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To cite this article: Francesco Sorge 2018 *J. Phys. Commun.* **2** 125012

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PAPER

OPEN ACCESS

RECEIVED

2 October 2018

REVISED

9 November 2018

ACCEPTED FOR PUBLICATION

23 November 2018

PUBLISHED

19 December 2018

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Gravitational waves and coherent phonon states in elastic media: 1D-analysis

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Keywords: gravitational waves, acoustic phonons, elastic string

Abstract

Longitudinal acoustic phonons in a one-dimensional elastic string are studied in the field of a weak gravitational wave. Working in the analogue gravity framework, a lagrangian describing longitudinal wave propagation along the string is derived in the background of a suitable *acoustic* metric, the latter encoding both the spacetime corrections due to the gravitational wave and the elastic properties of the string. After quantization, a phonon Hamiltonian is obtained, having the structure of the Luttinger liquid Hamiltonian. Such Hamiltonian is subsequently employed to analyze some effects induced by a gravitational wave, as time evolution of phonon vacuum state, gravitational vacuum squeezing and time evolution of two-mode coherent phonon states, with particular concern to ionic density fluctuations. These findings are in good agreement with similar results appeared in the literature, although in quite different material media. This suggests that the above cited effects do represent a quite general manifestation of the peculiar response to gravitational waves in macroscopic particle ensembles whose internal structure admits some kind of wave propagation.

1. Introduction

Recent experimental detection at LIGO [1, 2] of gravitational wave (GW) signals from several binary black hole mergers (the first of them was BBH GW150914) has strongly renewed the interest of researchers working in the field of General Relativity and observational astrophysics, both from a theoretical and experimental point of view.

The starting of such new exciting GW astronomy era could hopefully allow to discriminate among Einstein's theory and other alternative theories of gravity. Such issue has been anticipated and discussed in detail in [3], where it has been pointed out that an accurate analysis of gravitational wave signals from highly sensitive detectors could give a definitive test for General Relativity.

Over the years, a lot of effort has been devoted to the analysis of the interaction between gravitational waves and matter fields. Several papers have dealt with the interaction between gravitational waves and superfluids [4–7]. Coupling between superconductors and gravitational waves has been studied in [8–10]. Also, influence of electromagnetic, gravitational and gravitoelectromagnetic waves on superconductors has been investigated in [11–20]. In recent years some Authors have considered the behaviour of Bose–Einstein condensates (BEC) in presence of weak gravitational waves, discussing possible phonon excitation inside the condensate [21, 22] as an example of gravitationally induced particle creation [23]. Also, density fluctuations induced by gravitational waves in a BEC have been studied in [24].

However, to the best of our knowledge, little attention has been devoted to a general analysis of the interaction between gravitational waves and *elastic media* as e.g., crystal lattices. Such topic has been considered in the past [25] mainly through a classical approach, essentially based upon the geodesic deviation. In the present paper we will propose a quantum-mechanical approach, in which collective lattice vibrations are described in terms of *phonons* interacting with a gravitational wave. Such semi-classical approach seems indeed the most adequate, as the expected gravitational effects are so small to be comparable to other quantum fluctuations.

Elastic media have usually a discrete structure. However, as far as long wavelength phonons are taken into account, they can be considered a *continuous* media, obeying a linear dispersion relation ($\omega_k = c_L k$). We will

work in such long wavelength limit, also confining our analysis to a long, thin metallic string, modelled as a *one-dimensional, continuous elastic line*. In such a simplified model we will neglect transverse vibrational modes, taking into account the *longitudinal* acoustic phonons only. We will show that several effects predicted in quite different macroscopic particle ensembles (as BECs [21, 24]), such as phonon creation or parametric amplification induced by gravitational waves, arise quite generally in our 1D-elastic model. In that respect, wave propagation in material media does appear a relevant physical property allowing for an effective interaction between matter fields and gravitational waves.

The paper is organized as follows. In section 2 we focus on a one-dimensional *continuous elastic line*, described from a lagrangian point of view. In section 3 we generalize the theory to a weak gravitational wave background, introducing a suitable *acoustic* metric in the analogue gravity framework [21, 26, 27] and deriving the phonon Hamiltonian for the elastic line. In section 4 we quantize the Hamiltonian, analyzing the time evolution of the phonon vacuum state and particle creation effect in presence of a gravitational wave. In section 5 we study the interaction of a gravitational wave with a two-mode coherent phonon state. Finally, in section 6 we briefly consider ionic density fluctuations induced by a gravitational wave. Section 7 is devoted to some concluding remarks.

Throughout the paper, unless otherwise specified, use has been made of natural units: $c = 1$, $\hbar = 1$, $G = 1$. Greek indices take values from 0 to 3; latin ones take values from 1 to 3. The metric signature is -2 , with determinant g .

2. Elastic line in a GW background

According to the standard theory of ideal elastic media [28], all internal stresses are the result of strain. The most familiar class of elastic media is the one obeying Hooke's law: stress is proportional to strain. For a continuous medium, the resulting potential energy is translationally and rotationally invariant. In the case of a discrete medium (as a crystalline solid) these symmetries are broken at the lattice scales; they are—however—preserved in the long wavelength limit, in which we are mainly interested, as we will see. Before we proceed, let us point out some relevant assumptions:

- we will assume a *continuous*, elastic medium; this assumption is a good approximation as far as long wavelength phonons are taken into account;
- we will consider a 1D-model, namely an elastic *line*, hence neglecting the transverse dimensions; this allows us to discard possible transverse vibrational modes, hence simplifying the further analysis;
- in order to clearly disentangle the gravitational effects from other non-gravitational contributions, here we will not take into account dissipative effects, as higher order phonon-phonon or electron-phonon interactions. Such contributions will be considered in a future work.

