued integrals

There is at least one technical point to address. We will need a bit of machinery introduced by Gelfand (1936) [?] and Pettis (1938) [?]. Their construction produces integrals of continuous vector-valued functions with compact support. These integrals are not *constructed* using limits, in contrast to Bochner integrals, but instead are *characterized* by the desired property that they commute with linear functionals.

Let $V$ be a complex topological vector space. Let $f$ be a measurable $V$-valued function on a measure space $X$. A *Gelfand-Pettis integral* of $f$ is a vector $I_f \in V$ so that

$$\alpha(I_f) = \int_X \alpha \circ f$$

for all $\alpha \in V^*$. Assuming that it exists and is unique, the vector $I_f$ is denoted $I_f = \int_X f$.

Uniqueness and linearity of the integral follow from the fact that $V^*$ separates points by Hahn-Banach. Establishing the existence of Gelfand-Pettis integrals is more delicate.

**Theorem 44.** *Let $X$ be a compact Hausdorff topological space with a finite positive regular Borel measure. Let $V$ be a quasi-complete, locally convex topological vectorspace. Then continuous compactly-supported $V$-values functions $f$ on $X$ have Gelfand-Pettis integrals.*

The importance of the characterization of the Gelfand-Pettis integral is exhibited in the following corollary.

**Corollary 45.** *Let $T : V \to W$ be a continuous linear map of locally convex quasi-complete topological vector spaces and $f$ a continuous $V$-valued function on $X$. Then*

$$T\left(\int_X f\right) = \int_X T \circ f.$$

*Proof.* Since $W^*$ separates points, it suffices to show that

$$\mu\left(T\left(\int_X f\right)\right) = \mu\left(\int_X T \circ f\right).$$

Since $\mu \circ T \in V^*$, the characterization of Gelfand-Pettis integrals gives

$$\mu\left(T\left(\int_X f\right)\right) = (\mu \circ T)\left(\int_X f\right) = \int_X \mu(T \circ f) = \mu\left(\int_X T \circ f\right).$$

□

## 8.7 Holomorphic vector-valued functions

We will recall some basic facts about vector-values functions, most of which we will not prove here. However, for proofs and further explanation see Grothendieck's [**?**] for the original or Rudin's [**?**].

Let $f$ be a function of an open set $\Omega \subset \mathbb{C}$ taking values in a quasi-complete, locally convex space $V$. We say $f$ is *weakly holomorphic* when $\mathbb{C}$-valued functions $\lambda \circ f$ are holomorphic for all $\lambda \in V^*$.

Let $\mathrm{Hol}(\Omega, N)$ be the topological vector space of holomorphic $N$-valued functions on a fixed open $\Omega$.

**Theorem 46.** *For $V$ a locally convex quasi-complete topological vector space, weakly holomorphic $V$-valued functions $f$ are strongly holomorphic in the following senses.*

*First the usual Cauchy-theory integral formulas apply:*

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

*with $\gamma$ a closed path around $z$ having winding number 1. Second, the function $f(z)$ is infinitely differentiable, in fact strongly analytic, that is, expressible as a convergent power series $\sum_{n\geq 0} c_n(z - z_o)^n$ with coefficients $c_n \in V$ given by Gelfand Pettis integrals echoing Cauchy's formulas: $c_n = \dfrac{f^{(n)}(z_o)}{n!} = \dfrac{1}{2\pi i}\displaystyle\int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta$*

In [**?**], the proof also uses the fact that *weak* boundedness implies boundedness to first show that $f$ is continuous. Then recapulation in the vector-valued context is viable.

Now fix a non-empty open $\Omega \subset \mathbb{C}$. Let $V$ be quasi-complete, locally convex, with topology given by seminorms $\{\nu\}$. The space $\mathrm{Hol}(\Omega, N)$ of holomorphic $v$-values functions on $\Omega$ has a natural topology given by seminorms $\mu_{\nu.K}(f) = \sup_{z\in K} \nu(f(z))$ for compacts $K \subset \Omega$ seminorms $\nu$ on $V$.

**Corollary 47.** *$Hol(\Omega, N)$ is locally convex, quasi-complete.*

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