THE THEORY OF LINEAR ELECTRON ACCELERATORS

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By

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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ABSTRACT

The following work deals mainly with two basic subjects in the theory of linear electron accelerators, the accelerating field and the electron orbits. The discussion of the field problem begins with a resume of the existing theory of wave propagation in periodic structures with particular reference to the accelerator tube, that is, a disk-loaded waveguide; then it proceeds with different methods of solution, both rigorous and approximate, including the formulation of a variational method credited to Schwinger, and with numerical results. Both the longitudinal and the transverse orbits are discussed with neglect of space charge. The cases considered are classified according to two dimensionless parameters, the field amplitude α and the phase velocity β. Particular emphasis is placed on the most important and the simplest case α = constant, $\beta = 1$. Analytic solutions under different restrictive conditions are obtained, and numerical examples are given. The numerical solutions for several specific designs are also described. Various aspects of the bunching and focusing problems are discussed in detail, including a study of the case of rapidly varying parameters. Besides the basic subjects, the discussion covers the field energy and related quantities, such as the group velocity, energy velocity, attenuation, Q, and shunt impedance, thus connecting the theory with design and operation, and furnishing a clear picture of linear acceleration process. The loss in acceleration due to random constructional errors is also discussed.

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CHAPTER I

INTRODUCTION

Ever since Rutherford accomplished the disintegration of nuclei by alpha particles in 1919, physicists have been engaged in producing high energy particles for nuclear research work. There exist today at different laboratories various kinds of machines in constant use for accelerating charged particles, both positive ions and electrons, to energies as high as several hundred Mev. New huge machines, mostly having circular orbits, are being built at several places to produce more energetic particles.

According to present-day knowledge, the practical energy limit of circular machines that accelerate electrons is probably only a few Bev., since at higher energies they would be ineffective due to excessive radiative loss. On the other hand, the linear type of electron accelerator is free from the limitation due to radiative loss and may cost less at high energies. Besides, there are other technical considerations in favor of the linear type, such as larger beam currents and the greater ease with which the electrons can be injected to and ejected from the accelerator.

The two types are equally old in history, though the linear type has not been developed as much and used as often as the other type. Two principal reasons may be mentioned in this connection: first, the circular machines are quite satisfactory and economical enough in the moderate energy range; second, no suitable high-frequency power sources were available for linear accelerators. After some initial effort and success, the development work of the linear accelerator had been virtually suspended. The activity was revived at the end of the second World War, when the newly developed radar equipment and techniques became available.

Soon after serious efforts were begun, it became apparent that notwithstanding many advantages the linear accelerators are subject to one serious disadvantage, that is, the lack of orbit stability. Consider an equilibrium orbit in space-time coordinates along which a charged particle may travel in an electromagnetic field and continuously absorb energy from it. If this orbit is stable, the traveling particle should be able to resist any small perturbation from its equilibrium state of motion. In other words, if the particle happens to be displaced a small distance either longitudinally (in a direction parallel to its path) or transversely (in a direction perpendicular to its path), there should arise a restoring force to move the particle back to its equilibrium orbit. As proved by McMillan (see Section 6.1), the orbits in linear accelerators do not have first-order stability in all directions. By first-order it is meant that the variation in the particle velocity and the variation of position in the transverse plane in one cycle of the accelerating field are negligible in comparison with the respective quantities themselves. If the orbit has longitudinal stability it must be unstable in some transverse direction; and if it has stability in all transverse directions it must be unstable in the longitudinal direction. The early successes of linear accelerators are to be attributed to high-order effects. With respect to orbit stability the circular machines are much superior; they can possess first-order stabilizing effects in all directions.

Soon it also became apparent that although the problem of stability is very serious with heavy particles, it nevertheless is inconsequential with electrons. This fact was first stressed by Hansen. The reason may be briefly stated as follows. In an accelerating field of great strength the electrons, which have a small rest mass, will soon attain relativistic velocities. Once the velocity is near that of light, it can no longer be increased appreciably. The mass,momentum or energy of the electron will increase continuously while its velocity will be practically constant. If the electrons enter the accelerator with negligible radial velocities and near to the axis, they will not spread much laterally before spreading is no longer appreciable. If they enter in bunches with small axial extent, they will not be seriously debunched before debunching is practically negligible. Furthermore, besides the electric force there is magnetic force. The radial component of the resultant force tends to zero as the electron velocity tends to the velocity of light. Consequently the defocusing effect is in general small except at the very initial stage of acceleration, where the subsidiary means of stabilization can easily be applied. As far as the theory predicts, the linear electron accelerator is the most advantageous machine to be built for producing electrons with very high energies. And so far no doubt is cast by experimental evidence.

At Stanford University a project for the construction of a billion-volt linear electron accelerator is in progress. It was started immediately after the success of the fundamental research work on a 6 Mev. model accelerator. To check the final details for the big machine, another 50 Mev. machine was constructed. It has been tested and operated successfully. Now, an energy of about 150 Mev. has been obtained from the first few sections of the billion-volt machine. Man-made billion-volt electrons are just a question of time, and it is almost certain that electrons will soon be the most energetic of all charged particles accelerated in the laboratory.

This report is concerned with a number of mathematical problem arising in the theory of linear electron accelerators. The discussion will be limited to those subjects which directly pertain to the linear acceleration of electrons, although some of the results may also be applicable to heavy particles. There are then two fundamental problems: the field problem and the orbit problem. The subject matter of the field problem is the disk-loaded waveguide, a structure equivalent to the loaded transmission lines used for low frequencies. The phase velocity of wave propagation through such a waveguide varies with loading and can have any value from zero to infinity. For the purpose of accelerating electrons the field must have a strong longitudinal electric component traveling with a desired phase velocity which may or may not be equal to the electron velocity. The important question is how to load a waveguide to obtain the proper phase velocity; or what amounts to the same thing, what are the relations between the phase velocity and the various structural dimensions? The orbit problem deals with the action of the field on the electrons. Assume that the electrons are uniformly distributed in time at the injection end. Is it possible to bunch all the electrons in one wave cycle into a small phase angle such that they all gain practically the same maximum energy? Is it also possible to focus them transversely to a beam of very small cross section? If possible, is it also practicable? For a given accelerating field and known initial conditions, what are the actual orbits of the electrons? Such questions, of course, must be answered by solving the differential equations of the electron motion.

This study is made against a background of the rapid progress at Stanford in the actual development work on linear electron accelerators. Some earlier results of this study have already been published in several papers and part of these will be included here in order to give a general connected account of the subject. References to the published materials will be made at appropriate places in the text.

The next two chapters deal with the field problem. In Chapter II the general nature of the problem and the essential properties of its solutions are discussed without actually solving it. The discussion begins with the Maxwell equations and boundary conditions, proceeds with a comparison of various forms of differential equations satisfied by the wave functions then with such topics as may be answered by the existing theory of wave propagation in periodic structures, in particular, the Floquet theorem, the dependence of the wave number on frequency. passing and stopping bands, traveling and standing waves, etc., and is concluded with an outline of the procedure of solution.

Chapter III discusses the methods of solution together with numerical results. First, the problem is solved in a formal manner in two different ways by the method of matching functions. Formal solutions do not yield numerical results but help to understand the physical problem and form the basis of approximate solutions. Then various approximate solutions are discussed, each being valid under certain restrictive conditions. The results are rather simple though not very accurate. For the sake of accuracy, a very flexible variational method credited to Schwinger is formulated and applied. By this method both an upper bound and a lower bound of the exact value can be calculated and both bounds can be made to approach each other as closely as may be desired.

Chapters IV, V and VI deal with the orbit problem. The equations of motion are derived in Chapter IV, simply from the

Lorentz force equation. The simplifying conditions used in the derivation are very well satisfied when only the longitudinal part of the electron motion is considered. The latter subject is discussed in Chapter V. Separate cases are classified according to two dimensionless parameters, the field amplitude α and the phase velocity β. The most important case is α = constant, $\beta = 1$ and only this case is exactly soluble by elementary functions. Other cases have to be solved approximately or by numerical methods. Different methods of bunching are discussed and compared; best results may be obtained by increasing α and β simultaneously and properly. Numerical results of several concrete examples including one effective buncher design are given and discussed; some of the results were obtained on the differential analyzer at the University of California, Los Angeles.

The transverse motion and focusing of electrons are discussed in Chapter VI. Though dealing with a more complicated subject, this chapter follows the same general spirit as the last one, attempting to keep both theoretical and practical interests in view. The equation of motion is found to be a generalized Lamé equation having one regular and four elementary singular points. Approximate analytical solutions to a number of useful cases are obtained. Much emphasis is thrown on the most important case α = constant, $\beta = 1$. The differential analyzer solutions obtained at U.C.L.A. for two specific designs are described and discussed. With the inclusion of a focusing field the orbit problem is reformulated by the Hamiltonian method. A critical examination is given of the validity of various simplifying conditions and further extension made to cover the case of rapidly varying parameters. Various aspects of the focusing problem are discussed in detail.

Chapter VII discusses a number of characteristic quantities of a linear electron accelerator from the energy point of view. Chapter VIII discusses the effect of random constructional errors by utilizing the simple equivalent circuit concept. Supplementary remarks are given in the last chapter.

CHAPTER II

THE ACCELERATING FIELD

2.1. Introductory Remarks

In a linear electron accelerator, the moving electron is to take up energy along its path from an electromagnetic field. Maximum acceleration will be obtained if the electron always finds itself in such a phase relative to the field that maximum energy can be absorbed. Thus the field and the electron should travel together with exactly the same speed, save for some modifications during the initial bunching stage which appear to be highly desirable. In plain waveguides where only a single mode of electromagnetic wave is present and propagated, the velocity of propagation, or the phase velocity, is always greater than the velocity of light. Such simple fields in plain waveguides cannot therefore be used for accelerating electrons A rather complicated structure must be used in order to set up the rather complicated field needed. This field has infinitely many components or modes which superpose on each other to provide an accelerating component that is slowed down to the desired velocity. The velocity of the accelerating wave depends in a continuous fashion on the pattern of the field. By varying the structure, we may change the pattern and so adjust the phase velocity of the field at will; indeed, we may get any phase velocity that may be desired.

In this report we will consider exclusively one type of structure, which is called the disk-loaded waveguide.[1] The

1. E.L.Chu and W.W.Hansen, J.App.Phys., 8, 996-1008 (1947)

structure is shown in Fig. 2.1. It is not only the earliest but also the most promising type for future linear electron accelerators. The present billion-volt accelerator project at Stanford University is based on the loading-disk design. In fact, most of the linear electron accelerators now under construction or under experiment at various places in America[2] and in England[3] have more or less the same features. It does not seem likely that changes in design of a fundamental nature will take place. One particular structure is as good as any for theoretical discussion in so far as it involves the same fundamental principles. Our restriction to a particular model will not diminish the value of the results of our theoretical investigation.

Despite some modifications in the short bunching section and some very slight progressive changes in dimensions (a and d) in the first few feet of tube, the accelerator tube is basically a periodic structure. Wave propagation in the accelerator tube may be likened to the vibration of a weighted string, the passing of signals through a loaded transmission line or a band-pass filter, or the diffraction of electromagnetic or matter waves in crystal lattices.[4] All such periodic structures have essential properties in common. The most

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2. We may mention M.I.T. and Purdue University.

3. For example, Atomic Energy Research Establishment.

4. For a general discussion of various kinds of periodic structures see L. Brillouin, "Wave Propagation in Periodic Structures," (McGraw-Hill Book Co., Inc. 1946).

outstanding property is the relationship between the frequency υ and the wave number τ. As mentioned above, there exist in the field for a given frequency an infinite number of components arising from reflection and scattering by the discontinuities, which present themselves in a regular and periodic way. At any instant of time or any point in space, each component wave has a different value of τ, travels in a definite direction, positive or negative, and with a definite speed, which may be either greater or less than the velocity of light c. The desired objective is that the accelerating wave, having the desired phase velocity, should carry as large a fraction of the total input power as possible.

The problem is, of course, to solve Maxwell's equations subject to proper boundary conditions. Any theoretically possible solution can be realized in practice by properly feeding the accelerator tube but may not be useful for the acceleration of electrons, so we must look for a particular solution which is most suitable for this purpose. The solution need only contain the transverse magnetic (TM) modes and can be independent of the angular variable ϕ. This is obvious because it is only essential to have an axial electric field for accelerating electrons, and it is expedient to avoid as many complications as possible.

2.2 The Differential Equation and Boundary Conditions

In the Gaussian system of units, in which the dielectric constant and the permeability of vacuum are both numerically equal to unity, Maxwell's equations for free space are written as

$$\left.\begin{aligned} \operatorname{curl} \vec{H} &= \frac{1}{c} \frac{\partial \vec{E}}{\partial t} , \\ \operatorname{curl} \vec{E} &= -\frac{1}{c} \frac{\partial \vec{H}}{\partial t} , \\ \operatorname{div} \vec{H} &= 0 , \\ \operatorname{div} \vec{E} &= 0 , \end{aligned}\right\} \qquad (2.1)$$

where $\vec{E}(x,y,z;\ t)$ and $\vec{H}(x,y,z;\ t)$ are vectors representing the electric and magnetic field intensities respectively. For the boundary conditions we assume that the metallic walls are perfectly conducting. Thus the tangential component of $\vec{E}$ and the normal component of $\vec{H}$ are to vanish on the metallic boundaries, that is

and

$$\left.\begin{aligned} E_t &= 0 \\ H_n &= 0 \end{aligned}\right\} \quad \text{on boundary} \qquad (2.2)$$

This assumption of perfect conductivity is highly justifiable from the fact that the errors thus introduced in the frequency and in the relative distribution of the fields are negligible. We will consider finite conductivity when we come to the question of energy loss.

Maxwell's equations can be easily reduced to two vector wave equations

$$\left.\begin{aligned} \nabla^2\vec{E} - \frac{1}{c^2} \frac{\partial^2\vec{E}}{\partial t^2} &= 0 , \\ \nabla^2\vec{H} - \frac{1}{c^2} \frac{\partial^2\vec{H}}{\partial t^2} &= 0 \end{aligned}\right\} \qquad (2.3)$$

If we use the Lorentz vector and scalar potentials $\vec{A}$ and V, defined by

$$\left.\begin{aligned} \vec{H} &= \operatorname{curl} \vec{A} \\ \vec{E} &= -\operatorname{grad} V - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} , \end{aligned}\right\} \tag{2.4}$$

and

$$\operatorname{div} \vec{A} = -\frac{1}{c}\frac{\partial V}{\partial t} ,$$

we may reduce Maxwell's equations into one scalar and one vector wave equation as follows:

$$\left.\begin{aligned} \nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} &= 0 , \\ \nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} &= 0 , \end{aligned}\right\} \tag{2.5}$$

The time dependence factors in $\vec{E}(x,y,z;\ t)$, $\vec{H}(x,y,z;\ t)$, $V(x,y,z;\ t)$ and $\vec{A}(x,y,z;\ t)$ can be separated out at once. We see from the equations that if $\vec{H}$ depends on time as e^{-jkct}, i.e., $\vec{H}(x,y,z;\ t) = \vec{H}(x,y,z)\cdot e^{-jkct}$ where $k = 2\pi\upsilon/c = \omega/c$, $\vec{E}$ will depend on time as $e^{-j(kct - \pi/2)}$, i.e., $\vec{E}(x,y,z;\ t) = \vec{E}(x,y,z)\cdot e^{-j(kct - \pi/2)}$. $\vec{A}$ corresponds to $\vec{H}$ while V corresponds to $\vec{E}$ in regard to the time dependence. By separating out the time factors from (2.3) and (2.5) we obtain the following vector and scalar Helmholtz equations:

$$\left.\begin{aligned} \nabla^2 \vec{E} + k^2 \vec{E} &= 0 , \\ \nabla^2 \vec{H} + k^2 \vec{H} &= 0 , \end{aligned}\right\} \tag{2.6}$$

and

$$\left.\begin{aligned} \nabla^2 V + k^2 V &= 0 , \\ \nabla^2 \vec{A} + k^2 \vec{A} &= 0 . \end{aligned}\right\} \tag{2.7}$$

Here $\vec{E}$, $\vec{H}$, V and $\vec{A}$ are independent of t.

For TM modes, $H_z = 0$, i.e., $\text{curl}_z\vec{A} = 0$, we may take[5] in cylindrical coordinates

$$A_r = A_\phi = 0, \quad A_z = A_z(r,z) \neq 0 \tag{2.8}$$

and express all the other field quantities in terms of A_z. A_z is to depend on r and z only, since all field quantities are to be independent of ϕ. We obtain from equations (2.4):

$$V = -\frac{1}{k}\frac{\partial A_z}{\partial z}, \tag{2.9}$$

$$H_r = 0, \quad H_\phi = -\frac{\partial A_z}{\partial r}, \quad H_z = 0, \tag{2.10}$$

---

5. If $H_z = 0$, then $\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} = 0$; because $\text{div}\,\vec{H} = 0$. Thus H_x and H_y may be derived from a scalar function $\pi(x,y,z)$ such that

$$H_x = \frac{\partial \pi}{\partial y} \quad \text{and} \quad H_y = -\frac{\partial \pi}{\partial x} .$$

Comparing these expressions with $\vec{H} = \text{curl}\,\vec{A}$, i.e.

$$H_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \quad \text{and} \quad H_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x},$$

we may write $A_x = A_y = 0$ and $A_z = \pi(x,y,z)$. Hence $A_r = A_\phi = 0$ and $A_z = \pi(r, \phi, z)$. Since A_z is to be independent of ϕ, $A_z = A_z(r,z)$. $\vec{A}$ may be replaced by $\vec{A} + \text{grad}\,U$ ($U(r,\phi,z)$ being arbitrary function of position) without altering $\vec{H}$, but there is no advantage here in introducing such complications.

and

$$
\left.
\begin{aligned}
E_r &= -\frac{\partial V}{\partial r} = \frac{1}{k}\frac{\partial}{\partial r}\left(\frac{\partial A_z}{\partial z}\right) , \\
E_\phi &= 0 , \\
E_z &= -\frac{\partial V}{\partial z} + kA_z = \frac{1}{k}\left(\frac{\partial^2 A_z}{\partial z^2} + k^2 A_z\right) .
\end{aligned}
\right\} \qquad (2.11)
$$

$\vec{A}$ has only one component A_z and $\vec{H}$ only H_ϕ, but $\vec{E}$ has two components E_z and E_r. E_z is the useful component while E_r is indispensible as long as E_z exists, because we must have div $\vec{E} = 0$. This type of field is indeed the simplest possible type we can use for supplying an axial component of electric intensity. Since $\vec{A}$ and $\vec{H}$ have only one component, we can reduce the problem to a scalar one, so it is simpler to deal with $\vec{A}$ or $\vec{H}$ than to deal with $\vec{E}$. On the other hand, since $\vec{H}$ is a more fundamental field vector and has more direct physical significances than $\vec{A}$, it is better to solve for $\vec{H}$ directly than to solve for $\vec{A}$.

From $\nabla^2 A_z + k^2 A_z = 0$, i.e., $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_z}{\partial r}\right) + \frac{\partial^2 A_z}{\partial z^2} +$ $+ k^2 A_z = 0$, we obtain by differentiation

$$
\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial H_\phi}{\partial r}\right) + \frac{\partial^2 H_\phi}{\partial z^2} + \left(k^2 - \frac{1}{r^2}\right)H_\phi = 0 \qquad (2.12)
$$

In terms of H_ϕ the $\vec{E}$ components are:

$$
\left.
\begin{aligned}
E_r &= \frac{1}{k}\frac{\partial}{\partial z}\left(\frac{\partial A_z}{\partial r}\right) = -\frac{1}{k}\frac{\partial H_\phi}{\partial z} , \\
E_\phi &= 0 \\
E_z &= -\frac{1}{k}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_z}{\partial r}\right) = \frac{1}{kr}\frac{\partial}{\partial r}(rH_\phi) .
\end{aligned}
\right\} \qquad (2.13)
$$

Thus our problem is to solve a linear partial differential equation (2.12) in two dimensions, subject to the boundary conditions represented by (2.2). We may further reduce equation (2.12) to the standard Sturm-Lionville form by rearranging or by changing the dependent variable. We have by rearranging

$$\left.\begin{aligned}&\frac{\partial}{\partial r}\left(r\frac{\partial H_\phi}{\partial r}\right) + \frac{\partial}{\partial z}\left(r\frac{\partial H_\phi}{\partial z}\right) - \frac{1}{r}H_\phi + k^2 r H_\phi = 0 \;, \\ &\frac{\partial H_\phi}{\partial n} = 0 \quad \text{on plane boundaries ,} \\ &\frac{\partial H_\phi}{\partial n} + \frac{1}{r} H_\phi = 0 \quad \text{on cylindrical boundaries ,}\end{aligned}\right\} \qquad (2.14)$$

and by changing the variable

$$\left.\begin{aligned}&\frac{\partial^2}{\partial r^2}(\sqrt{r}\,H_\phi) + \frac{\partial^2}{\partial z^2}(\sqrt{r}\,H_\phi) + \left(k^2 - \frac{3}{4}\frac{1}{r^2}\right)\sqrt{r}\,H_\phi = 0 \;, \\ &\frac{\partial}{\partial n}(\sqrt{r}\,H_\phi) = 0 \quad \text{on plane boundaries ,} \\ &\frac{\partial}{\partial n}(\sqrt{r}\,H_\phi) + \frac{1}{2r}(\sqrt{r}\,H_\phi) = 0 \quad \text{on cylindrical boundaries,}\end{aligned}\right\} \qquad (2.15)$$

or

$$\left.\begin{aligned}&\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (rH_\phi)}{\partial r}\right) + \frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial (rH_\phi)}{\partial z}\right) + \frac{k^2}{r}(rH_\phi) = 0 \;, \\ &\frac{\partial}{\partial n}(rH_\phi) = 0 \quad \text{on all boundaries.}\end{aligned}\right\} \qquad (2.16)$$

The above three forms (2.14), (2.15) and (2.16) differ from each other in using different dependent variables; they use H_ϕ , $\sqrt{r}\,H_\phi$ and rH_ϕ respectively. The second form has the interesting feature that the differential equation has only a simple Laplacian operator and is exactly a two dimensional Schrödinger wave equation, but it has the same type of mixed boundary conditions as the first form. The third form is the simplest of the three, both as regards the differential equation and the boundary conditions. We will base our analysis on this. To simplify the form further, we introduce the notation

$$u = rH_{\phi} \quad , \tag{2.17}$$

then (2.16) becomes

$$\left.\begin{aligned} &\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial u}{\partial z}\right) + \frac{k^2}{r}u = 0 \quad , \\ &\frac{\partial u}{\partial n} = 0 \qquad \text{on all boundaries.} \end{aligned}\right\} \tag{2.18}$$

2.3. Two Linearly Independent Floquet's Solutions

Having obtained the differential equation for our problem, we will now discuss the general nature of the solution and show how an infinite number of modes arise and how the phase velocity of the accelerating field slows down due to the presence of the loading disks.

With the first few feet excepted, the loaded guide is a periodic structure, that is, the spacing between two neighboring disks is a constant. Let the length of the periodically loaded guide be regarded as infinite, and the loading disks be situated at $z = 0$ and $z = md$, m being positive and negative integers. If $u(r,z)$ is a solution, it is obvious from equation (2.18) that $u(r,z + md)$, $u(r, - z)$ and u(r, - z - md) are also solutions. All these functions are continuous for $r \leq a$ and twice differentiable for $r < a$, a being the disk hole radius.

For a fixed value of $r < a$, equation (2.18) may be written as

$$\frac{d^2}{dz^2} u(r,z) + F(r,z)\, u(r,z) = 0 \quad , \tag{2.19}$$

where

$$F(r,z) = k^2 + \frac{r}{u(r,z)} \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} u(r,z) \right) .$$

If u(r,z) is either even or odd in z, F(r,z) is even in z. If u(r,z + md) is a multiple of u(r,z), F(r,z) is periodic in z with period d. Then F(r, - z) is also periodic. In the special case where F(r,z) is both even and periodic, equation (2.19) is the well-known Hill's equation.[6]

The fact that such linearly dependent solutions u(r,z+md) actually exist follows from Floquet's theory.[6] The other set of such solutions is u(r, - z + md). And the complete solution of equation (2.18) for $r < a$ is obtained by a linear combination of any two such solutions, one from each set, because they are linearly independent. Thus the complete solution has the same general form as that of Hill's equation and is written as

$$u(r,z) = C_1 e^{\gamma z} v(r,z) + C_2 e^{-\gamma z} v(r, - z) \ , \tag{2.20}$$

where C_1 and C_2 are arbitrary constants, γ is a constant to be determined and v(r,z) is a periodic function of z with period d.

6. E.T.Whittaker and G.N.Watson, "A Course of Modern Analysis," (Cambridge University Press, 1927), Chapter XIX.

unless $e^{\gamma z}$ v(r,z) is an even or an odd function. Both γ and v(r,z) depend on the function F(r,z). The most general form of v(r,z) is a Fourier series expansion:

$$v(r,z) = \sum_{-\infty}^{\infty} a_n U_n(r) e^{j2\pi nz/d} \qquad (2.21)$$

where $U_n(r)$ are functions of r only and a_n are constant coefficients. $e^{\gamma z}$ v(r,z) can be even or odd only when $\gamma = jm\pi/d$, m being any positive or negative integer or zero. In such special cases $e^{\gamma z}$ v(r,z) is periodic with period d or 2d, being even or odd according as $a_n = a_{-n-m}$ or $a_n = -a_{-n-m}$, and u(r,z) will be either one of the two particular solutions. There exist two different frequencies, one for each solution.

To show that u(r,z) expressed by equation (2.20) indeed satisfies the differential equation (2.18) for r being a variable less than a, it is sufficient to consider one single term of the series. Let the term be

$$U_n(r)\, e^{(\gamma + j\frac{2\pi n}{d})z} = U_n(r)\, W_n(z) = u_n(r,z).$$

Substituting $u_n(r,z)$ for u in equation (2.18), we obtain

$$\frac{1}{U_n}\left(\frac{d^2U_n}{dr^2} - \frac{1}{r}\frac{dU_n}{dr}\right) + \frac{1}{W_n}\frac{d^2W_n}{dz^2} + k^2 = 0 \quad .$$

This equation may be separated at once. The resulting equations are

$$\left.\begin{aligned}&\frac{d^2 W_n}{dz^2} + k_{zn}^2 W_n = 0\\ \text{and}\quad&\\ &\frac{d^2 U_n}{dr^2} - \frac{1}{r}\frac{dU_n}{dr} + k_{rn}^2 U_n = 0 \quad ,\\ \text{with}\quad&\\ &k_{zn}^2 + k_{rn}^2 = k^2\end{aligned}\right\} \tag{2.22}$$

where k_{zn}^2 and k_{rn}^2 are separation constants.

In order that u(r,z) in the form of (2.20) may satisfy the differential equation (2.18) we must have

$$k_{zn}^2 = - (\gamma + j\,\frac{2\pi n}{d})^2 \quad ,$$

i.e.,

$$\pm\, jk_{zn} = \gamma + j\frac{2\pi n}{d} \quad . \tag{2.23}$$

The ambiguity in sign of k_{zn} arises from the fact that it is k_{zn}^2, not k_{zn}, which is defined in the separation process. It does not matter whether we choose the positive or the negative sign as long as we adhere to one of them. Let us choose the positive sign and write $\gamma = jk_{zo}$, then

$$\left.\begin{aligned}&k_{zn} = k_{zo} + \frac{2\pi n}{d} \quad ,\\ \text{so}\quad&\\ &\tau_{zn} = \tau_{zo} + n/d \quad ,\end{aligned}\right\} \tag{2.24}$$

where the τ's are the corresponding wave numbers.

Restoring the time factor we may write the general solution for our problem as

$$u(r,z)\ e^{-j\omega t} = C_1 \sum_{-\infty}^{\infty} a_n U_n(r)\ e^{-j(\omega t - k_{zn} z)}$$

$$+ C_2 \sum_{-\infty}^{\infty} a_n U_n(r)\ e^{-j(\omega t + k_{zn} z)}$$

or

$$u(r,z)\ e^{-j\omega t} = C_1 \sum_{-\infty}^{\infty} a_n U_n(r)\ e^{-j2\pi\left[\upsilon t - (\tau_{zo} + n/d) z\right]}$$

$$+ C_2 \sum_{-\infty}^{\infty} a_n U_n(r)\ e^{-j2\pi\left[\upsilon t + (\tau_{zo} + n/d) z\right]}$$

$$(r \leqq a) \qquad (2.25)$$

Each term of the above series represents a particular mode propagating in the forward or in the backward direction. Since $k^2 = k_r^2 + k_z^2$ for any mode and the k_z's differ from one another by an integral multiple of $2\pi/d$, $\omega = kc$ must be a periodic function of k_z with a period of $2\pi/d$. So $\upsilon = \omega/2\pi$ is a periodic function of τ_z with a period of $1/d$. Furthermore, since k is an even function of k_z, υ must be an even function of τ_z. Since n ranges from $-\infty$ to $+\infty$, we may consider k_{zo} and τ_{zo} as to be restricted in the intervals $-\pi/d \leqq k_{zo} \leqq \pi/d$ and $-1/2d \leqq \tau_{zo} \leqq 1/2d$ respectively without missing any mode. Hereafter we will designate a given mode having $\tau_z = \tau_{zo} + n/d$ by the value of n.

2.4 Frequency versus Wave Number

The $\upsilon - \tau_{zo}$ curve for the disk-loaded waveguide will be shown later to have the general form of Fig. 2.2, just like the curve for a periodically weighted continuous string or for some other similar periodic structure. The curve consists of an infinite number of branches, each representing a certain passing band. For frequencies between any two neighboring branches or below the lower cut-off frequency of the first branch, $\upsilon_{1\ell}$, no waves can propagate through the loaded guide. Such bands are called the stopping bands. The cut-off frequency $\upsilon_{1\ell}$ is related to the tube radius b by the same relation $\upsilon_{1\ell}/c = 2.405/2\pi b$ (assuming zero disk thickness) as that for an unloaded guide and is not affected by loading. The upper stopping bands are peculiar to the loaded structure. Their widths depend on the extent of loading; they become wider as the loading becomes heavier. The lower cut-off frequency of any passing-band

$$\upsilon_{m\ell} = c\left[\left(\frac{2.405}{2\pi b}\right)^2 + \left(\frac{m-1}{2d}\right)^2\right]^{1/2}$$

does not depend on the disk hole radius a, while the upper cut-off frequency υ_{mu} depends on a. υ_{mu} approaches $\upsilon_{m\ell}$ as a approaches zero. If the loading disks have finite thickness, $\upsilon_{m\ell}$ will be different from that given by the above simple expression. The variation of the band widths with loading, however, will still have the same general nature.

From the $\upsilon - \tau_{zo}$ curve we may easily obtain the $\upsilon - \tau_z$ curve by extending the former curve in both directions of the τ_z axis such that $\upsilon(\tau_{zo} + n/d) = \upsilon(\tau_{zo})$. The phase velocity for any mode is $v_{zn} = kc/k_{zn}$, so $v_{zn}/c = \frac{\upsilon}{c\tau_{zn}}$ or is the slope of the radius vector out to the point (υ, τ_{zn}) on the $\upsilon - \tau_z$ curve. The slope is positive for $\tau_z > 0$

and negative for $\tau_z < 0$. Thus the sign of τ_z determines the direction of propagation of the wave. It is to be noted that the zero-th interval $-1/2d \leq \tau_{zo} \leq 1/2d$ includes all passing frequencies and contains all waves with wave lengths greater than or equal to 2d, i.e., $\infty \geq \lambda_g \geq 2d$ and propagating in either direction.

As long as we work in the lowest passing band, the zero-th mode will have the largest amplitude (see later analysis). It is our purpose to make the phase velocity $-2d\upsilon_{1\ell} \leq v_{zo} \leq \infty$ of the zero-th mode wave to be less than c. We use a wave number $\tau_{zo} = 1/4d$ for our model accelerator tube, thus placing the operating point at the middle of the passing band. This is a convenient choice for good practical reasons[7] but not the best from the power point of view. A somewhat larger value of τ_{zo}, e.g., $\tau_{zo} = 1/3d$ is advisable because it enables some reduction of the energy loss by using fewer loading disks and gives a somewhat smaller phase velocity. With $\tau_{zo} = 1/4d$, the phase velocity v_{zo}/c is given by $4d\upsilon/c$, the slope of the vector OP shown in Fig. 2.2. If the disk spacing d is fixed, we may reduce the phase velocity by moving down the point P toward the τ_z-axis along the line PQ. This can be done either by lowering the cut-off frequency $\upsilon_{1\ell}$, i.e. by increasing the tube radius b while keeping the passing band width constant or by increasing the loading to narrow the passing band, i.e., by reducing the

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7. E.L. Ginzton, W.W. Hansen, and W.R. Kennedy, "A Linear Electron Accelerator," Rev. Sci. Inst., 19, 89 - 108 (1948)

hole radius a of the disks while keeping $\upsilon_{1\ell}$ fixed or by making both changes. It is clear from Fig. 2.2 that in any case the vector OP can never have a slope less than $\frac{\upsilon_{1\ell}}{c} / \frac{1}{4d} = 4d/\lambda_c$, $\lambda_c = c/\upsilon_{1\ell}$ being the free space cut-off wave length. In order to get very low phase velocity,we have to use very small disk spacing, that is, we must use a large number of disks per unit distance. Decreasing d moves the points P and Q towards the right. The phase velocity is directly proportional to d and can be made as small as we please by making d sufficiently small. Thus we see that for a given size of tube (b fixed) the phase velocity can only be reduced by an increase in loading. As the loading becomes heavier, the scattering of waves becomes more prominent, so the energy dissipated in other undesirable modes will be larger. This sort of scattering loss can be reduced, but not completely avoided, by proper design of the loaded guide.

In regard to the designation of the modes, we wish to point out that there is some arbitrariness in choosing the numbering system. It is most natural from a mathematical point of view to designate $-1/2d \leqq \tau_{zo} \leqq 1/2d$ as the zero-th mode and $-1/2d + n/d \leqq \tau_{zn} \leqq 1/2d + n/d$ as the n-th mode for all passing bands, but it is not a necessity. For example, we may designate the strongest mode propagating in the positive direction as the zero-th mode[8], thus

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8 J.C.Slater, "The Design of Linear Accelerators," M.I.T. Research Laboratory of Electronics Tech.Rep. No. 47, Sept. 2, 1947 or Rev. Mod.Phys., 20, 473-518 (1948)

$-1/2d \leqq \tau_{zo} \leqq 1/2d$ for the first (lowest) passing band, $0 \leqq \tau_{zo} \leqq 1/d$ for the second, $1/2d \leqq \tau_{zo} \leqq 3/2d$ for the third, etc. In the system of numbering we have adopted, these modes correspond to the zero-th mode for the first passing band, first mode for the second and third, second mode for the fourth and fifth, etc. It is immaterial in which way we designate the modes but it is important to know what are their relative intensities. These can be determined either by mathematical analysis, which will be described later in this section, or by experimental measurements. Experimentally, we need to measure the variation of the amplitude of a field variable, say $u(r,z)$ as a function of z (r fixed). We need to measure at sufficiently many points spaced at a distance not greater than half the wave length to be measured. Otherwise we will miss the higher modes of the spectrum. For instance, if we measure the amplitudes at points along the z-axis equally spaced by a distance d, we will not be able to distinguish whether a given mode is within or outside the zero-th interval. Any two wave numbers differing from each other by an integral multiple of $1/d$ will give the same amplitudes at the same sampling points. $u(r,z)$ must be known as a function of z (r fixed) with sufficient accuracy in order to make a reliable Fourier analysis.

2.5. The Representation of the Periodically Loaded Waveguide by Multiple Transmission Lines

To show that the $\upsilon - \tau_{zo}$ curve has the general form of

Fig. 2.2, let us consider the transmission line representation of the loaded waveguide. Strictly speaking, the loaded guide should be represented by an infinite number of loaded transmission lines in parallel, each admitting of only one mode (characterized by a certain eigenfunction for the unloaded guide) and being coupled to all the other lines at every discontinuity presented by loading, but practically a few lines will suffice because the higher modes will invariably be attenuated to a negligible value before traversing appreciable distances. One such representation having s lines in parallel, is shown in Fig. 2.3.

Let $\gamma_1, \gamma_2, \ldots \gamma_s$ and $Y_1, Y_2, \ldots Y_s$ be the propagation constants and the characteristic admittances of the s lines (unloaded) respectively, we have from conventional circuit analysis

$$\left.\begin{aligned} i'_{h,n} &= -V_{h,n} Y_h \coth \gamma_h d + V_{h,n+1} Y_h \operatorname{csch} \gamma_h d \;, \\ i_{h,n} &= -V_{h,n-1} Y_h \operatorname{csch} \gamma_h d + V_{h,n} Y_h \coth \gamma_h d \;, \\ \text{and} \quad i_{h,n} - i'_{h,n} &= \sum_{\ell=1}^{s} Y_{h\ell} V_{\ell,n} \;, \quad h = 1, 2, \ldots s. \end{aligned}\right\} \tag{2.26}$$

Eliminating $i_{h,n}$ and $i'_{h,n}$ from (2.26), we obtain

$$(V_{h,n-1} + V_{h,n+1}) Y_h \operatorname{csch} \gamma_h d$$

$$= 2V_{h,n} Y_h \coth \gamma_h d - \sum_{\ell=1}^{s} Y_{h\ell} V_{\ell,n}$$

or

$$V_{h,n-1} + V_{h,n+1}$$

$$= 2V_{h,n} \cosh \gamma_h d - \frac{\sinh \gamma_h d}{Y_h} \sum_{\ell=1}^{s} Y_{h\ell} V_{\ell,n} \quad . \tag{2.27}$$

These are linear difference equations with constant coefficients, so possess solutions of exponential form

$$V_{h,n} = V_h \, e^{n\gamma d}$$

Substituting this expression for $V_{h,n}$ in (2.27) and transforming, we obtain

$$V_h(\cosh \gamma d - \cosh \gamma_h d) + \frac{\sinh \gamma_h d}{2Y_h} \sum_{\ell=1}^{s} Y_{h\ell} V_\ell = 0 \tag{2.28}$$

Thus we have s linear homogeneous equations connecting s variables V_h. In order to have non-vanishing solutions, the determinant of the coefficients must be zero. The determinental equation is of the s-th degree in cosh γd, which therefore can have s different values. For each value of cosh γd, we can determine V_h's to some arbitrary constant factor. The complete solution will be a linear combination of various terms associated with different values of cosh γd. If the boundary conditions are properly chosen, that is, if we feed and terminate the guide in a proper manner, the solution may consist of only those terms arising from any one characteristic value of cosh γd. For any such solution

$$B_h = j \sum_{\ell=1}^{s} Y_{h\ell} \frac{V_\ell}{V_h}$$

has a unique value and may be defined as the equivalent shunt susceptance of loading in the h-th line.

Assuming perfect conductivity for the guide walls, $\gamma_h = j2\pi\tau_h$, τ_h being the wave number for the h-th mode (TM_{oh}) in the unloaded guide. τ_h is related to the free space wave number τ by the following equation

$$\tau^2 = \tau_h^2 + \tau_{hc}^2 \quad ,$$

where τ_{hc} is the free space cut-off wave number for the h-th mode. Since the guide is cylindrical and has radius b, $\tau_{hc} = \chi_h/2\pi b$, χ_h being the h-th root of $J_o(\chi) = 0$. The characteristic admittance Y_h for TM_{oh} mode is defined as

$$Y_h = \frac{(H_\phi(t))_{oh}}{(E_r(t))_{oh}} = \frac{(rH_\phi)_{oh}}{-\frac{j}{k}\frac{\partial}{\partial z}(rH_\phi)_{oh}}$$

Since $(rH_\phi)_{oh} \sim e^{\gamma_h z}$, so $\partial/\partial z(rH_\phi)_{oh} \sim \gamma_h e^{\gamma_h z}$ and $Y_h = j\frac{k}{\gamma_h} = \frac{k}{2\pi\tau_h} = \frac{\tau}{\tau_h}$. Evidently the unit for Y_h is such as to make the free space admittance numerically equal to unity.

Substituting these expressions for γ_h and Y_h in (2.28) and writing $\gamma = jk_z = j2\pi\tau_z$, we obtain

$$\cos 2\pi\tau_z d = \cos 2\pi\tau_h d - \frac{B_h}{2Y_h} \sin 2\pi\tau_h d \quad . \qquad (2.29)$$

To find $\upsilon - \tau_z$ relation, it is sufficient to consider one propagating mode because all other modes including attenuated ones must give the same relation. If there are several propagating modes, it is preferable to consider the most prominent one. From equation (2.29) the following results may be noted:

(a) If $B_h \ll Y_h$, $\tau_z = \tau_h \pm n/d = \sqrt{\tau^2 - \tau_{hc}^2} \pm n/d$.

(b) If $B_h \gg Y_h$, $\tau_z = \frac{1}{2\pi d} \cos^{-1}(\pm 1 \mp \frac{\epsilon B_h}{2 Y_h})$,

where $|\epsilon| = \sin 2\pi \tau_h d \ll 1$ and $0 \leqq \epsilon B_h/2Y_h \leqq 2$.

(c) When $\tau_h = m/2d$, $\tau_z = \tau_h \pm n/d$, m and n being integers.

(d) When $\tau_z = m/2d$, $\frac{d\tau}{d\tau_z} = \frac{\tau_h}{\tau}\frac{d\tau_h}{d\tau_z} = 0$.[9]

In general, B_h changes in a rather complicated way as frequency υ changes. Let us write $B_h = kC$ or $= -1/kL$ according as it is positive or negative. C or L will also change with frequency and may be called the equivalent loading capacity or the equivalent loading inductance respectively. If we plot equation (2.29) with $B_h = kC$ and C being considered as a constant, the graph for TM_{01} mode is as shown in Fig. 2.2. The lower cut-off frequency for the m-th pass-band is, as stated before $\upsilon_{m\ell} = c\sqrt{(\frac{2.405}{2\pi b})^2 + (\frac{m-1}{2d})^2}$, while the upper

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9. Assuming $\frac{dB_h}{d\tau}$ to be finite, it can easily be shown by differentiating equation (2.29) that if $\frac{d\tau_h}{d\tau_z} = 0$ then $\tau_z = m/2d$ and if $\tau_h = m/2d$, that is $\tau_z = m/2d \pm n/d$, then $\frac{d\tau_h}{d\tau_z} = 0$. Since τ_h and $d\tau_h/d\tau_z$ are continuous even periodic functions of τ_z, we must have $\frac{d\tau_h}{d\tau_z} = 0$ when $\tau_z = m/2d \pm n/2d$.

cut-off frequency is determined by the condition

$$\left| B_h \frac{\tau_h}{2\tau} \sin 2\pi \tau_h d \right| = 2 \qquad \left(\tau^2 = \tau_h^2 + \left(\frac{2.405}{2\pi b}\right)^2 \right) .$$

When loading is slight ($a/b \cong 1$), the upper cut-off frequency $\upsilon_{mu} \cong \upsilon_{(m+1)\ell}$; but when loading is excessive ($a/b \ll 1$), the width of the pass band

$$\delta\tau = \frac{\tau_h}{\tau} \delta \tau_h = \frac{1}{B_h} \frac{2}{\pi d} = \frac{1}{kC} \frac{2}{\pi d}$$

is inversely proportional to C. If C is not constant, the exact form of the $\upsilon - \tau_{zo}$ curve will be somewhat different from that shown in Fig.2.2, but the general features will remain the same as long as C does not change sign from positive to negative in the frequency range we are interested in.

2.6. Traveling Waves versus Standing Waves

Referring to equations (2.20) and (2.25), the two terms $C_1 e^{\gamma z} v(r,z)$ and $C_2 e^{-\gamma z} v(r,-z)$ represent two opposite sets of traveling waves; the former set has the zero-th component traveling in the positive direction while the latter set has the opposite sense. If we write $C_1 = (C_c + C_s)/2$ and $C_2 = (C_c - C_s)/2$, we obtain from (2.25)

$$u(r,z) = C_c \sum_{-\infty}^{\infty} a_n U_n(r) \cos k_{zn} z$$

$$+ jC_s \sum_{-\infty}^{\infty} a_n U_n(r) \sin k_{zn} z$$

$$= C_c u_c(r,z) + jC_s u_s(r,z) \quad . \qquad (2.30)$$

Thus the general solution may also be considered as a linear combination of two types of standing waves, one cosine type and one sine type. In fact, one set of traveling waves may be considered as the superposition of two types of standing waves in time quadrature and, conversely, one type of standing waves as the superposition of two sets of traveling waves running in opposite directions. It is evident that we may likewise describe the solution in terms of both traveling waves and standing waves. For discussing $\upsilon - \tau_{zo}$ relation, it is sufficient to consider either cosine type or sine type standing waves. Except for the special case where $k_{zo}d = \pi$, both types of solution yield the same frequency υ for the same τ_{zo}.

2.7. The Reduction of the Problem of an Infinite Structure to that of a Unit Cell

Since the guide is assumed to be infinitely long, it may be assumed without loss of generality that $\frac{k_{zo}d}{\pi} = \frac{q}{p} \leqq 1$, q and p being integers and having no common factor. By taking q and p sufficiently large, $\frac{k_{zo}d}{\pi}$ may be made to approach any number $\leqq 1$ as accurately as may be desired. Thus a guide with p cells or cavities will contain q half-waves. The guide of p cells is to be closed at $z = 0$ and $z = pd$, either with electric walls ($\frac{\partial u}{\partial n} = 0$ in this problem) or with magnetic walls ($u = 0$) according to whether the cosine type or the sine type solution is being considered. As far as the evaluation of frequency is concerned, the problem of an infinite guide is equivalent to the boundary value problem of a finite one with an integral number of cells.

If the guide is closed with electric walls,

$$u = u_c = \sum_{-\infty}^{\infty} a_n U_n(r) \cos(k_{zo} + \frac{2\pi n}{d})z \; . \qquad (2.31)$$

For the other case,

$$u = u_s = \sum_{-\infty}^{\infty} a_n U_n(r) \sin(k_{zo} + \frac{2\pi n}{d})z \; . \qquad (2.32)$$

Since $\cos\left[(k_{zo} + \frac{2\pi n}{d})(z + md)\right] = \cos k_{zo}md \cos(k_{zo} + \frac{2\pi n}{d})z$

$$- \sin k_{zo}md \sin(k_{zo} + \frac{2\pi n}{d})z$$

and $\sin\left[(k_{zo} + \frac{2\pi n}{d})(z + md)\right] = \cos k_{zo}md \sin(k_{zo} + \frac{2\pi n}{d})z$

$$+ \sin k_{zo}md \cos(k_{zo} + \frac{2\pi n}{d})z \; ,$$

so $u_c(z + md) = \cos k_{zo}md \cdot u_c(z) - \sin k_{zo}md \cdot u_s(z)$

and

$$\left.\begin{aligned} u_c(z + md) &= \cos k_{zo}md \cdot u_c(z) - \sin k_{zo}md \cdot u_s(z) \\ u_s(z + md) &= \sin k_{zo}md \cdot u_c(z) + \cos k_{zo}md \cdot u_s(z) \end{aligned}\right\} \qquad (2.33)$$

In any case, the solution for the (m+1)-th cell can be obtained by a linear superposition of the cosine type and the sine type solution for the very first cell.

2.8. The Equivalence Property of the Periodically Loaded Waveguide

The property that the electrically-shorted and the magnetically-shorted guide resonator have the same frequency for the same value of τ_{zo} will be referred to as the equivalence property of the periodically loaded guide. The two types of resonators are shown in Fig. 2.4a and 2.4b. To them may be added the structures shown in Fig. 2.4c and 2.4d. The boundary value problems for these four different resonators all give the same frequency υ for the same mode (TM_{01}) and the same wave number $k_{zo} = \frac{q\pi}{pd} < \frac{\pi}{d}$.

If q is even, then p must be odd. The resonators shown in Fig. 2.4a and 2.4c must evidently resonate at the same frequency. The same holds true for the structures shown in Fig. 2.4b and 2.4d. If q is odd, then p may be either even or odd. If both are odd, Fig. 2.4a is equivalent to Fig. 2.4d and Fig. 2.4b equivalent to 2.4c as far as frequency is concerned. If q is odd while p is even the equivalence property is again true. The following proof is valid for all these different cases.