2.1. Flat background analysis

For a crystalline solid, considered as an elastic medium, the action S_0 up to quadratic order can be written in terms of the symmetric combination of the spatial derivatives of the displacement field \vec{u} ,

$$S_0 = \frac{1}{2} \int dt d^3x [\rho (\partial_t u_i)^2 - 2\mu u_{ij} u^{ij} - \lambda (u_{kk})^2], \quad (2.1)$$

where ρ is the density of the solid, μ and λ are the Lamé coefficients and

$$u_{ij} = \frac{1}{2} (\partial_i u_j + \partial_j u_i) \quad (2.2)$$

is the *strain* tensor. Assume the solid to have, e.g., a cylindrical shape—an *elastic string*—with section A and length $L \gg \sqrt{A}$. Let us introduce a reference frame having the x -axis along the elastic line. Limiting ourselves to longitudinal strain, we may discard the transverse displacement, hence retaining only the u_x component. In such a case, we may integrate out the remaining spatial coordinates in (2.1), thus obtaining

$$S_0 = \int dt dx \mathcal{L}_0 = \frac{A}{2} \int dt dx [\rho (\partial_t u_x)^2 - Y (\partial_x u_x)^2], \quad (2.3)$$

where $Y = (2\mu + \lambda)$ appears as the Young modulus¹. Equation (2.3) can be regarded as the action for a one-dimensional continuous elastic *line*. The corresponding 1 + 1 lagrangian density \mathcal{L}_0 is [29]

¹ Recall the general relationship between the Lamé coefficients and the Young modulus: $2\mu + \lambda = \frac{Y(1-\sigma)}{(1+\sigma)(1-2\sigma)}$, where $\sigma = -\frac{u_{zz}}{u_{xx}}$ is the Poisson coefficient. In the present 1D case, $u_{zz} = 0 \Rightarrow \sigma = 0 \Rightarrow 2\mu + \lambda = Y$.

$$\mathcal{L}_0 = \frac{1}{2} \Lambda [c_L^{-2} (\partial_t u_x)^2 - (\partial_x u_x)^2], \tag{2.4}$$

where $u_x = u_x(t, x)$, $\Lambda = AY$ and

$$c_L = \sqrt{\frac{Y}{\rho}} = \sqrt{\frac{\Lambda}{A\rho}} \tag{2.5}$$

is the speed of propagation of the longitudinal sound waves along the elastic line². Actually, from the Euler–Lagrange equations the following equation of motion is readily deduced

$$c_L^{-2} \partial_t^2 u_x - \partial_x^2 u_x = 0, \tag{2.6}$$

just describing wave propagation along an elastic medium with velocity c_L . Recall that, in the assumed long wavelength limit, the following linear dispersion relation holds

$$\omega_k = c_L k. \tag{2.7}$$

2.2. Weak gravitational wave background

The above results hold in a flat space-time. In a slightly curved background, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, the lagrangian (2.4) will acquire a contribution stemming from the interaction between the matter field and gravity. In this paper we are interested in the interaction with a weak gravitational plane wave.

Let us begin considering a gravitational (plane) wave propagating along an arbitrary direction $\vec{\Omega} = (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta)$ in the reference frame $\{x, y, z\}$ of a static observer \mathbf{u} , with θ and ϕ being the usual polar coordinates on the 2-sphere [30]. In the TT-gauge frame $\{\bar{x}, \bar{y}, \bar{z}\}$ of the wave, in which \hat{z} is assumed as the direction of propagation $\vec{\Omega}$, the line element is

$$ds^2 = g_{\mu\nu} d\bar{x}^\mu d\bar{x}^\nu = d\bar{t}^2 - d\bar{x}^2 + \bar{h}_{\mu\nu} d\bar{x}^\mu d\bar{x}^\nu, \tag{2.8}$$

where

$$\bar{h}_{\mu\nu} = -h_+(\bar{z}, \bar{t}) \bar{e}_{\mu\nu}^+ - h_\times(\bar{z}, \bar{t}) \bar{e}_{\mu\nu}^\times \tag{2.9}$$

and

$$\begin{aligned} \bar{e}_{\mu\nu}^+ &= \bar{m}_\mu \bar{m}_\nu - \bar{n}_\mu \bar{n}_\nu \\ \bar{e}_{\mu\nu}^\times &= \bar{m}_\mu \bar{n}_\nu + \bar{n}_\mu \bar{m}_\nu, \end{aligned} \tag{2.10}$$

are the polarization tensors of the wave. In the TT frame we have

$$\bar{m}_\mu = (0, 0, -1, 0), \quad \bar{n}_\mu = (0, 1, 0, 0). \tag{2.11}$$

Defining $R(\phi, \theta) \equiv R_z(-\phi)R_y(\theta)$, where

$$R_y(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & 0 & \sin \theta \\ 0 & 0 & 1 & 0 \\ 0 & -\sin \theta & 0 & \cos \theta \end{pmatrix}, \quad R_z(-\phi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi & 0 \\ 0 & \sin \phi & \cos \phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.12}$$

we transform (2.8) from the TT frame to the \mathbf{u} reference frame $\{x, y, z\}$. We get, in matrix form [$x \equiv (t, \vec{x})$]

$$\begin{aligned} ds^2 &= (Rdx)^T g (Rdx) = (Rdx)^T (\eta + \bar{h}) (Rdx) \\ &= dx^T \eta dx + dx^T (R^T \bar{h} R) dx = dt^2 - d\vec{x}^2 + dx^T h dx, \end{aligned} \tag{2.13}$$

where

$$h = R^T \bar{h} R = -h_+(z, t) e^+ - h_\times(z, t) e^\times, \tag{2.14}$$

and the polarization tensors $e^+ = (R^T \bar{e}^+ R)$ and $e^\times = (R^T \bar{e}^\times R)$ have still the form (2.10), but now

$$\begin{aligned} m_\mu &= (0, \sin \phi, -\cos \phi, 0), \\ n_\mu &= (0, \cos \phi \cos \theta, \sin \phi \cos \theta, -\sin \theta). \end{aligned} \tag{2.15}$$

3. Gravitational interaction in the analogue gravity formalism

In this section we will describe the gravitational interaction following the so-called *analogue gravity* approach [21, 26, 27, 31]. Let us start observing that (2.4) can be written (hereafter we will suppress the subscript in the

² Notice that, in the adopted natural units, c_L appears here as an *adimensional* quantity. Furthermore, $c_L \ll 1$.

displacement field u_x , simply writing it u)

$$\mathcal{L}_0 = \frac{1}{2} \tilde{\eta}^{\mu\nu} \partial_\mu u \partial_\nu u, \quad (3.1)$$

where the *acoustic* flat metric $\tilde{\eta}$, according to an observer \mathbf{u} at rest with respect to the elastic line, is (see [31] and the almost exhaustive review [26] for details)

$$\tilde{\eta}_{\mu\nu} = \Lambda^{-1} \begin{pmatrix} c_L^2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \tilde{\eta}^{\mu\nu} = \Lambda \begin{pmatrix} c_L^{-2} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (3.2)$$

The perturbed acoustic metric \tilde{g} in the \mathbf{u} frame reads

$$\tilde{g}_{\mu\nu} = \tilde{\eta}_{\mu\nu} + \Lambda^{-1} h_{\mu\nu}, \quad (3.3)$$

$$\tilde{g}^{\mu\nu} = \tilde{\eta}^{\mu\nu} - \Lambda h^{\mu\nu}. \quad (3.4)$$

3.1. Phonon Hamiltonian

In what follows we will suppose that the typical wavelength of the gravitational wave is much larger than the size (L) of the elastic line. Expanding $h_{+, \times}(t, z)$ around $z = 0$ we can consider $h_{+, \times}(z, t)$ as functions of time t only, setting $h_{+, \times}(z, t) \simeq h_{+, \times}(0, t) \equiv h_{+, \times}(t)$ all along the line.

In linearized gravity, it is customary to write $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_g$; taking into account (3.4), the gauge-invariant interaction lagrangian density \mathcal{L}_g is

$$\mathcal{L}_g = -\frac{1}{2} \delta \tilde{g}_{\mu\nu} T_0^{\mu\nu} = -\frac{1}{2} \Lambda^{-1} h_{\mu\nu} T_0^{\mu\nu}, \quad (3.5)$$

where $T_0^{\mu\nu}$ is the stress-energy tensor of the matter field in absence of gravitational perturbation, namely

$$T_0^{\mu\nu} = \frac{\partial \mathcal{L}_0}{\partial (\partial_\mu u)} \partial^\nu u - \tilde{\eta}^{\mu\nu} \mathcal{L}_0. \quad (3.6)$$

From (3.5) and (3.6) we get

$$\begin{aligned} \mathcal{L}_g &= -\frac{1}{2} \Lambda^{-1} h_{xx}(t) \left[\frac{\partial \mathcal{L}_0}{\partial (\partial_x u)} \partial^x u - \tilde{\eta}^{xx} \mathcal{L}_0 + \tilde{\eta}^{yy} \mathcal{L}_0 \right] \\ &= -\frac{1}{2} \Lambda h_{xx}(t) (\partial_x u)^2. \end{aligned} \quad (3.7)$$

Introducing the amplitude polarizations A_+ and A_\times such that $h_+(t) = A_+ h(t)$ and $h_\times = A_\times h(t)$, we have from (2.14) $h_{xx} = h(t) \Xi(\Omega)$, where

$$\Xi(\Omega) = -A_+ (\sin^2 \phi - \cos^2 \phi \cos^2 \theta) - 2A_\times (\sin \phi \cos \phi \cos \theta). \quad (3.8)$$

So, the total lagrangian in presence of a gravitational wave reads

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_g = \frac{1}{2} \tilde{\eta}^{\mu\nu} \partial_\mu u \partial_\nu u + \frac{1}{2} \Lambda \Xi(\Omega) h(t) (\partial_x u)^2. \quad (3.9)$$