As explained before, if $u_c(r,z)$ is a solution, then $u_c(r,z + d)$ is also a solution. Let $z = (p - 1/2)d + z'$ and consider the solution

$$u_c(r,z) + u_c(r,z + d)$$

$$= u_c(r,(p - 1/2)d + z') + u_c(r,(p + 1/2)d + z')$$

$$= 2 \sum_{-\infty}^{\infty} a_n U_n(r) \cos\left[(k_{zo}+ \frac{2\pi n}{d})(pd+z')\right] \cos(k_{zo}+ \frac{2\pi n}{d})\cdot\frac{d}{2}$$

$$= 2(-1)^q \cos \frac{q\pi}{2p}\cdot \sum_{-\infty}^{\infty} (-1)^n a_n U_n(r) \cos(k_{zo} + \frac{2\pi n}{d})z'$$

$$= 2(-1)^q \cos \frac{q\pi}{2p} \cdot u_c'(z') \quad .$$

This solution satisfies the boundary condition $\frac{\partial u}{\partial n} = 0$ at $z' = 0$ and $z' = pd$ where z' is measured from a plane mid-way between two neighboring disks. It solves exactly the problem for the resonator shown in Fig. 2.4c. Thus Fig. 2.4a and

and Fig. 2.4c are equivalent as far as frequency is concerned. Therefore all four structures are equivalent.

The only exceptional case is when q = p, i.e. $k_{zo}d = \pi$. In this case $e^{\gamma z}$ v(r,z) and $e^{-\gamma z}$ v(r,-z) are no longer independent solutions. The cosine type solution gives a frequency different from that for the sine type solution.[10] One is the upper cut-off frequency of the pass-band under consideration while the other is the lower cut-off frequency for the next higher pass-band. Therefore, only one type of standing wave solution can be used. In effect, no traveling waves can pass through the guide with $k_{zo}d = \pi$.

2.9. Summary

Before proceeding to solve the field problem, it is desirable to summarize the general aspects of the problem discussed so far. The phase velocity of the main (strongest) component of waves may be slowed down to any desired value by reducing d. For a given phase velocity $\beta c = \frac{k}{k_z} c$, k_{zo} is known in terms of k and the mode of operation, defined by the value of $k_{zo}d$, may be chosen at will by assigning fixed integral values to q and p ($q < p$), and making $d = q\pi/pk_{zo}$. The problem of the infinite guide is to be replaced by a boundary value problem of a finite one having p cells. The frequency may be found from any one of the four equivalent structures shown in Fig. 2.4. It is usually preferred, however, to con-

10. L. Brillouin, loc.cit. footnote 4; J.C.Slater, Loc.cit. footnote 8.

sider those with totally enclosed electric walls. The problem of each structure may in turn be reduced to a problem of a single cell. The solution for the π-mode should be of the sine type (Fig. 2.4b) and, if the disk thickness is negligible, can be derived from the cosine type solution for the $\pi/2$ mode with d replaced by d/2. Among all different cases the $\pi/2$ mode or the equivalent π-mode is numerically the simplest.

The general form of the solution for $r \lesssim a$ is given by equation (2.31). The unknown coefficients and the frequency will be determined by the condition that the solution for $r \gtrsim a$ will satisfy the boundary condition on the cylinder and the disks and will be equal to u_c and have the same normal derivative as u_c at $r = a$. The mathematical problem is to match the solutions and their derivatives for the two different regions on their common surface.

Instead of matching solutions on a cylindrical surface, another pair of solutions may be taken and matched on a transverse plane. The closed guide structure may be considered as p cavities coupled in tandem by holes of radius a. The field in each cavity may be expressed by a single function which is valid everywhere inside but not outside of it. Two such functions and also their derivatives for any two neighboring cavities are to be matched on their common surface. The latter process, however, is not as convenient as the former, because the functions on a transverse plane are represented by a series of Bessel functions instead of the simple Fourier series.

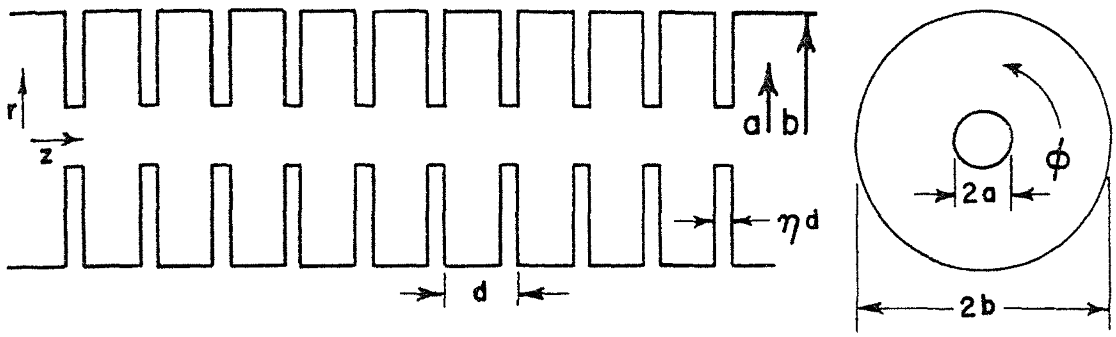

Fig. 2.1 - A disk-loaded waveguide.

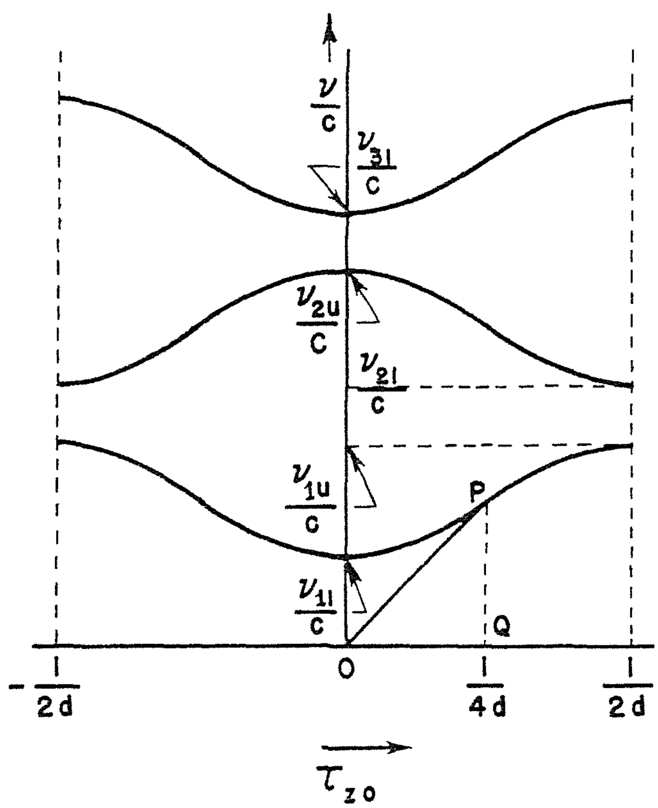

Fig. 2.2 - Frequency ν as a function of τ_{zo} for a disk-loaded waveguide, the loading being periodic.

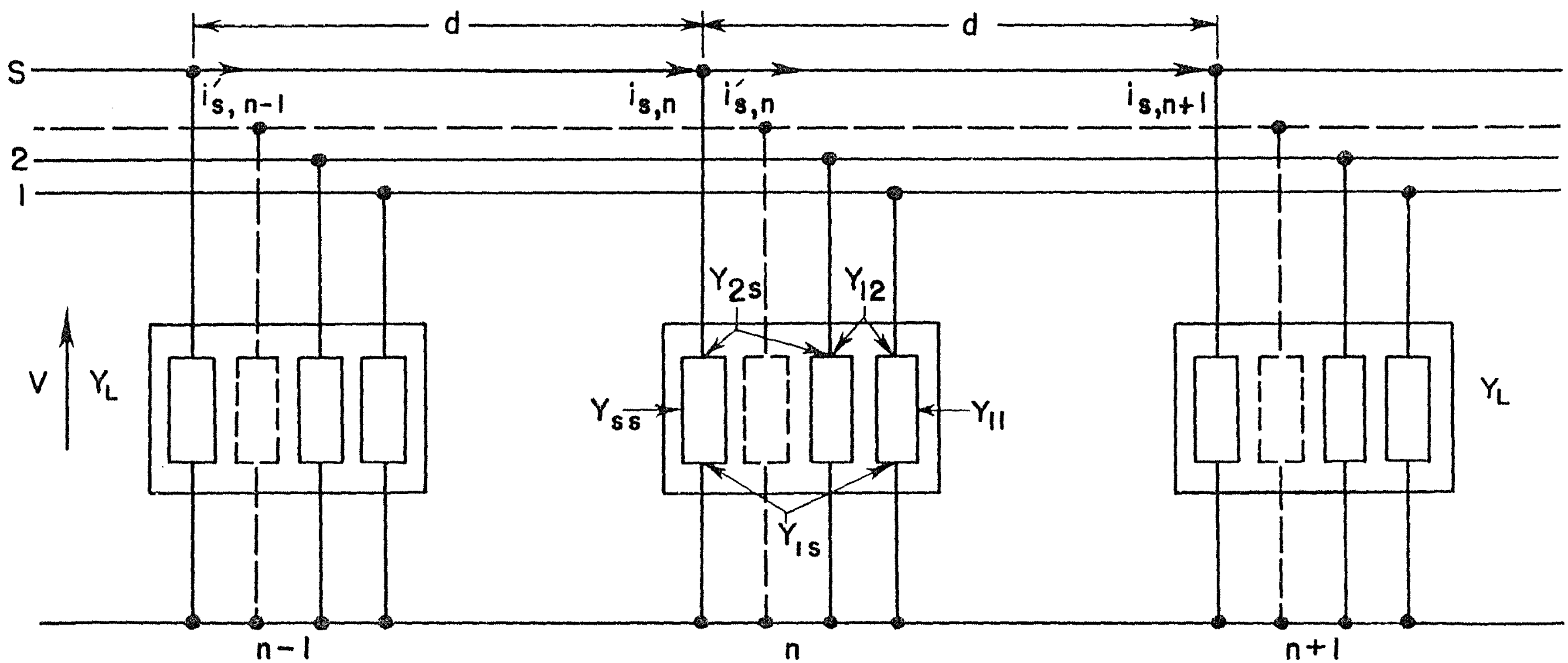

Fig. 2.3 - Equivalent circuit for the loaded waveguide consisting of s transmission lines in parallel. The shunt admittances used for loading are represented by the matrix

$$Y_L = \begin{Bmatrix} Y_{11} & Y_{12} & \cdots & Y_{1s} \\ Y_{21} & Y_{22} & \cdots & Y_{2s} \\ \cdots & \cdots & \cdots & \cdots \\ Y_{s1} & Y_{s2} & \cdots & Y_{ss} \end{Bmatrix}$$

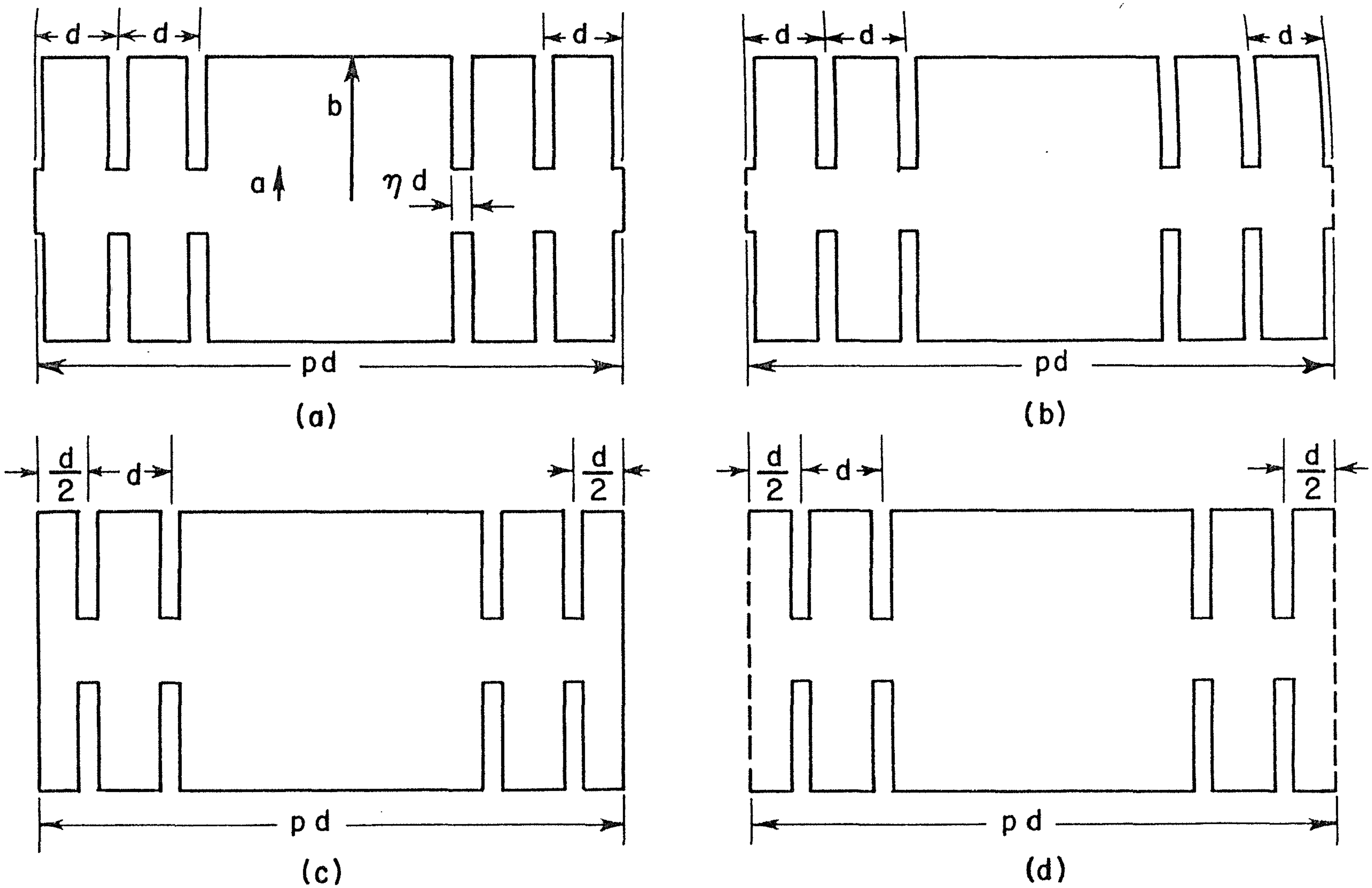

Fig. 2.4 - Four equivalent resonators of length pd, all having the same frequency for the same wave number ($k_{zo} = q\pi/pd$, $q<p$). Solid lines represent metallic or electric walls; dotted lines represent magnetic walls.

CHAPTER III

THE EIGENVALUE PROBLEM OF DISK-LOADED WAVE-GUIDES

3.1. Formal Analytical Solutions

Though analytical solutions for this problem cannot be evaluated to give exact numerical results, they are helpful in understanding the physical picture and form the basis of numerical approximations. It is desirable to have these solutions at hand before numerical methods are considered.

In the last section it has been pointed out that the problem may be solved by matching solutions either on the cylindrical surfaces or on transverse planes. In either case it is only necessary to match solutions for a single cell. The solution for any one cell may be decomposed into a cosine-type and a sine-type solution (see equations (2.33)). Apart from a constant factor one type of solution may be obtained from the other by changing sines into cosines or vice versa. Therefore, it is expedient to choose a cell in which only one type of solution is needed. Thus we will consider the very first cell of Fig. 2.4a and discuss both methods of matching solutions separately.

(i) Matching Solutions on a Cylindrical Surface

Let u^I and u^{II} denote the solutions which are valid for regions I and II respectively (see Fig. 3.1). The solution is of the cosine type so u^I is given by equation (2.31) with $U_n(r) = rJ_1(k_{rn}r)$ and $k_{rn}^2 + k_{zn}^2 = k^2$. Thus

$$u^{I} = u_c = \sum_{-\infty}^{\infty} a_n \, rJ_1(k_{rn}r) \cos k_{zn} z \quad ; \tag{3.1}$$

$$\frac{\partial u^{I}}{\partial r} = \frac{\partial u_c}{\partial r} = \sum_{-\infty}^{\infty} a_n(k_{rn}r) \, J_o(k_{rn}r) \cos k_{zn} z \quad . \tag{3.2}$$

The boundary condition $\frac{\partial u^{I}}{\partial z} = 0$ at $z = 0$ is automatically satisfied. The condition

$$\left(\frac{\partial u^{I}}{\partial r}\right)_{r=a} = 0 \qquad \text{for } 0 \leqq z \leqq \eta d/2$$

$$\text{and } d - \eta d/2 \leqq z \leqq d$$

can also be satisfied because $\frac{\partial u^{I}}{\partial r}$ at $r = a$ can be made to approximate any arbitrary function even in z by choosing a_n.

Writing $u^{II} = \sum b_n U_n^{II}(r) \, W_n^{II}(z)$, the boundary conditions are:

$$\frac{\partial}{\partial r} U_n^{II}(r) = 0 \qquad \text{at } r = b$$

and

$$\frac{\partial}{\partial z} W_n^{II}(z) = 0 \qquad \text{at } z = \eta d/2 \text{ and } d - \eta d/2.$$

Hence we may set[11]

$$U_n^{II}(r) = r\left[N_o(K_n b)J_1(K_n r) - J_o(K_n b)N_1(K_n r)\right]$$

and

$$W_n^{II}(z) = \cos \frac{n\pi}{(1 - \eta)d}\left(z - \frac{d}{2}\right) \quad \text{if n is even,}$$

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11. The Neumann functions $N_o(K_n r)$ and $N_1(K_n r)$ cannot be used for u^{I} because they have a singularity at $r = 0$.

$$= \sin \frac{n\pi}{(1 - \eta)d} (z - \tfrac{d}{2}) \qquad \text{if } n \text{ is odd,}$$

where
$$K_n^2 + \left[\frac{n\pi}{(1 - \eta)d} \right]^2 = k^2 \quad . \tag{3.3}$$

If we denote

$$Z_1(K_n r) \equiv N_o(K_n b)\, J_1(K_n r) - J_o(K_n b)\, N_1(K_n r) \tag{3.4}$$

and

$$Z_o(K_n r) \equiv N_o(K_n b)\, J_o(K_n r) - J_o(K_n b)\, N_o(K_n r) \ , \tag{3.5}$$

we may then write

$$u^{II} = \sum_{m=0}^{\infty} c_m r\, Z_1(K_{2m} r) \cos \frac{2m\pi}{(1 - \eta)d} (z - \tfrac{d}{2})$$

$$+ \sum_{m=1}^{\infty} s_m r\, Z_1(K_{2m-1} r) \sin \frac{(2m - 1)\pi}{(1 - \eta)d} (z - \tfrac{d}{2}) \tag{3.6}$$

and

$$\frac{\partial u^{II}}{\partial r} = \sum_{m=0}^{\infty} c_m(K_{2m} r)\, Z_o(K_{2m} r) \cos \frac{2m\pi}{(1 - \eta)d} (z - \tfrac{d}{2})$$

$$+ \sum_{m=1}^{\infty} s_m(K_{2m-1} r)\, Z_o(K_{2m-1} r) \sin \frac{(2m - 1)\pi}{(1 - \eta)d} (z - \tfrac{d}{2}) \ . \tag{3.7}$$

Since both u and $\partial u/\partial r$ are to be matched at $r = a$, $u^I(a,z) = u^{II}(a,z)$ and $(\partial u^I/\partial r)_{r=a} = (\partial u^{II}/\partial r)_{r=a}$, i.e.,

$$\sum_{-\infty}^{\infty} a_n\ J_1(k_{rn}a)\ \cos k_{zn}z$$

$$= \sum_{m=0}^{\infty} c_m\ Z_1(K_{2m}a)\ \cos \frac{2m\pi}{(1 - \eta)d}\ (z - \frac{d}{2})$$

$$+ \sum_{m=1}^{\infty} s_n\ Z_1(K_{2m-1}a)\ \sin \frac{(2m - 1)\pi}{(1 - \eta)d}(z - \frac{d}{2}) \qquad (3.8)$$

and

$$\sum_{-\infty}^{\infty} a_n(k_{rn}a)\ J_o(k_{rn}a)\ \cos k_{zn}z$$

$$= \sum_{m=0}^{\infty} c_m(K_{2m}a)\ Z_o(K_{2m}a)\ \cos \frac{2m\pi}{(1 - \eta)d}\ (z - \frac{d}{2})$$

$$+ \sum_{m=1}^{\infty} s_n(K_{2m-1}a)Z_o(K_{2m-1}a)\ \sin \frac{(2m - 1)\pi}{(1 - \eta)d}\ (z - \frac{d}{2}) \qquad (3.9)$$

Multiply both sides of equation (3.8) by $\cos \frac{2m\pi}{(1 - \eta)d}(z - \frac{d}{2})$ and $\sin \frac{(2m - 1)\pi}{(1 - \eta)d}(z - \frac{d}{2})$ respectively and integrating the resulting equations over the region $\frac{\eta d}{2} \leqq z \leqq d - \frac{\eta d}{2}$, we obtain

$$\frac{1}{2}(1 - \eta)d\ c_m\ Z_1(K_{2m}a) = \sum_{-\infty}^{\infty} a_n\ J_1(k_{rn}a)\ C_{nm}$$

and

$$\frac{1}{2}(1 - \eta)d\ s_m\ Z_1(K_{2m-1}a) = \sum_{-\infty}^{\infty} a_n\ J_1(k_{rn}a)\ S_{nm}\ , \qquad (3.10)$$

where

$$C_{nm} = \int_{\eta d/2}^{d-\eta d/2} \cos k_{zn} z \cos \frac{2m\pi}{(1-\eta)d}\left(z - \frac{d}{2}\right) dz$$

and

$$S_{nm} = \int_{\eta d/2}^{d-\eta d/2} \cos k_{zn} z \sin \frac{(2m-1)\pi}{(1-\eta)d}\left(z - \frac{d}{2}\right) dz \qquad (3.11)$$

Similarly we obtain from (3.9)

$$\frac{1}{2}(1-\eta)d\ c_m(K_{2m}a)\ Z_o(K_{2m}a) = \sum_{-\infty}^{\infty} a_n(k_{rn}a) J_o(k_{rn}a) C_{nm}$$

and

$$\frac{1}{2}(1-\eta)d\ S_m(K_{2m-1}a) Z_o(K_{2m-1}a) = \sum_{-\infty}^{\infty} a_n(k_{rn}a) J_o(k_{rn}a) S_{nm}\ .$$

But since the left-hand side of equation (3.9) should vanish over the regions $0 \leqq z \leqq \eta d/2$ and $d - \eta d/2 \leqq z \leqq d$ on account of the boundary condition, the result from this side of (3.9) should not change if the integration is carried over $0 \leqq z \leqq d$ instead of $\eta d/2 \leqq z \leqq d - \eta d/2$. In other words, if we denote

$$C'_{nm} = \int_0^d \cos k_{zn} z \cos \frac{2m\pi}{(1-\eta)d}\left(z - \frac{d}{2}\right) dz$$

and

$$S'_{nm} = \int_0^d \cos k_{zn} z \sin \frac{(2m-1)\pi}{(1-\eta)d}\left(z - \frac{d}{2}\right) dz\ , \qquad (3.12)$$

then

$$\sum_{-\infty}^{\infty} a_n(k_{rn}a)\ J_o(k_{rn}a) C_{nm} = \sum_{-\infty}^{\infty} a_n(k_{rn}a)\ J_o(k_{rn}a)\ C'_{nm}$$

and

$$\sum_{-\infty}^{\infty} a_n(k_{rn}a)\ J_o(k_{rn}a)\ S_{nm} = \sum_{-\infty}^{\infty} a_n(k_{rn}a)\ J_o(k_{rn}a) S'_{nm}\ .$$

Hence

$$\left.\begin{aligned} \tfrac{1}{2}(1 - \eta)d\ c_m(K_{2m}a)Z_o(K_{2m}a) &= \sum_{-\infty}^{\infty} a_n(k_{rn}a)J_o(k_{rn}a)C'_{nm} \\ \tfrac{1}{2}(1 - \eta)d\ s_m(K_{2m-1}a)Z_o(K_{2m-1}a) &= \sum_{-\infty}^{\infty} a_n(k_{rn}a)J_o(k_{rn}a)S'_{nm} \end{aligned}\right\} \quad (3.13)$$

Finally by substituting (3.13) into (3.10) we obtain

$$\left.\begin{aligned} &\sum_{-\infty}^{\infty} a_n\ J_1(k_{rn}a)\ \cdot \\ &\left[\frac{J_o(k_{rn}a)}{J_1(k_{rn}a)}\ (k_{rn}a)C'_{nm} - \frac{Z_o(K_{2m}a)}{Z_1(K_{2m}a)}\ (K_{2m}a)C_{nm}\right] = 0 \\ &\qquad\qquad m = 0,\ 1,\ 2,\ \ldots \\ \text{and} \\ &\sum_{-\infty}^{\infty} a_n\ J_1(k_{rn}a)\cdot \\ &\left[\frac{J_o(k_{rn}a)}{J_1(k_{rn}a)}\ (k_{rn}a)S'_{nm} - \frac{Z_o(K_{2m-1}a)}{Z_1(K_{2m-1}a)}(K_{2m-1}a)S_{nm}\right] = 0 \\ &\qquad\qquad m = 1, 2,\ \ldots \end{aligned}\right\} \quad (3.14)$$

This is an infinite set of simultaneous linear homogeneous equations. In order to have non-vanishing solutions, the determinant formed by the coefficients of a_n in this set of equations must be equal to zero. The solution of this determinental equation will give an infinite number of eigenvalues for the frequency and for any eigenvalue the Fourier amplitudes a_n may be determined relative to any one amplitude.

The evaluation of such infinite determinants is, of course, impossible. The obvious method of approximation is to take a few prominent terms from the series and consider a finite determinant. Walkinshaw[12] took as many as four terms and obtained a result which is different by about 0.1 per cent from the measured value. Slater,[13] on the other hand, used an asymptotic expansion to replace the infinite series and obtained very accurate results. Such methods, however, have the drawback that, unless checked by measurements or by other calculations with known accuracy, the magnitude and sign of error can not be stated.

(ii) Matching Solutions on Transverse Planes

Referring to Fig. 3.2, the three regions I, II and III are all circular cylinders, $0 \leqq r \leqq a$ for I and III and $0 \leqq r \leqq b$ for II. Since $r = 0$ is included, only Bessel functions, not Neumann functions, can enter the expressions for u. The expressions for u and $\partial u/\partial z$ for the different regions are:

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12. W. Walkinshaw, "Wave Guides for Slow Waves," J. App. Phys., 20, 634 (1949)

13. J.C. Slater, "Electromagnetic Waves in Iris-Loaded Waveguides," M.I.T. Research Lab. of Electronics Tech. Rep. No. 48, Sept. 19, 1947.

$$u^{I} = \sum_{1}^{\infty} C_n^{I} \cos(x_{zn} z)\; rJ_1(x_{rn} r) \quad ,$$

$$\frac{\partial u^{I}}{\partial z} = -\sum_{1}^{\infty} C_n^{I}\, x_{zn} \sin(x_{zn} z)\; rJ_1(x_{rn} r) \quad ; \tag{3.15}$$

$$u^{II} = \sum_{1}^{\infty} A_n \cos K_{zn}\left(z - \frac{\eta d}{2}\right) rJ_1(K_{rn} r)$$

$$+ \sum_{1}^{\infty} B_n \cos K_{zn}\left(z - d + \frac{\eta d}{2}\right) rJ_1(K_{rn} r) \quad ,$$

$$\frac{\partial u^{II}}{\partial z} = -\sum_{1}^{\infty} A_n K_{zn} \sin K_{zn}\left(z - \frac{\eta d}{2}\right) rJ_1(K_{rn} r) \tag{3.16}$$

$$-\sum_{1}^{\infty} B_n K_{zn} \sin K_{zn}\left(z - d + \frac{\eta d}{2}\right) rJ_1(K_{rn} r) \quad ;$$

and

$$u^{III} = \sum_{1}^{\infty} c_n^{III} \cos x_{zn}(z - d)\; rJ_1(x_{rn} r)$$

$$+ \sum_{1}^{\infty} s_n^{III} \sin x_{zn}(z - d)\; rJ_1(x_{rn} r) \quad ,$$

$$\frac{\partial u^{III}}{\partial z} = -\sum_{1}^{\infty} c_n^{III}\, x_{zn} \sin x_{zn}(z - d)\; rJ_1(x_{rn} r) \tag{3.17}$$

$$+ \sum_{1}^{\infty} s_n^{III}\, x_{zn} \cos x_{zn}(z - d)\; rJ_1(x_{rn} r) \quad .$$

x's and K's are defined by

$$x_{rn}^2 + x_{zn}^2 = K_{rn}^2 + K_{zn}^2 = k^2 \quad . \tag{3.18}$$

On account of the boundary conditions, it is to be demanded that

$$\left.\begin{aligned} &J_o(x_{rn}a) = 0 \quad , \\ &J_o(K_{rn}b) = 0 \quad , \\ &\frac{\partial u^{II}}{\partial z} = 0 \qquad \text{for } a \leqq r \leqq b \text{ at } z = \eta d/2 \\ &\qquad\qquad \text{and at } z = d - \eta d/2 \quad . \end{aligned}\right\} \tag{3.19}$$

Furthermore, on account of the periodic nature of the structure, the coefficients for regions I and III are related by the following equations:

$$C_n^{III} = C_n^I \cos k_{zo}d \ . \tag{3.20}$$

Matching u and $\partial u/\partial z$ on the transverse planes $z = \eta d/2$ and $z = d - \eta d/2$, we obtain

$$\sum_1^\infty C_n^I \cos(x_{zn}\frac{\eta d}{2})\ rJ_1(x_{rn}r)$$

$$= \sum_1^\infty A_n\ rJ_1(K_{rn}r) + \sum_1^\infty B_n \cos K_{zn}(d - \eta d)rJ_1(K_{rn}r) \ , \tag{3.21}$$

$$-\sum_1^\infty C_n^I\ x_{zn} \sin(x_{zn}\frac{\eta d}{2})\ rJ_1(x_{rn}r)$$

$$= \sum_1^\infty B_n\ K_{zn} \sin K_{zn}(d - \eta d)\ rJ_1(K_{rn}r) \ , \tag{3.22}$$

$$\sum_1^\infty A_n \cos K_{zn}(d-\eta d)\ rJ_1(K_{rn}r) + \sum_1^\infty B_n\ rJ_1(K_{rn}r)$$

$$= \sum_1^\infty C_n^I \cos k_{zo}d \cos(x_{zn}\frac{\eta d}{2})\ rJ_1(x_{rn}r)$$

$$- \sum_1^\infty S_n^{III} \sin(x_{zn}\frac{\eta d}{2})\ rJ_1(x_{rn}r) \tag{3.23}$$

and

$$- \sum_{1}^{\infty} A_n \, K_{zn} \sin K_{zn}(d - \eta d) \; rJ_1(K_{rn}r)$$

$$= \sum_{1}^{\infty} c_n^{I} \; (\cos k_{zo}d) \; x_{zn} \sin(x_{zn}\frac{\eta d}{2}) \; rJ_1(x_{rn}r)$$

$$+ \sum_{1}^{\infty} s_n^{III} \; x_{zn} \cos(x_{zn}\frac{\eta d}{2}) \; rJ_1(x_{rn}r) \quad . \qquad (3.24)$$

Multiply equations (3.21), (3.22), (3.23) and (3.24) by $J_1(x_{rm}r)$ and integrate over the region $0 \lesseqgtr r \lesseqgtr a$. If we denote

$$\left.\begin{aligned} J_{nm} &= \int_0^a rJ_1(K_{rn}r) \; J_1(x_{rm}r) \; dr \\ \text{and} \quad J'_{nm} &= \int_0^b rJ_1(K_{rn}r) \; J_1(x_{rm}r) \; dr, \end{aligned}\right\} \qquad (3.25)$$

the results are:

$$c_m^{I} \cos(x_{zm}\frac{\eta d}{2}) \; \frac{a^2}{2} \; J_1^2(x_{rm}a) = \sum_{1}^{\infty} A_n \; J_{nm}$$

$$+ \sum_{1}^{\infty} B_n \cos K_{zn}(d - \eta d) \; J_{nm} \quad , \qquad (3.26)$$

$$- c_m^{I} \; x_{zm} \sin(x_{zm}\frac{\eta d}{2}) \; \frac{a^2}{2} \; J_1^2(x_{rm}a)$$

$$= \sum_{1}^{\infty} B_n \; K_{zn} \sin K_{zn}(d - \eta d) \; J'_{nm} \quad , \qquad (3.27)$$

$$c_m^{I} \cos k_{zo}d \cos(x_{zm}\frac{\eta d}{2}) \; \frac{a^2}{2} \; J_1^2(x_{rm}a)$$

$$- s_m^{III} \sin(x_{zm}\frac{\eta d}{2}) \; \frac{a^2}{2} \; J_1^2(x_{rm}a)$$

$$= \sum_{1}^{\infty} A_n \cos K_{zn}(d - \eta d) \; J_{nm} + \sum_{1}^{\infty} B_n \; J_{nm} \quad , \qquad (3.28)$$

and

$$c_m^I \, x_{zm} \cos k_{zo}d \sin(x_{zm}\frac{\eta d}{2}) \, \frac{a^2}{2} \, J_1^2(x_{rm}a)$$

$$+ s_m^{III} \, x_{zm} \cos(x_{zm}\frac{\eta d}{2}) \, \frac{a^2}{2} \, J_1^2(x_{rm}a)$$

$$= -\sum_1^{\infty} A_n \, K_{zn} \sin K_{zn}(d - \eta d) \, J'_{nm} \quad . \qquad (3.29)$$

In equations (3.27) and (3.29), J_{nm} is replaced by J'_{nm} on account of the third condition of (3.19).

Finally, by eliminating c_m^I and s_m^{III} from the above equations, (3.26) to (3.29) inclusive, we obtain

$$\left.\begin{aligned}
&\sum_1^{\infty} A_n \, J_{nm} + \sum_1^{\infty} B_n \, K_{zn} \sin K_{zn}(d-\eta d)\left[\frac{\cot K_{zn}(d-\eta d)}{K_{zn}} J_{nm} + \frac{\cot x_{zm}\frac{\eta d}{2}}{x_{zm}} J'_{nm}\right] = 0 \\
&\text{and} \\
&\sum_1^{\infty} A_n \, K_{zn} \sin K_{zn}(d - \eta d) \cdot \left[\frac{\cot K_{zn}(d-\eta d)}{K_{zn}} J_{nm} - \frac{\tan x_{zm}\frac{\eta d}{2}}{x_{zm}} J'_{nm}\right] \\
&+ \sum_1^{\infty} B_n \, K_{zn} \sin K_{zn}(d-\eta d) \cdot \left[\frac{\csc K_{zn}(d-\eta d)}{K_{zn}} J_{nm} + \frac{\csc x_{zm}\,\eta d}{x_{zm}} \, 2\cos k_{zo}d \, J'_{nm}\right] = 0.
\end{aligned}\right\} \qquad (3.30)$$

These form an infinite set of simultaneous linear homogeneous equations. Like equations (3.14), they represent the formal exact solution for the problem.

If η approaches zero, we see from equation (3.30) that B_n/A_n approach zero. Putting $\eta = 0$ and $B_n = 0$ in equation (3.16) and proceeding as before, we obtain

$$\sum_{1}^{\infty} A_n(\cos k_{zo}d - \cos K_{zn}d)\, J_{nm} = 0$$

and

$$\sum_{1}^{\infty} A_n\, K_{zn} \sin K_{zn}d(J'_{nm} - J_{nm}) = 0$$

instead of (3.30). The above equations may be combined into

$$\sum_{1}^{\infty} A_n\, K_{zn} \sin K_{zn}d \;\cdot$$

$$\left[\left(\frac{\cos k_{zo}d \csc K_{zn}d}{K_{zn}} - \frac{\cot K_{zn}d}{K_{zn}}\right)J_{nm} + \alpha_m(J'_{nm} - J_{nm})\right] = 0 \quad , (3.31)$$

where α_m is an infinite set of numbers such that the determinant formed by the coefficients of A_n has a rank one lower than its degree.

Without going into numerical demonstrations it may be pointed out that the two methods are complementary to each other in the sense that when the series converge slowly in one method they usually converge fast in the other. Yet, for the reason mentioned before, the first method is in general to be preferred.

3.2. Simple Approximate Solutions

(i) For Small Disk Spacing by Approximate Matching of Functions

Since the accelerator tube is to be operated within the first passing band, the zero-th component is the strongest. Let us assume that the other components are negligible, which would be true if $d \ll a$, and take only one term of the series (3.1) and (3.2) to approximate u^I and $\partial u^I/\partial r$ respectively, i.e.,

$$\left.\begin{aligned} u^I &= a_o r J_1(k_{ro} r) \cos k_{zo} z \\ \frac{\partial u^I}{\partial r} &= a_o\, k_{ro} r J_o(k_{ro} r) \cos k_{zo} z \end{aligned}\right\} \qquad (3.32)$$

If, in addition, $d \ll b - a$ and $k_{zo} d \ll \pi$ it is plausible to make the same approximation to the series (3.6) and (3.7), i.e.,

$$\left.\begin{aligned} u^{II} &= c_o r\, Z_1(kr) \\ \frac{\partial u^{II}}{\partial r} &= c_o\, kr\, Z_o(kr) \end{aligned}\right\} \qquad (3.33)$$

As noted before $u = rH_\phi$ and $\partial u/\partial r = krE_z$. The corresponding approximations for $\partial u/\partial z = -\, krE_r$ in the two regions are

$$\frac{\partial u^I}{\partial z} = -\, a_o\, k_{zo} r\, J_1(k_{ro} r) \sin k_{zo} z \quad ,$$

$$\frac{\partial u^{II}}{\partial z} = 0 \quad .$$

Thus E_r is given as being finite for $r < a$ and zero for $r > a$. E_r^I and E_r^{II} are not matched anywhere on $r = a$. $u^I \sim \cos k_{zo} z$ with $k_{zo} d < \pi$ while u^{II} is independent of z; they can be matched on r = a only for a single value of z. The same is true for $\partial u/\partial r$. Notwithstanding its crudeness, this approximation can give reliable qualitative results for $d \ll a$, $d \ll b - a$ if the fields are matched by some averaging process; it is certainly correct for the limiting case $k_{zo}d = 0$.[14]

If $\eta = 0$, it is obvious that good approximation can be had by matching their average values. On the other hand, if η is finite, while it is still plausible to match $u_{ave.}$ it does not seem justified to match $(\partial u/\partial r)_{ave.}$ because $\partial u/\partial r$ given by (3.32) does not satisfy the boundary condition on the edge of the disks (r = a). In this connection it is expedient to look at the disks as having some slight taper on their edges, shown exaggerated in Fig. 3.3. This would not change the problem appreciably if the taper is negligible. It is clear then that $(\partial u/\partial r)_{ave.}$ on r = a and $r = a + \delta r$ are not equal but the integrated values of $\partial u/\partial r$ on these two surfaces are. With the latter condition satisfied the average value of $\partial u/\partial r$ vanishes on the edge of the disks. E_r, on the other hand, cannot be so matched for a single cell, but will be if all cells are to be averaged.

Thus we demand

$$a_o J_1(k_{ro}a) \frac{1}{d} \int_o^d \cos k_{zo}z \, dz = c_o \, Z_1(ka)$$

and

$$a_o(k_{ro}a)J_o(k_{ro}a)\int_o^d \cos k_{zo}z \, dz = c_o(ka)Z_o(ka)(1-\eta)d \ .$$

14. For a detailed discussion of this approximate theory, refer to Chu and Hansen, "The Theory of Disk-Loaded Wave Guides," J.App.Phys., 18, 996 (1947)

Dividing one equation by the other we obtain

$$\frac{1}{k_{ro}a} \frac{J_1(k_{ro}a)}{J_o(k_{ro}a)} = \frac{1}{1 - \eta} \frac{1}{ka} \frac{Z_1(ka)}{Z_o(ka)} , \tag{3.34}$$

from which k may be determined. Let us denote

$$\left.\begin{aligned} \frac{1}{k_{ro}a} \frac{J_1(k_{ro}a)}{J_o(k_{ro}a)} &= \emptyset(k_{ro}a) \\ \frac{1}{ka} \frac{Z_1(ka)}{Z_o(ka)} &= \alpha(ka, kb) \quad , \end{aligned}\right\} \tag{3.35}$$

then the equation to be solved is

$$\emptyset(k_{ro}a) = \frac{1}{1 - \eta} \alpha(ka, kb) \quad . \tag{3.36}$$

The function of $\emptyset$ is plotted for both real and imaginary values of $k_{ro}a$ in Fig. 3.4 and a contour map of α in the ka,kb plane is given in Fig. 3.5. Several easily derived approximate expressions for $\emptyset$ and α, which will be found useful for various future purposes, are written as follows:

$$\left.\begin{aligned} &\emptyset \cong \frac{1}{k_{ro}a} \frac{1}{2.405 - k_{ro}a} , \; k_{ro}a \text{ real}, \\ &\qquad\qquad 0 < 2.405 - k_{ro}a < 1 \; . \\ &\emptyset \cong \frac{1}{2}(1 + \frac{1}{8} k_{ro}^2 a^2) , \quad |k_{ro}^2 a^2| < 1 \; . \\ &\emptyset \cong j \frac{1}{k_{ro}a} , \quad - jk_{ro}a \text{ real} \gg 1 \; . \end{aligned}\right\} \tag{3.37}$$

$$\left.\begin{aligned}
\alpha &\cong \frac{1}{k^2a^2 \log \frac{b}{a}} , \quad ka \ll 1 , \quad kb \ll 1 . \\
\alpha &\cong \frac{1}{k^2a^2 \log \frac{2}{\gamma ka}} , \ \gamma = 1.781, \quad ka \ll 1, \quad J_o(kb) \neq 0 . \\
kb &\cong 2.405 + 1.545(\tfrac{1}{2} - \alpha)\, k^2a^2 , \ ka \ll 1, \ kb \cong 2.405 \\
\alpha &\cong \frac{1}{ka} \cot(kb - ka) , \quad ka \gg 1, \quad kb \gg 1 .
\end{aligned}\right\} \quad (3.38)$$

For given b and a, and therefore fixed b/a, α is now a function of k or of ka, which can be determined from Fig. 3.5. A typical curve of α vs. ka for kb/ka = a is plotted in Fig. 3.6. On this same graph, we have plotted $\emptyset$ as a function of $k_{ro}a$.

To solve (3.36) for given k and $\eta = 0$, we enter the chart at ka and follow the vertical dotted line to an intersection with the α- curve, proceed horizontally to find an equal value of $\emptyset$, and then drop down to the axis to find the value of $k_{ro}a$. Finally, we compute $\frac{k_{zo}}{k} = \sqrt{1 - (\frac{k_{ro}a}{ka})^2}$ and so are able to construct a graph of k_{zo}/k vs. ka or kb.

It is instructive to follow the results qualitatively as ka increases from zero. For ka small, α and so $\emptyset$ are large and we easily find $\frac{k_{zo}}{k} \cong j\frac{2.405}{ka}\sqrt{1 - (\frac{ka}{2.405})^2}$, i.e., the guide acts as an attenuator, and with the attenuation characteristic of a tube of radius a. The wave number $k_{zo}/2\pi$ first becomes real when $k_{ro}a = ka$, i.e., at the intersection of the two curves, where α and $\emptyset$ are usually about 0.5. Moreover, at this low frequency cut-off, which occurs at kb = 2.405, k_{ro} is a linear function of k and so k_{zo}/k has the approximate form

$$\frac{k_{zo}}{k} \cong \sqrt{2\frac{k - k_{ro}}{k}} = \sqrt{-\,2\frac{\delta k_{ro}}{k}}$$

$$= \sqrt{-\,2\frac{\delta k}{k}\,\frac{dk_{ro}}{dk}} = \sqrt{\frac{2(kb - 2.405)}{2.405}}\,\sqrt{-\,\frac{dk_{ro}}{dk}}\;,$$

and so the curve has a vertical tangent at cut-off. As we further increase ka, $k_{ro}a$ decreases and the next interesting point is at $k_{ro}a = 0$, where $k_{ro}a$ changes from real to imaginary, and k_{zo}/k passes through unity. Nothing striking happens to k_{zo}/k; how k_{zo} varies with k can best be understood by examining the derivative dk_{zo}/dk.

$$\frac{dk_{zo}}{dk} = \frac{k}{k_{zo}} - \frac{k_{ro}}{k_{zo}}\,\frac{dk_{ro}}{dk}\;.$$

By (3.36) $\quad \dfrac{dk_{ro}}{dk} = \dfrac{1}{1 - \eta}\,\dfrac{d\alpha/dk}{d\phi/dk_{ro}}$

so

$$\frac{dk_{zo}}{dk} = \frac{k}{k_{zo}} - \frac{k_{ro}}{k_{zo}}\,\frac{d\alpha/d(ka)}{d\phi/d(k_{ro}a)}\,\frac{1}{1 - \eta} \qquad (3.39)$$

In particular, if $k_{ro}a \cong 0$ we obtain with the help of the second expression of (3.37)

$$\frac{dk_{zo}}{dk} = 1 - \frac{1}{1 - \eta}\,\frac{8}{ka}\,\frac{d\alpha}{d(ka)} \qquad (3.40)$$

Formulae (3.39) and (3.40) are of some importance, for they give the reciprocal of the group velocity, and this quantity is often of direct interest, especially in the particular case $k_{ro} = 0$, $k_{zo} = k$ where the wave velocity is c.

As ka increases further, α and so ϕ approach zero, k_{ro} becomes large and imaginary, and k_{zo} large and real. In this region the wave velocity is less than c. As long as $k_{zo}d$ is small in comparison with π, the present approximate theory can be relied upon.

To illustrate the above results we plot in Fig. 3.7, k_{zo}/k as a function of kb for kb/ka = 2, $\eta = 0$, together with several approximate formulae. In Fig. 3.8, k_{zo}/k is plotted as a function of kb for several values of kb/ka. Both these figures assume that $k_{zo}d$ is small.

The case $k_{zo}/k = 1$, i.e., phase velocity equal to that of light, is of particular importance in accelerator theory. We plot for this particular case ka vs. kb in Fig. 3.9. Two approximations are also shown, one for small ka and the other for large ka.

For small ka, $\phi(k_{ro}a) = \frac{1}{1-\eta}\,\alpha(ka,kb) \cong 1/2$ and $kb \cong 2.405$. Inserting approximate values for Neumann and Bessel functions in α, we easily obtain

$$kb \cong 2.405 + \frac{\pi}{4}\,\frac{N_0(2.405)}{J_1(2.405)}\left(\eta k^2a^2 - \eta\frac{k^4a^4}{4} + \frac{k^4a^4}{8}\right),$$

$$ka \ll 1, \quad kd \ll 1 . \qquad (3.41)$$

For large ka, $\phi(k_{ro}a) = \frac{1}{1-\eta}\,\alpha(ka,kb)$

$$\cong \frac{1}{1-\eta}\,\frac{\cot(kb - ka)}{ka} \cong 1/2, \quad \text{so}$$

$$kb \cong ka + \cot^{-1}\frac{ka}{2}(1-\eta), \quad kd \ll 1 \ll ka. \qquad (3.42)$$

It is to be pointed out that if a approaches zero while d is fixed, ka must inevitably become smaller than kd, in which case our approximate analysis should fail. Nevertheless, if $d \ll b$ there is a region in which (3.41) is useful.

(ii) For Small Disk Holes by Perturbation Method. An Associated Electrostatic Problem Solved in Oblate Spheroidal Coordinates

We may now develop an approximate theory for the case $a \ll d$. The idea behind the approximation is to consider the loaded guide as derived from a sequence of uncoupled cavities perturbed by the introduction of coupling holes. It is rather an approximation to the second method of matching solutions than to the first.

For this and other perturbation methods calculation is based on an equation connecting the perturbed and unperturbed field quantities, which is derived from Gauss' theorem. Let $\vec{E}_1$, $\vec{H}_1$ be the real electric vector and the real magnetic vector for the unperturbed problem and $\vec{E}_2$, $\vec{H}_2$ the corresponding vectors for the perturbed problem. According to Maxwell's equations we have

$$\mathrm{curl}\ \vec{E}_i = k_i\ \vec{H}_i\ , \quad \mathrm{curl}\ \vec{H}_i = k_i\ \vec{E}_i\ , \quad i = 1 \text{ or } 2\ .$$

From the surface integral $\int_\Gamma (\vec{E}_1 \times \vec{H}_2 - \vec{E}_2 \times \vec{H}_1)_n\, d\sigma$ and transform

$$\int_\Gamma (\vec{E}_1 \times \vec{H}_2 - \vec{E}_2 \times \vec{H}_1)_n\, d\sigma = \int_R \mathrm{div}(\vec{E}_1 \times \vec{H}_2 - \vec{E}_2 \times \vec{H}_1)\, d\tau$$

$$= \int_R (\vec{H}_2 \cdot \mathrm{curl}\ \vec{E}_1 - \vec{E}_1 \cdot \mathrm{curl}\ \vec{H}_2 - \vec{H}_1 \cdot \mathrm{curl}\ \vec{E}_2 + \vec{E}_2 \cdot \mathrm{curl}\ \vec{H}_1)\, d\tau\ ,$$

i.e.,

$$\int_{\Gamma} (\vec{E}_1 \times \vec{H}_2 - \vec{E}_2 \times \vec{H}_1)_n \, d\sigma$$

$$= (k_1 - k_2) \int_R (\vec{E}_1 \cdot \vec{E}_2 + \vec{H}_1 \cdot \vec{H}_2) \, d\tau , \qquad (3.43)$$

where R denotes the volume enclosed by Γ. No dielectric or conducting material is to be within R. If the perturbation is small, $\vec{E}_2$ and $\vec{H}_2$ will not differ appreciably from $\vec{E}_1$ and $\vec{H}_1$ except in a very small region of R. Thus

$$\int_R (\vec{E}_1 \cdot \vec{E}_2 + \vec{H}_1 \cdot \vec{H}_2) \, d\tau \cong \int_R (E_1^2 + H_1^2) \, d\tau ,$$

so we obtain the perturbation formula

$$\delta k = \int_{\Gamma} (\vec{E}_2 \times \vec{H}_1 - \vec{E}_1 \times \vec{H}_2)_n d\sigma \Big/ \int_R (E_1^2 + H_1^2) \, d\tau \qquad (3.44)$$

Now the unperturbed problem is to deal with a set of uncoupled cavities, each having a totally enclosed boundary. $\vec{E}_1$, $\vec{H}_1$ are known. Referring to Fig. 2.4c and assuming that there is no hole on the disks, the field quantities in the m-th cavity (two end ones are half as wide as the others) are:

$$E_{1z}(m) = E_o \cos k_{zo} md \; J_o\left(\frac{2.405}{b} r\right) ,$$

$$H_{1\phi}(m) = E_o \cos k_{zo} md \; J_1\left(\frac{2.405}{b} r\right) ,$$

and all other components equal to zero.

On metallic boundary $\vec{E} \times \vec{n} = 0$, so the surface integral vanishes. We need only consider the area, say γ, covered by the coupling hole. On γ, $E_{1r} = 0$, so $(\vec{E}_1 \times \vec{H}_2) \cdot \vec{n} = 0$ and

$\int_{\gamma}(\vec{E}_2 \times \vec{H}_1 - \vec{E}_1 \times \vec{H}_2)_n \, d\sigma = \int_{\gamma} E_{2r} \cdot H_{1\phi} \, d\sigma$. Among the perturbed field quantities only E_{2r} at the coupling holes is needed for computing δk.

For $a \ll d$ and $a \ll b$, the electric field at the coupling holes may be approximated by the solution of a static problem concerning an infinite thin plane conducting sheet with a circular hole. The conditions at infinity for this static problem are

$$E_z = E_o \quad , \quad E_r = 0 \qquad \text{at } z = -\infty$$

$$E_z = E_o \cos k_{zo}d \ , \quad E_r = 0 \qquad \text{at } z = +\infty \ .$$

This problem may be solved in closed form in oblate spheroidal coordinates ξ, ζ, ϕ $(r = a\left[(1+\zeta^2)(1-\xi^2)\right]^{1/2}$, $z = a\zeta\xi$, $-\infty < \zeta < \infty$, $0 \leqq \xi \leqq 1)$.[15] The solutions are

$$\left.\begin{aligned} V &= -E_o a \xi \left[\zeta \cos k_{zo}d + \frac{1-\cos k_{zo}d}{\pi}(\zeta \cot^{-1}\zeta - 1)\right], \\ E_z &= -\frac{\partial V}{\partial z} \\ &= E_o\left[\cos k_{zo}d + \frac{1-\cos k_{zo}d}{\pi}\left(\cot^{-1}\zeta - \frac{\zeta}{\xi^2+\zeta^2}\right)\right], \\ E_r &= -\frac{\partial V}{\partial r} = E_o \frac{1-\cos k_{zo}d}{\pi} \frac{\xi}{\xi^2+\zeta^2}\sqrt{\frac{1-\xi^2}{1+\zeta^2}} \end{aligned}\right\} \quad (3.45)$$

At the hole $(z = 0)$, $\zeta = 0$ and $r = a(1-\xi^2)^{1/2}$, so

$$E_r = E_o \frac{1-\cos k_{zo}d}{\pi} \frac{r}{\sqrt{a^2 - r^2}} \ .$$

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15. W.R. Smythe, "Static and Dynamic Electricity," (McGraw Hill Book Co., Inc., 1939) p.159.

Substituting this value for E_{2r} and $\vec{E}_1(0)$, $\vec{H}_1(0)$ for $\vec{E}_1$, $\vec{H}_1$ in (3.44) and integrating the volume integral over the first half-cavity (m = 0), we obtain

$$k_2 - k_1 = k_2 - \frac{2.405}{b} \cong \frac{2}{3\pi} \frac{2.405}{J_1^2(2.405)} \frac{a^3}{b^3 d}(1 - \cos k_{zo}d) \ .$$

Writing k for k_2 and simplifying, the result is

$$kb = 2.405\left[1 + .787 \frac{a^3}{b^2 d}(1 - \cos k_{zo}d)\right] \ .$$

The result is exactly the same if we consider any other cavity instead of the two end ones. The above assumes that $\eta = 0$. If η is finite but small, a first order approximation to the above expression for kb is obtained by replacing d with $(1 - \eta)d$. Thus

$$kb \cong 2.405\left[1 + .787 \frac{a^3}{b^2 d(1 - \eta)} (1 - \cos k_{zo}d)\right]$$

$$\eta d \ll a \ll d, \quad a \ll \lambda \ . \quad (3.46)$$

The latter result is also plotted in Fig. 3.9.

(iii) For Large Disk Holes by Perturbation Method. An Associated Electrostatic Problem Solved by Conformal Transformation

On the other end of the possible range of a we may have $b - a \ll b$, $b - a \ll d$ and in this case it is obviously best to take as the unperturbed problem an unloaded guide and introduce the loading disks as perturbations. There are then two cases, $\eta d \ll b - a \ll d$ and $\eta d \gg b - a$.

In the first case we take Γ in (3.44) to be the metallic boundary surface of the loaded guide. Thus $\vec{E}_2 \times \vec{n} = 0$ on Γ.

Furthermore $\vec{E}_1 \times \vec{n} = 0$ on the cylindrical part of Γ, so

$$\int_{\Gamma} (\vec{E}_2 \times \vec{H}_1 - \vec{E}_1 \times \vec{H}_2)_n \, d\sigma = \int_{\text{disks}} \vec{H}_2 \times \vec{E}_1 \cdot \vec{n} \, d\sigma \quad .$$

$\vec{H}_2$ is to be derived from $\vec{E}_2$ and $\vec{E}_2$ is again to be derived from the solution of a static problem.

On account of the smallness of b - a we may assume that the cylindrical problem may be unrolled into a plane one. Thus we consider an infinite conducting plane (y = 0) on which a thin conducting wall of height h is erected in the plane x = 0.

This electrostatic problem has the following boundary conditions

$$E_y = E_\infty = \text{constant} \quad \text{at } |Z| = \sqrt{x^2 + y^2} = \infty \ ,$$

$$E_y = 0 \quad \text{at } x = 0, \quad 0 \leqq y \leqq h,$$

$$E_x = 0 \quad \text{at } y = 0 \text{ and at } x = 0,\ y > h,$$

and can be solved in closed form by the following transformation

$$CZ = jh(C^2 + W^2)^{1/2} \ ,$$

where $Z = x + jy$ and $W = u + jv$ are complex variables and C is a constant to be determined by the boundary conditions. From the relations

$$E_x = -\frac{\partial u}{\partial x} = \text{R.P.}\left(\frac{C^2 Z}{h^2 W}\right)$$

and

$$E_y = -\frac{\partial u}{\partial y} = \text{I.P.}\left(\frac{C^2 Z}{h^2 W}\right)$$

we find

$$\left.\begin{aligned} E_x &= E_\infty R^{-1/4}\left(y \cos\frac{\theta}{2} - x \sin\frac{\theta}{2}\right) \quad , \\ E_y &= E_\infty R^{-1/4}\left(x \cos\frac{\theta}{2} - y \sin\frac{\theta}{2}\right) \quad , \end{aligned}\right\} \tag{3.47}$$

where

$$R = (x^2 - y^2 + h^2)^2 + 4x^2y^2 \quad ,$$

$$\theta = \tan^{-1} \frac{2xy}{x^2 - y^2 + h^2} \quad ,$$

On the wall $E_y = 0$ but E_x has two values, one for $x = 0^+$ and the other for $x = 0^-$. They are

$$E_x^+ = - E_x^- = \frac{E_\infty y}{\sqrt{h^2 - y^2}} \tag{3.48}$$

We then take $E_{2z}^{\pm}$ on the disk surfaces as equal to $E_x^{\pm}$ with E_∞ equal to the value of $- E_{1r}(b)$ at the position of the disk and take $h = b - a$. Since $\vec{E}_2 = (1/k_2)\,\mathrm{curl}\,\vec{H}_2$,

$$E_{2z}^+ = \frac{1}{k_2}\frac{\partial H_{2\phi}^+}{\partial r} + \frac{H_{2\phi}^+}{k_2 r} \quad , \quad \text{so}$$

$$\int_a^r k_2\, E_{2z}^+\, dr = H_{2\phi}^+(r) - H_{2\phi}^+(a) + \int_a^r \frac{H_{2\phi}^+}{r}\, dr$$

$$\cong H_{2\phi}^+(r) - H_{2\phi}^+(a) \quad ,$$

i.e.,

$$H_{2\phi}^+(r) \cong \int_a^r \frac{k_2\, E_\infty (b - r)}{\sqrt{(b - a)^2 - (b - r)^2}}\, dr \quad .$$

Thus

$$H_{2\phi}^+ = - H_{2\phi}^- = k_2\, E_\infty \sqrt{(b - a)^2 - (b - r)^2} \tag{3.49}$$

For the unperturbed problem the non-vanishing field components are

$$H_{1\phi} = E_o \, J_1(k_{ro}r) \cos k_{zo}z \quad ,$$

$$E_{1z} = E_o \frac{k_{ro}}{k_1} J_o(k_{ro}r) \cos k_{zo}z$$

and

$$E_{1r} = E_o \frac{k_{zo}}{k_1} J_1(k_{ro}r) \sin k_{zo}z \ ,$$

where $k_{ro}b = 2.405$, $k_{zo}d = q\pi/p$, $k_1^2 = k_{ro}^2 + k_{zo}^2$ and $0 \leqq z \leqq pd$. Referring again to Fig. 2.4c we consider the surface integral over the m-th disk

$$\int_{\text{m-th disk}} \vec{H}_2 \times \vec{E}_1 \cdot \vec{n} \, d\sigma = \int_{\text{m-th disk}} 2H_{2\phi}^{+} \cdot E_{1r} \, d\sigma \ .$$

Now since $E_{\infty}(m) = - E_o \frac{k_{zo}}{k_1} J_1(2.405) \sin k_{zo}(m - \frac{1}{2}) \, d$,

we easily obtain

$$\int_{\text{m-th disk}} \vec{H}_2 \times \vec{E}_1 \cdot \vec{n} \, d\sigma$$

$$= - E_o^2 \, \pi^2 b(b-a)^2 \, k_2 \frac{k_{zo}^2}{k_1^2} J_1^2(2.405) \sin^2 k_{zo}(m - \tfrac{1}{2}) \, d$$

and

$$\delta k = \int_{\substack{\text{all} \\ \text{disks}}} \vec{H}_2 \times \vec{E}_1 \cdot \vec{n} \, d\sigma \Bigg/ \int_{\substack{\text{all} \\ \text{cells}}} (E_1^2 + H_1^2) \, d\tau$$

$$\cong - k_2 \left(\frac{k_{zo}}{k_1}\right)^2 \frac{\pi}{2} \frac{(b - a)^2}{bd} \left[1 - \frac{1}{p} \sum_{1}^{p} \cos(2m - 1)\frac{q\pi}{p} \right] \ .$$

The finite series

$$-\frac{1}{p}\sum_{1}^{p}\cos(2m-1)\frac{q\pi}{p} = -\frac{1}{2p}\sum_{1}^{p}\left[e^{i\frac{q\pi}{p}}\ \frac{1-e^{i2q\pi}}{1-e^{i\frac{2q\pi}{p}}} + \text{c.c.}\right]$$

$$= \begin{cases} 0 & \text{if } p \neq q\,, \\ 1 & \text{if } p = q\,, \quad \text{which is the exceptional } \pi\text{-mode case.} \end{cases}$$

Thus

$$\delta k \cong -k_2\left(\frac{k_{zo}}{k_1}\right)^2 \frac{\pi}{2}\,\frac{(b-a)^2}{bd} \cong -\frac{k_{zo}^2}{k_2}\ \frac{\pi}{2}\,\frac{(b-a)^2}{bd}\ .$$

Writing k for k_2 we finally obtain

$$k \cong k_{zo}\left[1 + \left(\frac{2.405}{k_{zo}b}\right)^2\right]^{1/2}\left[1 - \frac{\pi}{2}\left(\frac{k_{zo}}{k}\right)^2 \frac{(b-a)^2}{bd}\right],$$

$$\eta d \ll b - a \ll d\ ,\quad b - a \ll b,\ k_{zo}d < \pi. \qquad (3.50)$$

In the other case where $b - a \ll \eta d$, a first order approximation is possible by assuming $H_{2\phi} \cong H_{1\phi}$. Thus

$$\int_{\substack{\text{all}\\ \text{disks}}} \vec{H}_2 \times \vec{E}_1 \cdot \vec{n}\, d\sigma \cong \int_{\substack{\text{all}\\ \text{disks}}} \vec{H}_1 \times \vec{E}_1 \cdot \vec{n}\, d\sigma$$

$$= k_1 \int_{\substack{\text{all}\\ \text{disks}}} (H_1^2 - E_1^2)\, d\tau = k_1 \int_{\substack{\text{all}\\ \text{disks}}} (H_{1\phi}^2 - E_{1r}^2)\, d\tau$$

(since $E_{1z} \cong 0$ for $r \cong b$) and

$$\delta k \cong k_1 \int_{\substack{\text{all}\\ \text{disks}}} (H_{1\phi}^2 - E_{1r}^2)\, d\tau \Big/ 2 \int_{\substack{\text{all}\\ \text{cells}}} H_{1\phi}^2\, d\tau\,.$$

The result of the calculation is

$$k \cong k_{zo}\left[1 + \left(\frac{2.405}{k_{zo}b}\right)^2\right]^{1/2}\left[1 + (2.405)^2\ \eta k\ \frac{b - a}{(kb)^3}\right]$$

$$b - a \ll b\ , \quad b - a \ll \eta d \qquad (3.51)$$

This discussion of approximate calculations has covered almost all limiting cases. Such methods, though interesting and illustrative, are not of much use for practical cases where a, b, d are of the same order of magnitude. Putting them together we may form a clear view of the general nature of the problem but none of them is supposed to give good accuracy beyond their respective ranges of applicability. In the next section a variational method is to be described by which one upper and one lower limit for the exact value can be calculated and these two limits can be made to approach each other as closely as may be desired.

3.3 The Variational Method

(i) Introductory Remarks

In the calculus of variations it is a common-place to calculate an upper bound but not at all easy to get a lower one, at least not with comparable accuracy. The method to be discussed has the advantage of being able to give both types of bounds as easily and as accurately. It differs from the ordinary (Rayleigh-Ritz) variational method in that the trial functions used in this method satisfy the differential equation but not all the restrictive conditions.

The original idea of this is due to Courant[16] who first suggested expressly that it is feasible to calculate a lower bound by loosening the boundary conditions or the conditions of continuity. How this can be done was later shown by Trefftz.[17] Though much to be preferred in comparison with the then existing methods for lower bound calculations, the Courant-Trefftz method is still more involved than that of Ritz, not to mention the simplified versions of Ritz's method known as Galerkin's[18] and Grammel's[19] methods.

More recently, Schwinger[20] has further developed the method and made full use of it in his work on waveguide discontinuities, scattering cross sections and other related subjects. He calculated both the upper and the lower bounds of various quantities and in all cases the two bounds are almost equally good. His method combines the flexibility of the Courant-Trefftz method with the simplicity of Grammel's. It resembles the former method in spirit in perturbing restrictive conditions and resembles the latter in form in using integral equations.

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16. R. Courant, Math.Ann., 97, 711 (1927).

17. E. Trefftz, Math.Ann., 100, 503-521 (1928).

18. See F. Pfeiffer, Handbuch der Physik, 6,345 (1928); also Biezeno-Grammel, Tech. Dynamik (Verlag. Julius Springer, Berlin, 1939) 167-169.

19. R. Grammel, Proc. 5th Intern. Congress for App.Mech., 691-693 (1938); also Biezeno-Grammel, loc.cit.footnote 18

20. J. Schwinger, "Discontinuities in Waveguides," (lecture notes prepared by D.S. Saxon.

Schwinger's work, thus far, has covered only the boundary value problems. He has shown how a certain quantity can be expressed in the form of an infinite series which has the property of a definite quadratic form. As the trial function is varied over a set of functions, the value of the quadratic form is changed but remains greater or less than the true value, as the case may be. It is still interesting to ask why and under what conditions such definite forms may exist.

His process can also be used for the calculation of eigenvalues. Application to our particular problem has already been reported in a paper entitled "Disk-Loaded Wave Guides,"[21] in which detailed qualitative discussion has been given as to how it can give a lower bound for the eigenvalue. It has later been found possible to give a general formulation of the method in precise terms.

Despite the stringent condition that the trial functions are to satisfy the differential equation, this method can be applied to quite a large number of important physical problems. The disk-loaded waveguide is one example of a general class of regions, the composite type, for which the present method is particularly useful. A region is called composite if it can be divided by simple surfaces into two or more regions, in each of which the problem can be exactly solved once we impose suitable boundary conditions on those dividing surfaces. In view of the wide applicability of this method in both theoretical and applied physical problems, it seems desirable to give

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21. E.L.Chu and W.W. Hansen, J. App. Phys., 20, 280-285 (1949).

a brief account of the formulation of the method before we describe the specific analysis and numerical results.

For brevity it is desirable to restrict ourselves to our particular problem, which is concerned with the fundamental mode and has a simple boundary condition. Yet the method will be formulated in terms that are valid for the general case of the homogeneous boundary condition, and can be applied to any mode if proper consideration is given to orthogonality requirements. The generalization is not difficult but needs relatively lengthy discussion which we will not go into. For such considerations reference may be made to an article entitled "Upper and Lower Bounds of Eigenvalues for Composite-Type Regions."[22]

(ii) Formulation of the Method: Upper and Lower Bounds of Eigenvalues

For the sake of generality it is expedient to consider instead of equation (2.16) the two-dimensional Sturm-Liouville differential equation

$$L[u] + \lambda\rho u = (pu_r)_r + (pu_z)_z - qu + \lambda\rho u = 0 \qquad (3.52)$$

for a region R with boundary Γ. Here p, q and ρ are functions of r and z, $p > 0$, $\rho > 0$ in R. Equation (3.52) reduces to (2.16) if $p = 1/r$, $q = 0$, $\rho = 1/r$ and $\lambda = k^2$.

Let λ_1 and u_1 be the lowest eigenvalue and the corresponding eigenfunction of equation (3.52). u_1 will have no

22. E.L.Chu, J. App. Phys., 21, 454-467 (1950)

zeros in R, $u_1 \cdot \frac{\partial u_1}{\partial n} = 0$ everywhere on Γ. If we denote the boundary condition as $\frac{\partial u_1}{\partial n} + \sigma u_1 = 0$, then either $\sigma = \infty$ or $\sigma = 0$.

u_1 and λ_1 are to be found by minimizing the expression

$$J[u] = D[u] + \int_\Gamma p\sigma u^2 \, ds \qquad (3.53)$$

where D[u] is the abbreviation for D[u,u] which is defined by putting v = u in

$$D[u,v] = \iint_R p(u_r v_r + u_z v_z)\, drdz + \iint_R quvdrdz \qquad (3.54)$$

The minimization is made under the boundary condition that $u = 0$ at least on that part of Γ where $u_1 = 0$ and also under the normalizing condition $\iint_R \rho u^2 drdz = 1$. u should be continuous and have piecewise continuous first derivatives in R but is otherwise arbitrary.

Instead of (3.53) J[u] may be written as

$$J[u] = D[u] - \int_\Gamma p \frac{u^2}{u_1} \frac{\partial u_1}{\partial n} \, ds \qquad (3.55)$$

which is sensible because $u = 0$ wherever $u_1 = 0$ on Γ. J[u] in the form of (3.55) may be transformed by Green's theorem to obtain

$$D[u] - \int_\Gamma p \frac{u^2}{u_1} \frac{\partial u_1}{\partial n} \, ds = D[u] - D\left[u_1, \frac{u^2}{u_1}\right] - \iint_R \frac{u^2}{u_1} L[u_1]\, drdz$$

$$= \iint_R \frac{p}{u_1^2} (u_1 \vec{\nabla} u - u \vec{\nabla} u_1)^2 drdz - \iint_R \frac{u^2}{u_1} L[u_1]\, drdz$$

Since $L[u_1] + \lambda_1 \rho u_1 = 0$, we get

$$D[u] - \int_{\Gamma} p \frac{u^2}{u_1} \frac{\partial u_1}{\partial n} ds = \iint_R \frac{p}{u_1^2} (u_1 \vec{\nabla} u - u \vec{\nabla} u_1)^2 drdz$$

$$+ \lambda_1 \iint_R \rho u^2 \, drdz \, . \tag{3.56}$$

The above transformation is legitimate because u_1 has no zeros inside R and u is tacitly assumed to have first derivatives everywhere in R. If this assumption is not true, we may consider R as being composed of several regions in each of which u is everywhere differentiable. Equation (3.56) brings the variational principle explicitly in view. It states that λ_1 is the minimum value of J[u] for any admissible u that is properly normalized, because

$$\iint_R \frac{p}{u_1^2} (u_1 \vec{\nabla} u - u \vec{\nabla} u_1)^2 \, drdz \geq 0 \quad .$$

Now we restrict ourselves to use only those trial functions which satisfy the differential equation $L[u] + \lambda \rho u = 0$. From (3.56) and

$$D[u] = - \iint_R u L[u] \, drdz + \int_{\Gamma} pu \frac{\partial u}{\partial n} ds \quad ,$$

we obtain

$$\int_{\Gamma} pu \frac{\partial u}{\partial n} ds - \int_{\Gamma} p \frac{u^2}{u_1} \frac{\partial u_1}{\partial n} ds$$

$$= \iint_R \frac{p}{u_1^2} (u_1 \vec{\nabla} u - u \vec{\nabla} u_1)^2 drdz - (\lambda - \lambda_1) \iint_R \rho u^2 \, drdz \, . \quad (3.57)$$

If we demand $-\int_\Gamma pu \frac{\partial u}{\partial n} ds - \int_\Gamma p \frac{u^2}{u_1} \frac{\partial u_1}{\partial n} ds \geqq 0$,

then $\lambda \geqq \lambda_1$. In order that we may approach the limiting case $u = u_1$ as closely as may be desired, we should use the equality sign, i.e. $-\int_\Gamma pu \frac{\partial u}{\partial n} ds - \int_\Gamma p \frac{u^2}{u_1} \frac{\partial u_1}{\partial n} ds = 0$. It may be noted here, from the procedure of our proof, that σ does not have to be so restricted as to be either ∞ or 0. In fact σ may be any given piecewise continuous function of s. In our case the above condition reduces simply to $\int_\Gamma pu \frac{\partial u}{\partial n} ds = 0$.

Now u satisfies the differential equation

$\frac{\partial u}{\partial n} = -\int_{\Gamma'} pu \frac{\partial}{\partial n} \frac{\partial G_+(\lambda)}{\partial n'} ds'$, where $G_+(r,z; r', z'; \lambda)$ is the Green's function of the equation $L[u] + \lambda\rho u = 0$ such that $G_+ = 0$ on that part of Γ, say Γ', where the boundary condition is perturbed, and $\frac{\partial G_+}{\partial n} + \sigma G_+ = 0$ on the rest of Γ. So

$$- \int_\Gamma pu \frac{\partial u}{\partial n} ds - \int_\Gamma p\sigma u^2 ds$$

$$= \int_{\Gamma'} pu(s)ds \int_{\Gamma'} pu(s') \frac{\partial}{\partial n} \frac{\partial G_+(\lambda)}{\partial n'} ds' - \int_{\Gamma'} p\sigma u^2 ds .$$

Thus we formulate a simple rule for calculating an upper bound for λ_1 as follows:

Choose a function $u(s)$ such that $u(s) = 0$ on Γ' wherever σ is infinite ($u_1(s) = 0$). Derive $\partial u/\partial n$ from $u(s)$, i.e. $\frac{\partial u}{\partial n} = -\int_{\Gamma'} pu \frac{\partial}{\partial n} \frac{\partial G_+(\lambda)}{\partial n'} ds'$. Solve the equation

$$J_+(\lambda) = - \int_{\Gamma'} pu \frac{\partial u}{\partial n} ds - \int_{\Gamma'} p\sigma u^2 ds = 0 \tag{3.58}$$

for λ. The value obtained is always an upper bound.

Interchanging u with u_1 and λ with λ_1 in equation (3.57) we obtain an equation which is the mate of (3.57):

$$+\int_\Gamma p u_1 \frac{\partial u_1}{\partial n}\, ds - \int_\Gamma p \frac{u_1^2}{u} \frac{\partial u}{\partial n}\, ds \tag{3.59}$$
$$= \iint_R \frac{p}{u^2} (u \vec{\nabla} u_1 - u_1 \vec{\nabla} u)^2 \, drdz - (\lambda_1 - \lambda) \iint_R \rho u_1^2 \, drdz$$

Here we must have the condition that $u_1(s) = 0$ wherever $u(s)=0$. In other words, u can be zero on Γ only where u_1 is known to be zero. Furthermore u can have no zeros inside R. The latter restriction implies that u must be the solution for the fundamental mode of the modified problem.

If we demand $\int_\Gamma p u_1 \frac{\partial u_1}{\partial n} ds - \int_\Gamma p \frac{u_1^2}{u} \frac{\partial u}{\partial n} ds \lesseqgtr 0$ we must have $\lambda \lesseqgtr \lambda_1$. Since either $u_1 = 0$ or $\partial u_1/\partial n = 0$ on Γ, this condition will certainly be satisfied if $\partial u/\partial n = 0$ wherever $\partial u_1/\partial n = 0$. As long as the latter condition is satisfied we may impose any other condition on u(s) or $\partial u/\partial n$ without affecting the inequality $\lambda \lesseqgtr \lambda_1$. Nevertheless the condition to be imposed should be compatible with u_1 in order that we may be able to approach u_1 in the limiting case. Thus we demand $\int_\Gamma p u \frac{\partial u}{\partial n} ds = \int_\Gamma p u_1 \frac{\partial u_1}{\partial n} ds = 0$. This condition has the same form as that for the upper bound case but they have quite different implications. In the upper bound case u(s) is the trial function and in this it is $\partial u/\partial n$; each has its own boundary restriction to be satisfied.

To generalize this to the case where the boundary condition has the general homogeneous form, we should demand

$$-\int_{\Gamma} pu \frac{\partial u}{\partial n} ds - \int_{\Gamma} \frac{p}{\sigma} \left(\frac{\partial u}{\partial n}\right)^2 ds = 0$$

subject to the restriction that $\partial u/\partial n = 0$ wherever $\sigma = 0$. This also applies to the higher modes provided the orthogonality conditions are properly taken care of.

Now $u = \int_{\Gamma'} p \frac{\partial u}{\partial n} G_-(\lambda)\, ds'$, where $G_-(r,z; r',z'; \lambda)$ is the Green's function of the equation $L[u] + \lambda\rho u = 0$, such that $\partial G_-/\partial n = 0$ on Γ' where the boundary condition is perturbed and G_- is appropriate on $\Gamma - \Gamma'$, and

$$-\int_{\Gamma} pu \frac{\partial u}{\partial n} ds - \int_{\Gamma} \frac{p}{\sigma} \left(\frac{\partial u}{\partial n}\right)^2 ds$$

$$= -\int_{\Gamma'} p \frac{\partial u}{\partial n} ds \int_{\Gamma'} p \frac{\partial u}{\partial n'} G_-(s,s'; \lambda)\, ds' - \int_{\Gamma'} \frac{p}{\sigma}\left(\frac{\partial u}{\partial n}\right)^2 ds \quad .$$

The rule for calculating a lower bound for λ_1 is as follows:

Choose a function $\partial u/\partial n$ such that $\partial u/\partial n = 0$ on Γ' wherever $\sigma = 0$. Derive $u(s)$ from $\partial u/\partial n$, i.e., $u(s) = \int_{\Gamma'} p \frac{\partial u}{\partial n'} G_-(\lambda) ds'$. Solve the equation

$$J_-(x) = -\int_{\Gamma'} pu \frac{\partial u}{\partial n} ds - \int_{\Gamma'} \frac{p}{\sigma} \left(\frac{\partial u}{\partial n}\right)^2 ds = 0 \qquad (3.60)$$

for λ. The value obtained is always a lower bound.

It must be understood here that $\partial u/\partial n$ should be reasonably close to $\partial u_1/\partial n$, otherwise we might get a solution for a higher mode.

$J_+(\lambda)$ and $J_-(\lambda)$ have the following extremal properties:

$$\left.\begin{aligned} J_+(\lambda_1) &\lesseqgtr 0 \text{ for any admissible } u, \\ J_-(\lambda_1) &\lesseqgtr 0 \text{ for any admissible } \partial u/\partial n, \end{aligned}\right\} \tag{3.61a}$$

$$\left.\begin{aligned} \frac{\partial}{\partial\lambda} J_+ (\lambda) &> 0 \quad \text{for fixed } u, \\ \frac{\partial}{\partial\lambda} J_- (\lambda) &< 0 \text{ for fixed } \partial u/\partial n, \end{aligned}\right\} \tag{3.61b}$$

(iii) Application of the Method to the Problem of Disk-Loaded Waveguides: Variational Expressions, Trial Functions, Summation of Series, and Numerical Results.

To apply the above rules to the loaded waveguide problem, we consider the dividing surfaces shown in Fig. 3.1 or 3.2 as Γ'. Thus Γ consists of both the actual and the virtual boundaries, the former being traversed once while the latter is traversed twice in opposite directions.

For the sake of convenience we will adopt the first scheme of division (Fig. 3.1) and assume that the loading disks have zero thickness ($\eta = 0$). We will confine ourselves to the case $\beta = 1$ (neglecting the first few feet of tube) and take $k_{zo}d = \pi/2$ which is the design value for the Stanford billion-volt accelerator tube. The alternative scheme of division and other values of $k_{zo}d$ and η may of course be chosen; they do not present mathematical difficulties but introduce some complexity in the numerical calculations.

Also for convenience's sake we will consider ka, kd as fixed and calculate kb. It can be easily verified that kb is an upper bound in one case and a lower bound in the other just as k for fixed dimensions.

With these specializations the domain of the problem is reduced to a two-cell cavity (p = 2, q = 1 in Fig. 2.4a) which may be further reduced to a single-cell domain shown in Fig. 3.10 by specifying proper boundary conditions. They are: $\partial u_I/\partial n = 0$ on to $\partial u_I/\partial n = 0$ and $u_I = 0$ on op, $u_I = 0$ on pq, $\sigma_I = -\sigma_{II}$ on qt and tq and $\partial u_{II}/\partial n = 0$ on qrst.

Let $G^I_\pm$ and $G^{II}_\pm$ be the two types of Green's functions of regions I and II respectively. $G^{I,II}_+ = 0$, $\partial G^{I,II}_-/\partial n = 0$ on $\Gamma'_{I,II}$. $G^{I,II}_+$ and $G^{I,II}_-$ are appropriate on $\Gamma_{I,II}$. Since $\sigma_I = -\sigma_{II}$ on Γ' we have $\int_{\Gamma'} p\sigma u^2 ds = 0$ in the upper bound case and $\int_{\Gamma'} \frac{p}{\sigma}\left(\frac{\partial u}{\partial n}\right)^2 ds = 0$ in the lower bound case. Thus the equations to be solved in the two different cases are

$$J_+(k) = \int_{\Gamma'} pu(s)ds \int_{\Gamma'} pu(s')\, \frac{\partial}{\partial n}\frac{\partial}{\partial n'}\, G_+(s,s';\, k)\, ds' = 0 \,, \tag{3.62a}$$

$$J_-(k) = -\int_{\Gamma'} p\, \frac{\partial u}{\partial n}\, ds \int_{\Gamma'} p\, \frac{\partial u}{\partial n'}\, G_-(s,s';\, k)\, ds' = 0 \tag{3.62b}$$

By the usual method we obtain the Green's functions as follows:

$$G^I_+(r,z;\, r',z';\, k) = \sum_0^\infty \frac{\pi}{d}\, \frac{1}{J_1(k_{2n+1}a)} \cos\frac{(2n+1)\pi z}{2d} \cos\frac{(2n+1)\pi z'}{2d}\, F^I_n(r,r') \tag{3.63}$$

with

$$F_n^I(r,r') = \begin{cases} rJ_1(k_{2n+1}r)\ r'\overline{\xi}_1(k_{2n+1}r') & r \leqq r' \\ r\overline{\xi}_1(k_{2n+1}r)\ r'J_1(k_{2n+1}r') & r \geqq r' \end{cases}$$

where

$$\overline{\xi}_1(k_m r) = N_1(k_m a)\ J_1(k_m r) - J_1(k_m a)\ N_1(k_m r) \quad ,$$

$$\overline{\xi}_o(k_m a) = N_1(k_m a)\ J_o(k_m a) - J_1(k_m a)\ N_o(k_m a)$$

$$= -\ 2/\pi\ k_m a$$

and

$$k_{2n+1}^2 + \left[\frac{(2n+1)\pi}{2d}\right]^2 = k^2 \ .$$

$$G_+^{II}(r,\ z;\ r',\ z';\ k) =$$

$$-\sum_o^\infty \frac{\pi}{2d}\ \frac{2-\delta_{on}}{Z_1(K_n a)} \cos\frac{n\pi z}{d}\ \cos\frac{n\pi z'}{d}\ F_n^{II}(r,r') \tag{3.64}$$

with

$$\delta_{on} = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

$$F_n^{II}(r,r') = \begin{cases} r\ Z_1(K_n r)\ r'\overline{\xi}_1(K_n r') & r \geqq r' \\ r\overline{\xi}_1(K_n r)\ r'\ Z_1(K_n r') & r \leqq r' \end{cases}$$

where ξ_1 is defined as above,

$$Z_1(K_n r) = N_o(K_n b)\ J_1(K_n r) - J_o(K_n b)\ N_1(K_n r) \ ,$$

$$Z_o(K_n b) \equiv 0$$

and

$$K_n^2 + (\frac{n\pi}{d})^2 = k^2 .$$

$$G_-^I(r,z;\ r',\ z';\ k) =$$

$$= \sum_0^\infty \frac{\pi}{d} \frac{1}{J_o(k_{2n+1}a)} \cos\frac{(2n+1)\pi z}{2d} \cos\frac{(2n+1)\pi z'}{2d} f_n^I(r,r') \tag{3.65}$$

with

$$f_n^I(r,r') = \begin{cases} rJ_1(k_{2n+1}r)\ r'\xi_1(k_{2n+1}r') & r \leqq r' \\ r\xi_1(k_{2n+1}r)\ r'J_1(k_{2n+1}r') & r \geqq r' \end{cases}$$

where

$$\xi_1(k_m r) = N_o(k_m a)\ J_1(k_m r) - J_o(k_m a)\ N_1(k_m r) ,$$

$$\xi_1(k_m a) = 2/\pi\ k_m a$$

and

$$\xi_o(k_m a) \equiv 0 .$$

$$G_-^{II}(r,z;\ r',\ z';\ k)$$

$$= -\sum_0^\infty \frac{\pi}{2d} \frac{2 - \delta_{on}}{Z_o(K_n a)} \cos\frac{n\pi z}{d} \cos\frac{n\pi z'}{d}\ f_n^{II}(r,r') \tag{3.66}$$

with

$$f_n^{II}(r,r') = \begin{cases} r\ Z_1(K_n r)\ r'\xi(K_n r') & r \geqq r' \\ r\xi(K_n r)\ r'Z_1(K_n r') & r \leqq r' . \end{cases}$$

Substituting these expressions and $p = 1/r = 1/a$ in equations (3.62) we obtain

$$J_+(k) = \int_0^d\int_0^d \frac{1}{a}u(z)\left[- \sum_0^\infty \frac{2}{d}\, k_{2n+1}a \cdot \frac{J_o(k_{2n+1}a)}{J_1(k_{2n+1}a)} \cdot \right.$$

$$\cdot \cos\frac{(2n+1)\pi z}{2d}\cos\frac{(2n+1)\pi z'}{2d}$$

$$\left. + \sum_0^\infty \frac{2 - \delta_{on}}{d}\, K_n a \cdot \frac{Z_o(K_n a)}{Z_1(K_n a)}\cos\frac{n\pi z}{d}\cos\frac{n\pi z'}{d}\right] \cdot$$

$$\cdot \frac{1}{a}\, u(z')\, dz\, dz' = 0 \tag{3.67a}$$

$$J_-(k) = \int_0^d\int_0^d \frac{1}{a}\frac{\partial u}{\partial r}\left[- \sum_0^\infty \frac{2a}{k_{2n+1}d} \cdot \frac{J_1(k_{2n+1}a)}{J_o(k_{2n+1}a)} \cdot \right.$$

$$\cdot \cos\frac{(2n+1)\pi z}{2d}\cos\frac{(2n+1)\pi z'}{2d}$$

$$\left. + \sum_0^\infty \frac{(2 - \delta_{on})a}{K_n d} \cdot \frac{Z_1(K_n a)}{Z_o(K_n a)}\cos\frac{n\pi z}{d}\cos\frac{n\pi z'}{d}\right] \cdot$$

$$\cdot \frac{1}{a}\frac{\partial u}{\partial r'}\, dz\, dz' = 0 \tag{3.67b}$$

If we denote

$$\left.\begin{aligned} \sigma^{I}_{2n+1} &= - k_{2n+1}a\,\frac{J_o(k_{2n+1}a)}{J_1(k_{2n+1}a)} \\ \sigma^{II}_{n} &= K_n a\,\frac{Z_o(K_n a)}{Z_1(K_n a)} \end{aligned}\right\} \tag{3.68a}$$

$$\left.\begin{aligned} u^{I}_{2n+1} &= \sqrt{\frac{2}{d}} \int_{o}^{d} \frac{1}{a} \cos \frac{(2n+1)\pi z}{2d}\, u(z)\, dz \\ u^{II}_{n} &= \sqrt{\frac{2-\delta_{on}}{d}} \int_{o}^{d} \frac{1}{a} \cos \frac{n\pi z}{d}\, u(z)\, dz \end{aligned}\right\} \tag{3.68b}$$

and

$$\left.\begin{aligned} (u_r)^{I}_{2n+1} &= \sqrt{\frac{2}{d}} \int_{o}^{d} \cos \frac{(2n+1)\pi z}{2d} \frac{\partial u}{\partial r}\, dz \\ (u_r)^{II}_{n} &= \sqrt{\frac{2-\delta_{on}}{d}} \int_{o}^{d} \cos \frac{n\pi z}{d} \frac{\partial u}{\partial r}\, dz\ , \end{aligned}\right\} \tag{3.68c}$$

$J_{\pm}(k)$ may be written in the following simple quadratic forms:

$$J_{+}(k) = \sum_{o}^{\infty} \sigma^{I}_{2n+1} \left[u^{I}_{2n+1}\right]^2 + \sum_{o}^{\infty} \sigma^{II}_{n} \left[u^{II}_{n}\right]^2 \ , \tag{3.69a}$$

$$J_{-}(k) = \sum_{o}^{\infty} \frac{1}{\sigma^{I}_{2n+1}} \left[(u_r)^{I}_{2n+1}\right]^2 + \sum_{o}^{\infty} \frac{1}{\sigma^{II}_{n}} \left[(u_r)^{II}_{n}\right]^2 \ . \tag{3.69b}$$

Since $\beta = 1$, we have $kd = k_{zo}d = \frac{\pi}{2}$,

$$k_{2n+1} = \begin{cases} 0 & n = 0 \\ \text{imaginary} & n \neq 0 \ , \end{cases}$$

and

$$\left.\begin{aligned} \sigma^{I}_{2n+1} &< 0 \qquad \text{for any } n \\ \sigma^{II}_{n} &< 0 \qquad n \neq 0 \end{aligned}\right\} \tag{3.70}$$

Thus all but one term (σ^{II}_{o}) in $J_{+}(k)$ and $J_{-}(k)$ are negative. In order that the equation $J_{\pm}(k) = 0$ will have a solution we must have $\sigma^{II}_{o} > 0$, i.e.,

$$ka \frac{Z_0(ka)}{Z_1(ka)} > 0 \quad . \tag{3.71}$$

To solve these equations, we first devise functions for $u(a,z) \approx H_\phi(a,z)$ and $u_r(a,z) \approx E_z(a,z)$ that correspond as closely as possible to the actual fields and are analytically tractable.

Information useful in this connection may be obtained by considering the limiting cases of $ka \to 0$ and $ka \to kb$, both of which have been discussed in Section 3.2. The former may be considered as two cylindrical cavities end to end, perturbed by a hole through the common end, in which case the field will be qualitatively like Fig. 3.11a. When $ka \to kb \to \infty$, we can consider that we have a cylindrical cavity with one axial node and perturbed by a ring of small radial extent at $z = d$, in which case a field plot like Fig. 3.11c is indicated. For intermediate values, a field like Fig. 3.11b would be anticipated. In all cases, E_z must approach infinity like $\frac{1}{|d - z|^{1/2}}$ near the edge of the disk, just as it would in a static problem, since near an edge the field curvatures imposed by satisfying the boundary conditions dominate the curvature due to the k^2 term in the wave equation, so that the wave equation is well approximated by Laplace's equation. For small ka, E_z is nearly constant except near $z = d$; for large ka, E_z is nearly sinusoidal except near $z = d$. In the long guide, E_z is even about $z = 2md$ and odd about $z = (2m - 1)d$, m being integers.

Guided by the above considerations we choose the following functions for E_z : (1) $E_z = 1$, (2) $E_z = \cosh \alpha z$, α being a variable parameter, and (3) $E_z = (d^2 - z^2)^{-1/2}$. The first function is good in the limit $ka \to 0$. The last one has the right form of singularity, so may be expected to be good for intermediate values of ka, where the field is largely controlled by the disk edge. The second function has properties intermediate between the other two and has one variable parameter to be adjusted for best results. For large ka, the function $\cos \frac{\pi z}{2d}$ should be good, but this has not been tried because work with this function for H_ϕ shows that the values of ka for which this is a good choice are considerably above the range of practical interest.

For approximations to the magnetic fields, similar arguments lead us to try (1) $H_\phi = (d^2 - z^2)^{1/2}$ for intermediate ka and (2) $H_\phi = \cos \frac{\pi z}{2d}$ for large ka. For small ka, the obviously suggested H_ϕ = const. does not work, the discontinuity in current at $z = d$ implied by this choice causing the series to diverge. But although a suitable function can be devised, it turns out not to be needed, for on trial it is found that the first function works very well indeed for small ka. Moreover, even for ka at the other end of the range of interest, the square root function is better than the cosine, so that actually only one function is needed for H_ϕ.

We then calculate the Fourier coefficients defined in (3.68b) and (3.68c) and obtain the series for $J_\pm(k)$. All these series converge rather slowly. Numerical calculation is made possible by the help of Kummer's transformation, which consists simply of adding and subtracting from the given series one which is summable and has terms as similar in construction as possible to those of the given one.

Let us write $J(k) = J_I(k) + J_{II}(k)$ and denote the respective asymptotic series by $J'(k) = J'_I(k) + J'_{II}(k)$. The results may be summarized as follows:

U.B. Case 1. $\qquad H_\phi = \sqrt{d^2 - z^2}$

$$J_+(k) = \sum_0^\infty \sigma^I_{2n+1} \frac{J_1^2(n+\frac{1}{2})\pi}{(n+\frac{1}{2})^2} + \sum_0^\infty \sigma_n^{II}(1 - \frac{\delta_{on}}{2}) \frac{J_1^2(n\pi)}{n^2} \tag{3.72a}$$

$$J'_I(k) = \sum_0^\infty \Big[\frac{2ka}{\pi^2} \frac{1}{(n+\frac{1}{2})^2} + \frac{1}{2\pi^2} \frac{1}{(n+\frac{1}{2})^3}$$

$$+ (\frac{3}{16ka}\frac{1}{\pi^2} - \frac{ka}{4\pi^2}) \frac{1}{(n+\frac{1}{2})^4} \Big] ,$$

$$J'_{II}(k) = \sum_1^\infty \Big[\frac{2ka}{\pi^2} \frac{1}{n^2} - \frac{1}{2\pi^2}\frac{1}{n^3} + (\frac{3}{16ka}\frac{1}{\pi^2} - \frac{ka}{4\pi^2})\frac{1}{n^4} \Big] .$$

U.B. Case 2. $\qquad H_\phi = \cos \frac{\pi z}{2d}$.

$$J_+(k) = \frac{\pi^2}{8} + \sum_0^\infty \sigma_n^{II} (1 - \frac{\delta_{on}}{2}) \frac{1}{(4n^2-1)^2} . \tag{3.72b}$$

$$J'_{II}(k) = \sum_1^\infty \Big[\frac{ka}{8}\frac{1}{n^3} - \frac{1}{32}\frac{1}{n^4} + (\frac{3}{256}\frac{1}{ka} + \frac{3ka}{64}) \frac{1}{n^5} \Big] .$$

L.B. Case 1. $E_z = 1$.

$$J_-(k) = \frac{1}{\sigma_o^{II}} \frac{\pi^2}{8} + \sum_o^\infty \frac{1}{\sigma_{2n+1}^I} \frac{1}{(2n+1)^2} \tag{3.72c}$$

$$J_I'(k) = \sum_o^\infty \left[\frac{1}{8ka} \frac{1}{(n+\frac{1}{2})^3} - \frac{1}{32} \frac{1}{(ka)^2} \frac{1}{(n+\frac{1}{2})^4} - \right.$$

$$\left. - \left(\frac{1}{256} \frac{1}{(ka)^3} - \frac{1}{64ka}\right) \frac{1}{(n+\frac{1}{2})^5}\right] .$$

L.B. Case 2. $E_z = \cosh \alpha z, \alpha = \ell\pi/d$,

ℓ being variable.

$$J_-(k) = \sum_o^\infty \frac{1}{\sigma_{2n+1}^I} \left[\frac{(2n+1)\ell}{(2n+1)^2 + 4\ell^2} \pi \coth \ell\pi\right]^2$$

$$+ \sum_o^\infty \frac{1}{\sigma_n^{II}} \left(1 - \frac{\delta_{on}}{2}\right) \left[\frac{\ell^2}{n^2 + \ell^2} \frac{\pi}{2}\right]^2 \tag{3.72d}$$

$$J_I'(k) = \sum_o^\infty \left[\frac{1}{8ka} \frac{1}{(n+\frac{1}{2})^3} - \frac{1}{32} \frac{1}{(ka)^2} \frac{1}{(n+\frac{1}{2})^4} - \right.$$

$$\left. - \left(\frac{1}{256} \frac{1}{(ka)^3} - \frac{1}{64ka} + \frac{\ell^2}{4ka}\right) \frac{1}{(n+\frac{1}{2})^5}\right] \ell^2\pi^2 \coth^2 \ell\pi ,$$

$$J'_{II}(k) = \sum_{1}^{\infty} \left[\frac{1}{2ka}\frac{1}{n^5} + \frac{1}{8}\frac{1}{(ka)^2}\frac{1}{n^6} - \right.$$

$$\left. - \left(\frac{1}{64}\frac{1}{(ka)^3} - \frac{1}{16ka} + \frac{\ell^2}{ka}\right)\frac{1}{n^7}\right] \frac{\ell^4}{4}\pi^2 \quad .$$

L.B. Case 3. $\qquad E_z = (d^2 - z^2)^{-1/2}$

$$J_-(k) = \sum_{o}^{\infty} \frac{1}{\sigma^{I}_{2n+1}} J_o^2(n + \tfrac{1}{2})\pi$$

$$+ \sum_{o}^{\infty} \frac{1}{\sigma^{II}_{n}} \left(1 - \frac{\delta_{on}}{2}\right) J_o^2(n\pi) \quad . \tag{3.72e}$$

$$J'_{I}(k) = \sum_{o}^{\infty} \left[\frac{1}{2\pi^2}\frac{1}{ka}\frac{1}{(n+\frac{1}{2})^2} - \frac{1}{8\pi^2}\frac{1}{(ka)^2}\frac{1}{(n+\frac{1}{2})^3} - \right.$$

$$\left. - \left(\frac{1}{64\pi^2}\frac{1}{(ka)^3} - \frac{1}{16\pi^2}\frac{1}{ka}\right)\frac{1}{(n+\frac{1}{2})^4}\right] ,$$

$$J'_{II}(k) = \sum_{1}^{\infty} \left[\frac{1}{2\pi^2}\frac{1}{ka}\frac{1}{n^2} + \frac{1}{8\pi^2}\frac{1}{(ka)^2}\frac{1}{n^3} - \right.$$

$$\left. - \left(\frac{1}{64\pi^2}\frac{1}{(ka)^3} - \frac{1}{16\pi^2}\frac{1}{ka}\right)\frac{1}{n^4}\right] \quad .$$

In passing it may be noted that if we take only the first term from the series in equation (3.72c) we obtain

$$\frac{1}{ka}\frac{Z_1(ka)}{Z_o(ka)}\frac{\pi^2}{8} = \frac{1}{k_{ro}a}\frac{J_1(k_{ro}a)}{J_o(k_{ro}a)} \quad ,$$

which is the same as equation (3.34) in the approximate theory with $\eta = 0$ if the factor $\pi^2/8$ is replaced by unity.