The hamiltonian density \mathcal{H} with respect to the static observer $\mathbf{u} \equiv u^\mu = \delta_t^\mu (\tilde{g}_{tt})^{-1/2}$ reads

$$\mathcal{H} = u^\mu u^\nu T_{\mu\nu} = \tilde{\eta}_{tt} T^{tt}, \quad (3.10)$$

where

$$T^{tt} = \frac{\partial \mathcal{L}}{\partial (\partial_t u)} \partial^t u - \tilde{g}^{tt} \mathcal{L} = \frac{1}{2} \Lambda^2 c_L^{-2} [c_L^{-2} (\partial_t u)^2 + (\partial_x u)^2 - \Xi(\Omega) h(t) (\partial_x u)^2]. \quad (3.11)$$

From (3.10) we get

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_g, \quad (3.12)$$

where \mathcal{H}_0 is the flat hamiltonian density

$$\mathcal{H}_0 = \frac{1}{2} \Lambda [c_L^{-2} (\partial_t u)^2 + (\partial_x u)^2], \quad (3.13)$$

and

$$\mathcal{H}_g = -\frac{1}{2}\Lambda\Xi(\Omega)h(t)(\partial_x u)^2 \quad (3.14)$$

represents the contribution due to the interaction with the gravitational wave. We notice that \mathcal{H}_g generally depends *both* on the polarization states of the gravitational wave through the quantity $\Xi(\Omega)$.

4. Quantization

We now move to quantize the theory [32]. The displacement field u becomes an operator³, \hat{u}

$$\hat{u} = \sum_k N_k (\hat{c}_k(t) e^{ikx} + \hat{c}_k^\dagger(t) e^{-ikx}), \quad (4.1)$$

where the creation (annihilation) operators \hat{c}_k^\dagger (\hat{c}_k) satisfy the usual commutation relations

$$[\hat{c}_k, \hat{c}_{k'}^\dagger] = \delta_{k,k'}, \quad (4.2)$$

and $[\hat{c}_k, \hat{c}_{k'}] = [\hat{c}_k^\dagger, \hat{c}_{k'}^\dagger] = 0$. The Heisenberg evolution of canonical operators is

$$\hat{c}_k(t) = e^{-i\omega_k t} \hat{c}_k(0), \quad \hat{c}_k^\dagger(t) = e^{i\omega_k t} \hat{c}_k^\dagger(0). \quad (4.3)$$

Making use of (4.1) the Hamiltonian \hat{H}_0 in absence of gravitational perturbation reads

$$\begin{aligned} \hat{H}_0 = \int_L dx \hat{\mathcal{H}}_0 = & -\frac{1}{2}\Lambda \int_L dx \sum_{k,k'} N_k N_{k'} (|k||k'| + kk') \\ & \times (\hat{c}_k \hat{c}_{k'} e^{i(k+k')x} - \hat{c}_k \hat{c}_{k'}^\dagger e^{i(k-k')x} - \hat{c}_k^\dagger \hat{c}_{k'} e^{i(k-k')x} + \hat{c}_k^\dagger \hat{c}_{k'}^\dagger e^{i(k+k')x}). \end{aligned} \quad (4.4)$$

Integrating all over the elastic line and making the replacement $\sum_{k'} \rightarrow \frac{L}{2\pi} \int dk'$ we get

$$\hat{H}_0 = 2\Lambda L \sum_k N_k^2 k^2 \left(\hat{c}_k^\dagger \hat{c}_k + \frac{1}{2} \right). \quad (4.5)$$

Defining

$$N_k = N_{-k} = (2LA\rho\omega_k)^{-1/2}, \quad (4.6)$$

we obtain the well-known expected result

$$\hat{H}_0 = \sum_k \omega_k \left(\hat{c}_k^\dagger \hat{c}_k + \frac{1}{2} \right). \quad (4.7)$$

Following the same steps, we get the gravitational correction \hat{H}_g to the Hamiltonian

$$\hat{H}_g = \int_L dx \hat{\mathcal{H}}_g = -\frac{1}{2}\Lambda\Xi(\Omega)h(t) \int_L dx (\partial_x \hat{u})^2, \quad (4.8)$$

thus obtaining

$$\hat{H}_g = -\frac{1}{2}\Xi(\Omega)h(t) \sum_k \omega_k \left(\hat{c}_k^\dagger \hat{c}_k + \frac{1}{2} \hat{c}_k \hat{c}_{-k} + \frac{1}{2} \hat{c}_k^\dagger \hat{c}_{-k}^\dagger + \frac{1}{2} \right). \quad (4.9)$$

The full Hamiltonian finally reads

$$\hat{H} = \sum_k \omega_k \left[\left(1 - \frac{1}{2}\Xi(\Omega)h(t) \right) \left(\hat{c}_k^\dagger \hat{c}_k + \frac{1}{2} \right) - \frac{1}{4}\Xi(\Omega)h(t) (\hat{c}_k \hat{c}_{-k} + \hat{c}_k^\dagger \hat{c}_{-k}^\dagger) \right]. \quad (4.10)$$

The above form recalls that of the Luttinger liquid hamiltonian [33–36].