The numerical results are exhibited in Table I and Fig. 3.12 and 3.13. In the body of Table I are given approximate values of kb for various ka, the various columns corresponding to trial functions as specified at the column heads. For each value of ka, the highest lower bound and the lowest upper bounds are underlined - the true value surely lying between these limits. An upper limit for the uncertainty in kb is given in the last column.

Comparing the various trial functions we note the following. Values of ka used do not run high enough for the cosine approximation for H_ϕ to better the root function, and the discrepancy between the two is never larger than 2 percent. The cosh αz function is always better than the constant as it should be, since the former function includes the latter. But E_z = const. is quite good for small ka. The root function is better for large ka and might in turn be supplanted by a cosine function for still larger ka.

As to the uncertainty in kb, this is very small even if we simply take the numbers as given. And a still lower estimate is probably safe, if we note that near ka = 2.0, where the uncertainty is a maximum, the cosh αz function changes from better to worse than the root function, so that the estimate from E_z is certainly too low, while on the other hand the same H_ϕ function works on both sides of this point, so the estimate from H_ϕ is probably close. We therefore believe that the values in the first column are everywhere the best, and that the uncertainty is about 0.01 or less.

Slater's work[23] mentioned before covers exactly the same problem (π and $\pi/2$ modes are equivalent) and essentially the same range of numbers, so that a comparison is both desirable and possible. We have therefore plotted Slater's results with ours on both Fig. 3.12 and 3.13. Direct comparisons are impossible because Slater worked with simple values of kb while we used simple values of ka. It may be seen that, to within the accuracy of either the comparison or of either calculation, the two sets of results are identical. A further and nearly direct comparison is possible at one point, where Slater finds ka = 1.50 for kb = 2.8, while we find for ka = 1.5, $2.78 < kb < 2.80$ with the upper value preferred. Again the agreement is perfect.

Better precision may be achieved by using trial functions which have two or more terms with variable parameters. The following ones would be quite suitable:

$$H_\phi = (d^2 - z^2)^{1/2} + \alpha(d^2 - z^2) + \beta(d^2 - z^2)^{3/2} + \ldots$$

$$E_z = (d^2 - z^2)^{-1/2} + \alpha + \beta(d^2 - z^2)^{1/2} + \ldots ,$$

or

$$H_\phi = (d^2 - z^2)^{1/2} + \alpha \cos k_{z0} z + \beta \cos k_{z1} z + \ldots$$

$$E_z = (d^2 - z^2)^{-1/2} + \alpha \cos k_{z0} z + \beta \cos k_{z1} z + \ldots ,$$

α, β, . . . being variable parameters.

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23. J.C. Slater, loc.cit. footnote 13.

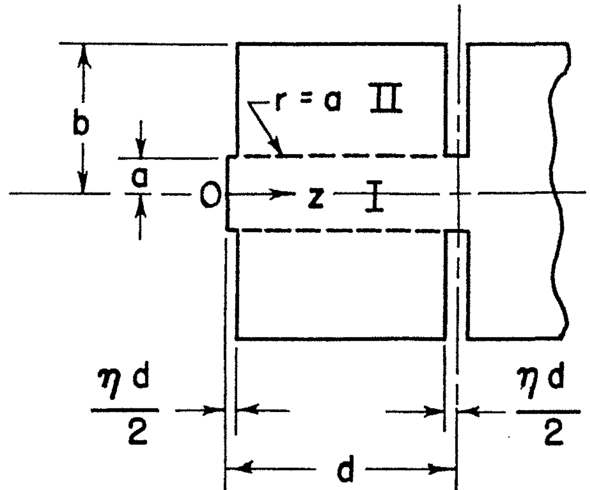

Fig. 3.1 - The first cell of the closed guide ($0 \leq z \leq d$) is shown divided into two simple regions I and II by the cylindrical surface r = a.

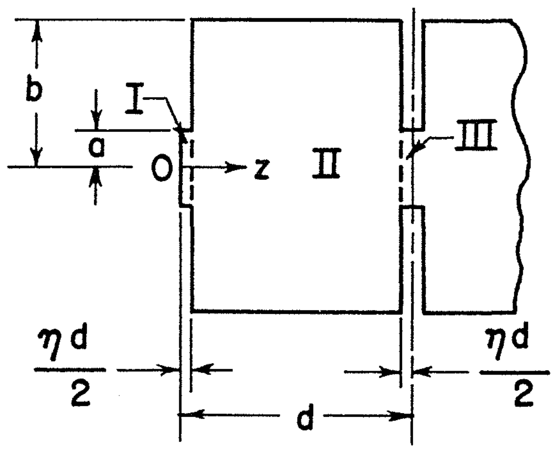

Fig. 3.2 - The first cell of the closed guide ($0 \leq z \leq d$) is shown divided into three simple regions by two transverse planes $z = \frac{\eta d}{2}$ and $z = d - \frac{\eta d}{2}$.

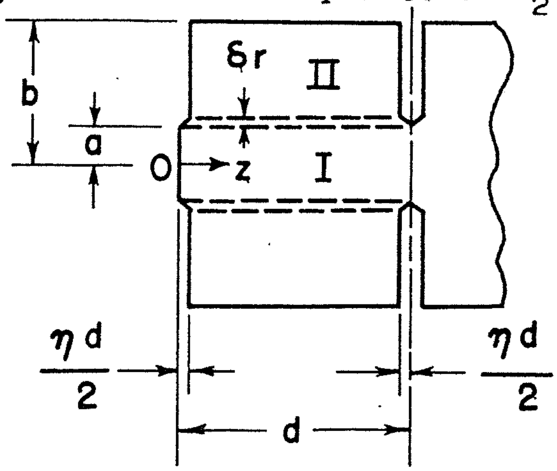

Fig. 3.3 - Same as Fig 3.1 but with disk edge shown tapered. δr is infinitesimal.

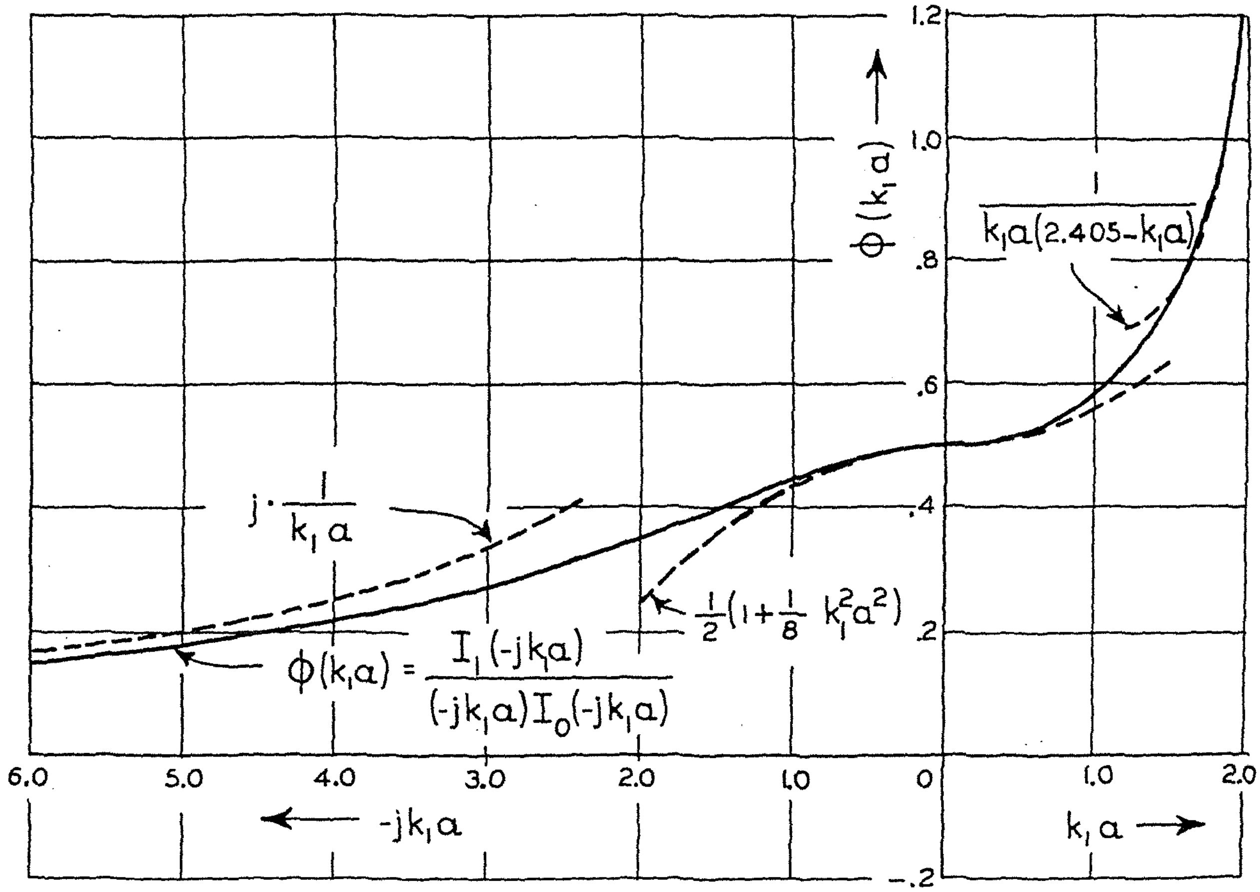

Fig. 3.4 - This shows ϕ as a function of $k_1 a$ ($k_1 a \equiv k_{ro} a$), real values of the argument being on the right-hand side, and imaginary values on the left. Various approximate formulae are shown in dotted lines.

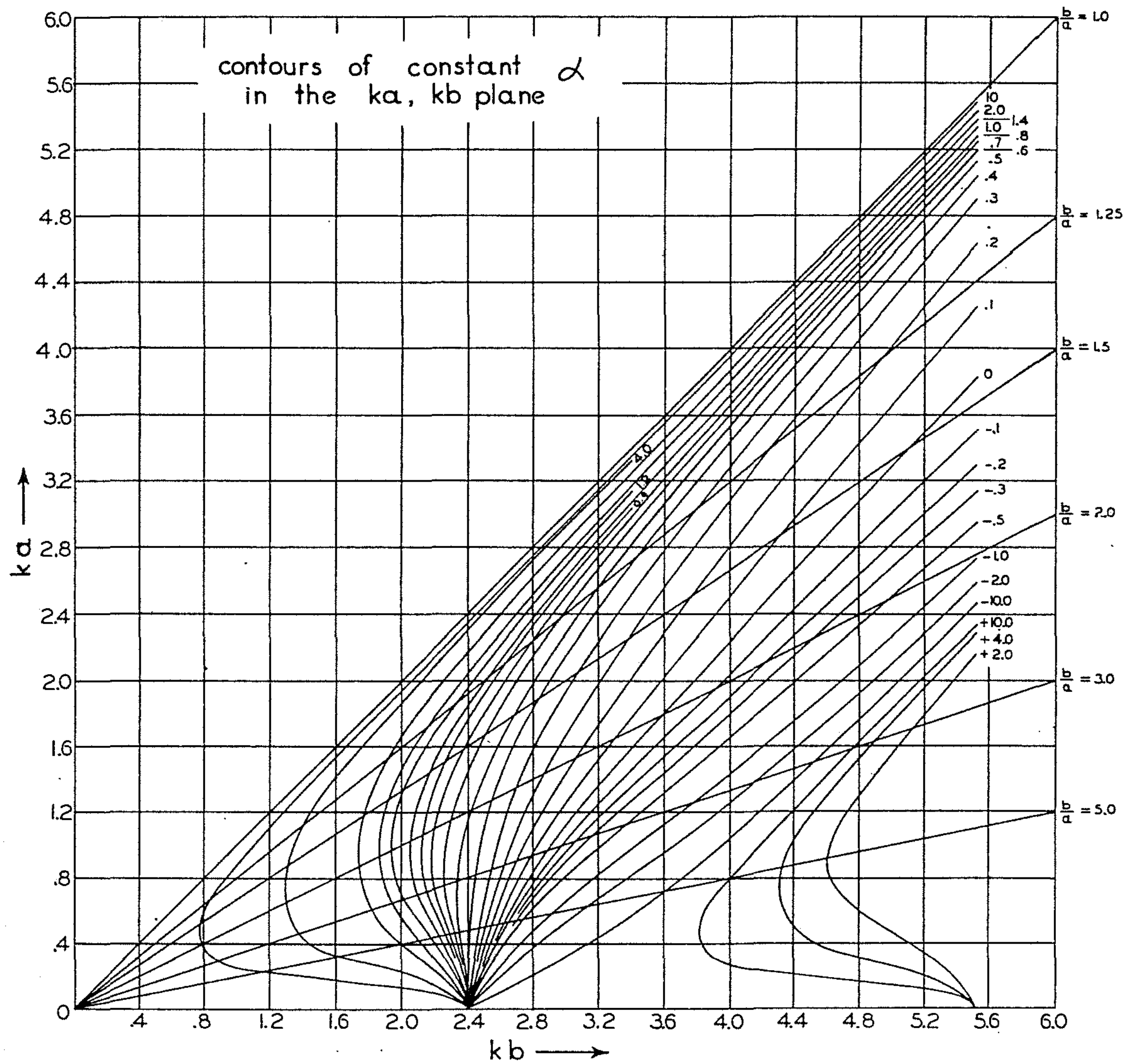

Fig. 3.5 - Shows contours of constant α in the ka, kb plane.

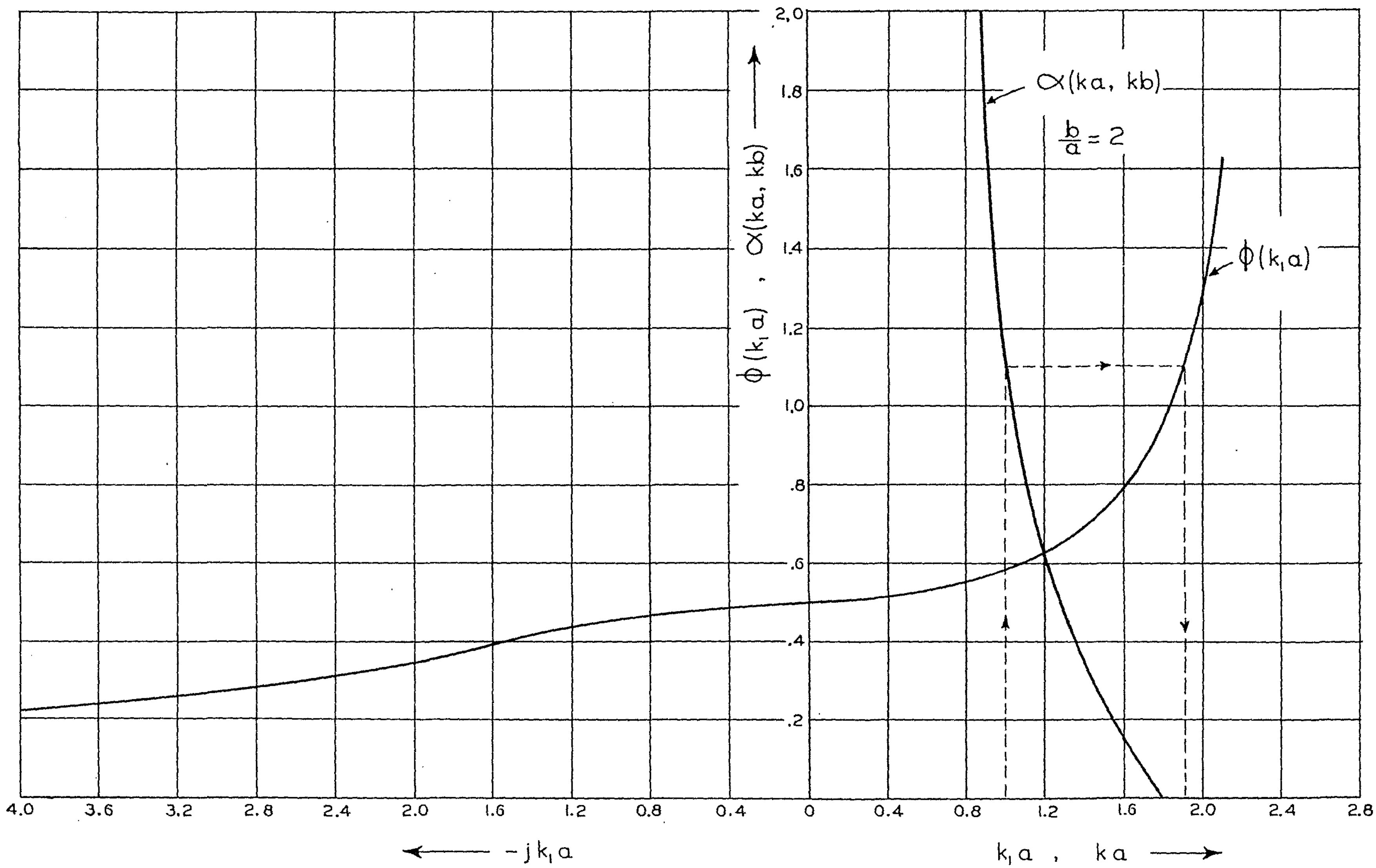

Fig. 3.6 - The functions ϕ and α are plotted against k_1a ($k_1a \equiv k_{ro}a$) and ka for b/a = 2.

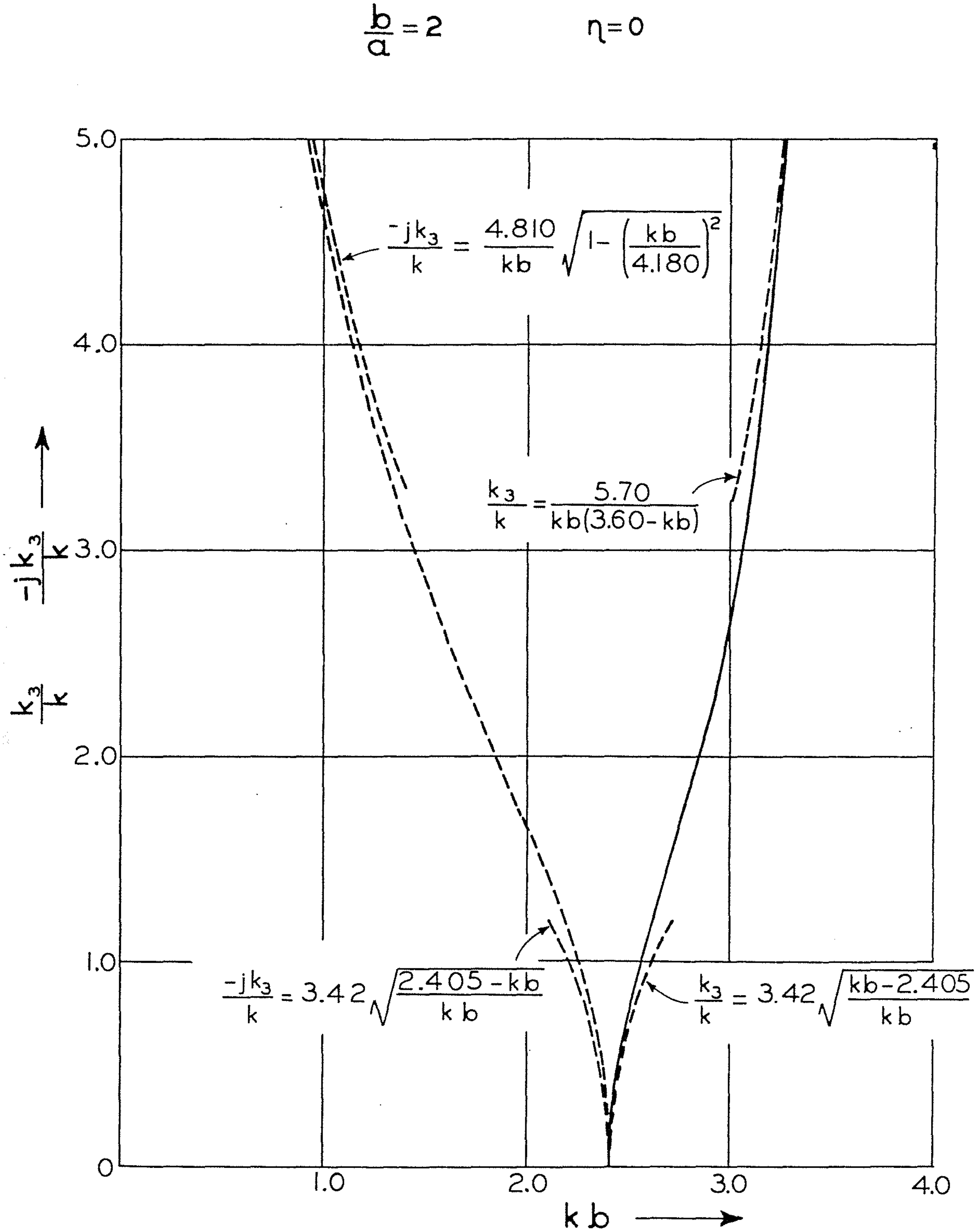

Fig. 3.7 - Shows k_3/k as a function of kb ($k_3 = k_{zo}$) for $b/a = 2$, $\eta = 0$, $kd \cong 0$. Also shown are a number of approximate formulae.

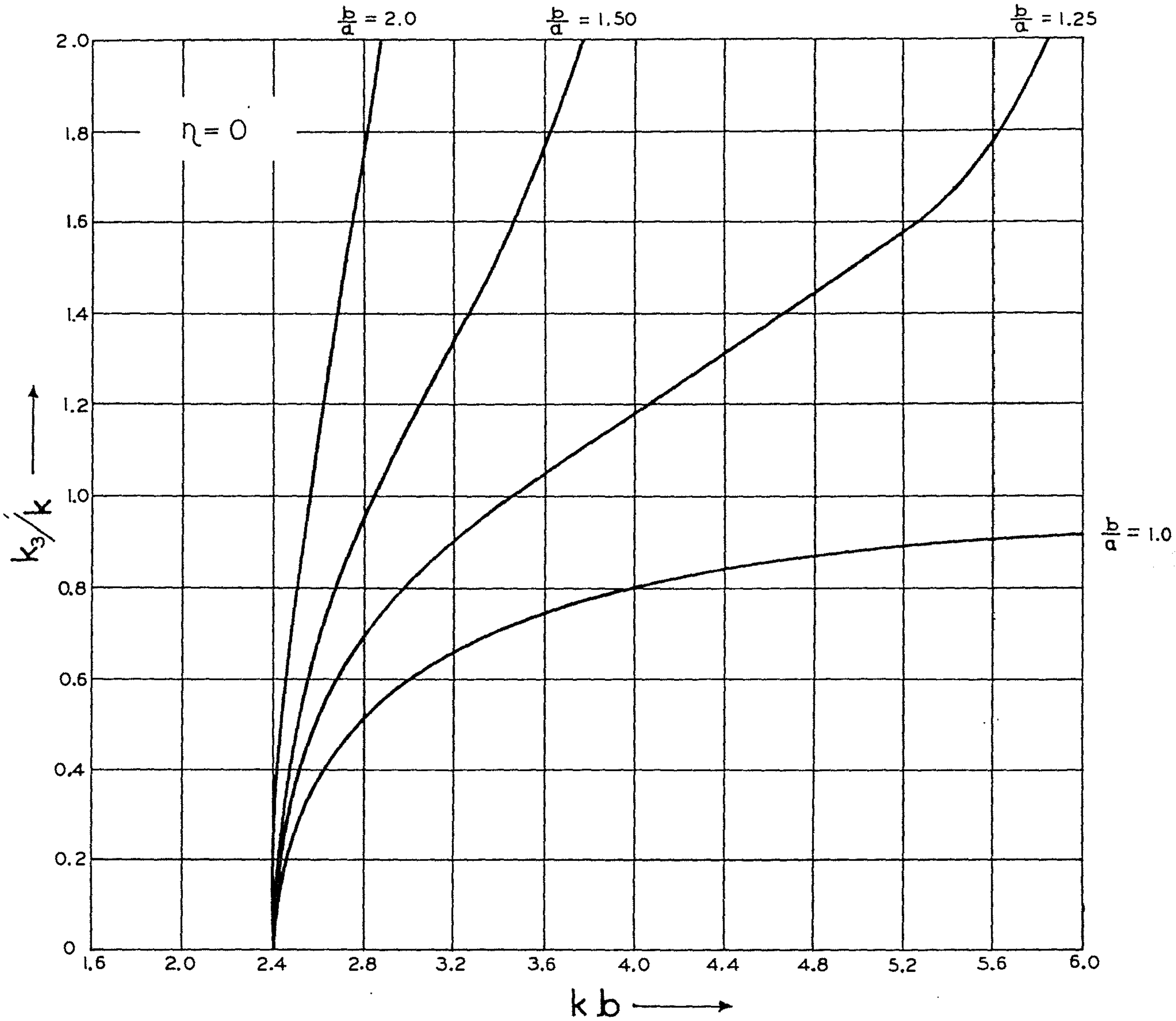

Fig. 3.8 - Shows k_3/k as a function of kb ($k_3 \equiv k_{zo}$) for a variety of ratios b/a.

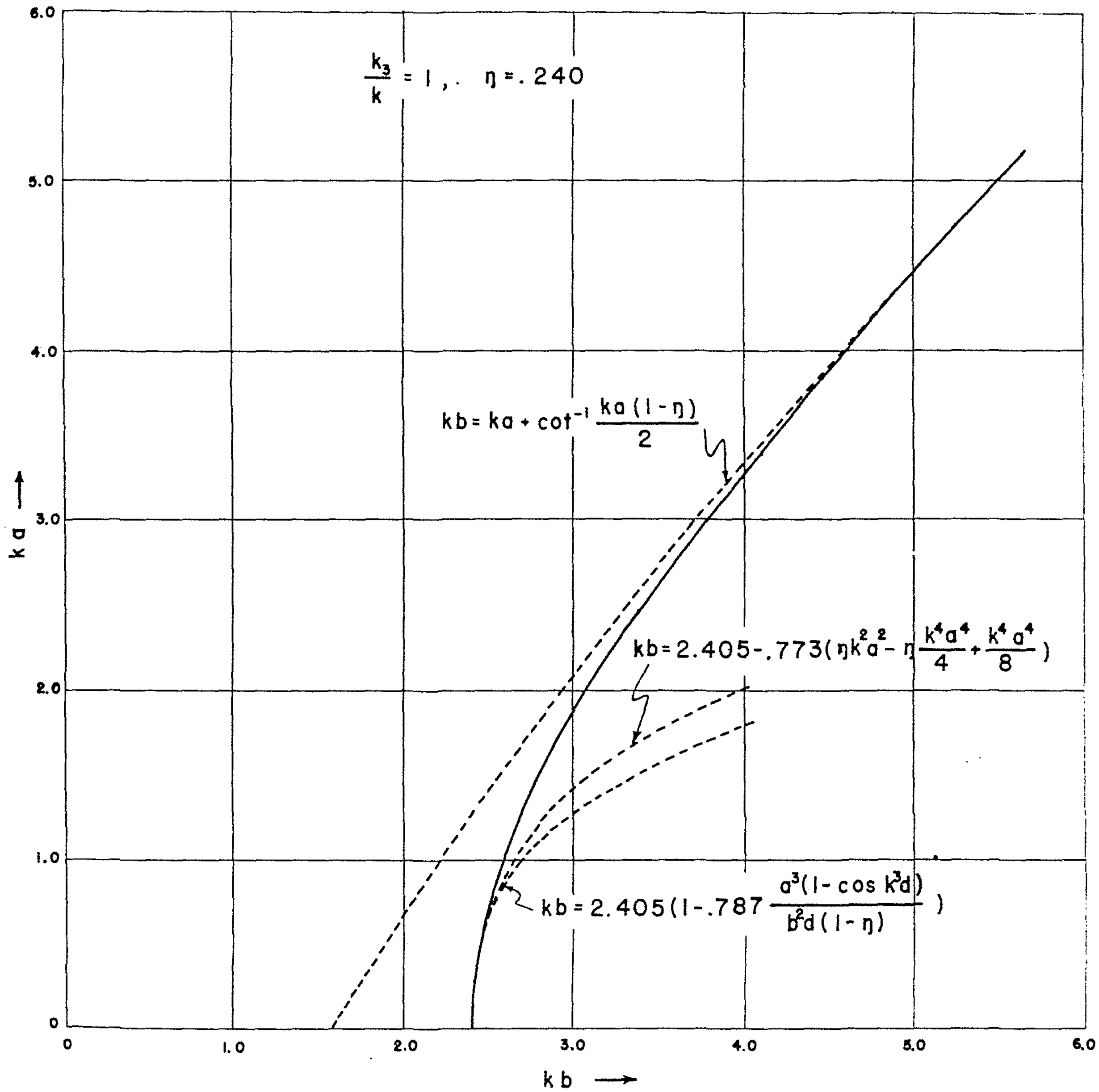

Fig. 3.9 - The value of ka that makes $k_3 = k$ is plotted against kb ($k_3 \equiv k_{zo}$). The full curve is based on the assumption that $d \ll a$, $d \ll b-a$.

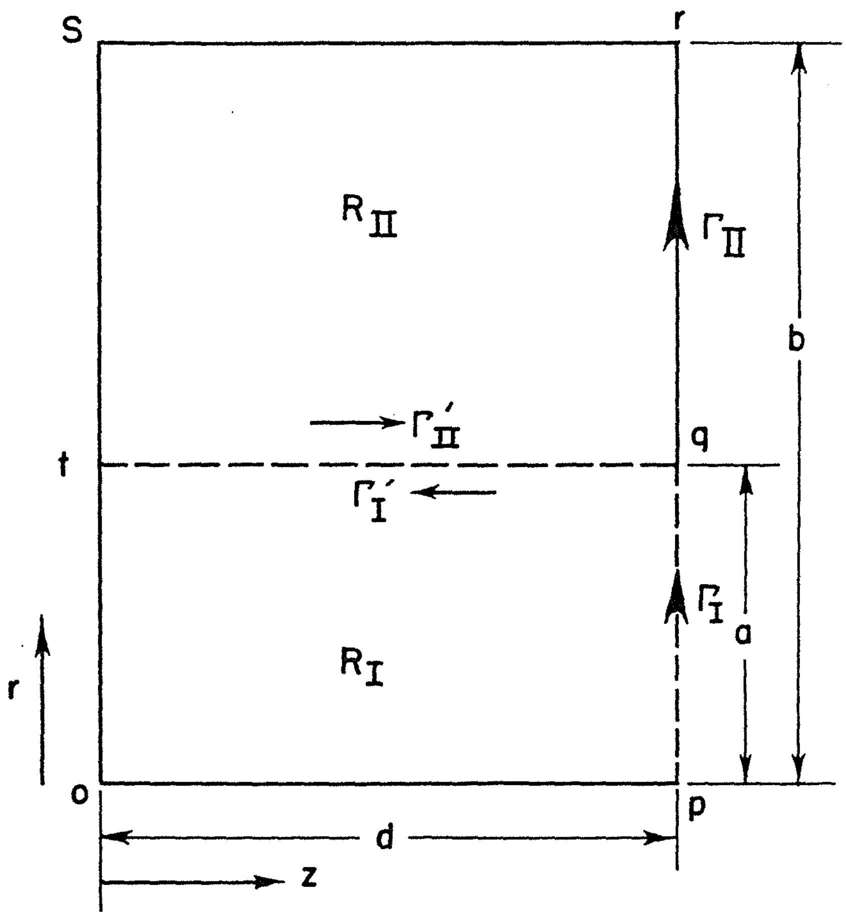

Fig. 3.10 - A composite region R representing the domain of the problem is shown divided into two simple regions R_I and R_{II} by the surface Γ'. The open boundary topq is denoted by Γ_I, qt by Γ'_I, tq by Γ'_{II} and qrst by Γ_{II}. $R = R_I + R_{II}$, $\Gamma = \Gamma_I + \Gamma'_I + \Gamma'_{II} + \Gamma_{II}$, and $\Gamma' = \Gamma'_I + \Gamma'_{II}$. A rotation of the figure about op gives the volume of the unit cell.

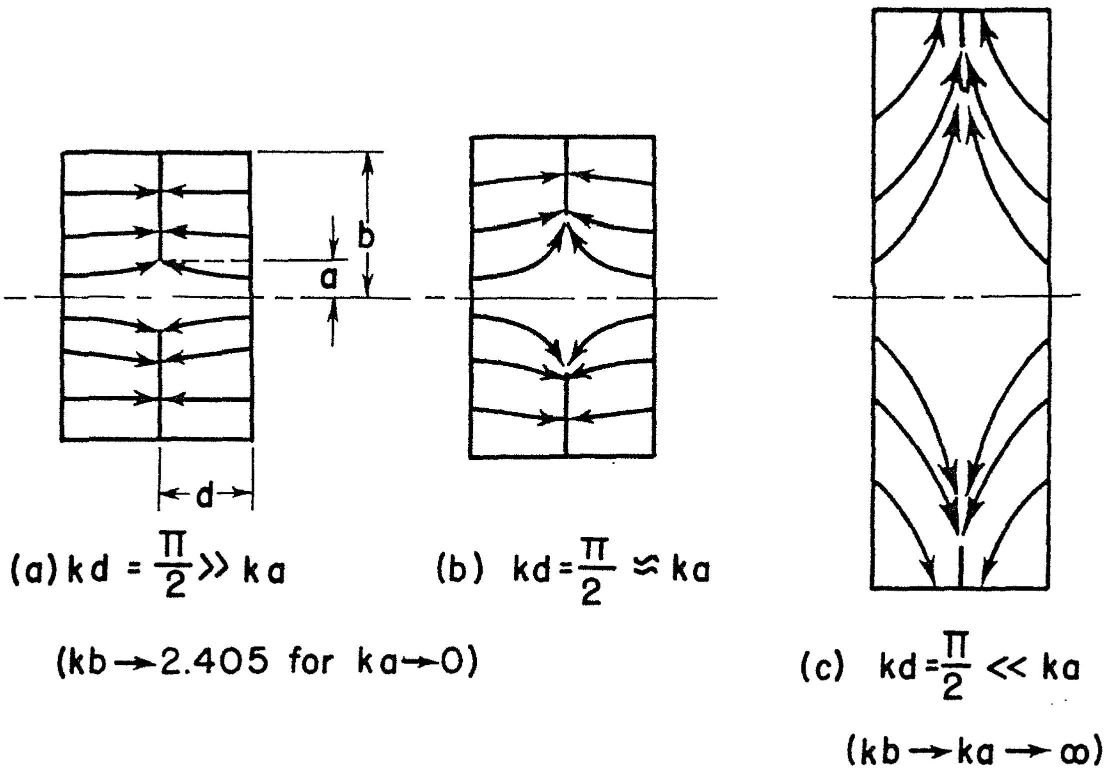

Fig. 3.11 - Qualitative picture of the field lines for various cavity shapes. The cavities are to scale.

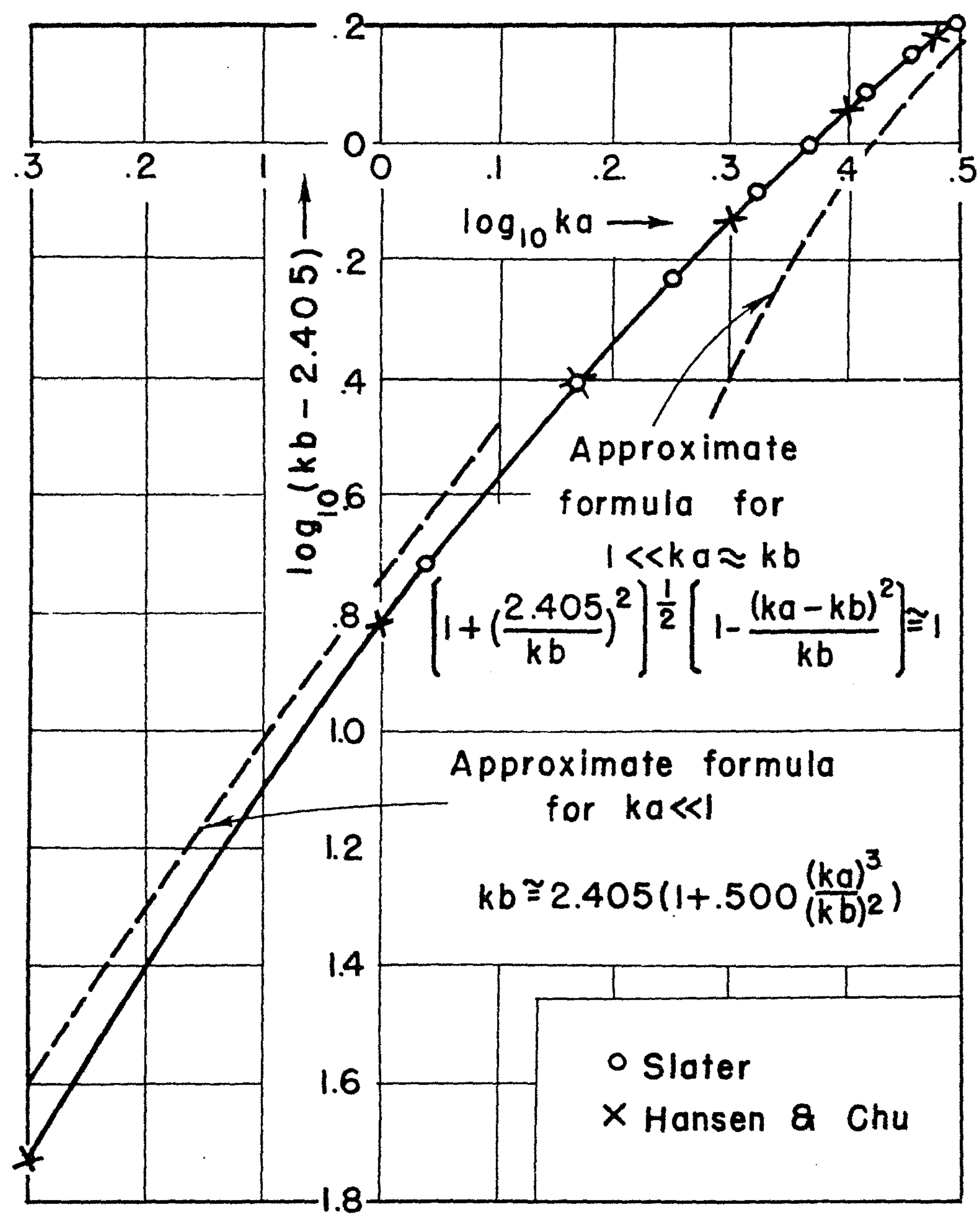

Fig. 3.12 - Plot of $\log_{10}$ (kb - 2.405) vs. $\log_{10}$ ka. Present calculations are shown as crosses, Slater's results as circles. Also shown are two approximate formulae derived in Section 3.2.

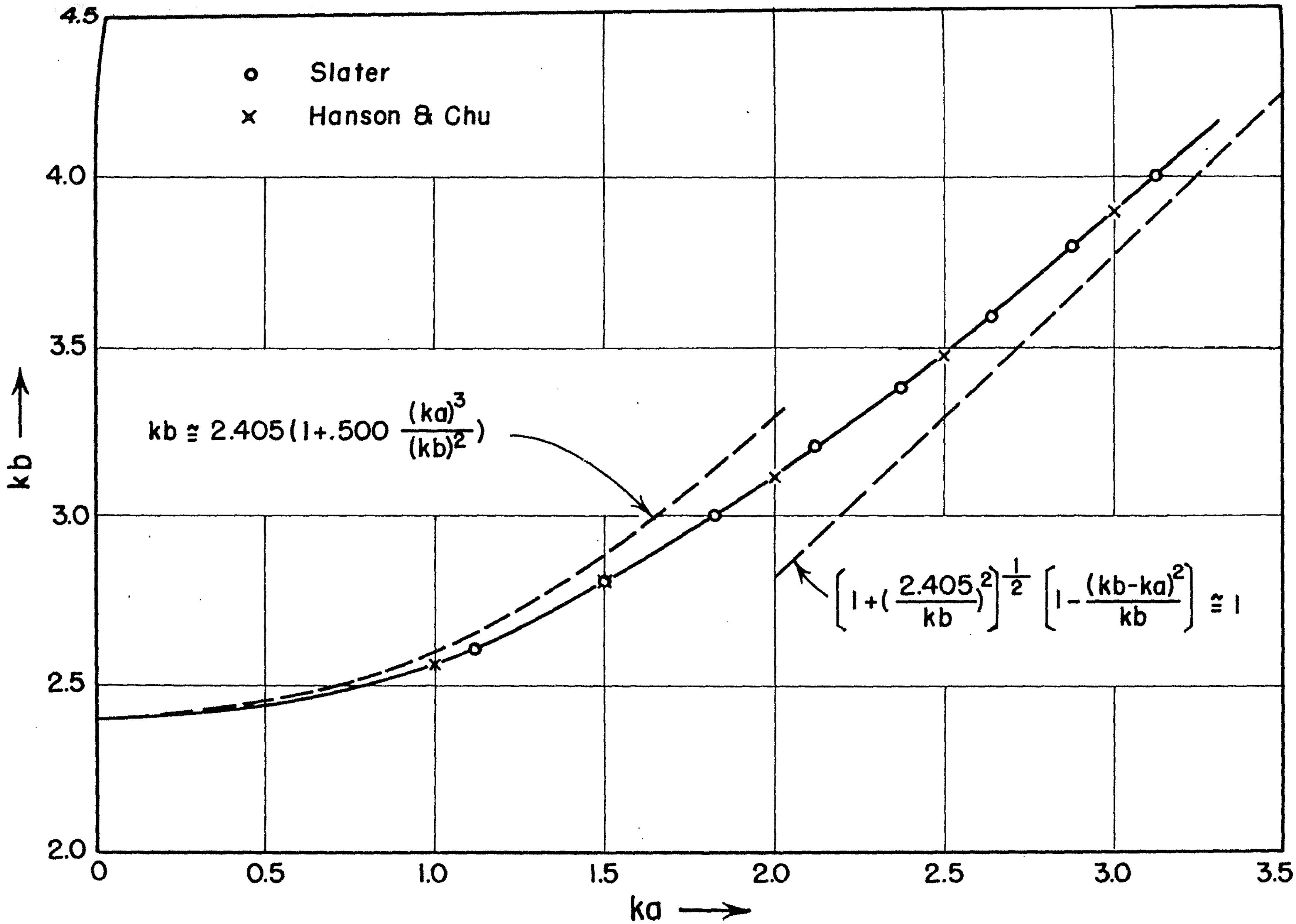

Fig. 3.13 - Data of Fig. 3.12, but using linear scales.

TABLE I

Calculated Values of kb are Given for Various ka, the Various Columns Corresponding to Trial Functions at the Column Heads.

ka	$B_\phi = (2dz - z^2)^{\frac{1}{2}}$	$B_\phi = \sin(\pi z/2d)$	$E_z =$ const.	$E_z = \cosh az$	$E_z = (2dz - z^2)^{-\frac{1}{2}}$	Un-certainty
0.50	2.42	2.43	2.42	2.42	2.29	0.00
1.00	2.56	2.61	2.53	2.55	2.46	0.01
1.50	2.80	2.86	2.73	2.78	2.75	0.02
2.00	3.12	3.19	3.02	3.09	3.10	0.02
2.50	3.50	3.57	3.36	3.47	3.49	0.01
3.00	3.90	3.98	3.75	3.86	3.90	0.00

CHAPTER IV

EQUATIONS OF ELECTRON MOTION

To simplify the discussion we shall neglect the minor effects on acceleration due to space charge: the loading effect on the accelerating field and the spreading effect on the electron beam. Thus we shall consider a single electron passing through the electromagnetic field in the disk-loaded waveguide. Presently, we shall derive the equations of electron motion simply from the Lorentz force equation. In a later chapter, where the transverse focusing of electron beams is considered, a general formulation of the problem by the Hamiltonian method will be given. The question of non-uniform loading of waveguides will be discussed briefly; further details will also be found later.

4.1. The Lorentz Force Equation

There are two kinds of forces acting on the electron, the electric force $e\vec{E}$ and the magnetic force $\frac{e}{c}\vec{v} \times \vec{H}$. The dynamic equation is

$$\frac{d(m\vec{v})}{dt} = e\vec{E} + \frac{e}{c}\vec{v} \times \vec{H} \quad . \tag{4.1}$$

$\vec{E}$ has E_z and E_r components, $\vec{H}$ only H_ϕ . The velocity vector of the electron can have two components, v_z and v_r ($v_r \ll v_z$). Splitting the above equation into its components we obtain

$$\frac{d(mv_z)}{dt} = eE_z + \frac{e}{c} v_r H_\phi \tag{4.2a}$$

$$\frac{d(mv_r)}{dt} = eE_r - \frac{e}{c} v_z H_\phi \tag{4.2b}$$

From the second equation it may be noted that the radial force is the algebraic sum of the electric and the magnetic force and is small in comparison with either of them, because, as we shall see later, they tend to cancel each other completely when $v_z \rightarrow c$.

4.2. Equations of Electron Motion

The different components of the traveling field may be easily found from the expression of u_I given by equation (2.25). Thus

$$H_\phi = \sum_{-\infty}^{\infty} a_n J_1(k_{rn}r) e^{j(k_{zn}z - \omega t)}$$

$$E_z = \sum_{-\infty}^{\infty} j a_n \frac{k_{rn}}{k} J_o(k_{rn}r) e^{j(k_{zn}z - \omega t)} \tag{4.3}$$

$$E_r = \sum_{-\infty}^{\infty} a_n \frac{k_{zn}}{k} J_1(k_{rn}r) e^{j(k_{zn}z - \omega t)}$$

or

$$E_z = \sum_{-\infty}^{\infty} E_n J_o(k_{rn}r) e^{j(k_{zn}z - \omega t + \phi_n)}$$

$$E_r = -\sum_{-\infty}^{\infty} j E_n \frac{k_{zn}}{k_{rn}} J_1(k_{rn}r) e^{j(k_{zn} - \omega t + \phi_n)} \tag{4.3'}$$

$$H_\phi = -\sum_{-\infty}^{\infty} j E_n \frac{k}{k_{rn}} J_1(k_{rn}r) e^{j(k_{zn}z - \omega t + \phi_n)} ,$$

where E_n, ϕ_n are real quantities denoting the amplitude and phase of the n-th Fourier component. Among the infinite number of Fourier components only one component which remains more or less stationary with respect to the electron will contribute to useful acceleration. Forces due to other components will average out in the long run.

Since $v_r \ll v_z < c$ and H_ϕ has an amplitude ($\sim k_{rn} r$ as $r \to 0$) small in comparison with that of E_z, the longitudinal magnetic force $\frac{e}{c} v_r H_\phi$ is negligible in comparison with the electric force eE_z. So equation (4.2a) may be approximated by

$$\frac{d(mv_z)}{dt} = eE_z = e \sum_{-\infty}^{\infty} E_n \, J_o(k_{rn} r) \cos(k_{zn} z - \omega t + \phi_n) \, .$$

If $A_n \ll A_o$ ($n \neq 0$), the above equation may further be approximated by

$$\frac{d(mv_z)}{dt} = eE_o \, J_o(k_{ro} r) \cos(k_{zo} z - \omega t + \phi_o) \, . \qquad (4.4a)$$

Otherwise, higher components must be considered in order to get a true picture of the motion. Nevertheless, if we are interested only in the calculation of the total change or the average rate of change of momentum (mv_z) over long distances, equation (4.4a) is correct in so far as results are concerned, because the contribution from any other component is negligible. In fact, if $\omega / v_z = k_e \cong k_{zo}$, the increment of mv_z in distance d will depend almost entirely on the zeroth component of the accelerating field.

Under the same restrictions we may also neglect the higher field components in equation (4.2b) and obtain

$$\frac{d(mv_r)}{dt} = (k_{zo} - k\frac{v_z}{c})\frac{eE_o}{k_{ro}}J_1(k_{ro}r)\sin(k_{zo}z - \omega t + \phi_o) \tag{4.4b}$$

4.3. Extension to the Case of Non-Uniformly Loaded Waveguides

So far it has been tacitly assumed that the loaded guide is a periodic structure, in other words, k_{zo} is a constant. This approximation is not good for the starting section of the accelerator where the wave velocity is continuously changed in order that the wave and the electron may keep pace with each other or varied in some particular fashion so as to get desirable bunching effect on the electron beam. Even in the case where the assumption is approximately true, it is desirable to know what are the effects of small regular or random variations.

Regardless of the existence of periodicity, the solution $u(r,z)$ can be expressed for fixed $r < a$ in the form of a Fourier integral,

$$u(r,z) = \int_{-\infty}^{\infty} r\cdot h(k_z)\frac{k}{k_r}J_1(k_r r)\,e^{jk_z z}\,dk_z$$

$$(k_z^2 + k_r^2 = k^2)\quad , \tag{4.5}$$

and may be represented in an exponential form with varying amplitude and varying wave number,

$$u(r,z) = |u(r,z)| \cdot e^{j\theta(rz,)}$$

$$= |u(r,z)| \cdot e^{j\int \frac{\partial\theta}{\partial z} dz} \qquad (4.6)$$

If the structure is exactly periodic with period d, $u(r,z)$ is a periodic function, i.e. $|u(r,z)| = |u(r,z + md)|$. From this relation we easily find

$$k_{zo} = \overline{\left(\frac{\partial\theta}{\partial z}\right)} \quad , \qquad (4.7)$$

where $\overline{\left(\frac{\partial\theta}{\partial z}\right)}$ denotes the average value of $\frac{\partial\theta}{\partial z}$ over any integral number (m) of periods. Although $\overline{\left(\frac{\partial\theta}{\partial z}\right)}$ is independent of z and m for the periodic case, we will take specifically

$$\overline{\left(\frac{\partial\theta}{\partial z}\right)} = \frac{1}{d} \int_{z - d/2}^{z + d/2} \frac{\partial\theta}{\partial z} dz$$

and consider equation (4.7) as a definition for k_{zo} for non-periodic cases.

If the structure varies but the variation is slow, $|u(r,z)|$, though not periodic will be approximately equal to $|u(r,z \pm d)|$, here d changes slowly from cavity to cavity, and $\overline{\left(\frac{\partial\theta}{\partial z}\right)}$ will remain practically constant over distance d. And

$$\frac{u(r,z \pm d)}{u(r,z)} \cong e^{j\int_{z}^{z \pm d} \frac{\partial\theta}{\partial z} dz}$$

$$\cong e^{j\int_{z}^{z \pm d} \overline{\left(\frac{\partial\theta}{\partial z}\right)} dz}$$

$$= e^{j\int_{z}^{z \pm d} k_{zo} dz}$$

If we write

$$u(r,z) = e^{j\int k_{zo} dz} \cdot v(r,z) \ ,$$

then $v(r,z)$ will be nearly periodic in the sense that $v(r,z) \cong v(r,z \pm d)$ and can therefore be represented approximately by a Fourier series having slowly varying coefficients and periodicity. $\overline{v(r,z)} = \frac{1}{d} \int_{z - d/2}^{z + d/2} v(r,z)\, dz$ is the amplitude of the k_{zo} component of $u(r,z)$ and, like $\overline{(\frac{\partial\theta}{\partial z})}$, remains practically constant over d.

If the variation is rapid, both $\overline{(\frac{\partial\theta}{\partial z})}$ and $\overline{v(r,z)}$ can still be defined as above but they will change rapidly with z. $v(r,z)$ can no longer be considered as nearly periodic, so cannot properly be analyzed into discrete Fourier components. Furthermore, since $\int^{z} \left[\frac{\partial\theta}{\partial z} - \overline{(\frac{\partial\theta}{\partial z})} \right] dz$ is in general not small, the concept of an average wave number cannot be useful even if $k_e = \overline{(\frac{\partial\theta}{\partial z})}$. For fast varying structures it is necessary to consider directly the varying wave number $\frac{1}{2\pi} \frac{\partial\theta}{\partial z}$ and the varying amplitude $|u(r,z)|$.

Without carrying the discussion further we may state that as long as the variation of structure from cell to cell is slow, the equations of motion can be written as (4.4a) and (4.4b), provided that $k_{zo}z$ is replaced by $\int^{z} k_{zo}\, dz$ and k_{zo}, k_{ro} and E_o are considered to be variable functions. On the other hand, for fast varying structures the equation (4.4a) will remain true if k_{zo} is understood as $\frac{\partial\theta}{\partial z}$ and $E_o J_o(k_{ro}r)$ as $\left| \frac{1}{kr} \frac{\partial}{\partial r} u(r,z) \right|$, but the equation (4.4b) will have to be revised. Further discussion on this point will be taken up in a later chapter, until then we will restrict ourselves when we discuss the transverse motion of electrons to periodic and nearly periodic structures.

The equations (4.4a) and (4.4b) may be simplified by choosing a suitable origin for z or t such that ϕ_o may be put equal to zero, by substituting $\int^z k_e \, dz$ for ωt and by using the relations $J_o(k_{ro}r) \cong 1$, $J_1(k_{ro}r) \cong \frac{1}{2} k_{ro}r$ for small r. Thus we may re-write these equations as

$$\frac{d(mv_z)}{dt} = eE_o \cos \int (k_{zo} - k_e) \, dz \quad , \tag{4.8a}$$

$$\frac{d(mv_r)}{dt} = \frac{eE_o}{2} k_{zo} r \left(1 - \frac{k^2}{k_e k_{zo}}\right) \sin \int (k_{zo} - k_e) \, dz \tag{4.8b}$$

4.4. Equations of Motion in Dimensionless Units

Now let us define a set of dimensionless quantities:

$\xi = z/\lambda$ = longitudinal or axial distance in unit of free space wavelength,

$\eta = r/\lambda$ = transverse or radial distance in unit of free space wavelength,

$\tau = \nu t$ = time in number of cycles,

$\alpha = \frac{eE_o\lambda}{m_r c^2}$ = maximum energy which a traveling wave of amplitude E_o can give to a moving electron over a distance λ divided by the electron rest energy,

$\beta = \frac{v_{zo}}{c}$ = phase velocity of the k_{zo} component of wave divided by the velocity of light,

$\beta_e = \frac{v_e}{c}$ ($\cong \frac{v_z}{c}$) = velocity of electron divided by velocity of light,

$\gamma = \frac{m}{m_r}$ = mass of electron in unit of electron rest mass ($\gamma = (1 - \frac{v_z^2 + v_r^2}{c^2})^{-1/2} \cong (1 - \frac{v_z^2}{c^2})^{-1/2}$, since $v_r \ll v_z$) ,

Δ = phase of electron in number of cycles with respect to the crest of the traveling wave (k_{zo} component), positive if it is ahead and negative if it is behind the crest, $2\pi\Delta = \int k_{zo}\, dz - \omega t = \int (k_{zo} - k_e)\, dz$.

With these notations we may write equations (4.8) as

$$\frac{d}{d\tau}(\gamma\dot{\xi}) = \alpha \cos 2\pi\Delta \tag{4.9a}$$

$$\frac{d}{d\tau}(\gamma\dot{\eta}) = \frac{\pi\alpha\eta}{\beta}(1 - \beta\dot{\xi}) \sin 2\pi\Delta\ , \tag{4.9b}$$

where $\dot{x}$ denotes $dx/d\tau$. These equations can further be transformed by means of the defining relations $\dot{\xi} \cong \beta_e$ and $\frac{d}{d\tau}(\gamma\dot{\xi}) \cong \frac{d\gamma}{d\xi}$. We finally obtain the equations of motion as follows:

$$\frac{d\gamma}{d\xi} = \alpha \cos 2\pi\Delta \tag{4.10a}$$

$$\frac{d\Delta}{d\xi} = \frac{1}{\beta} - \frac{1}{\dot{\xi}} \tag{4.10b}$$

$$\frac{d\tau}{d\xi} = \frac{1}{\beta_e} = \pm \frac{\gamma}{\sqrt{\gamma^2 - 1}} \tag{4.10c}$$

$$\frac{d}{d\xi}(\gamma\dot{\xi}\frac{d\eta}{d\xi}) = \pi\alpha\eta\,(\frac{1}{\beta\dot{\xi}} - 1) \sin 2\pi\Delta\,. \tag{4.10d}$$

Equations (4.10b) and (4.10c) are obtained directly from the defining relations. The longitudinal motion is described by the first three equations, and the transverse motion by all four. Besides the initial conditions, α and β must be known in order to solve these equations. Both of them may be varying functions of ξ.

CHAPTER V

LONGITUDINAL MOTION AND BUNCHING OF ELECTRONS

5.1. Introductory Remarks

In the last chapter we have derived the equations of electron motion. The longitudinal motion of electrons is described by three equations (4.10a) - (4.10c), all of which are independent of the transverse coordinate η and the radial velocity $\dot{\eta}$. Thus we are able to discuss exclusively the longitudinal part of the electron motion, including the problem of bunching, without dealing with the transverse motion. This is a great simplification and is based on the restriction that $\dot{\eta} \ll \dot{\xi}$. Since this restriction is actually very well satisfied, no appreciable error would be incurred through this approximation.

We shall first consider the case of α = constant, $\beta = 1$. This case is exactly soluble and is most important. If high injection voltages are available, linear electron accelerators of any length may be designed with $\beta = 1$ exclusively.

The case α = constant, β = constant $\neq 1$ will come next. The solution for this case contains an elliptic integral of the third kind which cannot be evaluated in terms of tabulated functions. We will discuss useful approximate solutions obtained under certain not too stringent conditions and will give results obtained by numerical integration for several typical cases. It has been found that short constant -β sections may either provide good bunching effect or introduce suitable phase shifting.

Then we shall consider the case α = constant, β = variable. If mono-energetic electrons could be injected only within a small phase angle in the first quadrant $(0 < 2\pi\Delta < \pi/2)$ and near the crest of the traveling wave, it should be expedient to accelerate them by varying β in synchronism with β_e of some one electron of the bunch. For then the bunch will remain practically at rest with respect to the wave and receive almost maximum energy. Let the synchronized electron be situated at $\Delta = \Delta_c$. The electrons with $\Delta > \Delta_c$ will receive less energy, so will travel slower than if they were at Δ_c, while those with $\Delta < \Delta_c$ will receive greater energy and travel faster. Thus the phase spread of the bunch will become narrower as the bunch moves on. If the bunch is not near the crest of the wave but remains in the first quadrant, the situation is similar but the acceleration is less. The first quadrant is the phase-stable region with positive accelerating field. The smaller is Δ_c, the more rapid the increase of β.

By assuming $\Delta \cong \Delta_c$ and $\beta_e \cong \beta$, the equation of motion is reduced to a second order linear differential equation. We will discuss the analytic solutions of this equation for both the oscillatory and non-oscillatory cases by the WKB approximation. The motion of individual electrons of the bunch has the simple feature of a damped oscillation around the equilibrium position Δ_c superimposed on the motion which is the same as that of the electron at Δ_c. When they get heavy or energetic enough, they will approach Δ_c asymptotically.

On the other hand, if $|\Delta - \Delta_c|$ is not small the motion should still have the same qualitative feature. But now the differential equation is non-linear, it seems not easy to obtain accurate analytic results. However, a qualitative discussion of the bunching process of varying β can easily be made without actually solving the equation. It may be said that this type of bunching is not quite effective. Unless the buncher is very long, quite an appreciable fraction of the total injected electrons will be lost through retrogression and the phase spread of the bunch cannot be very sharp. At the worst, however, the result should still be better than we would obtain if we used a uniformly loaded waveguide with $\beta = 1$ and used the same injection voltage. To illustrate, the solution for a particular case, where β varies so rapidly that Δ_c is everywhere zero, obtained on the differential analyzer at the University of California at Los Angeles will be given and discussed.

Lastly, the alternative case β = constant, α = variable will be discussed. We will show that it is far superior to the previous case as far as bunching is concerned. A still better method of bunching is to increase α and β simultaneously and in the proper way. For such complicated cases reliable results can only be obtained by numerical integration. Carter and Hansen[24] calculated a typical example with

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24. D. Carter and W. W. Hansen, unpublished results.

$$\alpha = \left(\frac{2.86}{\xi - 6.40}\right)^2 \quad ,$$

$$\frac{1}{\sqrt{1 - \beta^2}} - \frac{1}{\sqrt{1 - \beta_o^2}} = 8\,\xi$$

and

$$\gamma_o = \frac{1}{\sqrt{1 - \beta_o^2}} = 2.0$$

and obtained excellent results with a buncher length of only 5.5 wavelengths. The rate of increase of β, however, is quite rapid; it is advisable to reduce it, especially at the beginning, in order to minimize the loss of retrograde electrons. This idea has been incorporated in the design of a buncher intended for the Stanford billion-volt accelerator. The design functions and the bunching characteristics will be shown by figures.

5.2. α = constant, β = constant

Combining equations (4.10a), (4.10b) and (4.10c) we obtain

$$\left(\frac{1}{\beta} \mp \frac{\gamma}{\sqrt{\gamma^2 - 1}}\right) d\gamma = \alpha \cos 2\pi\Delta \, d\Delta \, .$$

This can be integrated at once to give

$$\left(\frac{\gamma}{\beta} \mp \sqrt{\gamma^2 - 1}\right) - \frac{\alpha}{2\pi} \sin 2\pi\Delta = H \quad , \tag{5.1}$$

where H is a constant to be determined from the initial values of γ and Δ , i.e., γ_o and Δ_o. From this equation we find that we always have $\frac{2\pi H}{\alpha} > -1$ if $\beta \lesseqgtr 1$, and γ can only approach infinity if $\beta = 1$ and $\left|\frac{2\pi H}{\alpha}\right| \lesseqgtr 1$.

H may be considered as the Hamiltonian in dimensionless units for the longitudinal motion of the electron. Thus, if we denote $\pm\sqrt{\gamma^2 - 1} = \gamma\beta_e$ by Γ, equation (5.1) becomes

$$\left(\frac{1}{\beta}\sqrt{1 + \Gamma^2} - \Gamma\right) - \frac{\alpha}{2\pi} \sin 2\pi\Delta = H \quad , \tag{5.2}$$

from which the Hamiltonian equations of motion

$$\frac{\partial H}{\partial \Gamma} = \frac{d\Delta}{d\tau} \quad , \quad -\frac{\partial H}{\partial \Delta} = \frac{d\Gamma}{d\tau} \tag{5.3}$$

can at once be derived.

Various pieces of information about the motion may be obtained from a $\Gamma - \Delta$ plot, the so-called phase diagram, with H as a parameter. Two such diagrams, one for $\beta = 1/2$ and one for $\beta = 1$, have been shown and discussed by Slater.[25] We refer to his paper for such discussion.

On the other hand, phase diagrams are not directly useful for determining the phase distribution and the energy spectrum of the exit electrons, both of which are questions of primary importance. Except for the case where $\beta = 1$ and $\xi \rightarrow \infty$, for which simple calculations will suffice, these questions must be answered by determining the $\Delta - \xi$ and $\gamma - \xi$ relations, so we have to solve equations (4.10a) and (4.10b) completely.

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25. J.C. Slater, loc cit. footnote 8.

(i) $\beta = 1$

We first consider $\beta = 1$, this being the simplest and most important case. Denoting $A = \frac{2\pi H}{\alpha}$, we write equation (5.1) as

$$\gamma \mp \sqrt{\gamma^2 - 1} = \frac{\alpha}{2\pi} (A + \sin 2\pi\Delta) , \tag{5.4}$$

which gives a unique value for γ

$$2\gamma = \frac{\alpha}{2\pi} (A + \sin 2\pi\Delta) + \frac{2\pi}{\alpha} \frac{1}{A + \sin 2\pi\Delta} \tag{5.4'}$$

and transform (4.10b) to the following form

$$\frac{d\xi}{d\Delta} = \frac{1}{2} - \frac{1}{2} \frac{1}{(\frac{\alpha}{2\pi})^2 (A + \sin 2\pi\Delta)^2} \tag{5.5}$$

As $\xi \to \infty$, any electron which can be bound to the wave will have $\gamma \to \infty$. From equation (5.4) or (5.4') it is evident that binding will occur at a phase angle $2\pi\Delta_\infty = -\sin^{-1} A$ provided $|A| \leq 1$. Hence

$$\sin 2\pi\Delta_\infty = \sin 2\pi\Delta_o - \frac{2\pi}{\alpha} (\gamma_o - \sqrt{\gamma_o^2 - 1}) \tag{5.6}$$

$$= \sin 2\pi\Delta - \frac{2\pi}{\alpha} (\gamma \mp \sqrt{\gamma^2 - 1})$$

In Fig. 5.1 we plot Δ_∞ vs. Δ_o with α as a parameter and $\gamma_o = 2$. $\alpha/2\pi = 1.266$ ($\alpha = 7.958$) is the design value for the Stanford billion-volt accelerator. The number of the bound electrons increases with α. The effect would be the same if α is fixed and γ_o is increased.

For large values of α, electrons injected in the negative half-cycle ($\pi/2 < 2\pi\Delta_o \lesseqgtr 3\pi/2$) may come to a stop and move in the reversed direction before they come to the accelerating region. Some electrons may even be driven out of the accelerator tube through the injection end. Under such circumstance, it is not justifiable to consider only one field component. In other words, α and β can no longer be properly considered as being constant. Nevertheless, if the other field components have only small amplitudes in comparison with the main one, the approximation will not be seriously in error.

Let $\Delta = \Delta_\infty + \epsilon$, $2\pi\epsilon \ll 1$, then by equation (5.5) we obtain

$$\frac{d\xi}{d\Delta} \cong \frac{1}{2(\alpha \cos 2\pi\Delta_\infty)^2} \frac{1}{\epsilon^2} \quad .$$

As $\epsilon \to 0$, $d\Delta/d\xi = d\epsilon/d\xi \to 0$ as $-\epsilon^2$. If the electron travels a distance ξ in reducing ϵ to half its value, the same electron will have to travel $2^n\xi$ to reduce the phase difference from $\epsilon/2^n$ to $\epsilon/2^{n+1}$. If binding can occur, it takes place almost entirely in the very initial stage of acceleration. The electron will not change its phase appreciably after it has gained sufficient energy.

Equation (5.5) can be integrated to give results expressible in terms of elementary functions. Thus we obtain

$$\xi = \frac{\Delta}{2} - \frac{\pi}{\alpha^2} \frac{1}{A^2 - 1} \left[\frac{\cos 2\pi\Delta}{A + \sin 2\pi\Delta} - \frac{2A}{\sqrt{A^2 - 1}} \tan^{-1} \frac{\sqrt{A^2 - 1}\ \tan \frac{\pi}{4}(1 - 4\Delta)}{A + 1} \right] + \text{const.}$$

$$A > 1,\ 3/4 > \Delta > -1/4;$$

$$\xi = \frac{\Delta}{2} + \frac{\pi}{3\alpha^2} \tan \frac{\pi}{4}(1 - 4\Delta) \frac{2 + \sin 2\pi\Delta}{1 + \sin 2\pi\Delta} + \text{const.},$$

$$A = 1 ;$$

$$\xi = \frac{\Delta}{2} + \frac{\pi}{\alpha^2} \frac{1}{1 - A^2} \left[\frac{\cos 2\pi\Delta}{A + \sin 2\pi\Delta} - \frac{2A}{\sqrt{1 - A^2}} \cdot \tanh^{-1} \frac{\sqrt{1 - A^2} \tan \frac{\pi}{4}(1 - 4\Delta)}{A + 1} \right] + \text{const.},$$

$$1 > A > -1, \quad 3/4 > \Delta > -1/4 \qquad (5.7)$$

We have plotted in Fig. 5.2a and 5.2b $\Delta - \xi$ curves for two different sets of conditions, one being $\alpha/2\pi = .10$, $\gamma_o = 5$ and the other $\alpha/2\pi = 1.266$, $\gamma_o = 2$. In the former case where α is relatively small, binding progresses over quite a long distance, more than 50 wavelengths, and no electrons ever stop and move in the reverse direction; in the other case with large α, electrons which can be bound to the wave have binding practically completed in the first few wavelengths, and some of those which cannot be bound are turned back and driven out of the accelerator tube through the injection end. A condition which insures that the direction of motion cannot be reversed is $A < (2\pi/a) - 1$. Thus if $\alpha/\pi < 1 - (\gamma_o - \sqrt{\gamma_o^2 - 1})$, no electrons will ever stop.

(ii) $\beta \neq 1$

Then we will consider $\beta \neq 1$. Instead of (5.4) and (5.5) we have

$$\frac{\gamma}{\beta} \mp \sqrt{\gamma^2 - 1} = \frac{\alpha}{2\pi} (A + \sin 2\pi\Delta) \tag{5.8}$$

and

$$\frac{d\xi}{d\Delta} = \frac{\beta}{1 - \beta^2} \cdot$$

$$\left\{ 1 \pm \frac{\beta (A + \sin 2\pi\Delta)}{\left[(A + \sin 2\pi\Delta)^2 - \left(\frac{2\pi}{\alpha}\right)^2 \frac{1 - \beta^2}{\beta^2} \right]^{1/2}} \right\} \tag{5.9}$$

Let

$$\left. \begin{aligned} A + \sin 2\pi\Delta &= x \\ \frac{2\pi}{\alpha} \frac{\sqrt{1 - \beta^2}}{\beta} &= x_s \end{aligned} \right\} \tag{5.10}$$

we obtain from (5.9)

$$\int_0^\xi d\xi = \frac{1}{2\pi} \frac{\beta}{1 - \beta^2} \int_{x_o}^{x} \frac{dx}{\sqrt{1 - (x-A)^2}} \pm \frac{1}{2\pi} \frac{\beta^2}{1 - \beta^2} \cdot$$

$$\cdot \int_{x_o}^{x} \frac{x\,dx}{\sqrt{1 - (x-A)^2} \sqrt{x^2 - x_s^2}} \tag{5.11}$$

At $x = x_s$, $\gamma = \gamma_s = \frac{1}{\sqrt{1 - \beta^2}}$, i.e. $\beta_e = \beta$, the electron velocity is the same as the phase velocity of the wave. Except for the special case A = 0, for which a closed-form solution can immediately be obtained, the integral

$$\int_{x_o}^{x} \frac{x\,dx}{\sqrt{1 - (x-A)^2} \sqrt{x^2 - x_s^2}}$$

is found to consist of an elementary integral and two elliptic integrals of the standard forms, the first and third kind. The first presents no difficulty. The third kind can be evaluated in terms of Jacobian Theta- and Zeta-functions[26] but, unfortunately, the arguments of these functions turn out to be imaginary irrespective of whether $\beta < 1$ or $\beta > 1$, and there exist no tables for such functions.

If we restrict ourselves to the case $x \cong x_s$, i.e. $\beta_e \cong \beta < 1$ such that $\sqrt{x - x_s} \ll \sqrt{x_s}$, we may neglect the first integral on the right-hand side of (5.11) in comparison with the second one and approximate the integrand of the latter by $\frac{\sqrt{x_s}}{\sqrt{2}} \frac{1}{\sqrt{1 - (x-A)^2} \cdot \sqrt{x - x_s}}$. We then have

$$\int d\xi \cong \pm \frac{1}{2\pi} \frac{\beta^2}{1 - \beta^2} \frac{\sqrt{x_s}}{\sqrt{2}} \frac{dx}{\sqrt{1 - (x-A)^2} \cdot \sqrt{x - x_s}} \qquad (5.12)$$

The last elliptic integral is listed in Pierce's Table (No. 545). The integrated result is

$$\xi(x) - \xi(x_s) = \frac{\sqrt{x_s}}{2\pi} \frac{\beta^2}{1 - \beta^2} \cdot$$

$$\mathrm{sn}^{-1}\left(\sqrt{\frac{2}{A + 1 - x_s} \frac{x - x_s}{x - A + 1}}, \sqrt{\frac{A + 1 - x_s}{2}}\right).$$

This may be written more concisely as

$$\xi(x) + \frac{\sqrt{x_s}}{2\pi} \frac{\beta^2}{1 - \beta^2} \cdot F(\phi, \kappa) = \text{const.},$$

$$\sqrt{x - x_s} \ll \sqrt{x_s}, \qquad (5.13)$$

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26. E.T. Whittaker and G.N. Watson, "A Course of Modern Analysis," (Cambridge Univ. Press, 1927) Chap. XXII, p.522.

where $\phi = \sin^{-1}\sqrt{\frac{A + 1 - x}{A + 1 - x_s}}$ and $\kappa = \sqrt{\frac{A + 1 - x_s}{2}}$.

Thus x is a periodic function of ξ . As $A + 1 \rightarrow x_s$, $A + 1 - x_s \geqq x - x_s \rightarrow 0$. For such very small oscillations, the sn function degenerates into a sine function. The period of oscillation is $\sqrt{x_s}\,\frac{\beta^2}{1 - \beta^2}$ in ξ or $\sqrt{x_s}\,\frac{\beta}{1 - \beta^2}$ in τ, so frequency is equal to $\frac{\upsilon}{\sqrt{x_s}}\,\frac{1 - \beta^2}{\beta} = \upsilon\sqrt{\frac{\alpha}{2\pi}\,\frac{(1-\beta^2)^{3/2}}{\beta}}$.

The last expression is the same as that given by Slater.[27] In fact, if we change x in the approximate relation (5.12) back to Δ , we can easily obtain the Hamiltonian for small oscillations. Thus we obtain

$$\left(\frac{d\Delta}{d\xi}\right)^2 = \frac{\alpha}{\pi}\left(\frac{\sqrt{1 - \beta^2}}{\beta}\right)^3 (A' + \sin 2\pi\Delta),$$

where

$$A' = A - x_s = A - \frac{2\pi}{\alpha}\,\frac{\sqrt{1 - \beta^2}}{\beta} . \quad \text{Since } \frac{d\Delta}{d\xi} \cong \frac{\beta_e - \beta}{\beta^2} ,$$

the above equation reduces to

$$\frac{(\beta_e - \beta)^2}{2\sqrt{(1 - \beta^2)^3}} - \frac{\alpha\beta}{2\pi}\sin 2\pi\Delta = \frac{\alpha\beta}{2\pi}A' = H' .$$

Setting $\Gamma' = \frac{\beta_e - \beta}{\sqrt{(1 - \beta^2)^3}}$, $\gamma_\ell = \frac{1}{\sqrt{(1 - \beta^2)^3}}$ and $\xi' = \beta\Delta$,

we get

$$\frac{\Gamma'^2}{2\gamma_\ell} - \frac{\alpha\beta}{2\pi}\sin\frac{2\pi}{\beta}\xi' = H' ,$$

from which the Hamilton equations of motion

$$\frac{\partial H'}{\partial \Gamma'} = \frac{d\xi'}{d\tau} , \qquad -\frac{\partial H'}{\partial \xi'} = \frac{d\Gamma'}{d\tau}$$

27. J.C. Slater, loc cit. footnote 8.

follow directly.

As the oscillation amplitude increases from zero but remains small, the period increases (rather slowly) and the frequency decreases. A better approximation may be made by retaining the first integral of (5.11) and a still better one by using one more term in Taylor's expansion of $\frac{x}{\sqrt{x + x_s}}$, i.e., $\frac{x}{\sqrt{x + x_s}} \cong \frac{x_s}{\sqrt{2x_s}} + \frac{3}{4}\frac{x - x_s}{\sqrt{2x_s}}$.

Then equation (5.11) becomes

$$\int d\xi = \frac{1}{2\pi}\frac{\beta^2}{1 - \beta^2}\int \frac{dx}{\sqrt{1 - (x-A)^2}} \cdot$$

$$\cdot \left[\pm \frac{\sqrt{x_s}}{\sqrt{2}\sqrt{x - x_s}} + \frac{1}{\beta} \pm \frac{3}{4}\frac{\sqrt{x - x_s}}{\sqrt{2x_s}} \right] .$$

The last term on the right-hand side is an elliptic integral of the second kind and is to be evaluated in terms of $E(\phi,k)$. The result of integration is

$$\xi - \frac{\beta\Delta}{1 - \beta^2} + \frac{1}{4\pi}\frac{\beta^2}{1 - \beta^2}\frac{1}{\sqrt{x_s}}\left\{\left[2x_s - 3(1 - k^2)\right] F(\phi,k) + \right.$$

$$\left. + 3E(\phi,k)\right\} = \text{const.}, \qquad (x - x_s)^2 \ll x_s^2 , \quad (5.14)$$

from which we find

$$\text{Period in } \tau = \frac{1}{\pi}\frac{\beta}{1 - \beta^2}\frac{1}{\sqrt{x_s}}\left\{\left[2x_s - 3(1 - k^2)\right] K + 3E\right\} ,$$

where $K = F(\pi/2, k)$, $E = E(\pi/2,k)$. While the relation (5.13) is valid only for $\sqrt{x - x_s} \ll \sqrt{x_s}$, the above relations can be accurate for a much wider range in which $(x - x_s)^2 \ll x_s^2$. The latter condition is generally satisfied, provided α is not too large. For if $\alpha/2\pi < 1$ such that $x_s = \frac{2\pi}{\alpha}\frac{\sqrt{1 - \beta^2}}{\beta} > 2$, then $x - x_s \lesseqgtr 2 < x_s$.

If α is large and the condition $(x - x_s)^2 \ll x_s^2$ is not satisfied, then it is simpler to integrate equation (5.9) directly by numerical means than to deal with the Jacobian Zeta functions with complex arguments.

We take for example $\alpha/2\pi = 1.266$, $\beta = .866$, $\gamma_o = 2$ and plot in Fig. 5.3 $d\xi/d\Delta$ as a function of Δ with Δ_o as a parameter. The curves are symmetrical with respect to the axis $\Delta = .25$, so only half-branches $(.25 \lesseqgtr \Delta \lesseqgtr .75)$ are shown. $d\xi/d\Delta$ is a double-valued function of Δ. Each curve has two branches, one being the reflection of the other with respect to the line $d\xi/d\Delta = \beta/(1 - \beta^2)$. Only one upper curve is shown in this figure. Δ is increasing on the upper branches and is decreasing on the lower ones. $d\xi/d\Delta \rightarrow \infty$ as $\Delta \rightarrow \Delta_s$, at which the electron and the wave have the same velocity. In this particular case, since $\gamma_o = \frac{1}{\sqrt{1 - \beta^2}}$, Δ_o coincides with one of the two values of Δ_s. We also note from this figure that the curves are relatively flat anywhere Δ is not too near Δ_s, so the greater part of the numerical work can be quite easy and accurate. When Δ is near to Δ_s, x is near x_s, numerical integration becomes somewhat difficult but then we may advantageously use the approximate solutions (5.13) and (5.14).

The integrated results are shown in Fig. 5.4 by plotting Δ against ξ. Electrons with Δ between $-.20$ and $.55$ at the injection end are bunched to a narrower stream with

$-.10 \leqq \Delta \leqq .28$ at $\xi = 1$ and with $.05 \leqq \Delta \leqq .36$ at $\xi = 2$. As ξ increases further, the range in Δ will first spread and then again contract and so on. The electrons oscillate back and forth around the equilibrium position $\Delta = .25$ with a period which is approximately equal to $\sqrt{x_s} \cdot \frac{\beta}{1 - \beta^2}$ for small oscillations and increases with increasing amplitudes. We get a bunching factor, which may be defined as the ratio of the initial phase spread to the value at a given ξ, of about 2 by using a constant -β injection section as short as one wavelength. This is the advantage of using large α. The disadvantage is that some of the electrons injected in the deceleration half-cycle (such as those with Δ_o greater than .55 and somewhat less than .70 in the case of Fig. 5.4) are turned back and driven out of the injection end.

For $\beta > 1$, Δ can only decrease with time and $d\xi/d\Delta$ is single-valued. We should in this case adhere to the negative sign in equation (5.9). $d\xi/d\Delta$ is everywhere regular, so it is not difficult to integrate numerically. Such calculations, however, will not be made because one example of this case ($\alpha/2\pi = 1.266$, $\beta = 1.177$, $\gamma_o = 2$) has already been solved by using the differential analyzer at U.C.L.A.[28] From the results obtained we observe that there are more electrons driven out of the injection end than in the previous case of $\beta = \sqrt{1 - 1/\gamma_o^2} < 1$, including some that started with a phase corresponding to positive

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28. A full account of the results obtained on the differential analyzer will be given in the next chapter.

acceleration. Despite the fact that Δ decreases steadily and the stream of electrons tends to spread more and more, there is some bunching action in a short tube of one to two wavelengths, though smaller than that of the previous case.

If we assume an infinitely long accelerator tube which has $\beta = 1$ throughout except the first wavelength of the injection end where β is constant ($\beta = \beta_o$) but not equal to 1, we may easily calculate the exit phases of electrons by combining the results for the short section and the infinite tube, which we have discussed above.

Thus, in Fig. 5.5 we have plotted Δ_∞ against Δ_o with $\beta_o = .866$, $\beta_o = 1$ and $\beta_o = 1.177$. The curve $\beta_o = 1$ is symmetrical about the line $\Delta_o = .25$ while the other two curves are not. The $\beta_o = .866$ curve is double-peaked, and a flat top is to be expected with some intermediate value of β_o between $\sqrt{1 - 1/\gamma_o^2}$ and 1. More electrons pass through the infinite tube with $\beta_o < 1$ than with $\beta_o > 1$, but the latter case has the advantage of having a peak nearer to the Δ_o-axis than the other two. The electron density is largest at the peak. If it is near to the Δ_o-axis, the dense part of the electron beam will have approximately the maximum energy. In this particular case, electrons entering the accelerator tube with $.15 \leqq \Delta_o \leqq .40$ (25 per cent of total) will exit with $-.045 \leqq \Delta \leqq .045$ and will have energy not less than about 96 per cent of the maximum value. This advantage, however, is rather questionable, because the higher peaks of the other two cases can easily be shifted down to the Δ_o-axis without appreciably affecting the form of the curves by inserting a short section with $\beta > 1$ at a somewhat later stage where the electrons have already gained moderate amounts of energy. So compensated, injection with $\beta < 1$ is definitely superior to that with $\beta > 1$. According to Fig. 5.5, more than half of the electrons in the case $\beta_o = .866$ have a final phase spread less than .09.

Having obtained the $\Delta_\infty - \Delta_o$ relations, it is straight-forward to calculate the energy spectrum of the output electrons. Consider, for example, the case $\beta_o = .866$ shown in Fig. 5.5. We assume that the curve has been shifted down such that the Δ_o-axis touches the minimum point of the curve at $\Delta_o = .25$ and that the accelerator has a total length of 250λ. The electron energy V is plotted as a function of the entrance phase Δ_o in Fig. 5.6a. From this curve we calculate the energy spectrum. The result is shown in Fig. 5.6b by plotting the distribution of electrons n(V) as a function of energy. Also shown in this figure is the integrated function $\int n(V)dV$ which gives the total number of electrons with energy greater than V.

5.3. α = constant, $\frac{1}{\sqrt{1-\beta^2}} = \alpha'\xi$

Unless otherwise stated, it is assumed that $\beta_e \cong \beta$ and that β is varied such that there exists a fixed equilibrium position Δ_c in the first quadrant ($\Delta_c = 1/4$

corresponds to the case of constant β), at which a moving electron may travel synchronously with the wave. Let γ_c denote the mass of the synchronized electron, $\gamma_c = \frac{1}{\sqrt{1 - \beta^2}}$. Thus Δ_c is defined by

$$\frac{d}{d\xi}\left(\frac{1}{\sqrt{1 - \beta^2}}\right) = \alpha \cos 2\pi\Delta_c = \alpha' \quad ,$$

i.e.,

$$\frac{1}{\sqrt{1 - \beta^2}} = \alpha'\xi \tag{5.15}$$

if a proper origin is chosen for ξ.

We obtain by differentiating equation (4.10b)

$$\frac{d^2\Delta}{d\xi^2} + \frac{1}{\beta^2}\frac{d\beta}{d\xi} - \frac{1}{\beta_e^2}\cdot\frac{d\beta_e}{d\xi} = 0$$

and from (4.10a)

$$\frac{\beta_e}{(1 - \beta_e^2)^{3/2}}\ \frac{d\beta_e}{d\xi} = \cos 2\pi\Delta ,$$

and then by combining these two equations with (5.15)

$$\frac{d^2\Delta}{d\xi^2} + \alpha'\left(\frac{\sqrt{1 - \beta^2}}{\beta}\right)^3 - \alpha \cos 2\pi\Delta \left(\frac{\sqrt{1 - \beta_e^2}}{\beta_e}\right)^3 = 0 \quad .$$

Expanding $\left(\frac{\sqrt{1 - \beta_e^2}}{\beta_e}\right)^3$ into Taylor's series and neglecting the second and higher order terms in $\frac{\beta_e - \beta}{1 - \beta^2}$, we have

$$\left(\frac{\sqrt{1-\beta_e^2}}{\beta_e}\right)^3 \cong \left(\frac{\sqrt{1-\beta^2}}{\beta}\right)^3 \cdot \left[1 - \frac{3}{\beta}\,\frac{\beta_e - \beta}{1-\beta^2}\right]$$

$$= \left(\frac{\sqrt{1-\beta^2}}{\beta}\right)^3 \cdot \left[1 - \frac{3\beta_e}{1-\beta^2}\,\frac{d\Delta}{d\xi}\right]$$

This, when substituted into the previous equation, gives

$$\frac{d^2\Delta}{d\xi^2} + 3\alpha \cos 2\pi\Delta \left(\frac{\sqrt{1-\beta^2}}{\beta}\right)^3 \frac{\beta_e}{1-\beta^2}\,\frac{d\Delta}{d\xi}$$

$$+ (\alpha' - \alpha \cos 2\pi\Delta) \left(\frac{\sqrt{1-\beta^2}}{\beta}\right)^3 = 0 \; ,$$

which may be written as

$$\frac{d^2\Delta}{d\xi^2} + 3\alpha' \left(\frac{\sqrt{1-\beta^2}}{\beta}\right)^3 \frac{\beta}{1-\beta^2}\,\frac{d\Delta}{d\xi} \left(1 + \frac{\beta_e - \beta}{\beta}\right)$$

$$+ (\alpha' - \alpha \cos 2\pi\Delta) \left(\frac{\sqrt{1-\beta^2}}{\beta}\right)^3 \cdot \left[1 - 3\,\frac{\beta_e - \beta}{\beta(1-\beta^2)}\right] = 0$$

On condition that $|\beta_e - \beta| \ll \beta$ and $|\beta_e - \beta| \ll 1 - \beta^2$ we finally obtain

$$\frac{d^2\Delta}{d\xi^2} + 3\alpha' \left(\frac{\sqrt{1-\beta^2}}{\beta}\right)^3 \frac{\beta}{1-\beta^2}\,\frac{d\Delta}{d\xi}$$

$$+ (\alpha' - \alpha \cos 2\pi\Delta) \left(\frac{\sqrt{1-\beta^2}}{\beta}\right)^3 = 0 \qquad (5.16)$$

It may be pointed out here that if we drop the second term in (5.16) and treat β as a constant we would get by integration exactly the same Hamiltonian function as that given by Slater.[29] On the other hand, if we drop the first

29. J.C. Slater, loc. cit, footnote 8.

term of equation (5.16), we would get essentially the same approximate result as that given by Ginzton, Hansen and Kennedy.[30] The latter approximation holds good for $\frac{d\Delta}{d\xi} \cong$ const., which is satisfied if both β and β_e vary slowly.

Let us write $\Delta = \Delta_c + \Delta'$ where Δ' measures the relative phase of the electron with respect to the equilibrium phase and assume that $|2\pi\Delta'| \ll 1$. The latter assumption of small Δ' does not necessarily mean small displacement in ξ. Equation (5.16) may then be reduced to

$$\frac{d^2\Delta'}{d\xi^2} + 3\alpha'\left(\frac{\sqrt{1-\beta^2}}{\beta}\right)^3 \cdot \frac{\beta}{1-\beta^2}\frac{d\Delta'}{d\xi}$$

$$+ (2\pi\alpha' \tan 2\pi\Delta_c)\left(\frac{\sqrt{1-\beta^2}}{\beta}\right)^3 \cdot \Delta' = 0 .$$

The latter equation becomes

$$\frac{d^2\Delta'}{d\xi^2} + 3\alpha' \frac{\alpha'\xi}{\alpha'^2\xi^2 - 1}\frac{d\Delta'}{d\xi} + \alpha'^2 \frac{a}{(\alpha'^2\xi^2 - 1)^{3/2}} \cdot \Delta' = 0 ,$$

$$2\pi\Delta' \ll 1 , \qquad (5.17)$$

when β is expressed in terms of ξ and the constant $2\pi/\alpha'$ tem $2\pi\Delta_c$ is denoted by a.

Equation (5.17) is a linear differential equation of the second order and may be likened to that of an alternating electric circuit having variable resistance and variable capacity. We may note that $\alpha' > 0$, $\alpha'\xi > 1$. And since Δ_c is in the first quadrant $a > 0$. Both coefficients of $d\Delta'/d\xi$ and Δ' are positive, so the motion will be damped. The

30. E.L.Ginzton, W.W.Hansen, and W.R. Kennedy, loc.cit., footnote 7.

electrons will either approach the equilibrium position $\Delta = \Delta_c$ asymptotically or oscillate about Δ_c with decreasing amplitudes. The equilibrium is stable at least for small enough oscillations. On the other hand, for electrons around $-\Delta_c$ in the fourth quadrant $a < 0$. The coefficient of $d\Delta'/d\xi$ is positive but that of Δ' is negative, so the equilibrium is obviously unstable. We shall limit our discussion to $a > 0$.

The condition for the existence of small oscillatory orbits is easily found to be

$$\frac{\sqrt{\alpha'^2\xi^2 - 1}}{\alpha'^2\xi^2} \geqq \frac{9}{4a} \quad , \quad \text{i.e.} \quad \beta\sqrt{1 - \beta^2} \geqq \frac{9}{4a} \; . \tag{5.18}$$

Since $\beta\sqrt{1 - \beta^2} \leqq 1/2$ and has its maximum value at $\beta = \sqrt{2}/2$, there can be no small oscillation unless $a > 9/2$. The latter condition implies that α' should be small, i.e. the variation of β should be slow.

Equation (5.17) can further be simplified by using the substitution

$$\alpha'\xi = \cosh \alpha'\xi' \; . \tag{5.19}$$

Thus we obtain

$$\frac{d^2\Delta'}{d\xi'^2} + 2\alpha' \coth \alpha'\xi' \frac{d\Delta'}{d\xi'} + \alpha'^2(\alpha \operatorname{csch} \alpha'\xi')\,\Delta' = 0$$

The first order term in the above equation may be removed by transforming Δ' into y with

$$y = \Delta' \sinh \alpha'\xi' \; . \tag{5.20}$$

The resulting equation is

$$\frac{d^2y}{d\xi'^2} + \alpha'^2(a \operatorname{csch} \alpha'\xi' - 1)y = 0 \quad ,$$

$$2\pi y \ll \sinh \alpha'\xi' \quad . \qquad (5.21)$$

If we make another transformation, i.e. $y = e^{\int u d\xi'}$, we would get a non-linear Riccati equation[31] which, as is known, is not integrable by quadratures. We will, henceforth, impose further restrictions and discuss the oscillatory and the non-oscillatory cases separately. In the former case we assume $\alpha' \ll 1 \ll a\cdot\operatorname{csch} \alpha'\xi'$; in the latter $a\cdot\operatorname{csch} \alpha'\xi' < 1$, $a\cdot\operatorname{csch} \alpha'\xi' \ll (1 - a\cdot\operatorname{csch} \alpha'\xi')/\alpha'$.

Let us first take up the oscillatory case. Since $a \operatorname{csch} \alpha'\xi' \gg 1$, equation (5.21) may properly be replaced by

$$\frac{d^2y}{d\xi'^2} + \alpha'^2(a \operatorname{csch} \alpha'\xi')\, y = 0 \; ;$$

and since $\alpha' \ll 1$, $a \operatorname{csch} \alpha'\xi'$ is a slow varying function of ξ', the above equation can most conveniently be solved by the WKB method. The solution to a second order of approximation is

$$y \cong (\operatorname{csch} \alpha'\xi')^{-1/4}\cdot\Big[Ae^{j\int \alpha'\sqrt{a \operatorname{csch} \alpha'\xi'}\, d\xi'}$$

$$+ Be^{-j\int \alpha'\sqrt{a \operatorname{csch} \alpha'\xi'}\, d\xi'}\Big].$$

The elliptic integral can be evaluated at once, thus,

$$\int \alpha'\sqrt{a \operatorname{csch} \alpha'\xi'}\, d\xi' = -\sqrt{a} \cdot F(\phi, \frac{1}{\sqrt{2}})$$

31. E.L. Ince, "Ordinary Differential Equations," (Dover Publications, New York) p.23.

where $\cos\phi = \dfrac{\operatorname{csch}\alpha'\xi' - 1}{\operatorname{csch}\alpha'\xi' + 1}$.

So

$$\Delta' \cong C(\operatorname{csch}\alpha'\xi')^{3/4}\cdot\cos\left[\sqrt{a}\cdot F(\phi, 1/\sqrt{2}) + D\right], \qquad (5.22)$$

where C and D are constants of integration.

The motion of electrons with respect to the equilibrium position $\Delta = \Delta_c$ is oscillatory; the amplitude of oscillation decreases, though rather slowly, as ξ' increases. The period of oscillation is to be computed from two values of ϕ, ϕ_1 and ϕ_2, such that

$$F(\phi_2, \frac{1}{\sqrt{2}}) - F(\phi_1, \frac{1}{\sqrt{2}}) = \frac{2\pi}{\sqrt{a}} .$$

Corresponding to any two fixed values of ϕ, or $\alpha'\xi'$, or $\alpha'\xi$, or β, the change in ξ' or ξ is inversely proportional to α' while the number of oscillations is proportional to $\sqrt{a}$. Therefore the average period in ξ is proportional to $\dfrac{1}{\alpha'\sqrt{a}} = \dfrac{1}{\sqrt{\alpha}\sin 2\pi\Delta_c}$. It can readily be shown that the average period is

$$\Lambda_{ave.} = \sqrt{\frac{2\pi}{\alpha}}\,\frac{1}{\sqrt{\sin 2\pi\Delta_c}}\left(\frac{\beta}{\sqrt{1-\beta^2}}\right)^{3/2}_{ave.} ,$$

so it increases with β or ξ. This agrees with the result $\Lambda = \sqrt{\dfrac{2\pi}{\alpha}}\left(\dfrac{\beta}{\sqrt{1-\beta^2}}\right)^{3/2}$ for the case of constant β, where $\sin 2\pi\Delta_c = 1$.

If $\sin 2\pi\Delta_c$ is small, $\Lambda_{ave.}$ may even be longer than that calculated by assuming β = const. = $\beta_{max.}$ of the interval. For example, if we assume $\alpha = .10$, $2\pi\Delta_c = 10^{\circ}$, then $\alpha' \cong .10$

and $a \cong 11$. As β increases from $1/2$ to $\sqrt{2}/2$, $\operatorname{csch} \alpha'\xi'$ decreases from $\sqrt{3}$ to 1 and $F(\phi, \frac{1}{\sqrt{2}})$ increases from 1.43 to 1.85. The calculated average period is approximately 12 while the period for β = const. = $\sqrt{2}/2$ is about 7.9. If a is sufficiently large and α' small, the solution (5.22) is accurate for quite a large range of ξ.