4.1. Time evolution

Using (4.10) in the Heisenberg equation

$$\partial_t \hat{c}_k = i[\hat{H}, \hat{c}_k], \quad (4.11)$$

we obtain

$$\partial_t \hat{c}_p = -i \left(1 - \frac{1}{2}\Xi(\Omega)h(t) \right) \omega_p \hat{c}_p + \frac{i}{2}\Xi(\Omega)h(t) \omega_p \hat{c}_{-p}^\dagger. \quad (4.12)$$

A similar a similar relation is obviously found for \hat{c}_p^\dagger . Limiting ourselves to the lowest approximation order, we substitute \hat{c}_p and \hat{c}_{-p}^\dagger in the r.h.s. of (4.12) with their freely evolving counterparts (4.3), thus writing

³ Throughout the paper, a caret will be used to characterize an *operator*.

$$\partial_t \hat{c}_p + i\omega_p \hat{c}_p = \frac{i}{2} \Xi(\Omega) h(t) \omega_p (\hat{c}_p(0) e^{-i\omega_p t} + \hat{c}_{-p}^\dagger(0) e^{i\omega_p t}), \quad (4.13)$$

whose solution is a Bogolubov transformation

$$\hat{c}_p(t) = \alpha_p(t) \hat{c}_p(0) + \beta_p(t) \hat{c}_{-p}^\dagger(0), \quad (4.14)$$

where we have defined (for $t \geq t_0$)

$$\alpha_p(t) = e^{-i\omega_p t} \left[1 + \frac{i}{2} \omega_p \Xi(\Omega) \int_{t_0}^t dt' h(t') \right] \quad (4.15)$$

$$\beta_p(t) = e^{-i\omega_p t} \left[\frac{i}{2} \omega_p \Xi(\Omega) \int_{t_0}^t dt' h(t') e^{2i\omega_p t'} \right]. \quad (4.16)$$

Notice that $\alpha_p(t) = \alpha_{-p}(t)$ and $\beta_p(t) = \beta_{-p}(t)$. Looking at (4.15) and (4.16) we see that, on lack of gravitational perturbation (i.e., $h(t) = 0$), $\alpha_p(t) = e^{-i\omega_p t}$, $\beta_p(t) = 0$ and the free evolution of the canonical operator \hat{c}_p (4.3) is recovered.

Armed with the above machinery, we will now move to explore the time evolution of two relevant phonon states, namely the vacuum state $|0\rangle$ and—in the next section—a two-mode coherent state (TMCS).

4.2. Vacuum evolution

Let us consider the phonon vacuum state evolution under the influence of a gravitational wave. The phonon vacuum state $|0\rangle$ is defined as $\hat{c}_k|0\rangle = 0$, $\forall k$. Consider the particle number operator \hat{N}

$$\hat{N} = \sum_k \hat{c}_k^\dagger \hat{c}_k. \quad (4.17)$$

Suppose we have initially no phonons in the elastic line at $t = 0$. Using (4.14) we evaluate the expectation value of the phonon number after the interaction with the gravitational wave ($t \rightarrow +\infty$)

$$\langle \hat{N} \rangle = \lim_{t \rightarrow +\infty} \langle 0 | \sum_k \hat{c}_k^\dagger(t) \hat{c}_k(t) | 0 \rangle = \lim_{t \rightarrow +\infty} \sum_k |\beta_k(t)|^2. \quad (4.18)$$

We see that the gravitational interaction caused the excitation of a small number of phonon out of the vacuum state. Looking at (4.16), we recognize this is a second order ($O(h^2)$) effect in the small gravitational wave strain h .

4.3. Gravitational squeezing of vacuum state

To gain further insight in the evolution of the phonon vacuum under the influence of a gravitational wave, let us consider the evolution operator $\hat{U}(t)$, which, in the present $O(h)$ approximation, reads [37]

$$\hat{U}(t) \simeq e^{-i\frac{\hat{H}_0 t}{\hbar}} \exp\left(-\frac{i}{\hbar} \eta(t) \hat{H}_0\right) \exp \sum_k (\zeta_k^*(t) \hat{c}_k \hat{c}_{-k} - \zeta_k(t) \hat{c}_k^\dagger \hat{c}_{-k}^\dagger), \quad (4.19)$$

where

$$\eta(t) = -\frac{1}{2} \Xi(\Omega) \int_0^t dt' h(t'), \quad (4.20)$$

and

$$\zeta_k(t) = \frac{i}{4} \Xi(\Omega) \int_0^t dt' h(t') e^{2i\omega_k t'}. \quad (4.21)$$

Inspection of (4.19) reveals that the effect of the gravitational interaction is encoded in the action of two-mode quadrature squeezing operators [38, 39] $\hat{S}_{\hat{c}_k, \hat{c}_{-k}} = \exp(\zeta_k^*(t) \hat{c}_k \hat{c}_{-k} - \zeta_k(t) \hat{c}_k^\dagger \hat{c}_{-k}^\dagger)$, followed by a phase rotation. The squeezing parameter $\zeta_k(t)$ is indeed very small, due to the intrinsic weakness of the gravitational wave.

5. Two-Mode Coherent State evolution

Our main goal is the analysis of the behaviour of a coherent phonon state interacting with a weak gravitational wave.