Then we consider the other case where we have $a \operatorname{csch} \alpha'\xi' < 1$, $a \operatorname{csch} \alpha'\xi' \ll (1 - \operatorname{csch} \alpha'\xi')/\alpha'$. These conditions should be satisfied during and after the somewhat later stage of bunching, because there $a = (2\pi/\alpha') \tan 2\pi\Delta_c$ should be made small by letting $\Delta_c \rightarrow 0$ in order to bunch the electrons onto the peak of the traveling wave. Even if a is not too small, they can still be satisfied, provided ξ' is sufficiently large. As ξ' is further increased $\beta \rightarrow 1$ and the variation of β becomes very slow, it will make no appreciable difference whether we consider β as constant or varying. Therefore, the result for large values of ξ' should be comparable to that for β = const. = 1.

Subject to the above conditions, $1 - a \operatorname{csch} \alpha'\xi'$ is again a slow varying function of ξ', so the WKB solution

$$y \sim (1 - a \operatorname{csch} \alpha'\xi')^{-1/4} \cdot e^{\pm \int \alpha' \sqrt{1 - a \operatorname{csch} \alpha'\xi'}\, d\xi'}$$

is good. Evaluating the integral, we obtain

$$\Delta'(\xi') \cong \frac{\operatorname{csch} \alpha'\xi'}{(1 - a \operatorname{csch} \alpha'\xi')^{1/4}} \cdot \Big[A e^{\alpha'\xi'} (\coth \frac{\alpha'\xi'}{2})^{a/2} + B e^{-\alpha'\xi'} (\tanh \frac{\alpha'\xi'}{2})^{a/2} \Big] \qquad (5.23)$$

As $\xi' \rightarrow \infty$, $\Delta' \rightarrow 2A$. We denote $2A = \Delta'(\infty)$, then

$$\frac{\Delta'(\xi_1') - \Delta'(\xi')}{\Delta'(\xi_1') - \Delta'(\infty)} \cong \frac{\coth \alpha'\xi_1' - \coth \alpha'\xi'}{\coth \alpha'\xi_1' - 1} ,$$

$$\xi_1' < \xi' < \infty , \qquad (5.24)$$

either for a = 0 or for $\operatorname{csch}^2 \frac{\alpha'\xi'}{2} \ll 1$, $a \neq 0$. We note that the derivative of the right-hand member of (5.24) is equal to

$$\frac{\operatorname{csch}^2 \alpha'\xi'}{\coth \alpha'\xi_1' - 1} = \frac{2 \operatorname{csch}^2 \alpha'\xi'}{\operatorname{csch}^2 \alpha'\xi_1'} ,$$

which is very small except where $\xi' \cong \xi_1'$. Thus $\Delta'(\xi')$ will approach the value $\Delta'(\infty)$ rapidly as ξ' increases and indeed will not change appreciably in the entire range from $\alpha'\xi' \cong \alpha'\xi_1' + 2$ up to infinity.

From (5.24) we easily find

$$\epsilon = \Delta'(\xi') - \Delta'(\infty) \xrightarrow[\xi' \rightarrow \infty]{} \left[\Delta'(\xi_1') - \Delta'(\infty)\right] \cdot \tfrac{1}{2} \operatorname{csch} \alpha'\xi' ,$$

i.e.

$$\epsilon \xrightarrow[\xi \rightarrow \infty]{} \text{const.} \frac{1}{(\alpha'\xi)^2} ,$$

so

$$\frac{d\epsilon}{d\xi} \xrightarrow[\xi \rightarrow \infty]{} \text{const.}\, \epsilon^{3/2} .$$

This may be compared with the case of $\beta = 1$ where we have found

$$\frac{d\epsilon}{d\xi} \xrightarrow[\xi \rightarrow \infty]{} \text{const.}\, \epsilon^2 .$$

Both $\Delta'(\infty)$ and $\Delta'(\xi_1')$ depend on the initial conditions and on the bunching system. If bunching is perfect, all electrons will have $\Delta_c + \Delta' \cong 0$, for then all of them will receive maximum energy. Otherwise, many electrons may end up with a phase Δ not quite small and with energy far less than the maximum. Electrons should be bunched properly before they get massive enough. After that, no effective bunching can be achieved.

In between the two cases discussed above, a csch $\alpha'\xi'$ is of the same order of magnitude as unity. In the small neighborhood of $a \cdot \operatorname{csch} \alpha'\xi' = 1$ the solution is trivial, and outside that the WKB solution is not good, because $a \cdot \operatorname{csch} \alpha'\xi'$ can no longer be considered as slow varying. For the latter regions, the perturbation method may be used to the best advantage but we will not discuss it. The results must, obviously, be intermediate between the above two cases.

5.4. Longitudinal Bunching by Slow Increase of β

The above analysis suffers from the drawback that we should restrict to $\beta_e = \beta$ and $2\pi\Delta' \ll 1$. Though it gives us a great deal of information about the motion of a narrow bunch of electrons about the equilibrium position, it does not tell us how electrons can be bunched to such a beam in order to satisfy the above-mentioned restrictions. If β varies slowly and α is not too large, so that no electrons will ever stop and move in the backward direction, the general nature of oscillation, small or large, will not be appreciably different from that of the oscillation with corresponding amplitudes in the case of constant β. As in the latter case, the large oscillations will have the same qualitative features as the small ones. Thus the periods of larger oscillations are longer and the amplitudes will decrease with time or distance. If we wait long enough, so that electrons oscillating with large amplitudes are allowed to travel a sufficiently long distance with slow varying β, they will eventually be bunched to within a small amplitude around the equilibrium position. Indeed, such is what we should expect from the invariant nature of the phase integral $\oint \Gamma d\Delta$ under adiabatic variations. Let $\Delta_s (-\pi/2 < 2\pi\Delta_s < \pi/2)$ be the value of Δ at which the electron and the wave have the same velocity, i.e., $\gamma_s = \dfrac{1}{\sqrt{1-\beta^2}}$, then we easily find

$$\oint \Gamma d\Delta = \frac{4}{\sqrt{1-\beta^2}} \int_{\Delta_s}^{1/4} \sqrt{\frac{\alpha\beta}{2\pi} \frac{1}{\sqrt{1-\beta^2}} (\sin 2\pi\Delta - \sin 2\pi\Delta_s)} \; .$$

$$\cdot \sqrt{\frac{\alpha\beta}{2\pi} \frac{1}{\sqrt{1-\beta^2}} (\sin 2\pi\Delta - \sin 2\pi\Delta_s) + 2} \cdot d\Delta, \qquad (5.25)$$

Though this elliptic integral is not expressible in terms of tabulated functions, it is obvious that the value of the integral increases with β if Δ_s is fixed and decreases as Δ_s increases if β is fixed. In order that the integral may be invariant as β increases slowly, Δ_s must increase so the amplitude of oscillation must decrease. It is also obvious that the same conclusion holds true if α varies slowly.

Adiabatic increase of β is of course impracticable, nor is it necessary. By assuming $|\beta_e - \beta| < \beta$ and $|\beta_e - \beta| < 1 - \beta$, one can show without much difficulty from the equations (4.10a) and (4.10b) that, if β increases monotonically, the amplitude of the phase oscillation around Δ_c will decrease continuously from cycle to cycle. The above assumptions imply that α is small and the increase of β is slow. They are sufficient conditions, but not necessary. However, if β increases rapidly, the electrons will not be able to oscillate more than a few cycles before β gets near unity, and so cannot be well bunched. Furthermore, as $d\beta/d\xi$ increases, more and more electrons will lead retrograde instead of oscillatory orbits and become lost to the bunch. Retrogression takes place whenever Δ decreases to less than $-\Delta_c$ in the negative quadrant. Thus, in a buncher of this kind, β can only increase so fast that a sufficient number of oscillation cycles can take place and that $-\Delta_c$ should be less than the minimum value of Δ of the whole bunch of electrons. The latter requirement imposes a severe condition on the input end of the buncher, where the electrons are uniformly distributed over the wave cycle. If the loss of electrons is to be minimized, the increase of β at the input end should be exceedingly slow.

5.5. A Numerical Example of Ineffective Bunching By Rapid Increase of β

Having discussed the limitations of the bunching process of increasing β, we now show by a specific example what results are to be expected from a short buncher of this kind under adverse conditions.

In Fig. 5.7 we compare, by plotting Δ_∞ vs. Δ_o, the bunching characteristics of two systems, one being a uniformly loaded waveguide with $\beta = 1$ and the other the same structure with the addition of a 1-1/2 λ variable -β section. $\alpha/2\pi = 1.266$ and $\gamma_o = 2.0$ are the same for both systems. α is large enough to turn back part of the injected electrons. β in the variable case increases from $\beta_o = .866$ so rapidly that Δ_c is everywhere zero, so that quite an appreciable fraction of the electrons will be lost through degeneration of orbits from oscillatory to retrograde. The solution for the variable -β section was obtained on the differential analyzer at U.C.L.A., and we have to correct the result for the addition of an infinitely long structure of $\beta = 1$ before we can plot Δ_∞. The two curves are almost coincident for Δ_o in the negative half cycle and almost parallel to each other for Δ_o in the positive half cycle, the variable -β curve being wider than the $\beta = 1$ curve by about 12^o in the Δ_o direction. The bunching action of this variable -β section is only slightly better than the uniformly loaded structure.

5.6. Longitudinal Bunching By Increase of α

We have discussed in some detail the bunching process with constant α and increasing β. The rate of increase of β should be slow from both the viewpoints that the phase spread of the bunch should be small and that the loss of electrons due to retrogression should be slight. The situation is different if β is fixed while α is made to increase monotonically. α can increase fast without causing the electrons to lead retrograde orbits and the ratio of the final value to initial value of α can be many times larger than the corresponding ratio of β. The bunching process of increasing α is practically free from the limitations of that of increasing β.

The effectiveness of bunching by increasing α is best told by the equation

$$\left(\frac{1}{\beta} - \frac{\gamma}{\sqrt{\gamma^2 - 1}}\right) d\gamma = \alpha \cos 2\pi\Delta \, d\Delta$$

which is obtained by combining the three equations (4.10) of longitudinal motion. We integrate this equation over a half period of oscillation from $\Delta = \Delta_{s,n}$ to $\Delta = \Delta_{s,n+1}$, the two consecutive values of Δ_s at which the electron and the wave have the same instantaneous velocity. Thus we obtain

$$\left(\frac{\gamma}{\beta} - \sqrt{\gamma^2 - 1}\right) \Bigg|_{\gamma_{s,n}}^{\gamma_{s,n+1}} = \int_{\Delta_{s,n}}^{\Delta_{s,n+1}} \alpha \cos 2\pi\Delta \, d\Delta ,$$

i.e., $$0 = \int_{\Delta_{s,n}}^{1/4} + \int_{1/4}^{\Delta_{s,n+1}} \alpha \cos 2\pi\Delta \, d\Delta \ .$$

Since α increases monotonically, we must have

$$\ldots < \left|1/4 - \Delta_{s,n+1}\right| < \left|1/4 - \Delta_{s,n}\right| < \left|1/4 - \Delta_{s,n-1}\right| < \ldots$$

The amplitude of oscillation decreases continuously from quarter-cycle to quarter-cycle; and the more rapidly α increases, the faster the amplitude decreases. If the initial value of α is small enough and the increase of α is not too rapid at the beginning, then all the electrons injected during one complete wave cycle will execute oscillations around the same equilibrium position $\Delta_c = 1/4$ and will readily be bunched to a very narrow beam in as short a distance as a few wavelengths.

In the above discussion, one restriction on β_e has been tacitly assumed, that is $\beta_e > 0$. If $\beta_e < 0$, the electron will certainly be lost. It is for this reason that α must not increase too rapidly at first.

A short buncher of this kind may be connected directly to an accelerator of uniformly loaded structure with $\beta = 1$ and $\alpha =$ const. If α is suitably chosen for the corresponding value of β in the buncher, the beam bunched at $\Delta = 1/4$ will shift to the peak of the accelerating field after traveling a short distance in the uniformly loaded guide and will practically maintain a constant phase throughout the subsequent travel. If conditions are not suitable, a short section of phase shifter with β equal to a constant, not unity, may be inserted at some proper place in the $\beta = 1$ accelerator tube. Better still, a section of variable-β phase shifter may be used to connect the buncher and the accelerator so as to avoid abrupt changes in loading by increasing β gradually from the bunching value to unity. It works in the same manner as the variable-β buncher which has been discussed in section 5.4. Since electrons entering the phase shifter have already been properly bunched, β in the phase shifter can be increased rapidly without incurring loss of electrons due to retrograde orbits.

It is quite obvious now that the most expedient way of bunching is to incorporate an effective buncher with a smooth phase shifter in a short section of loaded waveguide by increasing α and β simultaneously and properly. For such a design it is important to use a relatively small value for α at the injection end and increase α and β slowly at that end.

As discussed before, it has been found possible only under special conditions to get reliable analytic solutions to the equations of motion in terms of the tabulated functions. When α is also variable, the problem becomes more difficult; we have to rely on numerical methods for determining the bunching characteristics. Numerical processes not only have the disadvantage of being limited in accuracy but also of being tedious and almost impossible under certain adverse conditions. For example, if α is too large, some electrons will oscillate back and forth in space with very low average velocities; the calculation of such an orbit is lengthy and difficult for any moderate length of displacement. However, if conditions are favorable, all injected electrons will arrive at a phase $\Delta \cong 0$ in several wavelengths after a few oscillations in Δ (not ξ), and the numerical integration will be easy and reliable.

Carter and Hansen have calculated a typical example with variable-α and variable-β for a bunching length of about 5.5λ. The bunching action is very strong, but since β increases rather too fast, especially at the beginning, the retrogression loss is quite appreciable. More flexible forms for $\alpha(\xi)$ and $\beta(\xi)$ may be devised, e.g.,

$$\left.\begin{aligned} \alpha - \alpha_o &= A\left[\sin 2\pi \frac{\xi - c}{d} + \sin 2\pi \frac{c}{d}\right] , \\ \frac{1}{\sqrt{1-\beta^2}} - \frac{1}{\sqrt{1-\beta_o^2}} &= \frac{\alpha}{a} \left(b \sinh \frac{\xi}{b} - \xi\right) , \end{aligned}\right\} \qquad (5.26)$$

where α_o, β_o, A, a, b, c and d are constant parameters. By giving suitable values to these parameters, the functions $\alpha(\xi)$ and $\beta(\xi)$ can easily be made to conform to the general requirements discussed before. In fact, these parameters are just sufficient in number, because we have, besides the initial values α_o and β_o, five essential quantities to be selected properly, namely, the length of the buncher, the values of α and β at the output end of the buncher and the rates of increase of α and β at the injection end.

A study of such characteristic functions (5.26) has since been made to facilitate the design of a tapered bunching section for the Stanford billion-volt accelerator. The specific forms of $\alpha(\xi)$ and $\beta(\xi)$ are shown in Fig. 5.8.

$$\alpha = \begin{cases} 2.2 + 1.8 \sin \frac{\xi - 3}{3} \frac{\pi}{2}, & \xi \leqq 6 \\ 4.0 & \xi \geqq 6 \end{cases}$$

The initial part of the β-curve is defined by

$$\frac{d}{d\xi}\left(\frac{1}{\sqrt{1 - \beta^2}}\right) = \alpha(\xi) \sin\left[\frac{\pi}{2} \cdot \frac{\int_0^\xi \alpha(\xi) d\xi}{\int_0^6 \alpha(\xi) d\xi}\right],$$

$$0 \leqq \xi \leqq 4 .$$

From $\xi = 4$ on, β is increased more or less uniformly to a maximum value of about 1.1 at $\xi = 6$. Thereafter β is decreased uniformly to unity at $\xi = 7$. $\gamma_o = \frac{1}{\sqrt{1 - \beta_o^2}} = 1.155$.

The bunching characteristics of this section are shown in Fig. 5.9 by plotting Δ versus ξ . Its effectiveness may be noted by the fact that all electrons with $-1.5 \leqq 2\pi\Delta \leqq 4.5$ are bunched around the wave crest in seven wavelengths with a phase spread of about .50 ($.96 \leqq \cos 2\pi\Delta \leqq 1$).

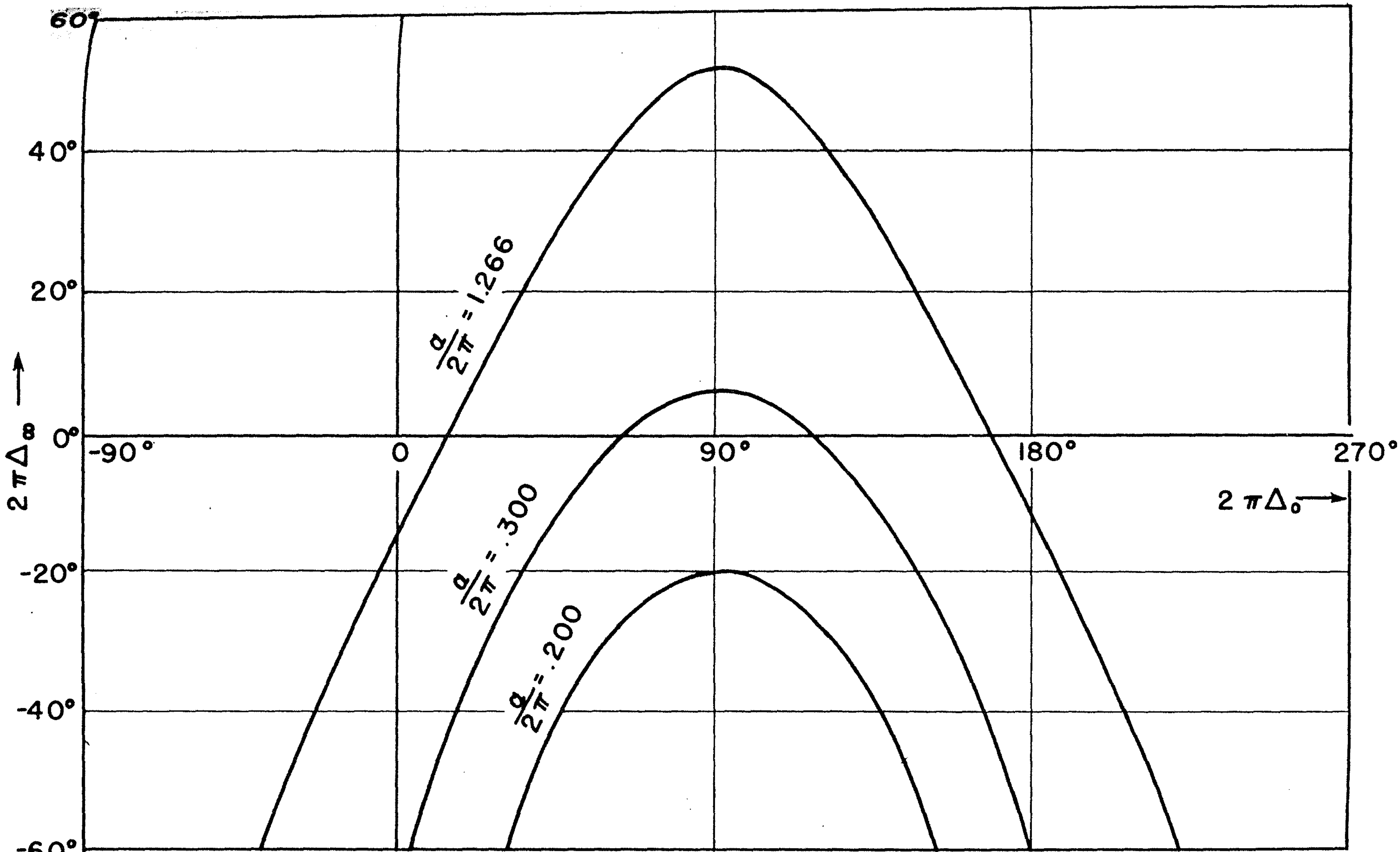

Fig. 5.1 - Δ_∞ vs. Δ_o with α as a parameter. $\beta = 1$, $\gamma_o = 2$.

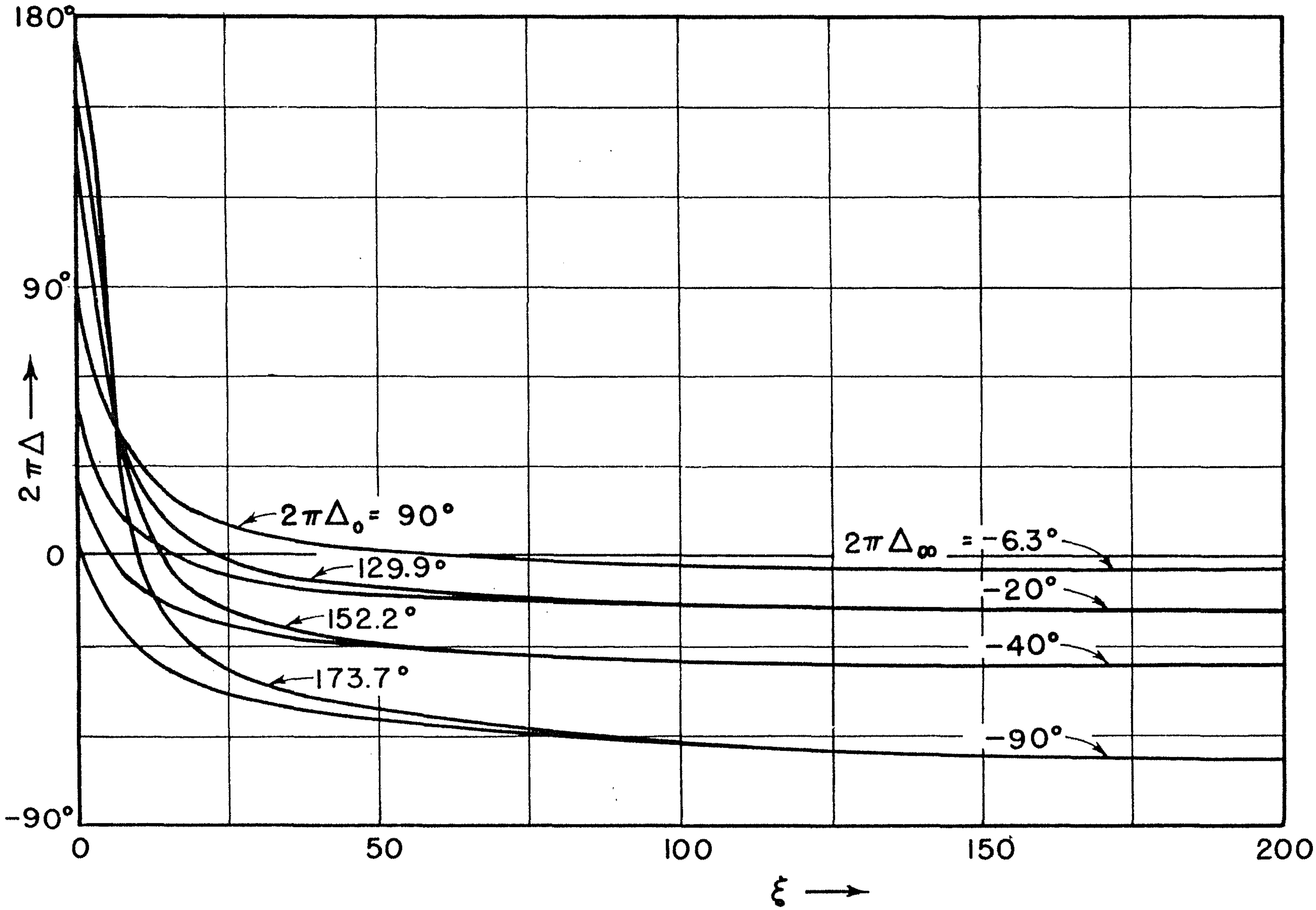

Fig. 5.2a - Δ vs. ξ for $\frac{\alpha}{2\pi} = .1$, $\beta = 1$, $\gamma_0 = 5$.

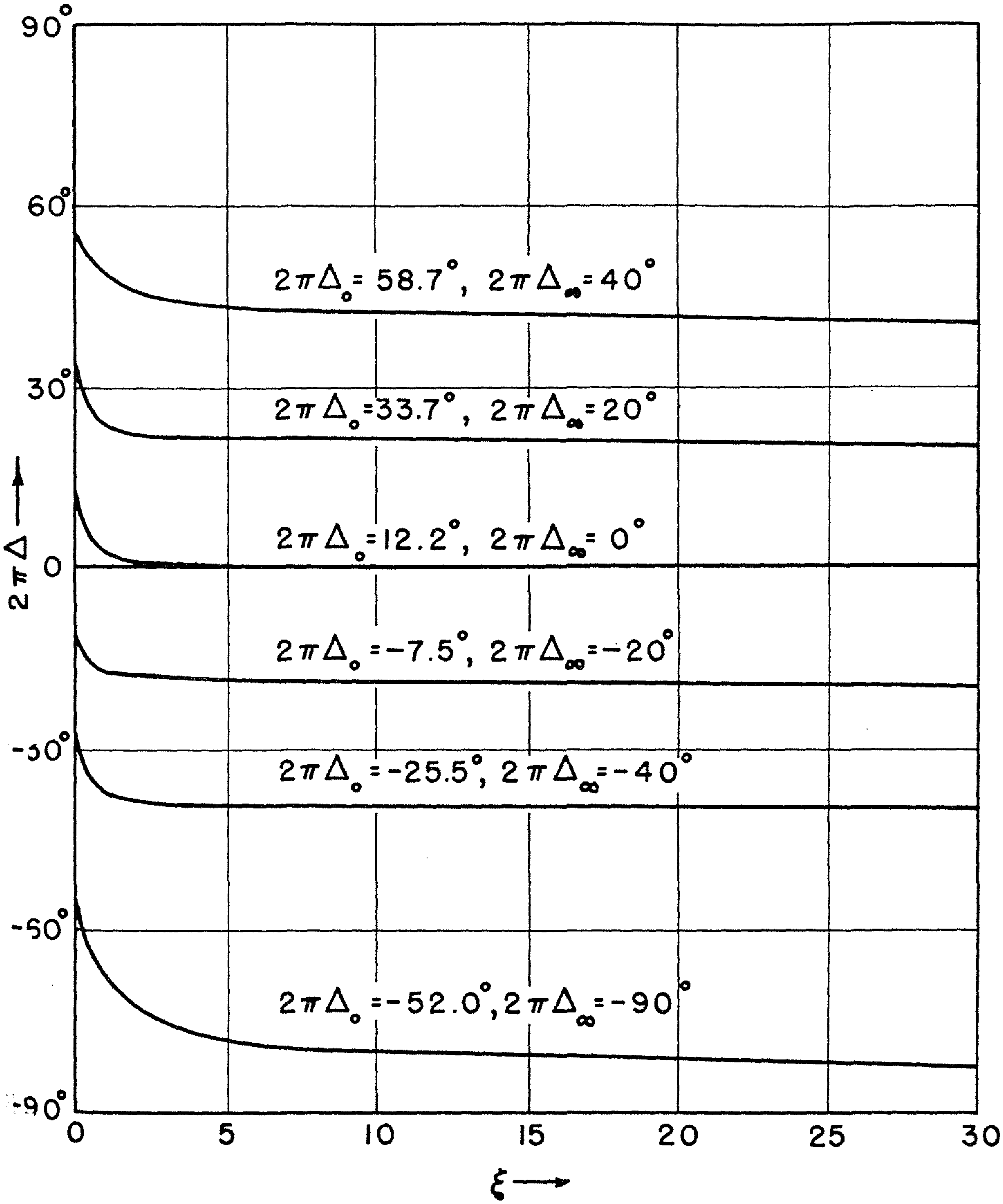

Fig. 5.2b - Δ vs. ξ for $\frac{\alpha}{2\pi} = 1.266$, $\beta = 1$, $\gamma_o = 2$.

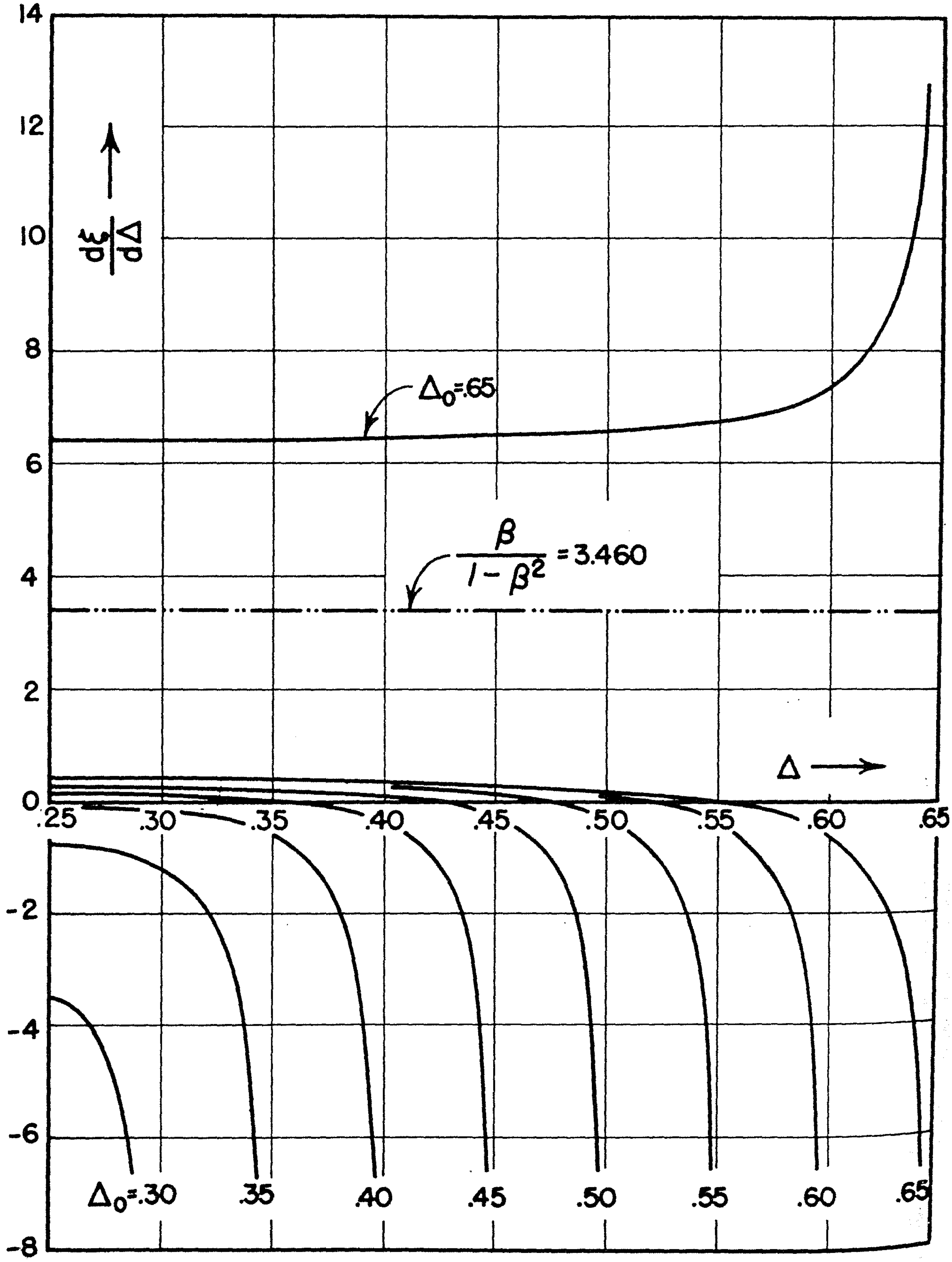

Fig. 5.3 - $\frac{d\xi}{d\Delta}$ vs. Δ with Δ_o as a parameter.

$\frac{a}{2\pi}$ = 1.266, β = .866, γ_o = 2.

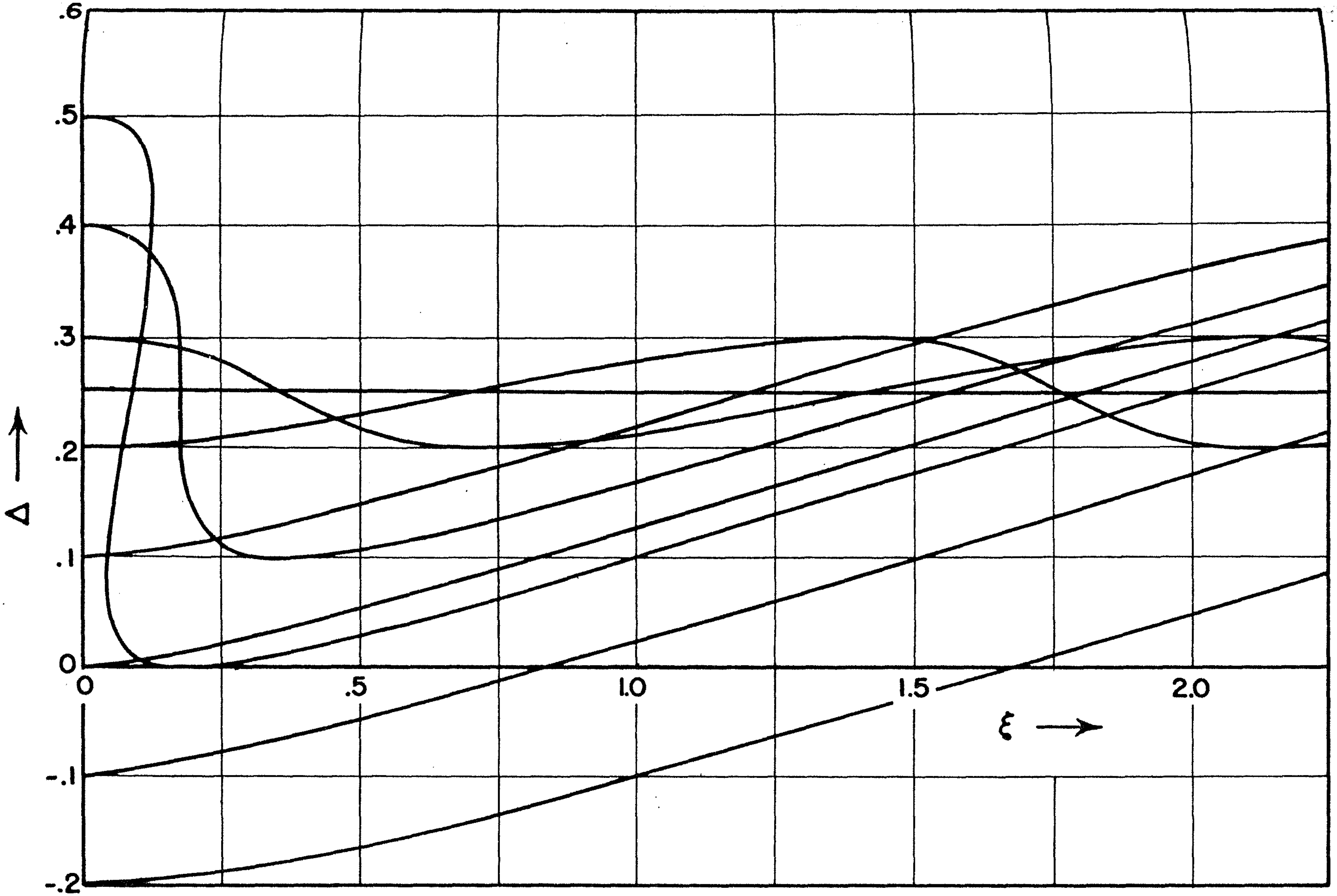

Fig. 5.4 - Δ vs. ξ with Δ_o as a parameter. $\frac{\alpha}{2\pi} = 1.266$, $\beta = .866$, $\gamma_o = 2$.

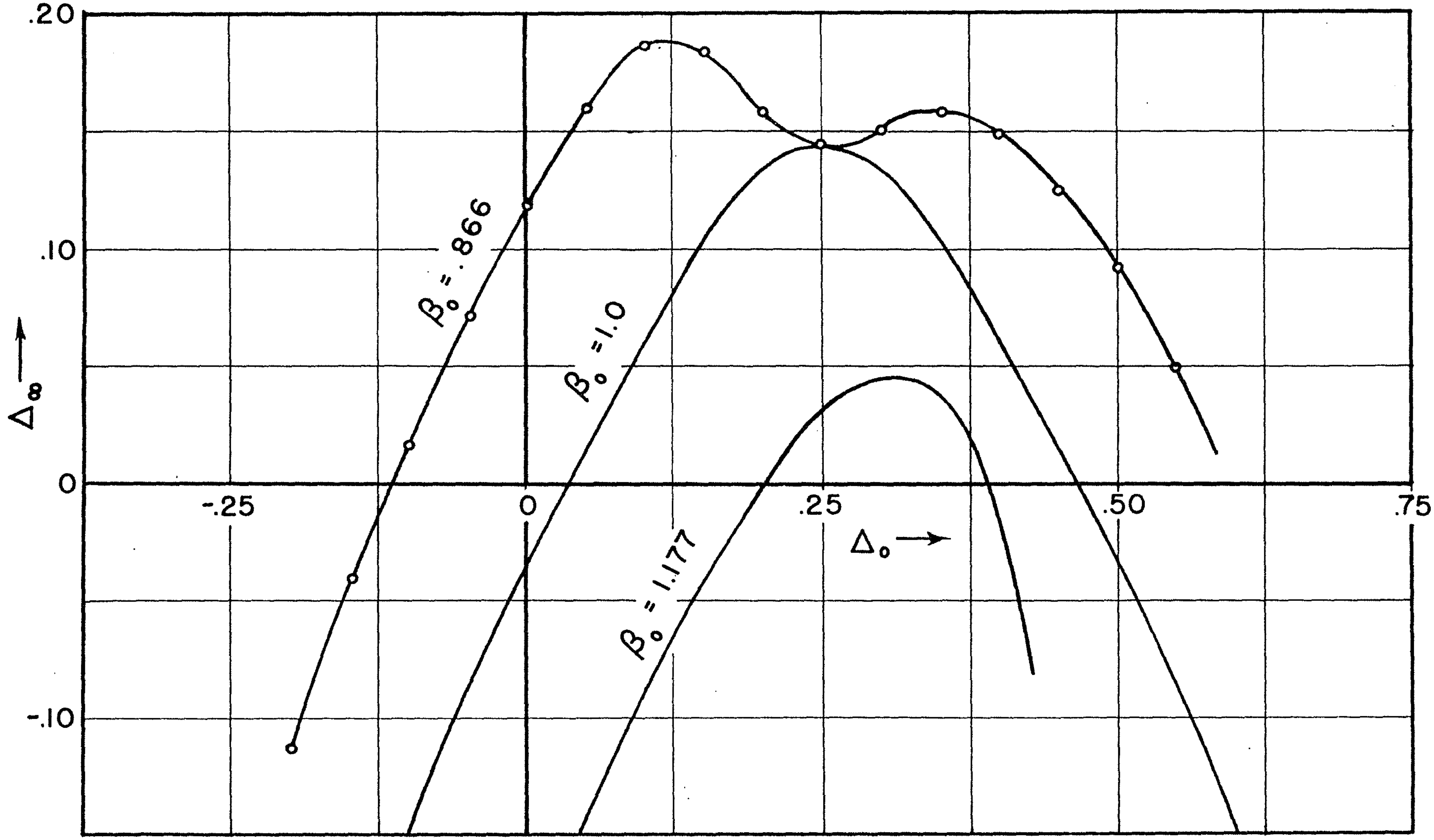

Fig. 5.5 - Δ_∞ vs. Δ_o with β_o as a parameter.

$\frac{a}{2\pi} = 1.266$, $\gamma_o = 2$, $\beta = \beta_o$ for $\xi < 1$; $\beta = 1$ for $\xi > 1$.

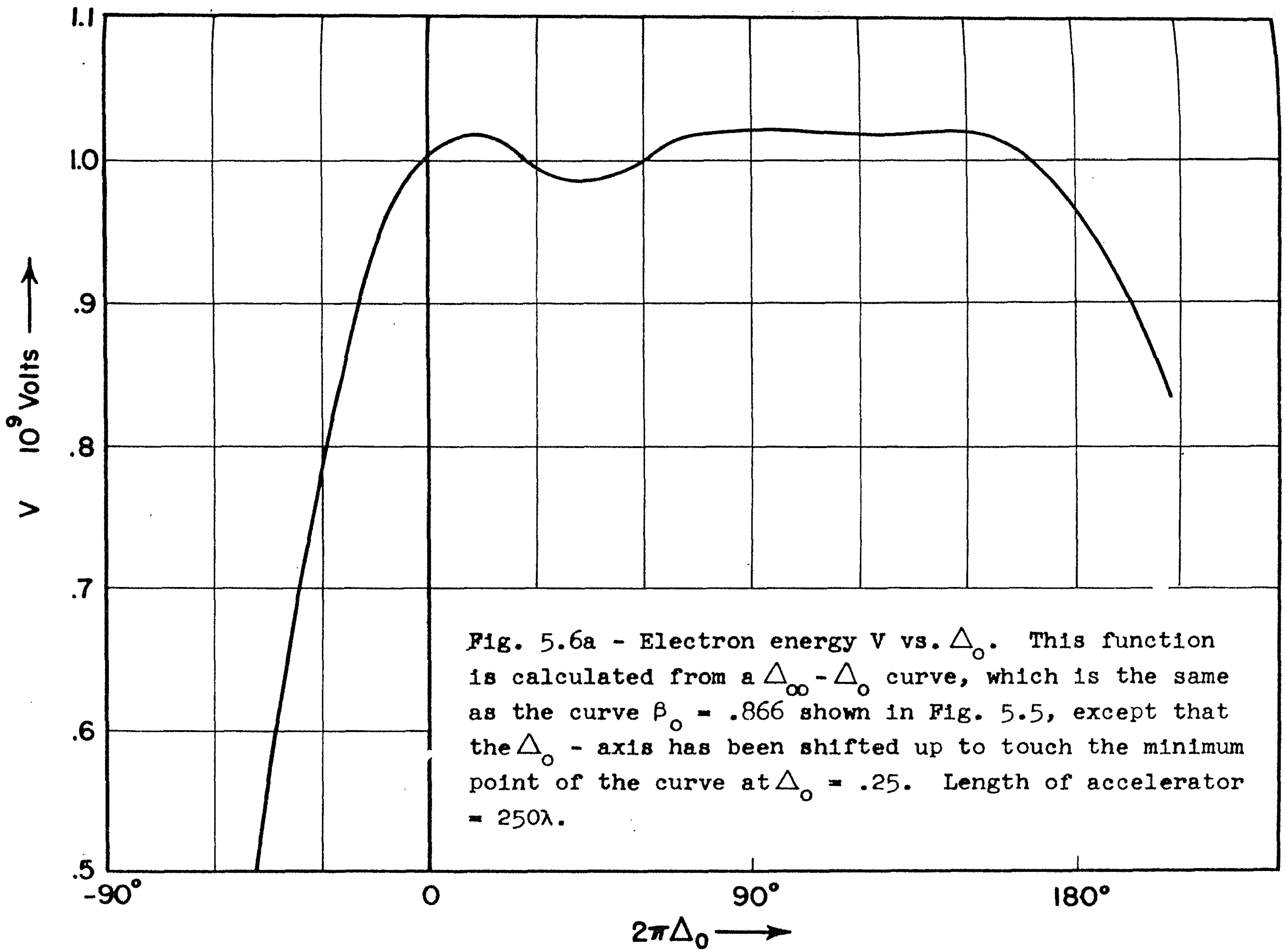

Fig. 5.6a - Electron energy V vs. Δ_o. This function is calculated from a $\Delta_{oo} - \Delta_o$ curve, which is the same as the curve β_o = .866 shown in Fig. 5.5, except that the Δ_o - axis has been shifted up to touch the minimum point of the curve at Δ_o = .25. Length of accelerator = 250λ.

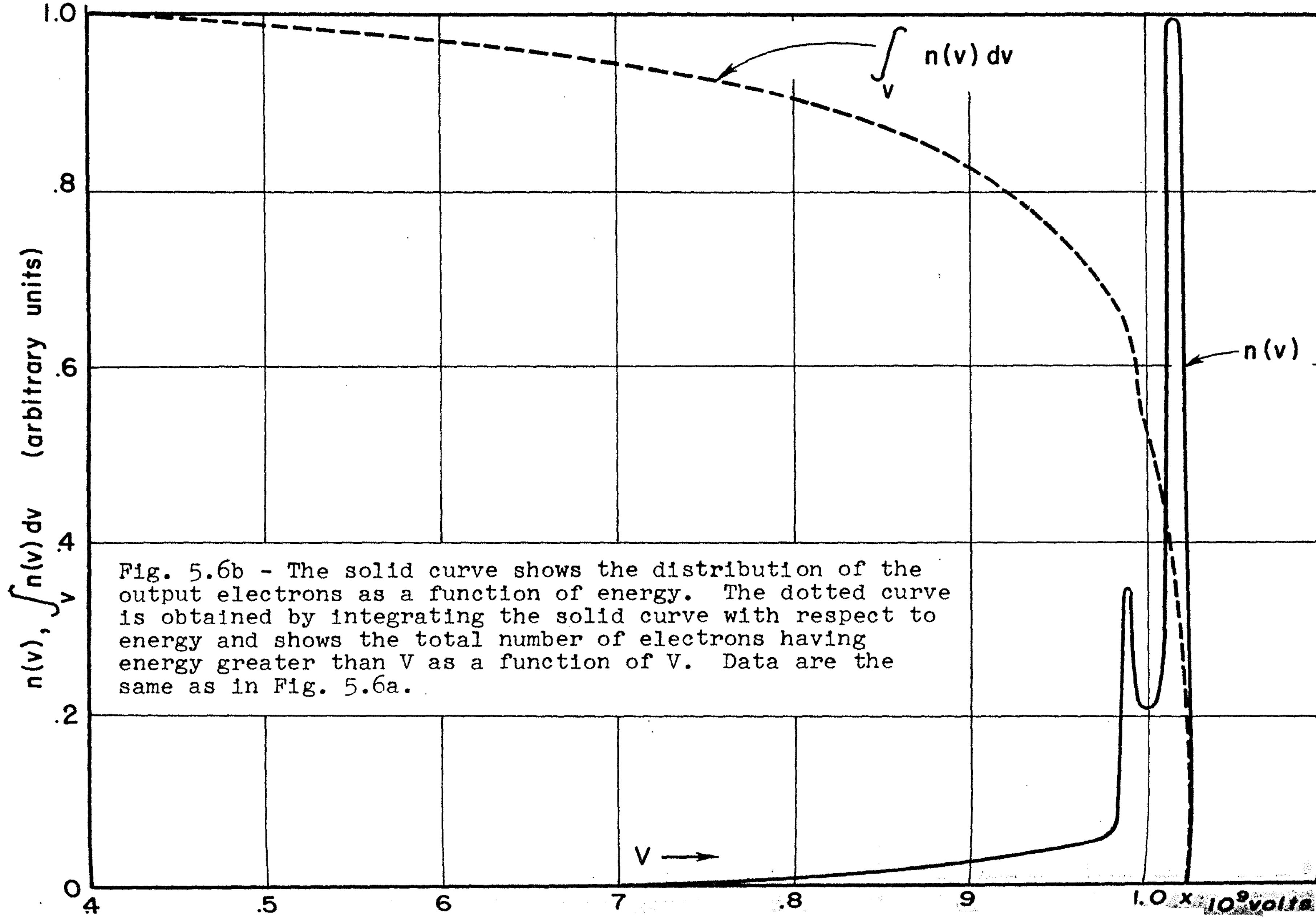

Fig. 5.6b - The solid curve shows the distribution of the output electrons as a function of energy. The dotted curve is obtained by integrating the solid curve with respect to energy and shows the total number of electrons having energy greater than V as a function of V. Data are the same as in Fig. 5.6a.

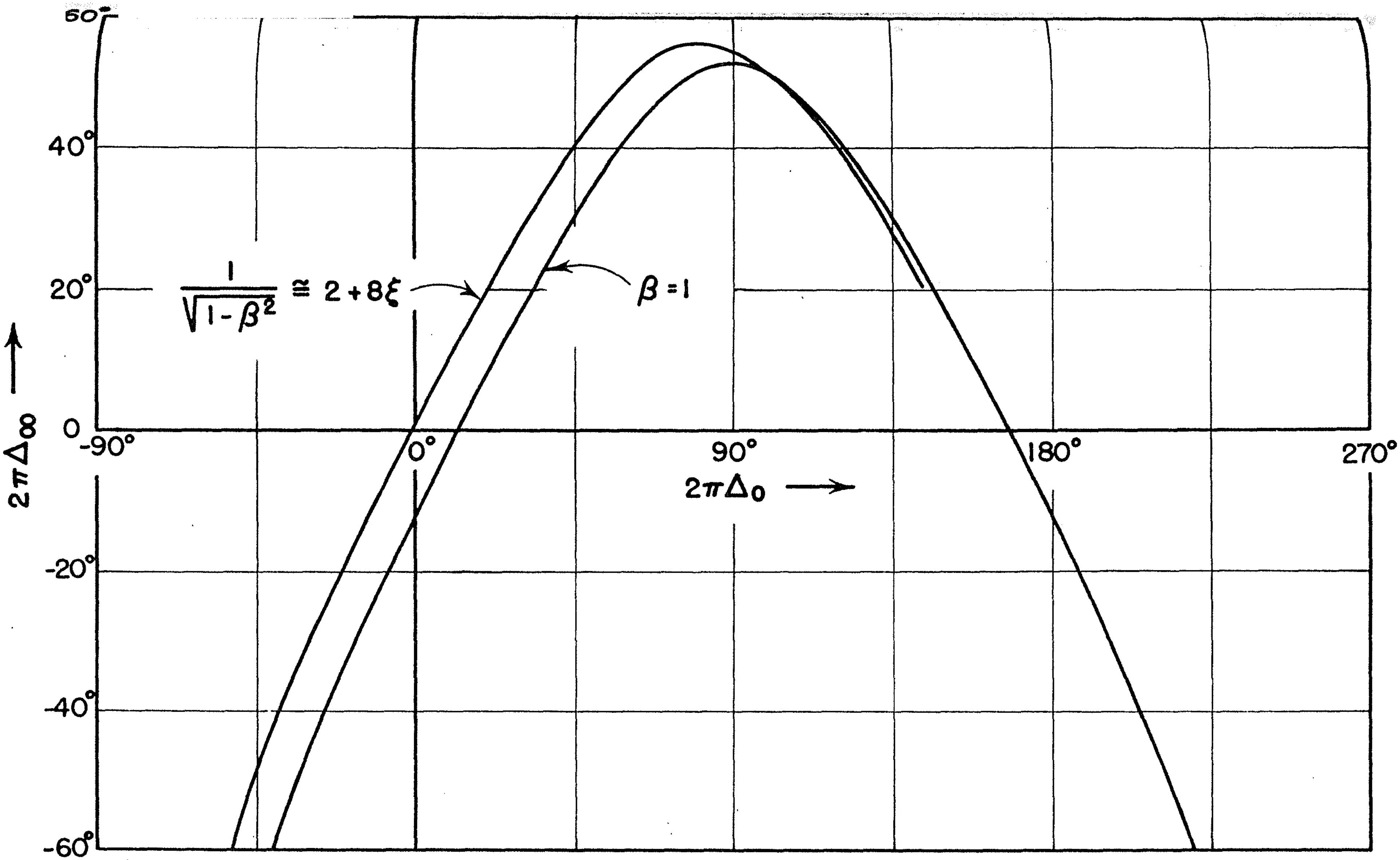

Fig. 5.7 - Δ_{oo} vs. Δ_o for two different starting sections. $\beta = 1$ in one case; $(1-\beta^2)^{-1/2} = \alpha\xi + (1-\beta_o^2)^{-1/2}$ in the other case. $\alpha/2\pi = 1.266$, $\gamma_o = (1-\beta_o^2)^{-1/2} = 2$.

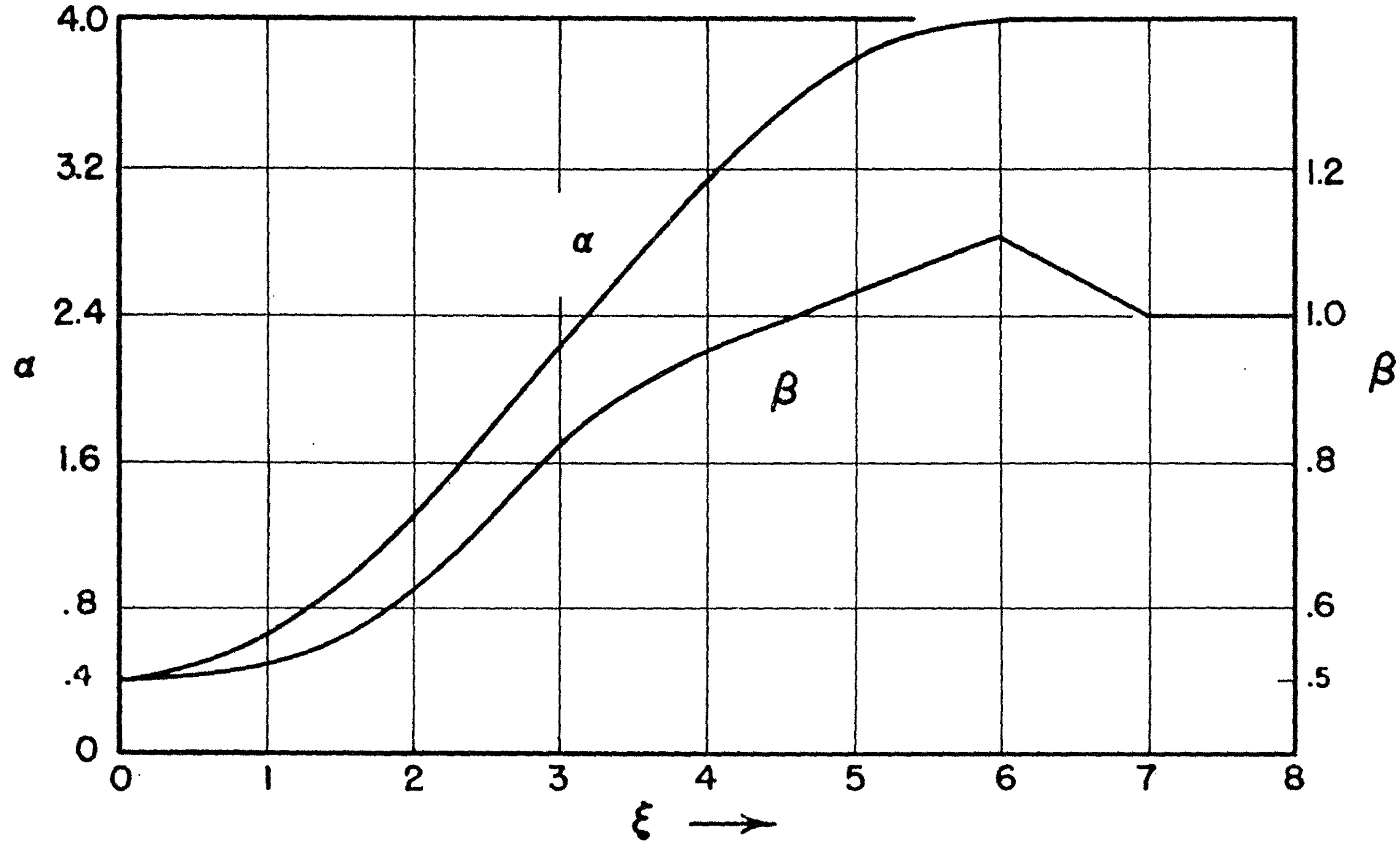

Fig. 5.8 - Shows functions of $\alpha(\xi)$ and $\beta(\xi)$ for a tapered bunching section intended for the billion-volt accelerator.

$$\alpha = \begin{cases} 2.2 + 1.8 \sin(\pi/2)(\xi-3)/3 , & \xi \leqq 6 \\ 4.0 & \xi \geqq 6 ; \end{cases}$$

$\beta = 1$ for $\xi \geqq 7$.

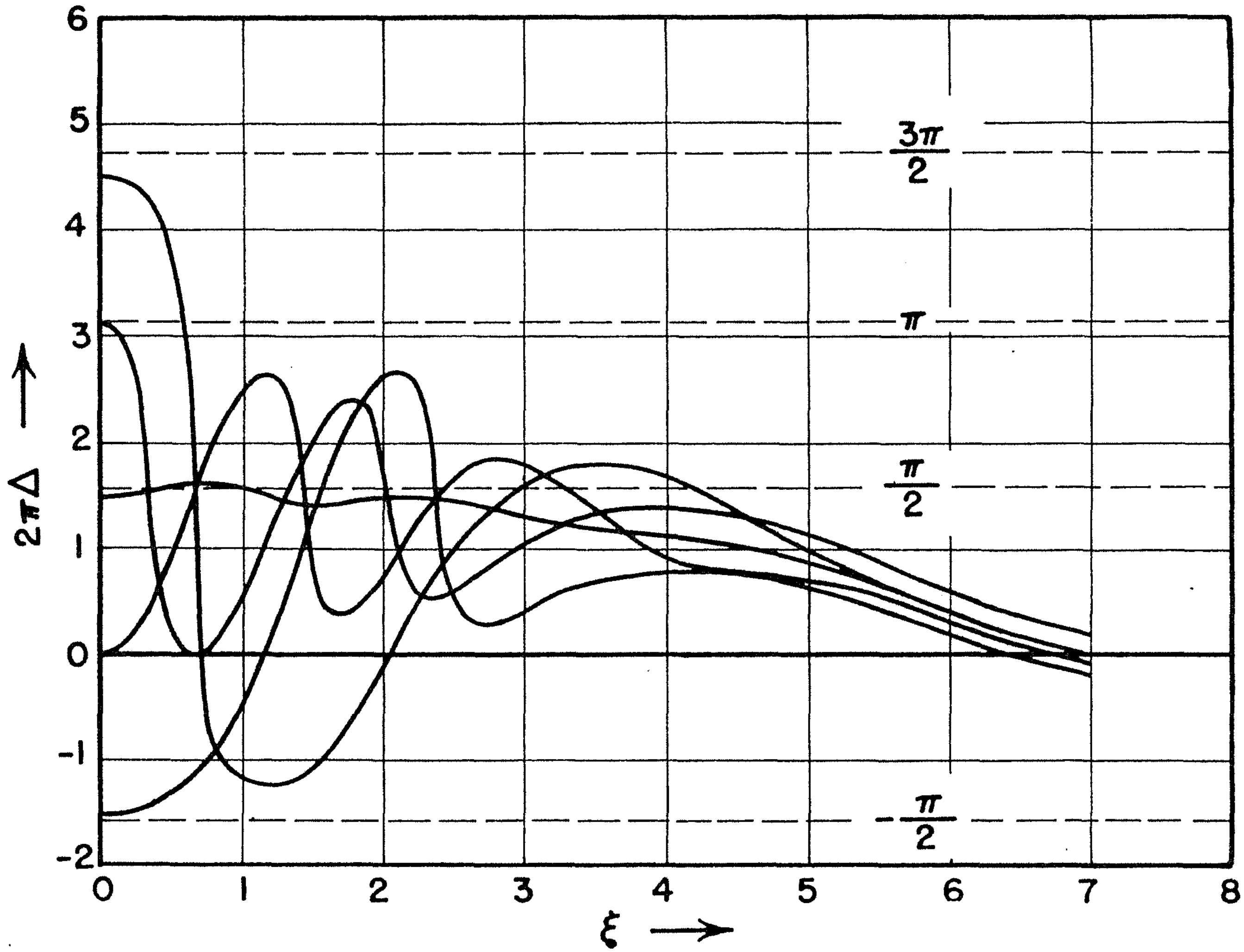

Fig. 5.9 - Shows the bunching characteristics of a tapered section. $\alpha(\xi)$ and $\beta(\xi)$ are shown in Fig. 5.8; $\gamma_o = (1-\beta_o^2)^{-1/2} = 1.155$.

CHAPTER VI

TRANSVERSE MOTION AND FOCUSING OF ELECTRONS

6.1. Introductory Remarks

The equation of transverse motion of electrons is given by (4.9b) or (4.10d). We see from these equations that the radial force acting on the electron is away from or towards the axis according to whether $(1 - \beta\dot{\xi}) \sin 2\pi\Delta$ is positive or negative. With $\beta \leq 1$, $1 - \beta\dot{\xi}$ is always positive, so the radial force has the same sign as $\sin 2\pi\Delta$. Thus the electron experiences a defocusing force in the bunching region $(0 < 2\pi\Delta < \pi)$ and a focusing force in the debunching region $(-\pi < 2\pi\Delta < 0)$. Bunching and focusing actions mutually exclude each other. With $\beta > 1/\dot{\xi} > 1$, i.e., $1 - \beta\dot{\xi} < 0$, the radial force has the opposite sign of $\sin 2\pi\Delta$, so bunching and focusing actions occur together. The stability, however, is only temporary because the electron travels slower than the wave and will soon slip out of the stable region. Any scheme of shifting the phase back into the stable region will introduce defocusing forces. Such fast waves can have certain short range applications for phase shifting purposes[32] but they are incapable of providing useful stability over long distances.[33]

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32. See Section 5.2(ii) for discussion of such applications.

33. It is true that, when $\dot{\xi} \cong 1$, β can be made slightly greater than $1/\dot{\xi}$, e.g. $\beta - 1/\dot{\xi} = (1/\dot{\xi}) - 1$ to give rise to a weak focusing action. But this is not very useful, because when $\dot{\xi} \cong 1$, the electrons are no longer sensitive to weak focussing forces. If the radial momentum of the electron has already become unfavorably large, it can hardly be affected during the subsequent travel.

For all these cases it can be said that the electron motion is inherently unstable. And this property of instability is possessed by all kinds of charged particles. The important question is whether a finite beam of charged particles can be made to pass through an electromagnetic field any integral number of repeat lengths or periods without being dispersed in any direction. This question, evidently, is to be decided by the integrated values of bunching and focusing effects. The answer is "no" with regard to the first order effects but "yes" in general. By first order we mean that the variation of the particle velocity and the variation of position in the transverse plane in one period are negligible in comparison with the respective quantities themselves. The incompatibility of the first order bunching and focusing requirements was proved to be true by McMillan.[34] It is a fundamental consequence of Maxwell's equations and is independent of the type of the electromagnetic field. In fact, McMillan's proof is a generalization of Earnshaw's theorem which is valid only in the realm of electrostatics.

This question of instability caused much concern in the early planning and development work on linear accelerators. Since linear orbits can only derive their stability from second and higher order effects, it seemed doubtful whether very long linear accelerators could be made practicable for supplying high energy beams of useful intensity and whether they could stand the competition of various machines having

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34. E.M. McMillan, Phys. Rev. 80, 493 (1950)

circular orbits. While the situation is indeed serious for heavy ions, there is no real difficulty for electrons. The difference lies in the rate at which the dispersion occurs. In this respect electrons stand in a uniquely advantageous position among all kinds of charged particles.

This unique aspect of the linear electron accelerator was first discussed by Hansen[35] and later by Slater[36] and several other writers.[37] Electrons have a rest energy of about 1/2-Mev. With accelerating fields being several Mev. per foot, α is of the order of unity, the electron will reach relativistic energy in a few wavelengths. The defocusing force experienced by the electron, which is the algebraic sum of the electric and the magnetic forces and is proportional to $1 - \beta\dot{\xi}$, will soon become negligibly small. Thus after a distance of one foot or so the transverse momentum of the electron will be practically constant while its logitudinal momentum will increase at a finite rate. The ratio of the two components of momentum tends to zero, as does the ratio of the two component velocities. As a result, the electron will soon afterwards move in a path which is practically parallel to the axis with its radial distance increasing only logarithmically. If the electrons succeed in passing through the first few wavelengths at small enough radii and with low enough transverse velocities, as

35. W.W.Hansen, Report of the Linear Electron Accelerator Project, Microwave Laboratory, Stanford Univ., Nov. 1946.

36. J.C.Slater, loc.cit., footnote 8.

37. See for example T.R.E. Reports on Linear Electron Accelerators.

they certainly can if a well-designed injection system is used, they will also succeed in passing through a relatively long accelerator tube and receiving high energy. Linear electron accelerators can be made to work without any supplementary focusing device, though naturally with some sacrifice in performance. And the latter defect can easily be avoided by means of the usual technique of D.C. magnetic focusing. The magnetic field is produced by solenoidal coils with their axis coinciding with that of the accelerator tube. The field first interacts with the electrons to set them into spiral motion and then interacts with the spiral motion to exert focusing action. This is a second order effect but the required magnetic field is of comfortable magnitude, being at most a few thousand gauss over a short distance, so no practical difficulty is to be encountered. The corresponding figures for heavy ions are prohibitively large.[38]

Notwithstanding the fact that the lack of the first order stabilizing effects sets no limit to the practicability and usefulness of linear electron accelerators, it would surely inflict certain loss in efficiency or effectiveness unless it is properly compensated. This needs rather exacting design and construction and cannot possibly be done without a detailed knowledge of the transverse part of the electron motion. But transverse motion is a more complicated subject than longitudinal motion, because the latter can be treated independently

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38. The focusing field strength increases as the square root of the rest mass of the charged particle and the distance over which the focusing field is needed increasesas its first power.

of the former as we did in Chapter V, while a simplification in the reverse direction is not possible. We cannot expect to get as much in the way of concrete results as in the longitudinal case without resorting to large scale numerical calculations. Nevertheless,we shall attempt to discuss this subject in the same general spirit as we did the last with both theoretical and practical interests in view.

We shall first transform the equation (4.9b) into a linear differential equation with η as a function of Δ by eliminating the variables γ and τ determined from the longitudinal part of the solution, and then transform the latter equation into canonical forms and discuss their analytic aspects. To simplify discussion α will be taken as constant and β may be either constant or variable; in the latter case the variation of β will be the same as in Section 5.3. Approximate analytic solutions will be given for various cases under the above simplifying conditions. The important case $\beta = 1$ will be emphasized. Next we will describe the early numerical work done on the U.C.L.A. differential analyzer for two specific designs ($(1 - \beta^2)^{-1/2} = \alpha\xi + 2$ and $1/\beta = .85$ for the starting section) intended for the Stanford billion volt accelerator; the results are summarized in several graphs. Finally we shall discuss the problem of D.C. magnetic focusing.

In this last connection, we shall re-formulate the problem of electron motion in a more general way by the Hamiltonian method, which is applicable to any type of field. The expressions for the accelerating wave field will be generalized to cover the case of rapidly varying parameters. An optimum condition will be derived which maximizes the focusing force for a given magnetic field strength. The required focusing field strength for given parameters will be estimated. The effectiveness of focusing depends on the disposition of the magnetic field and the electron gun system; investigation will be made to determine the essential requirements for effective focusing. The possibility of using a high frequency transverse electric type of field for focusing will also be discussed.

6.2. Approximate Analytical Solutions

(i) Restrictive Conditions

In Chapter V we discussed the problem of longitudinal motion by considering only those electrons which move along the axis and were content to say that the solutions we obtained are also good enough for those electrons which move slightly off the axis. This assumption greatly simplifies the discussion and, generally speaking, would not introduce serious error. Now we are considering the transverse motion, and since we naturally want to use the results of the last chapter, it seems advisable to make certain whether this approximation is really justified.

Let us refer to the two sets of equations of motion (4.9) and (4.10) again. Among them the three equations (4.9b),(4.10b) and (4.10d) are exact forms. The equation (4.10a) is derived from (4.2a) by neglecting the magnetic field term. The exact equation is

$$\frac{d}{d\tau}(\gamma\dot{\xi}) = \alpha \cos 2\pi\Delta + \pi\alpha\eta\dot{\eta} \sin 2\pi\Delta \ . \tag{6.1}$$

In equation (4.9a) there is also one term lacking. This equation should be replaced by

$$\frac{d\gamma}{d\xi} = \alpha \cos 2\pi\Delta + \frac{\pi\alpha\eta}{\beta}\frac{d\eta}{d\xi} \sin 2\pi\Delta \tag{6.2}$$

as can easily be verified to follow from the law of conservation of energy. The equation (4.10c) is accurate enough in comparison with the exact form

$$\frac{d\tau}{d\xi} = \pm \frac{\gamma}{\sqrt{\gamma^2 - (\gamma\dot{\eta})^2 - 1}} \tag{6.3}$$

as long as $(\dot{\eta}/\dot{\xi})^2 \ll 1$, a condition which is usually satisfied. Thus the approximation is good if both η and $\dot{\eta}$ are sufficiently small except possibly in the neighborhood of $\cos 2\pi\Delta = 0$, i.e. $|\sin 2\pi\Delta| = 1$. In the latter region we should, strictly speaking, use the exact equations.

From equations (6.1) and (4.10b) we get

$$\frac{\gamma}{\beta} - \gamma\dot{\xi} = \frac{\alpha}{2\pi}(A + \sin 2\pi\Delta) + \int_{\xi_0}^{\xi} \pi\alpha\eta \frac{d\eta}{d\xi}\frac{1-\beta^2}{\beta^2} \sin 2\pi\Delta \, d\xi \ .$$

This is reduced to

$$\frac{\gamma}{\beta} - \gamma\dot{\xi} = \frac{\alpha}{2\pi}(A + \sin 2\pi\Delta) \tag{6.4}$$

exactly if $\beta = 1$, and approximately if

$$2\pi^2(\xi - \xi_o)\left|\eta \frac{d\eta}{d\xi}\frac{1-\beta^2}{\beta^2}\right|_{max.} \ll |\sin 2\pi\Delta| . \qquad (6.5)$$

Since $(\dot{\eta}/\dot{\xi})^2 \ll 1$, equation (6.4) may further be reduced to equation (5.8), which we have described as the Hamiltonian equation for longitudinal motion. The condition (6.5) can certainly be satisfied in the neighborhood of $|\sin 2\pi\Delta| = 1$, provided that η and $\dot{\eta}$ are small enough and the distance $(\xi - \xi_o)$ considered is not too great. It is interesting to note that while the differential equations (4.9a) and (4.10b) are not valid for all phase angles, the approximate solution (6.4) can be used in general.

Instead of (6.5) we may state the condition as

$$\pi\alpha(\xi - \xi_o)\cdot\left|\eta \frac{d\eta}{d\xi}\frac{1-\beta^2}{\beta^2}\right|_{max.} \ll \left|\frac{\gamma}{\beta} - \gamma\dot{\xi}\right|_{min.} , \qquad (6.6)$$

which is independent of the phase angle. And if β = constant < 1, (6.6) may be written as

$$\pi\alpha\frac{1+\beta}{\beta}(\xi - \xi_o)\cdot\left|\frac{1}{\gamma}\eta\frac{d\eta}{d\xi}\right|_{max.} \ll 1 .$$

Assuming some specific values, e.g., $\alpha = 8$, $\beta = .5$, $\gamma_{min.} = 1$, $\eta_{max.} = .05$ and $|d\eta/d\xi|_{max.} = .01$, we may verify that the relation (6.4) or (5.8) is valid for at least a few wavelengths of distance.

(ii) Transformation of the Equation of Transverse Motion with Constant Parameters to the Generalized Lamé Equation

Now we proceed to discuss the transverse motion with constant parameters. As in the corresponding case of longitudinal motion it is most convenient to use Δ as the independent variable. By changing variable from τ to Δ we can easily show that

$$(1 + \dot{\eta}\,\frac{d\eta}{d\Delta})\,\frac{d}{d\tau}(\gamma\dot{\eta}) = \gamma(\frac{\dot{\xi}}{\beta} - 1)^2\,\frac{d^2\eta}{d\Delta^2}$$

$$- (\dot{\xi} - \frac{1}{\beta})\,\frac{d}{d\tau}\,(\gamma\dot{\xi})\,\frac{d\eta}{d\Delta}\,.$$

Substituting the values of $\frac{d}{d\tau}(\gamma\dot{\xi})$ and $\frac{d}{d\tau}(\gamma\dot{\eta})$ from equations (6.1) and (4.9b) into the above equation we obtain

$$\gamma(\frac{\dot{\xi}}{\beta} - 1)^2\,\frac{d^2\eta}{d\Delta^2} - (\dot{\xi} - \frac{1}{\beta})\,\alpha\cos 2\pi\Delta\,\frac{d\eta}{d\Delta}$$

$$+ (\dot{\xi} - \frac{1}{\beta})\,\pi\alpha\eta\sin 2\pi\Delta = 0 \qquad (6.7)$$

By equation (4) and

$$\gamma^2(\frac{\dot{\xi}}{\beta} - 1)^2 = \gamma^2(\dot{\xi} - \frac{1}{\beta})^2 + \gamma^2(1 - \dot{\xi}^2)(1 - \frac{1}{\beta^2})$$

equation (6.7) may be reduced to

$$\frac{(A + \sin 2\pi\Delta)^2 - (\frac{2\pi}{\alpha})^2\frac{1 - \beta^2}{\beta^2}}{A + \sin 2\pi\Delta}\,\frac{d^2\eta}{d\Delta^2} + 2\pi\cos 2\pi\Delta\,\frac{d\eta}{d\Delta}$$

$$- 2\pi^2\eta\sin 2\pi\Delta = 0 \qquad (6.8)$$

under the following restrictions

$$\dot{\eta}^2 \ll \dot{\xi}^2 \,, \quad \dot{\eta}^2 \ll 1 - \dot{\xi}^2 \quad . \tag{6.9}$$

The first derivative term in (6.8) may be eliminated by the substitution

$$y = \eta \cdot \left[(A + \sin 2\pi\Delta)^2 - \left(\frac{2\pi}{\alpha}\right)^2 \frac{1 - \beta^2}{\beta^2} \right]^{1/4} \quad , \tag{6.10}$$

thus we get

$$\frac{d^2y}{d\Delta^2} + \frac{1}{4}(2\pi \cos 2\pi\Delta)^2 \cdot$$

$$\cdot \frac{(A + \sin 2\pi\Delta)^2 + 2\left(\frac{2\pi}{\alpha}\right)^2 \frac{1 - \beta^2}{\beta^2}}{\left[(A + \sin 2\pi\Delta)^2 - \left(\frac{2\pi}{\alpha}\right)^2 \frac{1 - \beta^2}{\beta^2} \right]^2} \cdot y = 0 \quad , \tag{6.11}$$

which is a second order linear differential equation with periodic coefficient and constant Wronskian.

On the other hand, if we adopt the notations defined in equations (5.10) and transform equation (6.8) by changing the independent variable from Δ to x we obtain the following canonical form:

$$\frac{d^2\eta}{dx^2} + \left[\frac{1/2}{x + x_s} + \frac{1/2}{x - x_s} + \frac{1/2}{x - A + 1} + \frac{1/2}{x - A - 1} \right] \frac{d\eta}{dx}$$

$$+ \frac{1}{2} \frac{x(x - A)}{(x^2 - x_s^2) \cdot [(x - A)^2 - 1]} \eta = 0 \tag{6.12}$$

$$\left(x = A + \sin 2\pi\Delta, \quad x_s = \frac{2\pi}{\alpha} \cdot \frac{\sqrt{1 - \beta^2}}{\beta} \right) .$$

This is a generalized form of the Lamé equation[39] usually denoted by the scheme [4, 1, 0]. It has four elementary singularities with exponent difference 1/2 and one regular singularity at infinity with the exponent difference being imaginary. As one regular singularity can be generated by coalescing two elementary singularities, this equation is equivalent in complexity to an equation [6, 0, 0] with six elementary singularities.

The mathematical problem at hand is much more complicated than that of longitudinal motion. We have to impose further simplifying conditions in order to get useful solutions. We will discuss the two cases $\beta = 1$ and $\beta < 1$ separately.

(iii) α = constant, $\beta = 1$

For $\beta = 1$, the equations (6.8), (6.11) and (6.12) are reduced to

$$(A + \sin 2\pi\Delta)\frac{d^2\eta}{d\Delta^2} + 2\pi\cos 2\pi\Delta\frac{d\eta}{d\Delta} - 2\pi^2\eta\sin 2\pi\Delta = 0 \,, \tag{6.13}$$

$$\left.\begin{aligned} &\frac{d^2y}{d\Delta^2} + \left(\frac{\pi\cos 2\pi\Delta}{A + \sin 2\pi\Delta}\right)^2 y = 0 \\ &y = \eta(A + \sin 2\pi\Delta)^{1/2} \end{aligned}\right\} \tag{6.14}$$

and

$$\left.\begin{aligned} &\frac{d^2\eta}{dx^2} + \left[\frac{1/2}{x-A+1} + \frac{1}{x} + \frac{1/2}{x-A-1}\right]\frac{d\eta}{dx} + \frac{1}{2x}\,\frac{x - A}{(x-A)^2 - 1}\,\eta = 0 \\ &\qquad x = A + \sin 2\pi\Delta \end{aligned}\right\} \tag{6.15}$$

39. E.L. Ince., loc. cit. footnote 31, p. 502.

respectively. All these equations are exact because they are independent of the restrictive conditions (6.5) and (6.9). They look rather simple but still are not soluble.

We want to restrict ourselves to a small region of Δ around some fixed value so that simple approximate solutions may be obtained. We will use equation (6.13) because we find it convenient here to deal with η directly.

Of most importance is the relativistic case for which $\gamma \gg 1$ and $2\pi\Delta \to 2\pi\Delta_\infty = -\sin^{-1}A$. We denote $2\pi\Delta = 2\pi\Delta_\infty + 2\pi\epsilon$ with $2\pi\epsilon \ll 1$, and transform equation (6.13) by changing variable from Δ to ϵ and neglecting small terms $\epsilon^n \frac{d\eta}{d\epsilon}$ and $\epsilon^n\eta$ $(n \geq 2)$. Thus we obtain

$$\epsilon \frac{d^2\eta}{d\epsilon^2} + (1 - \pi\epsilon \tan 2\pi\Delta_\infty) \frac{d\eta}{d\epsilon} - \pi\eta\left[\tan 2\pi\Delta_\infty + \pi\epsilon(2 + \tan^2 2\pi\Delta_\infty)\right] = 0 \qquad (6.16)$$

The solution of this equation may be expressed in terms of the confluent hypergeometric function[40] and, if ϵ is sufficiently small, may be approximated by a simple series, i.e.,

$$\eta_I \sim 1 + \pi\epsilon \tan 2\pi\Delta_\infty + \ldots$$

As the indicial equation of (6.16) around $\epsilon = 0$ has two identical roots equal to zero, the other linearly independent solution can easily be found to be

$$\eta_{II} \sim \log\epsilon\,(1 + \pi\epsilon \tan 2\pi\Delta_\infty + \ldots) - \pi\epsilon \tan 2\pi\Delta_\infty - \ldots$$

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40. E. Kamke, Differentialgleichungen Methoden und Losungen, Band I (Chelsea Publishing Co., New York, 1948) p.475.

Thus the general solution is

$$\eta = C(1 + \pi\epsilon \tan 2\pi\Delta_{\infty} + \ldots)$$

$$+ D\,(1 + \pi\epsilon \tan 2\pi\Delta_{\infty} + \ldots) \log\epsilon - \pi\epsilon \tan 2\pi\Delta_{\infty} - \ldots \tag{6.17}$$

C and D may be expressed in terms of the initial values η_1 and $\left(\frac{d\eta}{d\xi}\right)_1$ at $\epsilon = \epsilon_1$, so we obtain

$$\eta \cong \eta_1 \frac{1 + \pi\epsilon \tan 2\pi\Delta_{\infty}}{1 + \pi\epsilon_1 \tan 2\pi\Delta_{\infty}}$$

$$+ \left[\eta_1 \frac{\pi\epsilon_1 \tan 2\pi\Delta_{\infty}}{1 + \pi\epsilon_1 \tan 2\pi\Delta_{\infty}} + \frac{\gamma_1 \dot{\xi}_1}{\gamma_1 (1 - \dot{\xi}_1)} \epsilon_1 \left(\frac{d\eta}{d\xi}\right)_1\right] \log \frac{\epsilon_1}{\epsilon} . \tag{6.18}$$

Since $\log \frac{\epsilon_1}{\epsilon} \cong \log \frac{\gamma}{\gamma_1} \cong \log \frac{\xi}{\xi_1}$, the increase of η is very slow, being proportional to the logarithm of the energy or distance. If we neglect the defocusing force altogether the solution would be

$$\eta = \eta_1 + \frac{\gamma_1 \dot{\xi}_1}{\gamma_1 (1 - \dot{\xi}_1)} \epsilon_1 \left(\frac{d\eta}{d\xi}\right)_1 \log \frac{\epsilon_1}{\epsilon} + O(\epsilon_1 - \epsilon).$$

The major effect of the defocusing force is represented by the other logarithmic term in equation (6.18), which is proportional to η_1 instead of $\left(\frac{d\eta}{d\xi}\right)_1$. Other characteristic properties of the solution can better be explained by referring to simple examples:

(a) If $2\pi\Delta_{\infty} = 0$,

$$\eta - \eta_1 \cong \frac{\gamma_1}{\alpha} \left(\frac{d\eta}{d\xi}\right)_1 \log \frac{\gamma}{\gamma_1} . \tag{6.18a}$$

η will remain constant if $(\frac{d\eta}{d\xi})_1 = 0$. η may either increase or decrease according as $(\frac{d\eta}{d\xi})_1 > 0$ or < 0.

For a given allowable spread of the beam and a certain final energy the maximum allowable value of the initial slope $(\frac{d\eta}{d\xi})_1$ is directly proportional to α and inversely to γ_1. If η_h and $(\frac{d\eta}{d\xi})_1$ are small enough, the spread of the beam will not be excessive even though the final energy reaches several billion volts. To illustrate, we take $\alpha = 8$, $\gamma_1 = 10$ ($\cong 5$ Mev.), $\gamma = 10^4$, $\eta_1 = .02$ and $(\frac{d\eta}{d\xi})_1 = .01$. We find $\eta \cong .106$, being smaller than the disk hole radius of the Stanford billion volt accelerator.

(b) If $(\frac{d\eta}{d\xi})_1 = 0$ but $\eta_h \neq 0$, then

$$\eta - \eta_h \cong \frac{\eta_h}{2\gamma_1} \frac{\pi \tan 2\pi\Delta_\infty}{\alpha \cos 2\pi\Delta_\infty} \log \frac{\gamma}{\gamma_1} \quad . \tag{6.18b}$$

η increases in the defocusing region ($\tan 2\pi\Delta_\infty > 0$) and decreases in the focusing region ($\tan 2\pi\Delta_\infty < 0$). Comparing two electrons with the same η_h the change of η is inversely proportional to γ_1 and directly to $|\sec 2\pi\Delta_\infty \cdot \tan 2\pi\Delta_\infty|$. The logarithmic increase of η arising from η_h has quite different relations to the various parameters compared to that arising from $(\frac{d\eta}{d\xi})_1$.

(c) If $\eta_h = 0$ but $(\frac{d\eta}{d\xi})_1 \neq 0$, then

$$\eta \cong \frac{\gamma_1 (\frac{d\eta}{d\xi})_1}{\alpha \cos(2\pi\Delta_\infty + \pi\epsilon_1)} \log \frac{\gamma}{\gamma_1} \tag{6.18c}$$

η increases in the same general manner in both the focusing and defocusing regions. The difference between the two cases is only slight; it is interesting to compare them carefully.

Let us consider two electrons, one in the defocusing region with $2\pi\Delta_1^+ = \theta$ and the other in the focusing region with $2\pi\Delta_1^- = -\theta$. They are to have the same initial values γ_1 and $(\frac{d\eta}{d\xi})_1$ and also the same final value of γ. We find

$$\frac{\eta^+}{\eta^-} \cong \frac{\cos(2\pi\Delta_\infty^- + \pi\epsilon_1^-)}{\cos(2\pi\Delta_\infty^+ + \pi\epsilon_1^+)} = \frac{\cos(\theta + \pi\epsilon_1^-)}{\cos(\theta - \pi\epsilon_1^+)}$$

$$\cong 1 - (\pi\epsilon_1^- + \pi\epsilon_1^+)\cdot\tan\theta < 1 \quad .$$

η increases by a smaller amount in the defocusing region than in the focusing region. The reason for this seemingly erroneous result is to be found in the slight difference of the longitudinal field strength experienced by the two electrons. The one in the focusing region experiences a slightly weaker field, so takes a slightly longer distance for the same energy increase. The focusing effect is more than balanced out by the spreading caused by traveling the greater distance.

Again let us consider two different electrons, one with $2\pi\Delta_1^+ = \theta$ and $2\pi\Delta_\infty^+ = \theta - 2\pi\epsilon_1$ and the other with $2\pi\Delta_1^- = -2\pi\Delta_\infty^+ = -\theta + 2\pi\epsilon_1$ **and** $2\pi\Delta_\infty^- = -\theta$. They are to have the same $(\frac{d\eta}{d\xi})_1$ and the same final energy. Since the average field strength on the two electrons are equal, they will have to travel approximately the same distance for the same final γ. The ratio of η^+ to η^- is found to be

$$\frac{\eta^+}{\eta^-} \cong \frac{\gamma_1^+}{\gamma_1^-} \cong \frac{\cos(\theta - \pi\epsilon_1)}{\cos(\theta + \pi\epsilon_1)} = 1 + 2\pi\epsilon_1 \cdot \tan\theta \ .$$

$\eta^+ > \eta^-$ as is to be expected.

Now it is clear that when the electron has gained several million volts energy, the transverse instability and stability will both be negligible. The spread is logarithmic and is mainly determined by the term arising from the initial slope $(\frac{d\eta}{d\xi})_1$. The cathode structure, the injection system and the bunching section must all be properly designed and adjusted, so that the electron stream will enter the main accelerator tube with high enough energy and small enough radial velocity. Moreover, the main accelerator tube will have to be aligned with an accuracy better than the angle indicated by $(\frac{d\eta}{d\xi})_1$. For a given allowable spread of the beam, the maximum allowable value of $(\frac{d\eta}{d\xi})_1$ varies inversely as the logarithm of the total energy. Let us again refer to the numerical example given in connection with equation (6.18a). If $\eta = .106$ is the maximum allowable spread, the whole length of the accelerator must be aligned with an accuracy better than .01 radian. Suppose that the final energy is 5×10^{12}v. instead of 5×10^9v., then the alignment should be within .0033 radian. While the final energy or the accelerator length is increased 1,000 times, the required accuracy is only increased three times.

One simple non-relativistic case with $\beta = 1$ which may advisably be discussed here is for $2\pi\Delta = \pm\, \pi/2$. Denoting $2\pi\Delta = \pm(\pi/2) + 2\pi\epsilon$, $(2\pi\epsilon)^2 \ll 1$ we may simply write equation

(6.13) as

$$\pm \left(\frac{A \pm 1}{4\pi^2}\right) \frac{d^2\eta}{d\epsilon^2} - \epsilon \frac{d\eta}{d\epsilon} - \frac{1}{2}\eta = 0 \tag{6.19}$$

The solution can at once be found; it is

$$\eta = C\left[1 \pm \frac{\pi^2 \epsilon^2}{A \pm 1} + O(\epsilon^4)\right] + D\left[\epsilon + O(\epsilon^3)\right] .$$

By evaluating the integration constants C and D in terms of the initial values we obtain

$$\eta \cong \eta_1 \left[1 \pm \frac{\pi^2}{A \pm 1} (\epsilon_1 - \epsilon)^2\right] + \left(\frac{d\eta}{d\xi}\right)_1 \frac{\gamma_1 \dot{\xi}_1}{\gamma_1 (1 - \dot{\xi}_1)} (\epsilon_1 - \epsilon) . \tag{6.20}$$

Remembering that $\epsilon_1 - \epsilon > 0$ and $A \pm 1 = \frac{2\pi}{\alpha}\left(\gamma - \sqrt{\gamma^2 - 1}\right) > 0$, we see clearly from (6.20) that if $\left(\frac{d\eta}{d\xi}\right)_1 = 0$, η increases around $2\pi\Delta \cong \pi/2$ and decreases around $2\pi\Delta \cong -\pi/2$, and if $\eta_1 = 0$, η increases or decreases in both regions according as $\left(\frac{d\eta}{d\xi}\right)_1 > 0$ or < 0. Since $\xi - \xi_1 = O(\epsilon_1 - \epsilon)$ for non-relativistic velocities, the beam spreading arising from η_1 is proportional to the square of the distance while that arising from $\left(\frac{d\eta}{d\xi}\right)_1$ is proportional directly to the distance. Though the distance considered is necessarily small on account of the restriction imposed on ϵ, the spreading per unit distance is great. Such results are indeed to be expected because at $2\pi\Delta \cong \pm\pi/2$ the longitudinal electric field is too small to have appreciable effect on the moving electron while the transverse electric and magnetic fields are at about their maximum values.

(iv) α = constant, β = constant $<$ 1.
Heun's Equation and the Jacobian Form of the Generalized Lamé Equation

Now we consider the case for β = constant $<$ 1. We shall only consider those electrons which have bound phase orbits; those with progressive or retrograde orbits are not as important and present quite formidable analytical difficulties. Thus we are to take

$$A - 1 < x_s = \frac{2\pi}{\alpha}\frac{\sqrt{1-\beta^2}}{\beta} < A + 1$$

in the following discussion on equations (6.8) and (6.12).

By similar algebraics as for $\beta = 1$ we may approximately reduce the equation (6.8) to

$$\left.\begin{aligned} &\epsilon\frac{d^2\eta}{d\epsilon^2} + \frac{1}{2}(1 - A_1\pi\epsilon)\frac{d\eta}{d\epsilon} - \frac{1}{2}\pi\eta(\tan 2\pi\Delta_s + A_2\pi\epsilon) = 0 \\ &\text{with} \\ &2\pi\Delta = 2\pi\Delta_s + 2\pi\epsilon \\ &x_s = A + \sin 2\pi\Delta_s \\ &A_1 = \tan 2\pi\Delta_s - \frac{\cos 2\pi\Delta_s}{x_s} \\ &A_2 = 2 + \tan^2 2\pi\Delta_s + \frac{\sin 2\pi\Delta_s}{x_s} \end{aligned}\right\} \quad (6.21)$$

under the conditions that $|2\pi\epsilon| \ll 1$, $|2\pi\epsilon \tan 2\pi\Delta_s| \ll 1$ and $(\sin 2\pi\Delta - \sin 2\pi\Delta_s)^{1/2} \ll x_s^{1/2}$. The solution is

$$\eta = C(1 + g\epsilon) + D\sqrt{\epsilon} + O(\epsilon^{3/2})$$

where $g = \pi \tan 2\pi\Delta_s$, C and D are integration constants. Referring to equation (5.9) we easily find

$$\frac{d\epsilon}{d\xi} \cong \pm \frac{1 - \beta^2}{\beta} h\sqrt{\epsilon}\ (1 - h\sqrt{\epsilon})$$

where $h = \frac{1}{\beta}\left(\frac{2}{x_s} 2\pi \cos 2\pi\Delta_s\right)^{1/2}$. Thus

$$\frac{d\eta}{d\xi} \cong \pm \frac{1 - \beta^2}{\beta} h(1 - h\sqrt{\epsilon})\left(\frac{D}{2} + Cg\sqrt{\epsilon}\right) \quad .$$

The solution may be expressed in terms of the initial values as

$$\eta \cong \eta_1 \left[1 + g(\sqrt{\epsilon} - \sqrt{\epsilon_1})^2\right]$$

$$\pm \left(\frac{d\eta}{d\xi}\right)_1 \frac{2\beta}{1 - \beta^2} \frac{1 + h\sqrt{\epsilon_1}}{h} (\sqrt{\epsilon} - \sqrt{\epsilon_1}) \quad .$$

According to equation (5.12) we have

$$\epsilon \cong \frac{\cos 2\pi\Delta_s}{4\pi} \cdot \mathrm{sn}^2(K - u, \kappa) = \frac{\cos 2\pi\Delta_s}{4\pi} \mathrm{cd}^2(u, \kappa) \quad ,$$

where $K - u = F(\frac{\pi}{2}, \kappa) - F(\phi, \kappa)$

$$= \frac{2\pi}{\sqrt{x_s}} \frac{1 - \beta^2}{\beta} \left[\xi(\Delta) - \xi(\Delta_s)\right] \quad ,$$

$$\sin \phi = \mathrm{sn}\ u$$

and

$$K = \sqrt{\frac{A + 1 - x_s}{2}} \quad ;$$

hence we obtain

$$\eta \cong \eta_1 \left[1 + \frac{1}{4} \sin 2\pi\Delta_s(\mathrm{cd}\ u - \mathrm{cd}\ u_1)^2\right]$$

$$+ \left(\frac{d\eta}{d\xi}\right)_1 \frac{\beta}{1 - \beta^2} \frac{1}{2\pi}(\beta\sqrt{x_s} + |\mathrm{cd}\ u_1 \cdot \cos 2\pi\Delta_s|) \cdot |\mathrm{cd}\ u - \mathrm{cd}\ u_1| \quad . \tag{6.22}$$

The η_1-dependent term of the solution increases in the defocusing region and decreases in the focusing region while the $(\frac{d\eta}{d\xi})_1$-dependent term has its sign determined by the sign of $(\frac{d\eta}{d\xi})_1$ in both regions. Both terms contain rather complicated elliptic functions of the distance. These functions, however, may usually be approximated by circular functions if $\kappa^2 < 1/2$. Thus, if $\frac{1}{4}(\frac{2}{\pi}K - 1) \ll 1$, we have[41]

$$\mathrm{sn}(K - u) \cong \frac{2\pi}{K} \frac{1}{5 - 2K/\pi} \sin \frac{\pi}{2K}(K - u)$$

and $$\mathrm{cd}\, u - \mathrm{cd}\, u_1 \cong \frac{2\pi}{K} \frac{1}{5 - 2K/\pi}\, 2\left[\sin \ell(\xi - \xi_s) - \sin \ell(\xi_1 - \xi_s)\right]$$

with $$\ell = \frac{\pi}{2K} \frac{2\pi}{\sqrt{x_s}} \frac{1 - \beta^2}{\beta^2} \quad .$$

In comparison with the logarithmic solution for $\beta = 1$, the beam spreading per unit distance here is very much greater.

For $2\pi\Delta \cong \pi/2$ the above solution is not valid. In this small region equation (6.8) may be reduced to

$$\frac{A' + 1}{4\pi^2} \frac{d^2\eta}{d\epsilon^2} - \epsilon \frac{d\eta}{d\epsilon} - \frac{1}{2}\eta = 0 \qquad (6.23)$$

with $2\pi\Delta = \pi/2 + 2\pi\epsilon$, $|2\pi\epsilon| \ll 1$, $(2\pi\epsilon)^2 \ll \frac{(A+1)^2 - x_s^2}{A + 1}$ and $A' = A - \frac{x_s^2}{A + 1}$. This has exactly the same form as equation (6.19) and $\frac{d\epsilon}{d\xi} = \frac{1}{\beta} - \frac{1}{\xi}$ is also approximately constant, so the solution will have the same general nature as that for the corresponding case of $\beta = 1$ and therefore needs no further

41. E.T.Whittaker and G.N.Watson, loc.cit., footnote 6, p.510; Janke and Emde, Funktionentafeln (Dover Publications, New York, 1945) p.74.

discussion.