A coherent state $|z_p\rangle$ is generated out from the vacuum under the action of the displacement operator

$$\hat{D}(z_p) = \frac{1}{2} \exp(z_p \hat{c}_p^\dagger - z_p^* \hat{c}_p), \quad z_p = |z_p| e^{i\phi_p}. \quad (5.1)$$

Indeed, using the Baker-Campbell-Hausdorff formula $e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}+[\hat{A},\hat{B}]/2}$, which holds true for any two operators \hat{A} and \hat{B} whose commutator is a c-number, we have

$$|z_p\rangle \equiv \hat{D}(z_p)|0\rangle = e^{-\frac{1}{2}|z_p|^2} e^{z_p \hat{c}_p^\dagger} |0\rangle = e^{-\frac{1}{2}|z_p|^2} \sum_n \frac{z_p^n}{\sqrt{n!}} |0\rangle, \quad (5.2)$$

representing a coherent state with amplitude $|z_p|$ and phase ϕ_p . Notice that $\hat{N}|z_p\rangle = |z_p|^2$, so that $|z_p|^2$ can be considered as the mean phonon number in the coherent state $|z_p\rangle$.

To gain more insight in the physical meaning of the state $|z_p\rangle$, let us introduce the ionic density operator $\hat{n} = -n_0 \partial_x \hat{u}$

$$\hat{n} = -\frac{i\rho}{m} \sum_k N_k (\hat{c}_k e^{ikx} - \hat{c}_k^\dagger e^{-ikx}), \quad (5.3)$$

with m being the ionic mass and n_0 the ionic number density ($\rho = n_0 m$). The expectation value of \hat{n} for the coherent state $|z_p\rangle$ is

$$\begin{aligned} \langle \hat{n} \rangle &= \langle z_p | \hat{n} | z_p \rangle = -\frac{i\rho}{m} [N_p z_p p e^{i(px - \omega_p t)} - \text{h.c.}] \\ &= \left(\frac{2\rho |p|}{m^2 ALc_l} \right)^{1/2} |z_p| \sin(px - \omega_p t + \phi_p). \end{aligned} \quad (5.4)$$

(5.4) has the form of a travelling density wave (i.e., an *acoustic* wave, corresponding to the coherent phonon state).

Let us now move to the two-mode coherent state (TMCS)

$$|z_p, z_{-p}\rangle \equiv \hat{D}(z_p) \hat{D}(z_{-p}) |0\rangle = \exp\left(-\frac{1}{2}(|z_p|^2 + |z_{-p}|^2)\right) \exp(z_p \hat{c}_p^\dagger) \exp(z_{-p} \hat{c}_{-p}^\dagger) |0\rangle. \quad (5.5)$$

Using the particle number operator (4.17) we get

$$\langle |z_p, z_{-p}\rangle | \hat{N} | z_p, z_{-p}\rangle = |z_p|^2 + |z_{-p}|^2. \quad (5.6)$$

Hence the state (5.5) represents a coherent state made of $|z_p|^2$ phonons having (quasi-) momentum p and $|z_{-p}|^2$ counterpropagating phonons with (quasi-) momentum $-p$.

Using again (5.3) let us evaluate

$$\langle |z_p, z_{-p}\rangle | \hat{n} | z_p, z_{-p}\rangle = -\frac{i\rho}{m} N_p p [|z_p| e^{i(px - \omega_p t + \phi_p)} - |z_{-p}| e^{-i(px + \omega_p t - \phi_{-p})} - \text{h.c.}]. \quad (5.7)$$

Suppose that the phonon amplitudes of the TMCS are equal (the same phonon number for each mode), $|z_p| = |z_{-p}|$. Then the initial total phonon number is $\langle \hat{N}(0) \rangle = 2|z_p|^2$, and (5.7) reads

$$\langle \hat{n} \rangle = \langle z_p, z_{-p} | \hat{n} | z_p, z_{-p} \rangle = \frac{4\rho N_p p}{m} |z_p| \sin\left(px + \frac{\phi_p - \phi_{-p}}{2}\right) \cos\left(\omega_p t + \frac{\phi_p + \phi_{-p}}{2}\right). \quad (5.8)$$

Clearly we are now facing a *standing* wave, due to the superposition of two coherent phonon states having opposite momenta.

Consider now the action of the gravitational wave. After the gravitational interaction, the expectation value of the particle number operator \hat{N} is

$$\begin{aligned} \langle \hat{N} \rangle &= \lim_{t \rightarrow +\infty} \langle |z_p, z_{-p}\rangle | \hat{N} | z_p, z_{-p} \rangle \\ &= \lim_{t \rightarrow +\infty} \langle |z_p, z_{-p}\rangle | \sum_k (|\alpha_k(t)|^2 \hat{c}_k^\dagger \hat{c}_k + |\beta_k(t)|^2 \hat{c}_{-k}^\dagger \hat{c}_{-k} \\ &\quad + \alpha_k^*(t) \beta_k(t) \hat{c}_k^\dagger \hat{c}_{-k} + \alpha_k(t) \beta_k^*(t) \hat{c}_k \hat{c}_{-k} | z_p, z_{-p} \rangle. \end{aligned} \quad (5.9)$$

Looking at (5.9) we see that the first two terms in round brackets yield

$$(|\alpha_p(t)|^2 + |\beta_p(t)|^2)(|z_p|^2 + |z_{-p}|^2) + \sum_k |\beta_k(t)|^2 = |z_p|^2 + |z_{-p}|^2 + O(\hbar^2). \quad (5.10)$$

The remaining terms in the last row of (5.9) give the following first-order $O(\hbar)$ contribution

$$\lim_{t \rightarrow +\infty} 2\Re \langle z_p, z_{-p} | \sum_k \alpha_k(t) \beta_k^*(t) \hat{c}_k \hat{c}_{-k} | z_p, z_{-p} \rangle + O(\hbar^2). \quad (5.11)$$

Recalling that $\langle \hat{N}(0) \rangle = 2|z|^2$, we put (5.10) and (5.11) together, using also the explicit forms (4.15) and (4.16). We get, with $\bar{\phi} = \phi_p + \phi_{-p}$

$$\langle \hat{N} \rangle = \langle \hat{N}(0) \rangle \left[1 + \Re e \left(i\omega_p \Xi(\Omega) e^{i\bar{\phi}} \int_0^{+\infty} dt' h(t') e^{2i\omega_p t'} \right) \right] + O(h^2). \quad (5.12)$$

We see that the gravitational interaction gives rise to a first-order ($O(h)$) change in the phonon number

$$\delta N = \Re e \left[i\omega_p e^{i\bar{\phi}} \Xi(\Omega) \int_0^{+\infty} dt' h(t') e^{2i\omega_p t'} \right] \langle \hat{N}(0) \rangle, \quad (5.13)$$

proportional to the initial phonon number $\langle \hat{N}(0) \rangle$.

5.1. Response to a continuous sinusoidal wave

As an example, consider the response of the elastic line to a sinusoidal gravitational wave, having frequency ω_g , produced by a steady source. In the TT reference frame we have

$$h(t) = \sin(\omega_g t + \phi_g), \quad (5.14)$$

$$h_+(t) = A_+ h(t), \quad h_\times(t) = A_+ h(t). \quad (5.15)$$

Assume that the measurement has started at $t = 0$ in the \mathbf{u} frame. Then, from (5.13)

$$\frac{\delta N(t)}{\langle N(0) \rangle} = \Re e \left[i\omega_p e^{i\bar{\phi}} \Xi(\Omega) \int_0^t dt' \sin(\omega_g t' + \phi_g) e^{2i\omega_p t'} \right]. \quad (5.16)$$

The most interesting condition is $\omega_g = 2\omega_p$. In this case, after an integration time $t = T$ we get

$$\frac{\delta N(t)}{\langle N(0) \rangle} = \frac{1}{4} \omega_g \Xi(\Omega) \left[\frac{1}{2\omega_g} \sin(2\omega_g t + \phi_g + \bar{\phi}) - \frac{1}{2\omega_g} (\phi_g + \bar{\phi}) - T \cos(\phi_g - \bar{\phi}) \right]. \quad (5.17)$$

If $T \gg \omega_g^{-1}$, the last term in square brackets becomes dominant with respect to the remaining terms. Hence

$$\frac{\delta N(t)}{\langle N(0) \rangle} \simeq -\frac{1}{4} \omega_g \Xi(\Omega) T \cos(\phi_g - \bar{\phi}), \quad T \gg \omega_g^{-1}. \quad (5.18)$$

The above result is a nice example of parametric amplification of a two-mode coherent phonon state induced by a gravitational wave. We point out that we are facing a first-order effect in the gravitational strain $h(t)$ [observe that $\Xi(\Omega) \sim O(h)$].

As a rough numerical estimate, take a steel string with $c_L \simeq 5 \times 10^3 \text{ ms}^{-1}$. Consider a steady gravitational source having a frequency $\omega_g = 10^3 \text{ s}^{-1}$ and amplitude $A_+ \simeq 10^{-21}$. If the gravitational wave propagates along the z -direction of the observer's frame, $\Xi(\Omega) = A_+$. Parametric resonance condition ($\omega_g = 2\omega_p$) requires a string with minimum length $L = \frac{\pi c_L}{\omega_p} \simeq 31.4 \text{ m}$. Assume a power source $P = 1 \text{ mW}$ @ $\omega_p = 500 \text{ s}^{-1}$, so that the mean phonon number in the standing wave along the string is $\langle N \rangle = \frac{PL}{c_L \hbar \omega_p} \simeq 1.3 \times 10^{26}$. Being a coherent state, the corresponding fluctuation is $\Delta N = \sqrt{\langle N \rangle}$. From (5.18) we have (assuming optimal phase conditions)

$$SNR = \left| \frac{\delta N(T)}{\Delta N} \right| = \frac{1}{4} A_+ \omega_g \sqrt{\langle N \rangle} T \simeq 2.8 \times 10^{-6} T. \quad (5.19)$$

After an integration time $T = 1$ month, we would get $SNR \simeq 7$. Obviously, this is a highly idealized scenario. We have discarded all the dissipative processes which unavoidably would cause a relevant attenuation in the SNR . Nevertheless, the result seems of some interest from a theoretical point of view, deserving—in our opinion—further careful investigation. In particular, we point out that the signal-to-noise ratio could be improved (even in a shorter integration time) provided the initial coherent state were prepared in a suitable phonon number-squeezed state. This latter issue is currently a field of intensive research (see [37, 40] and refs. cited therein).

6. Density fluctuations: gravitational wave-induced phase shift

We complete the present analysis studying the behaviour of the ionic density $\langle \hat{n} \rangle$ due to the interaction of a TMCS with a gravitational wave. We use again (5.3), where the operators \hat{c}_k and \hat{c}_k^\dagger evolve in time according to (4.15) and (4.16). Hence

$$\begin{aligned}
\langle \hat{n} \rangle &= \langle z_p, z_{-p} | \hat{n} | z_p, z_{-p} \rangle \\
&= \langle z_p, z_{-p} | -i \frac{\rho}{m} \sum_k N_k k [(\alpha_k(t) \hat{c}_k + \beta_k(t) \hat{c}_{-k}^\dagger) e^{ikx} - \text{h.c.}] | z_p, z_{-p} \rangle \\
&= -\frac{i\rho}{m} N_p p |z_p| [\alpha_p(t) (e^{i(px+\phi_p)} - e^{-i(px-\phi_p)}) + \beta_p(t) (e^{i(px-\phi_p)} - e^{-i(px+\phi_p)}) - \text{h.c.}]. \tag{6.1}
\end{aligned}$$

To be definite, suppose that the phases satisfy the following condition

$$\phi_{-p} = -\phi_p. \tag{6.2}$$

Then (6.1) reads

$$\begin{aligned}
\langle \hat{n} \rangle &= 4 \frac{\rho}{m} N_p p |z_p| \sin(px + \phi_p) \Re(\alpha_p(t) + \beta_p(t)) \\
&= 4 \frac{\rho}{m} N_p p |z_p| \sin(px + \phi_p) \left[\cos \omega_p t + \omega_p \Xi(\Omega) \Re \left(\int_0^t dt' h(t') (1 + e^{2i\omega_p t'}) \right) \sin \omega_p t \right]. \tag{6.3}
\end{aligned}$$

From (6.3) we see that the effect of the gravitational interaction is the appearance of a $O(h)$ standing wave, superimposed to the pre-existing phonon wave. The amplitude of such $O(h)$ wave varies in time according to the small $O(h)$ quantity in round brackets. We may also rewrite the above result as follows

$$\langle \hat{n} \rangle = 4 \frac{\rho}{m} N_p p |z_p| \sin(px + \phi_p) \cos(\omega_p t - \Phi(t)), \tag{6.4}$$

where

$$\Phi(t) = \omega_p \Xi(\Omega) \Re \left(\int_0^t dt' h(t') (1 + e^{2i\omega_p t'}) \right). \tag{6.5}$$

Hence, the effect of the gravitational wave interaction appears through a *time-varying* phase in the original phonon standing wave. It may be that future technological improvements allow for the detection of such small phase variations, e.g., exploiting acousto-optical properties of selected materials.

7. Concluding remarks

In this paper we have analyzed the influence of a weak gravitational wave on a one-dimensional elastic string (an *elastic line*). Working in the long wavelength limit we have considered the string as a continuous medium.

Following a lagrangian approach, we have described the longitudinal wave propagation due to the elastic string properties. We have then generalized the theory to the case of a slightly curved background, in presence of a weak gravitational wave. In the analogue gravity framework, we have represented the matter-gravity interaction by means of a suitable *acoustic* metric, hence obtaining the corresponding wave Hamiltonian for the elastic line.

After quantization, we have employed the resulting *phonon* Hamiltonian (having the structure of the Luttinger liquid Hamiltonian) to explore time evolution of phonon vacuum state in presence of a gravitational wave. We have thus proven that gravitational waves can be effective in phonon excitation out from the vacuum state. Although derived in a quite different physical context, such result basically agrees with those obtained elsewhere in the literature [21]. Also, we have shown that the gravitational interaction gives rise to a weak squeezing in the phonon vacuum. We have further considered the behaviour of a two-mode coherent phonon state, proving the occurrence of parametric amplification effects when suitable resonance conditions are met. Finally, we have investigated ionic fluctuations in a coherent phonon standing wave, in which gravitational waves induce a time-dependent phase shift.

Our results suggest that phonon creation and parametric amplification as well as other related effects induced by gravitational waves do represent a quite general manifestation of the peculiar response of macroscopic particle ensembles whose internal structure admits some kind of wave propagation. Actually, the above discussed effects are likely to be described not only in elastic media but also in other physical systems allowing for wavelike propagation of small perturbations, as (super-)fluids and BECs, although in the latter case several experimental challenges have to be overcome as, e.g., the typical very short condensate lifetime.

Experimental observation of the above described effects in a thin, elastic string is presumably still out of reach with the present technology. Also when ultra-low temperature are reached (in order to reduce thermal noise), the main challenge still remains handling the quantum noise. In that respect, phonon squeezing in coherent states [39, 40] could be a promising route, although recent claims about the experimental production of such squeezed phonon states [41] have not been confirmed yet.

Apart from present-day experimental observation, the proposed analysis seems however theoretically interesting, as it further contributes to shed light on the interaction between gravity and matter fields described

in a semi-classical framework. It could also suggest possible valuable GW-based tests for General Relativity [3] in the not-too-distant future.

A more exhaustive investigation, involving also dissipative effects in three-dimensional elastic media—both in the discrete and in the continuous regime—could represent the natural further step in a future work.

Acknowledgments

We would like to thank the referees for their helpful comments.

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