Having discussed the two extreme cases $2\pi\Delta \cong 2\pi\Delta_s$ and $2\pi\Delta \cong \pi/2$, we now proceed to find the transverse part of the solution for small phase oscillations. These latter cases may most expediently be discussed under a single restriction, i.e.

$$\frac{x - x_s}{x - A + 1} \ll 1 .$$

We first transform equation (6.12) into

$$\frac{d^2y}{dx^2} + \left[\frac{1/2}{x-A+1} + \frac{1/2}{x - x_s} + \frac{1/2}{x-A-1}\right]\frac{dy}{dx}$$

$$+ \frac{1}{8}\left[\frac{1}{x-A+1}\,\frac{1}{x - x_s} + \frac{1}{x - x_s}\,\frac{1}{x-A-1} - \frac{1}{x - x_s}\,\frac{1}{x + x_s}\right.$$

$$\left. + \frac{3}{2}\,\frac{1}{(x + x_s)^2}\right] y = 0 \qquad (6.24)$$

by means of the substitution

$$y = \eta(x + x_s)^{1/4} \quad , \qquad (6.24a)$$

and then to

$$\frac{d^2y}{dz^2} + \left[\frac{1/2}{z} + \frac{1/2}{z - 1/\kappa^2} + \frac{1/2}{z - 1}\right]\frac{dy}{dz} + \frac{1}{8}\,\frac{1}{\kappa^2 z}\,\frac{1}{(z - 1/\kappa^2)^2}\,\cdot$$

$$\cdot\left[1 + \frac{(1/\kappa^2) - 1}{z - 1} - \frac{(1/\kappa^2) - 1}{[(1/\kappa^2) - 1]\kappa^2 z - x_s(z - 1/\kappa^2)}\right.$$

$$\left. + \frac{3}{2}\,\frac{\kappa^2 z\left[(1/\kappa^2) - 1\right]^2}{\left\{[(1/\kappa^2) - 1]\kappa^2 z - x_s(z - 1/\kappa^2)\right\}^2}\right] y = 0 \qquad (6.25)$$

with $$z = \frac{2}{A + 1 - x_s} \frac{x - x_s}{x - A + 1} = \frac{1}{\kappa^2} \frac{x - x_s}{x - A + 1} \quad . \tag{6.25a}$$

Noting that

$$\kappa^2 z \ll 1 \tag{6.25b}$$

we may develop the coefficient of the y-term of equation (6.25) into a power series in $\kappa^2 z$. Thus (6.25) may be written as

$$\frac{d^2y}{dz^2} + \left[\frac{1/2}{z} + \frac{1/2}{z - 1/\kappa^2} + \frac{1/2}{z - 1}\right] \frac{dy}{dx}$$

$$+ \frac{1}{4} \frac{1}{z} \frac{1}{z - 1/\kappa^2} \frac{1}{z - 1} \frac{1}{\kappa^2} \left(A_o + A_1 \kappa^2 z + A_2 \kappa^4 z^2 + \ldots\right) y = 0 \tag{6.26}$$

where

$$A_o = -\frac{1}{2} + \kappa^2 (1 - \frac{1}{2x_s}) + \kappa^4 \frac{1}{2x_s}$$

$$A_1 = -(1 - \frac{1}{2x_s}) + \kappa^2 (1 - \frac{3}{2x_s} + \frac{5}{4x_s^2})$$

$$A_2 = -1 + \frac{1}{x_s} - \frac{5}{4x_s^2} \quad .$$

Transforming (6.26) again by substituting

$$z = \mathrm{sn}^2 (v, \kappa) \tag{6.27a}$$

i.e.,

$$v = \mathrm{sn}^{-1} (\sqrt{z}, \kappa) = \mathrm{sn}^{-1} \left(\sqrt{\frac{1}{\kappa^2} \frac{x - x_s}{x - A + 1}}, \kappa\right)$$

$$= \frac{2\pi}{\sqrt{x_s}} \frac{1 - \beta^2}{\beta^2} (\xi - \xi_s) \; , \tag{6.27b}$$

we finally obtain

$$\frac{d^2y}{dv^2} + \left[A_o + A_1 \kappa^2 \mathrm{sn}^2 v + A_2 \kappa^4 \mathrm{sn}^4 v + \ldots\right] y = 0 \tag{6.27}$$

For the limiting case of very small phase oscillations, $\kappa \to 0$, we may use the zero-th order approximation, i.e.

$$\frac{d^2y}{dv^2} - \frac{1}{2}y \cong 0 \quad . \tag{6.28}$$

So $\quad y \cong Ce^{v/\sqrt{2}} + De^{-v/\sqrt{2}}$ $\quad$; thus

$$\left.\begin{aligned} &\eta \cong (2x_s)^{-1/4}\cdot\left[Ce^{\xi/\Xi} + De^{-\xi/\Xi}\right] \\ &\text{where} \quad \frac{1}{\Xi} = \frac{1}{\sqrt{2}}\,\frac{2\pi}{\sqrt{x_s}}\,\frac{1-\beta^2}{\beta^2} \quad . \end{aligned}\right\} \tag{6.29}$$

This is the worst case of beam spreading, η increasing exponentially with ξ. Ξ is the distance required for a given value of η to increase in the ratio e; Ξ is equal to $\sqrt{2}/2\pi$ times the period in ξ of the phase oscillation which we found in the last Chapter as being equal to $\sqrt{x_s}\cdot\frac{\beta^2}{1-\beta^2}$.

For phase oscillations with moderate amplitudes it would be sufficient to take the following approximation:

$$\frac{d^2y}{dv^2} + (A_o + A_1\kappa^2 sn^2 v)\, y = 0 \quad . \tag{6.30}$$

This is a generalized Lamé equation written in the Jacobian form,[42] while the corresponding algebraic equation obtained by putting A_2 ... equal to zero in equation (6.26) is known as Heun's equation.[43] The solution may be found straightforwardly by the series method. But since the recursion

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42. E.T. Whittaker and G.N.Watson, loc. cit., footnote 6, p. 570.

43. E.Kamke, loc. cit. footnote 40, pp. 485-487.

formula has three terms, the algebra is much more involved. We will go no further into this here, because the results must lie somewhere amid the three extreme cases discussed above.

(v) α = constant, $\dfrac{1}{\sqrt{1-\beta^2}} = \alpha'\xi$

As the last analytical problem of the transverse motion we consider the case of the so-called synchronized operation, α = constant, β variable, $\dfrac{1}{\sqrt{1-\beta^2}} = \alpha'\xi$ as in Section 5.3. The equation (4.10d) may be written as

$$\frac{\dot{\xi}^2}{\sqrt{1-\dot{\xi}^2}}\frac{d^2\eta}{d\xi^2} + \alpha \cos 2\pi\Delta \frac{d\eta}{d\xi} - \frac{\pi\alpha\eta}{\beta}(1-\beta\dot{\xi}) \sin 2\pi\Delta = 0 \quad .$$

Under the same restrictions and by the same analysis as in deriving the equation (5.16) we transform the above equation into

$$\frac{d^2\eta}{d\xi^2} + \frac{\sqrt{1-\beta^2}}{\beta^2}\alpha \cos 2\pi\Delta \frac{d\eta}{d\xi} - \pi\alpha\eta\left(\frac{\sqrt{1-\beta^2}}{\beta}\right)^3 \sin 2\pi\Delta = 0 \ .$$

Since $2\pi\Delta \cong 2\pi\Delta_c$ = constant and β may be expressed in terms of ξ, so

$$\frac{d^2\eta}{d\xi^2} + \alpha' \frac{\alpha'\xi}{\alpha'^2\xi^2 - 1}\frac{d\eta}{d\xi} - \frac{\alpha'^2}{2}\frac{a}{(\alpha'^2\xi^2 - 1)^{3/2}}\eta = 0 \qquad (6.31)$$

where $a = \dfrac{2\pi}{\alpha'} \tan 2\pi\Delta_c$ as defined before. By changing the independent variable from ξ to ξ' with

$$\alpha'\xi = \cosh \alpha'\xi'$$

we obtain

$$\frac{d^2\eta}{d\xi'^2} - \frac{\alpha'^2}{2} (a \operatorname{csch} \alpha'\xi')\, \eta = 0 \,. \tag{6.32}$$

Before proceeding to solve this equation one simplest case may advisably be considered here, and this is for $2\pi\Delta_c = 0$, which means that the equilibrium phase is on the peak of the traveling E_z field. For this case the second term of the above equation vanishes, so we simply have

$$\frac{d^2\eta}{d\xi'^2} = 0$$

Thus $\eta = C\xi' + D$ or $\eta = C \cosh \alpha'\xi + D$, i.e.,

$$\eta = \eta_1 + \frac{1}{\alpha'}\sqrt{\alpha'^2\xi_1^2 - 1}\left(\frac{d\eta}{d\xi}\right)_1 \left[\cosh^{-1}\alpha'\xi - \cosh^{-1}\alpha'\xi_1\right]$$

$$= \eta_1 + \frac{1}{\alpha'}\sqrt{\alpha'^2\xi_1^2 - 1}\left(\frac{d\eta}{d\xi}\right)_1 \log \frac{\alpha'\xi + \sqrt{\alpha'^2\xi^2 - 1}}{\alpha'\xi_1 + \sqrt{\alpha'^2\xi_1^2 - 1}}$$

$$= \eta_1 + \frac{1}{2\alpha'}\frac{\beta_1}{\sqrt{1 - \beta_1^2}}\left(\frac{d\eta}{d\xi}\right)_1 \log \frac{1 - \beta_1}{1 + \beta_1}\,\frac{1 + \beta}{1 - \beta} \,. \tag{6.33}$$

The equation (6.32) can easily be solved by the WKB method. The result is

$$\left.\begin{aligned} &\eta \cong (\sinh \alpha'\xi')^{1/4} \cdot \left[C e^{\sqrt{\frac{a}{2}}\, F(\phi, \frac{1}{\sqrt{2}})} + D e^{-\sqrt{\frac{a}{2}}\, F(\phi, \frac{1}{\sqrt{2}})} \right] \\ &\text{with} \\ &\cos \phi = \left| \frac{\operatorname{csch} \alpha'\xi' - 1}{\operatorname{csch} \alpha'\xi' + 1} \right| \end{aligned}\right\} \tag{6.34}$$

This is valid as long as csch $\alpha'\xi'$ is a slowly varying function of ξ', so we should demand here

$$\left|\frac{d}{d\xi'}(\text{csch } \alpha'\xi')\right| \ll \text{csch } \alpha'\xi' \ ,$$

i.e.,

$$\alpha' \coth \alpha'\xi' = \alpha'/\beta \ll 1 \ .$$

This implies that β should vary slowly.

As $\beta \to 1$, $\phi \to 0$; so $F(\phi, 1/\sqrt{2}) \to 0$ and

$$\eta \sim (\sinh \alpha'\xi')^{1/4} = \left(\frac{\beta}{\sqrt{1-\beta^2}}\right)^{1/4} = \gamma^{1/4} \ .$$

η increases as $\gamma^{1/4}$ instead of $\log \gamma$ as for the case of $\beta = 1$; the difference is slight within the billion-volt range and it arises from the fact that here Δ stays constant at Δ_c instead of decreasing and approaching Δ_∞ asymptotically.

On account of the slow varying property of csch $\alpha'\xi'$ we may, for short distances where β is far different from unity, neglect the variation of $(\sinh \alpha'\xi')^{1/4}$ in comparison with that of the exponential factor. Then

$$\eta_2/\eta_1 \sim e^{\sqrt{\frac{a}{2}}\cdot\left|F(\phi_2, 1/\sqrt{2}) - F(\phi_1, 1/\sqrt{2})\right|} \ .$$

It can easily be shown that the latter expression may be written as a simple function of ξ, namely

$$e^{\sqrt{\frac{a}{2}}\cdot\left|F(\phi_2, 1/\sqrt{2}) - F(\phi_1, 1/\sqrt{2})\right|} = e^{(\xi_2 - \xi_1)/\Xi} \ ,$$

where $$\frac{1}{\Xi} = \sqrt{\pi\alpha \sin 2\pi\Delta_c}\left(\frac{\sqrt{1-\beta^2}}{\beta}\right)^{3/2}_{\text{ave.}}$$

This reduces to the expression of (6.29)for the limiting case β = constant.

In case α' is not small or β does not vary slowly, the WKB solution is usually unreliable and needs to be replaced by another one. Let us write equation (6.32) as

$$\frac{1}{\alpha'^2}\,\frac{1 - e^{-2\alpha'\xi'}}{e^{-\alpha'\xi'}}\,\frac{d^2\eta}{d\xi^2} - a\eta = 0$$

and transform it once more by changing the independent variable with $w = e^{-\alpha'\xi'} = \cosh\alpha'\xi' - \sinh\alpha'\xi'$. The resulting equation is

$$w(w^2 - 1)\,\frac{d^2\eta}{dw^2} + (w^2 - 1)\,\frac{d\eta}{dw} + a\eta = 0,$$

which is again Heun's equation and as mentioned before, can only be solved with much involved algebra. Instead of going into this, we shall be content to conclude the analytical discussion of the transverse motion by considering the following approximate equation:

$$\frac{d^2\eta}{d\xi'^2} - \alpha'^2 a e^{-\alpha'\xi'}\cdot\eta = 0,$$

i.e.,

$$w\,\frac{d^2\eta}{dw^2} + \frac{d\eta}{dw} - a\eta = 0 \qquad (6.35)$$

which is valid for larger values of β such that $e^{-2\alpha'\xi'} \ll 1$. Since (6.35) is a simple Bessel equation, we may write down the solutuon directly, i.e.

$$\eta = C\,I_o(2\sqrt{aw}) + DK_o(2\sqrt{aw})\ ,$$

I_o and K_o being the modified Bessel functions. By changing w back to ξ or β the expression becomes

$$\eta = CI_o\left[2\sqrt{a}(\alpha'\xi - \sqrt{\alpha'^2\xi^2 - 1})^{1/2}\right]$$

$$+ DK_o\left[2\sqrt{a}(\alpha'\xi - \sqrt{\alpha'^2\xi^2 - 1})^{1/2}\right]$$

$$= CI_o\left[2\sqrt{a}\left(\frac{\sqrt{1 - \beta^2}}{1 + \beta}\right)^{1/2}\right] + DK_o\left[2\sqrt{a}\left(\frac{\sqrt{1 - \beta^2}}{1 + \beta}\right)^{1/2}\right] . \qquad (6.36)$$

As $w \to 0$, $\eta \to D \log \frac{1}{1.781} \left(\frac{1}{a} \frac{1 + \beta}{\sqrt{1 - \beta^2}}\right)^{1/2} \to$

$\to \frac{D}{2} \log \frac{1}{\sqrt{1 - \beta^2}} \sim \log \gamma$. The approximation is probably too crude to tell the difference between the variable case $\beta \to 1$ and the constant case $\beta = 1$.

6.3. Differential Analyzer Solutions for Two Specific Cases: $\beta = \beta_s$ and $\beta = \beta_+$

The electron orbit problem for two possible designs for the billion-volt accelerator was studied on the differential analyzer at U.C.L.A. The two designs differ only in the function of β used for the starting or bunching section. One uses a synchronous type of buncher with $\Delta_c \cong 0$, while the other uses a short buncher with $\beta > 1$. The latter case is very simple and yet possesses useful bunching characteristics, so it was studied in greater detail than the synchronous case. In this section we will discuss the numerical solutions obtained for these two cases. The longitudinal orbits will also be included here, because the problem of the transverse motion alone is not soluble.

The equations used on the differential analyzer have the following specific forms:

$$\gamma = \int \alpha \cos 2\pi\Delta \frac{\sqrt{\gamma^2 - 1}}{\gamma}\, d\tau = \int \alpha \cos 2\pi\Delta \, d\int \frac{\sqrt{\gamma^2 - 1}}{\gamma}\, d\tau \quad (6.37a)$$

$$\Delta = \int \left(\frac{1}{\beta} \frac{\sqrt{\gamma^2 - 1}}{\gamma} - 1\right) d\tau = \int \frac{1}{\beta}\, d\int \frac{\sqrt{\gamma^2 - 1}}{\gamma}\, d\tau - \tau \quad (6.37b)$$

$$\xi = \int \frac{\sqrt{\gamma^2 - 1}}{\gamma}\, d\tau \quad (6.37c)$$

$$\eta = \int \frac{P}{\gamma}\, d\tau = \int P d \int \frac{1}{\sqrt{\gamma}}\, d \int \frac{1}{\sqrt{\gamma}}\, d\tau \quad (6.37d)$$

$$P = \int \pi\alpha \left(\frac{1}{\beta} - \frac{\sqrt{\gamma^2 - 1}}{\gamma}\right) \eta \sin 2\pi\Delta \, d\tau$$

$$= \int \pi\alpha \left(\frac{1}{\beta} - \frac{\sqrt{\gamma^2 - 1}}{\gamma}\right) d\int \eta \, d\int \sin 2\pi\Delta \, d\tau. \quad (6.37e)$$

These are the same as equations (4.10a) to (4.10d). The new variable P represents the radial component of momentum, i.e.,

$$P = \sqrt{\gamma^2 - 1} \; \frac{d\eta}{d\xi} = \gamma \frac{d\eta}{d\tau} \quad (6.38)$$

As discussed before, the error introduced by the approximations in equations (4.10a) and (4.10c) is considered to be negligible.

α = constant = 7.958 for both cases. This value of α corresponds to an E_z field equal to 11.83 M.v. per foot.

The function of β for the synchronous case will be denoted by $\beta = \beta_s$ while that for the other case by $\beta = \beta_+$. both β_s and β_+ are shown in Fig. 6.1. The β_s function is obtained from the exact synchronous solution

$\frac{1}{\sqrt{1 - \beta^2}} = \alpha\xi + \frac{1}{\sqrt{1 - \beta_o^2}}$ by modifying β slightly so that β_s reaches the value unity smoothly in 1.5λ and $2\pi\Delta_c$ is slightly greater than zero for $\xi < 1.25$. The exact relation is also shown in this figure for comparison.

The initial value of γ is fixed, $\gamma_o = 2$. This value of γ_o corresponds to an injection energy of .512 Mev.

We complete the specification of the problem by assigning initial values to η, $d\eta/d\xi$ or P and Δ. $\xi_o = 0$ and $\tau_o = 0$ naturally.

Nine runs were made for the synchronous case with $\eta_o = .040$, $(\frac{d\eta}{d\xi})_o = 0$ and Δ_o taking various values. The results are summarized in three figures: Fig. 6.2 showing Δ vs. ξ, Fig. 6.3, γ vs. ξ and Fig. 6.4, η vs. ξ, all with Δ_o as parameters.

Fifteen runs were made for the case $\beta = \beta_+$ with $\eta_o = .040$, $(\frac{d\eta}{d\xi})_o = 0$ and Δ_o taking various values. The corresponding results are summarized in Figs. 6.5, 6.6 and 6.7. Only ten curves are shown in these figures; others are omitted in order to avoid over-crowding the figures.

Another twelve runs were made for $\beta = \beta_+$ with $\eta_o = 0$, $(\frac{d\eta}{d\xi})_o = .060$ and Δ_o taking various values. For given Δ_o the longitudinal part of the solution is independent of η_o and $(\frac{d\eta}{d\xi})_o$, so here the $\Delta - \xi$ and $\gamma - \xi$ curves are the same as shown in Figs. 6.5 and 6.6. The solutions for η are shown in Fig. 6.8.

Once we know the corresponding values of Δ and γ for certain ξ with $\beta = 1$, we may calculate Δ_∞ corresponding to $\gamma = \infty$ by equation (5.6). $\Delta_\infty - \Delta_o$ curves have already been shown in Figs. 5.5 and 5.7.

Since the equations of motion are linear, η is directly proportional to the initial values η_o and $(\frac{d\eta}{d\xi})_o$ and the two values of η, one for $(\frac{d\eta}{d\xi})_o = 0$ and the other for $\eta_o = 0$, can be linearly combined. From the curves shown in Figs. 6.7 and 6.8 we can calculate η for any set of initial values for $\beta = \beta_+$. The assigned values $\eta_o = .040$ and $(\frac{d\eta}{d\xi})_o = .060$ seem to be intolerably large, so we calculate η for $\eta_o = .010$, $(\frac{d\eta}{d\xi})_o = 0$ and $\eta_o = 0$, $(\frac{d\eta}{d\xi})_o = .020$. The values of η for $\gamma = 15$ are shown plotted versus Δ_o in Fig. 6.9.

From $\gamma = 15$ on, η is given with sufficient accuracy (see equations (6.18b) and (6.18c)) by the following equation:

$$\eta(\gamma) \cong \eta_1 + \left[\frac{\eta_1}{30} \frac{\pi \tan 2\pi\Delta_\infty}{\alpha \cos 2\pi\Delta_\infty} + \frac{P_1}{\alpha \cos(\pi\Delta_\infty + \pi\Delta_1)} \right] \cdot \log \frac{\gamma}{15}$$

where the subscript 1 stands for $\gamma = 15$. If $\tan 2\pi\Delta_\infty < 1$ the above equation may further be approximated by

$$\eta(\gamma) \cong \eta_1 + \frac{P_1}{\alpha \cos 2\pi\Delta_\infty} \log \frac{\gamma}{15} \quad .$$

Using these relations we calculate η for $\gamma = 1{,}500$ and plot $\eta(1500)$ vs. Δ_o in Fig. 6.10. $\gamma = 1500$ corresponds to a final energy of about 750 Mev.

If a certain maximum value is assumed for the output beam radius we may calculate the maximum allowable values for

η_o and $(\frac{d\eta}{d\xi})_o$. In Fig. 6.11 we show $\eta_{o\ max.}$ for $(\frac{d\eta}{d\xi})_o = 0$ and $(\frac{d\eta}{d\xi})_{o\ max.}$ for $\eta_o = 0$ as functions of Δ_o with the condition that $\eta(1500) \lesseqgtr .100$.

We see from Figs. 6.10 and 6.11 that in the region between $\Delta_o = .100$ to $\Delta_o = .325$ the beam spread is small and from Fig. 5.7 that in the greater half of the same region the gain in energy is quite near the maximum. This simple design of the starting section with $\beta = \beta_+$ indeed has good overall characteristics. The electron gun system must be properly designed to give an electron beam of good intensity with both η_o and P_o small enough to meet the requirements at rather unfavorable phase angles. Otherwise part of the electrons with acceptable energy will be lost through transverse defocusing.

In Fig. 6.12 we show $\eta(1500)$vs. Δ_o curves, illustrating four examples of initially converging beams, $\eta_o = .010$ for all cases, $(\frac{d\eta}{d\xi})_o = -.004, -.006, -.008$ and $-.010$ respectively. The results are obtained simply by linear combination of the values taken from the curves shown in Fig. 6.10. For all these cases the spreading of the beam with Δ_o between 0 and .375 is quite small, the maximum value of final η being about .022. This shows clearly that it is possible to obtain a sharply defined electron beam of very high energy by linear acceleration provided that the beam is initially small and well focused and the accelerator alignment is accurate enough.

6.4. Transverse Focusing

As pointed out before, it is both practicable and essential to use a subsidiary field for transverse focusing in the initial stage of acceleration, where the electron motion is most unstable. Also in the initial stage the longitudinal bunching has to be achieved. Stronger bunching requires stronger focusing. The bunching process has already been discussed in the last Chapter; we see there that both α and β may need to be varied and the variations may not be slow. We have also seen there how such varying parameters introduce great analytical difficulties into the mathematical problem. The present subject is still more complicated due to the presence of the focusing field. Here again, only numerical methods are feasible for obtaining quantitative results. But the amount of the required numerical labor far exceeds that for any of the previous cases and is not greatly reduced by assuming constant parameters. Therefore we shall be content in this section to discuss some of the general aspects of the focusing problem without completely solving it. Such discussion is made possible simply by the axial-symmetric property of the problem. To suit future applications under various conditions we will lay great stress on rigor and generality rather than on simplicity. To present-day calculating machines, simplicity is only of secondary importance.

(i) Hamiltonian and Equations of Motion

Instead of making piecemeal revisions for the equations of motion we find it expedient to re-formulate the problem by the general Hamiltonian method. Now the system consists of a charged particle, the accelerating field and the focusing field. However complicated these fields may be, they can always be expressed together in terms of a scalar potential V and a vector potential $\vec{A}$, connected by the relation $\operatorname{div} \vec{A} = -\frac{1}{c}\frac{\partial v}{\partial t}$. In cylindrical coordinates, the relativistic version of the Hamiltonian is as follows:

$$H = eV + \left\{ m_o^2 c^4 + c^2 \cdot \left[(p_r - \tfrac{e}{c}A_r)^2 + \tfrac{1}{r^2}(p_\phi - \tfrac{e}{c}rA_\phi)^2 + (p_z - \tfrac{e}{c}A_z)^2 \right] \right\}^{1/2} \qquad (6.39)$$

where

$$\left. \begin{aligned} p_r &= m\dot{r} + \tfrac{e}{c}A_r \\ p_\phi &= mr^2\dot{\phi} + \tfrac{e}{c}rA_\phi \\ p_z &= m\dot{z} + \tfrac{e}{c}A_z \end{aligned} \right\} \qquad (6.40)$$

$$(\dot{q} = dq/dt)$$

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44. The Hamiltonian (6.39) is referred to the laboratory system with coordinates $(r,\phi,z;t)$. If referred to a moving system of coordinates $(r',\phi',z';t)$ such that

$r' = r$, $\phi' = \phi$, $z' = z - \int^t \beta c\,dt$, β being a function of time only, the Hamiltonian will have a slightly different form, namely,

$$H = eV + \left\{ m_o^2 c^4 + c^2 \left[(p_r - \tfrac{e}{c}A_r)^2 + \tfrac{1}{r^2}(p_\phi - \tfrac{e}{c}rA_\phi)^2 + (p_{z'} - \tfrac{e}{c}A_{z'})^2 \right] \right\}^{1/2} - p_{z'} \cdot \beta c \quad .$$

are the so-called "generalized momenta". H is independent of ϕ. Both $V = V(r,z;\ t)$ and $\vec{A} = \vec{A}(r,z;\ t)$ satisfy the wave equation, namely

$$\Box V = 0$$

and

$$\Box \vec{A} = 0,$$

i.e.,

$$-\nabla \times \nabla \times \vec{A} + \nabla(\nabla \cdot \vec{A}) - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = 0\ .$$

In particular, we have

$$A_r = 0 \tag{6.41a}$$

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right] + \frac{\partial^2 A_\phi}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 A_\phi}{\partial t^2} = 0 \tag{6.41b}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_z}{\partial r}\right) + \frac{\partial^2 A_z}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 A_z}{\partial t^2} = 0 \tag{6.41c}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{\partial^2 V}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = 0\ . \tag{6.41d}$$

A_ϕ specifies a focusing field of the T.E. type; it gives rise to two components of magnetic field: $H_r = -\frac{\partial A_\phi}{\partial z}$ and $H_z = \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)$. A_z together with V specifies the accelerating field of the T.M. type; they give rise to only one magnetic field component, i.e. $H_\phi = -\frac{\partial A_z}{\partial r}$. For D.C. focusing field, A_ϕ is indepndent of t.

From the Hamilton equations $-\frac{\partial H}{\partial q_i} = \frac{dp_i}{dt}$ and the Maxwell equations we obtain the following equations of motion:

$$\frac{d}{dt}(m\dot{z}) = eE_z + \frac{e}{c}\left(\dot{r}H_\phi - r\dot{\phi}H_r\right) \tag{6.42a}$$

$$\frac{d}{dt}(m\dot{r}) = eE_r + \frac{e}{c}(r\dot{\phi}H_z - \dot{z}H_\phi) + mr\dot{\phi}^2 \qquad (6.42b)$$

$$\frac{d}{dt}(mr^2\dot{\phi} + \frac{e}{c}rA_\phi) = 0 \ . \qquad (6.42c)$$

Here E_r, E_z and H_ϕ are the accelerating field components. H_r, H_z and E_ϕ are the components of the focusing field. It may be noted that E_ϕ does not enter into these equations except through A_ϕ. The last equation (6.42c) follows directly from the axial-symmetric property which demands that $\frac{dp_\phi}{dt} = -\frac{\partial H}{\partial \phi} = 0$. But for this simplification we would have to use

$$\frac{d}{dt}(mr^2\dot{\phi}) = erE_\phi + r\,\frac{e}{c}(\dot{z}H_r - \dot{r}H_z) \qquad (6.43)$$

instead of (6.42c) and a simple discussion of the problem would not be at all possible.

(ii) The Focusing Force

Let us examine the $\dot{\phi}$-dependent radial force terms in equation (6.42b), i.e.,

$$\frac{e}{c}\,r\dot{\phi}H_z + mr\dot{\phi}^2 = \frac{r}{m}\left(\frac{eH_z}{c}\right)^2\left[\frac{m\dot{\phi}}{\frac{e\,H_z}{c}} + \frac{(m\dot{\phi})^2}{\left(\frac{eH_z}{c}\right)^2}\right] \equiv F \qquad (6.44)$$

In order to have focusing action, the expression inside the square brackets must be negative. In other words we must have $\dot{\phi}/H_z < 0$. For a given focusing field H_z, the effect is greatest if the bracketed expression is minimum. The minimum value is reached at

$$m\dot{\phi} = -\frac{1}{2}\frac{eH_z}{c} , \qquad (6.45)$$

i.e.,
$$\dot{\phi} = -\frac{eH_z}{2mc} = -\frac{m_o}{m}\omega_L \qquad (6.45a)$$

where ω_L is the so-called Lamor frequency. If (6.45) is satisfied everywhere along the electron path,we achieve maximum focusing with

$$F = -\frac{1}{4}\frac{r}{m}\left(\frac{eH_z}{c}\right)^2 . \qquad (6.46)$$

The question is: can the optimum conditions be really satisfied?

From equation (6.42c) we have

$$mr^2\dot{\phi} + \frac{e}{c}rA_\phi = m_c r_c^2 \dot{\phi}_c + \frac{e}{c}r_c A_{\phi,c} \equiv p_{\phi,c}$$

where the subscript c indicates the initial value at the cathode surface. Thus

$$m\dot{\phi} = \frac{p_{\phi,c} - \frac{e}{c}rA_\phi}{r^2} = m_c\dot{\phi}_c \frac{r_c^2}{r^2} + \frac{e}{c}\frac{r_c A_{\phi,c} - rA_\phi}{r^2}$$

Since A_ϕ satisfies equation (6.41b) we may take for all practical purposes for small r

$$A_\phi \sim \text{R.P.}\ \frac{1}{h_r} J_1(h_r r)\ (Ce^{jh_z z} + De^{-jh_z z})\ e^{-jhct}$$

with $\quad h^2 = h_r^2 + h_z^2 \quad$ (h = 0 for **D.C.**),

except, of course, near both ends of the focusing field where A_ϕ must satisfy particular boundary conditions. H_z is determined from A_ϕ, i.e.,

$$H_z \sim \text{R.P. } J_0(h_r r)\,(Ce^{jh_z z} + De^{-jh_z z})\, e^{-jhct} .$$

Thus

$$A_\phi \cong \frac{1}{2} rH_z$$

and

$$m\dot{\phi} = m_c\dot{\phi}_c \frac{r_c^2}{r^2} - \frac{1}{2}\frac{eH_z}{c}\left(1 - \frac{H_{z,c}}{H_z}\frac{r_c^2}{r^2}\right) .$$

Substituting this value of $m\dot{\phi}$ in equation (6.44) we obtain

$$F = -\frac{r}{4m}\left(\frac{eH_z}{c}\right)^2 + \frac{1}{mr^3}\left(m_c r_c^2\dot{\phi}_c + \frac{e}{c}\frac{H_{z,c}}{2} r_c^2\right)^2 . \qquad (6.47)$$

From these relations it is clear that the optimum condition cannot be satisfied unless $(m_c\dot{\phi}_c + \frac{eH_{z,c}}{2c}) = 0$, i.e., $p_{\phi,c} = 0$.

According to the statistical theory of thermionic emission, only a few of the emitted electrons can have $p_{\phi,c} = 0$ and the greater $H_{z,c}$ is the fewer will be such electrons. The situation will be best when $H_{z,c}$ is made equal to zero by shielding the cathode from the focusing field so that no magnetic lines will penetrate its surface. Then

$$p_{\phi,c} = m_c r_c^2\dot{\phi}_c$$

and with Maxwellian distribution the expectation value of $p^2_{\phi,c}$ is

$$\langle p^2_{\phi,c}\rangle = \langle (m_c r_c^2\dot{\phi}_c)^2\rangle = m_c (r_c)^2_{max.} \frac{1}{2} kT . \qquad (6.48)$$

On most of the electrons there will be a defocusing force arising from the initial angular momentum which, most probably, is equal to $\frac{1}{mr^3}\; m_c (r_c)^2_{max.} \frac{1}{2} kT$. Except for vanishing values of r, this is generally negligible in comparison with the optimum focusing force $\frac{r}{4m}(\frac{eH_z}{c})^2$ for practical values of H_z. We will, henceforth, consider the case of the shielded cathode as the optimum case and consider $m_c r_c^2 \dot{\phi}_c$ as equal to zero unless the contrary is explicitly stated.

The $\dot{\phi}$-independent terms in expression (6.47) for F may be written as

$$- \frac{r}{4m} (\frac{eH_z}{c})^2 (1 - \frac{H_{z,c}^2}{H_z^2} \cdot \frac{r_c^4}{r^4}) \quad .$$

This expression has the same sign irrespective of whether H_z is positive or negative. So it is possible, at least theoretically, to use either A.C. or D.C. for transverse focusing. The H_z^2 factors in the above expression clearly indicate that the focusing action of H_z is a second order effect. Physically, the process may be visualized as taking place in two steps: first, the electron interacts with the focusing field to acquire $\dot{\phi}$ and then interacts with the field again through its $\dot{\phi}$ to create the focusing force. With A.C. focusing fields the frequency may have any value and the power may be pulsed. These features seem to be very attractive and may prove to be of value for certain applications. But one inherent limitation is also apparent and is due to the skin effect of metallic conductors. If the electron beam is enclosed by metallic walls as in the accelerator tube, the axial magnetic field cannot effectively be excited with A.C. currents flowing on the outside of the walls. If the frequency is increased high enough so that T.E. waves may be excited inside the walls, the wall skin depth becomes so thin that power loss on the walls, despite the reduction made possible by using short pulse times is prohibitively large. Unless new effective means for exciting such fields can be devised, it is not practicable to use them for focusing in linear accelerators. Hereafter, we will consider D.C. focusing fields exclusively.

Under the optimum condition $\dot{\phi}$ is directly proportional to A_ϕ or H_z. The electron will stop spiralling whenever $A_\phi = 0$. $\dot{\phi}/H_z$ is always negative and the radial force F arising from $\dot{\phi}$ is proportional to r and is always directed toward the axis. Both $\dot{\phi}$ and F are finite at r = o. The focusing field is only needed for a relatively short distance where $\dot{\xi}$ is appreciably different from unity, beyond which the defocusing effect will be tolerable. A_ϕ or H_z should be a properly chosen function of z such that the desired amount of focusing is obtained with the minimum amount of power.

As a sharp contrast to the optimum case, we may advisably consider a uniform focusing field, i.e., $H_{z,c} = H_z$ = constant, z being less than a certain fixed value. The initial thermal velocities will also be assumed to be negligible. Here

$$m\dot{\phi} = -\frac{1}{2}\frac{eH_z}{c}\left(1 - \frac{r_c^2}{r^2}\right)$$

and

$$F = -\frac{r}{4m}\left(\frac{eH_z}{c}\right)^2\left(1 - \frac{r_c^4}{r^4}\right) \quad .$$

When $r > r_c$, $\dot{\phi}/H_z$ is negative; and when $r < r_c$, $\dot{\phi}/H_z$ is positive. The radial force F, like $\dot{\phi}/H_z$, has opposite signs in the two circumstances. Both $\dot{\phi}$ and F approach infinity as r approaches zero. The focusing force tends to keep the electron at its initial radius, i.e. $r = r_c$. Notwithstanding these marked differences, the focusing force for $r \gg r_c$ is practically the same as the optimum value. If r_c is sufficiently small, the two cases differ very little; and all actual cases with imperfect cathode shielding will naturally fall in between them. In fact they all become identical for a point cathode.

However, if r_c is not quite small, another important point must be considered. By uniform field we do not intend to suggest that the field should cover the entire length of the accelerator, because this would mean tremendous waste of power. If the field is cut off somewhere before the end of the accelerator, the electron will have an angular momentum

$$mr^2\dot{\phi} = \frac{1}{2}\,\frac{eH_{z,c}}{c}\,r_c^2 \ ,$$

which is conserved along the remaining path and gives rise to a defocusing force

$$mr\dot{\phi}^2 = \frac{1}{mr^3}\left(\frac{eH_{z,c}}{2c}\,r_c^2\right)^2 \quad ,$$

which may be quite large for large $H_{z,c}$. Being proportional to $1/mr^3$, this defocusing force decreases more slowly and so will become, at some farther distance, greater than that arising from the accelerating field which is proportional to r/m^2, as long as r increases more slowly than $m^{1/4}$. Thus one would expect that r will increase at some rate between log m and $m^{1/4}$ and serious beam spreading will take place if the subsequent path is long. Further discussion on this point will be given in a later section. We shall see there that the spreading is nothing worse than logarithmic; it can indeed be many times as large as in the corresponding optimum case.

(iii) The Accelerating Field Intensities with Fast Varying Parameters

In Chapter IV we have pointed out that, if the loading of the waveguide changes rapidly from cavity to cavity, the field intensity expressions (4.3) or (4.3') will no longer be valid. We should, then, change the Fourier series into a Fourier integral and must consider the variation of the actual phase angle rather than an average wave number, since the latter concept will no longer be useful. Since in the region where focusing takes place the variation of parameters can be rather rapid, it seems desirable here to discuss the question of fast varying parameters in further detail.

The integral expression for the wave function $u = rH_\phi$ is given by equation (4.5). For the sake of simplicity we will consider H_ϕ directly:

$$H_{\phi,+}(r,z) = \int_{-\infty}^{\infty} h(k_z)\,\frac{k}{k_r}\,J_1(k_r r)\,e^{jk_z z}\,dk_z$$

$$\equiv A(r,z)\,e^{j\Theta(r,z)} \tag{6.49}$$

where $k_z^2 + k_r^2 = k^2$, $A(r,z)$ and $\Theta(r,z)$ are real functions and the subscript + is used to indicate that the field is a traveling wave running in the positive z-direction. Since $H_{\phi,+}(r,z)$ and $H_{\phi,+}(r,-z)$ are complex conjugates, $A(r,z)$ must be even while $\Theta(r,z)$ must be odd with respect to z. $h(k_z)$ is an unknown function, so $A(r,z)$ and $\Theta(r,z)$ are also unknown functions, formally representing the modulus and the argument of the complex integral. $h(k_z)$ is to be determined solely by the boundary conditions, because, regardless of the form of $h(k_z)$, the Fourier integral satisfies the wave equation identically. But whatever function $h(k_z)$ may be, $H_{\phi,+}(r,-z)$ represents a traveling wave running in the negative z-direction, i.e.,

$$H_{\phi,-}(r,z) = \int_{-\infty}^{\infty} h(k_z)\,\frac{k}{k_r}\,J_1(k_r r)\,e^{-jk_z z}\,dk_z$$

$$\equiv A(r,z)\,e^{-j\Theta(r,z)}\,. \tag{6.50}$$

From these traveling waves we obtain the cosine-type and the sine-type standing waves by the usual method of superposition. Let us denote

$$h(k_z) = h_e(k_z) + h_o(k_z) ,$$

where $h_e(k_z)$ and $h_o(k_z)$ are even and odd functions respectively. Then

$$H_{\phi,c} = \int_{-\infty}^{\infty} h_e(k_z) \frac{k}{k_r} J_1(k_r r) \cos k_z z \, dk_z$$

$$\equiv A(r,z) \cos \theta(r,z) \qquad (6.51)$$

and

$$H_{\phi,s} = \int_{-\infty}^{\infty} h_o(k_z) \frac{k}{k_r} J_1(k_r r) \sin k_z z \, dk_z$$

$$\equiv A(r,z) \sin \theta(r,z). \qquad (6.52)$$

Obviously, both $h_e(k_z)$ and $h_o(k_z)$ should be real, so $h(k_z)$ is also a real function. As to the specific form of $h(k_z)$, however, the boundary value problem is not even approximately soluble. Hence we will consider $A(r,z)$ and $\theta(r,z)$ instead. These quantities have greater physical significances, and can at least be determined by actual measurements.

By substituting $A(r,z)\, e^{j\theta(r,z)}$ for H_ϕ into the wave equation it can easily be shown that $A(r,z)$ and $\theta(r,z)$ are to satisfy the following simultaneous differential equations:

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rA)\right] + \frac{\partial^2 A}{\partial z^2} + \left[k^2 - \left(\frac{\partial\theta}{\partial r}\right)^2 - \left(\frac{\partial\theta}{\partial z}\right)^2\right] A = 0 \qquad (6.53a)$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\theta}{\partial r}\right) + \frac{\partial^2\theta}{\partial z^2} + \frac{2}{A}\left[\frac{\partial A}{\partial r}\frac{\partial\theta}{\partial r} + \frac{\partial A}{\partial z}\frac{\partial\theta}{\partial z}\right] = 0 . \qquad (6.53b)$$

From the physical picture of the problem we may ascertain that at least for small r the inequality $\frac{\partial\theta}{\partial r} \ll \frac{\partial\theta}{\partial z}$ holds true. Let us denote

$$\left.\begin{aligned} k_{ro} &\equiv \sqrt{k^2 - \left(\frac{\partial\theta}{\partial z}\right)^2} \equiv \sqrt{k^2 - k_{zo}^2} \\ \text{and} \qquad & \\ A(r,z) &\equiv r\,B(r,z)\ . \end{aligned}\right\} \qquad (6.54)$$

The equations (6.53) may then be transformed into

$$\frac{\partial^2 B}{\partial r^2} + \frac{3}{r}\frac{\partial B}{\partial r} + \frac{\partial^2 B}{\partial z^2} + k_{ro}^2\, B = 0 \qquad (6.55a)$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\theta}{\partial r}\right) + \frac{\partial^2\theta}{\partial z^2} + \frac{2}{B}\left[\frac{1}{r}\frac{\partial}{\partial r}(rB)\,\frac{\partial\theta}{\partial r} + \frac{\partial B}{\partial z}\frac{\partial\theta}{\partial z}\right] = 0. \qquad (6.55b)$$

Here we should note that k_{ro}, like $\partial\theta/\partial z$, is a variable function.

The general form of B(r,z) and θ(r,z) can be determined by solving the equations (6.55) approximately. Having obtained these forms we may then derive the field expressions. Without going into the detailed steps we may write down the results directly:

$$H_\phi(r,z;t) = E_o(z)\,\frac{kr}{2}\,\sin 2\pi\Delta + O(r^3) \qquad (6.56a)$$

$$E_z(r,z;t) = E_o(z)\,\cos 2\pi\Delta + O(r^2) \qquad (6.56b)$$

$$E_r(r,z;t) = E_o(z)\,\frac{kr}{2\beta}\,\sin 2\pi\Delta - \frac{r}{2}\frac{\partial E_o}{\partial z}\cos 2\pi\Delta + O(r^3) \qquad (6.56c)$$

with $\beta = \frac{k}{k_{zo}} = \frac{k}{\frac{\partial\theta}{\partial z}}$ and $2\pi\Delta = \int^{z} k_{zo}\,dz - \int^{t} \omega dt$

as defined before. We see here that H_ϕ and E_z have the same forms as those for the case of slow varying parameters but E_r, apart from the direct extension of the definition of β, has one correction term depending on $\frac{\partial E_o}{\partial z}$. If E_o increases monotonically, this term represents a focusing force in the acceleration region, i.e., $-\pi/2 < 2\pi\Delta < \pi/2$ and a defocusing force in the region of deceleration. This term predominates over the E_o-term for $2\pi\Delta \cong 0$ or π, and is in general important if $\frac{1}{E_o}\frac{\partial E_o}{\partial z}$ reaches the same order of magnitude as k/B.

(iv) Equations of Motion in Dimensionless Units with Fast Varying Parameters

By substituting the expressions (6.56) for the field intensities in equations (6.42) we obtain the most general set of equations describing the motion of an electron in an axially symmetric field; these equations are expressed in dimensionless units as follows:

$$\frac{d}{d\tau}(\gamma\dot{\xi}) = \alpha \cos 2\pi\Delta + \pi\alpha\eta\dot{\eta} \sin 2\pi\Delta + \eta\dot{\phi}\frac{\partial\pi_\phi}{\partial\xi} \tag{6.57a}$$

$$\frac{d}{d\tau}(\gamma\dot{\eta}) = \pi\alpha\eta\left(\frac{1}{\beta} - \dot{\xi}\right) \sin 2\pi\Delta - \frac{1}{2}\eta\frac{\partial\alpha}{\partial\xi}\cos 2\pi\Delta + \gamma\eta\dot{\phi}^2 + \dot{\phi}\frac{\partial}{\partial\eta}(\eta\pi_\phi) \tag{6.57b}$$

$$\frac{d}{d\tau}(\gamma\eta^2\dot{\phi} + \eta\pi_\phi) = 0 \; . \tag{6.57c}$$

All quantities in the above equations, except

$$\pi_\phi = \frac{eA_\phi}{m_o c^2} = \eta\frac{\omega_L}{v} \quad , \tag{6.58}$$

have already been defined.

When the two small terms on the right-hand side of (6.57a) are neglected and the $\dot{\phi}$-dependent terms in (6.57b) are evaluated by means of (6.57c), these equations finally become

$$\frac{d}{d\tau}(\gamma\dot{\xi}) = \alpha \cos 2\pi\Delta \tag{6.59a}$$

$$\begin{aligned}\frac{d}{d\tau}(\gamma\dot{\eta}) &= \pi\alpha\eta\left(\frac{1}{\beta} - \dot{\xi}\right) \sin 2\pi\Delta - \frac{1}{2}\eta\frac{\partial\alpha}{\partial\xi}\cos 2\pi\Delta \\ &\quad - \frac{\eta}{\gamma}\left(\frac{\omega_L}{v}\right)^2 + \frac{\eta}{\gamma}\frac{\eta_c^4}{\eta^4}\left(\frac{\omega_{L,c}}{v} + \gamma_c\dot{\phi}_c\right)^2\end{aligned} \tag{6.59b}$$

These equations together with the defining relation $\frac{d\Delta}{d\tau} = \frac{\dot{\xi}}{\beta} - 1$ constitute a simple set of general equations of electron motion which covers almost all cases likely to be met with in linear accelerators. The equation (6.59b) is reduced to

$$\begin{aligned}\frac{d}{d\tau}(\gamma\dot{\eta}) &= \pi\alpha\eta\left(\frac{1}{\beta} - \dot{\xi}\right) \sin 2\pi\Delta - \frac{1}{2}\eta\frac{\partial\alpha}{\partial\xi}\cos 2\pi\Delta \\ &\quad - \frac{\eta}{\gamma}\left(\frac{\omega_L}{v}\right)^2 .\end{aligned} \tag{6.59b'}$$

under the optimum focusing condition.

(v) The Required Focusing Field Strength

Although equations (6.59) can only be solved by numerical means, some useful estimation of the required focusing field strength is rather simple. It is evident that the focusing force will be strong enough as long as $\frac{d}{d\tau}(\gamma\dot{\eta}) \lesseqgtr 0$ and this, under the optimum condition, is equivalent to

$$(\frac{\omega_L}{\upsilon})^2 \geqq \pi\alpha\gamma(\frac{1}{\beta} - \dot{\xi}) \sin 2\pi\Delta - \frac{1}{2}\gamma \frac{\partial\alpha}{\partial\xi} \cos 2\pi\Delta , \qquad (6.60)$$

Since

$$\pi\alpha\gamma(\frac{1}{\beta} - \dot{\xi}) \sin 2\pi\Delta \leqq \pi\alpha(\frac{\gamma}{\beta} - \sqrt{\gamma^2(1 - \dot{\eta}^2) - 1})\cdot|\sin 2\pi\Delta|$$

$$\leqq \frac{\pi\alpha}{\beta}\left[1 + \gamma(1 - \sqrt{1 - \dot{\eta}^2})\right]\cdot|\sin 2\pi\Delta|$$

$$\cong \frac{\pi\alpha}{\beta}\cdot|\sin 2\pi\Delta|$$

$$-\frac{1}{2}\gamma \frac{\partial\alpha}{\partial\xi} \cos 2\pi\Delta \leqq \frac{1}{2}\gamma \left|\frac{\partial\alpha}{\partial\xi}\right|\cdot|\cos 2\pi\Delta|$$

and

$$\frac{\pi\alpha}{\beta}\cdot|\sin 2\pi\Delta| + \frac{1}{2}\gamma\left|\frac{\partial\alpha}{\partial\xi}\right|\cdot|\cos 2\pi\Delta|$$

$$\leqq \frac{\pi\alpha}{\beta}\cdot\sqrt{1 + (\frac{\beta}{\pi\alpha}\ \frac{\gamma}{2}\frac{\partial\alpha}{\partial\xi})^2} ,$$

we may replace the condition (6.60) by a bounded estimate:

$$(\frac{\omega_L}{\upsilon})^4 \geqq (\frac{\pi\alpha}{\beta})^2 + (\frac{\gamma}{2}\frac{\partial\alpha}{\partial\xi})^2 .$$

Furthermore, since according to equation (6.2)

$$\gamma = \gamma_o + \int_{\xi_o}^{\xi} (\alpha \cos 2\pi\Delta + \frac{\pi\alpha\eta}{\beta}\frac{d\eta}{d\xi} \sin 2\pi\Delta)\, d\xi ,$$

$$\gamma \leqq \gamma_o + \int_{\xi_o}^{\xi} \left[\alpha\,|\cos 2\pi\Delta| + \frac{\pi\alpha\eta}{\beta}\left|\frac{d\eta}{d\xi}\right|\cdot|\sin 2\pi\Delta|\right] d\xi$$

$$\leqq \gamma_o + \int_{\xi_o}^{\xi} \alpha\sqrt{1 + (\frac{\pi\alpha\eta}{\beta}\frac{d\eta}{d\xi})^2}\, d\xi \cong \gamma_o + \int_{\xi_o}^{\xi} \alpha\, d\xi ,$$

we may relate ω_L/υ solely with the known parameters:

$$\left(\frac{\omega_L}{v}\right)^4 \geq \left(\frac{\pi\alpha}{\beta}\right)^2 + \left[\frac{1}{2}\frac{\partial\alpha}{\partial\xi}\left(\gamma_o + \int_{\xi_o}^{\xi}\alpha\, d\xi\right)\right]^2 . \tag{6.61}$$

This estimation is valid for all phase angles (Δ_o), provided that $\dot{\xi} > 0$. For constant or slow varying parameters the condition is simply

$$\left(\frac{\omega_L}{v}\right)^2 \geq \frac{\pi\alpha}{\beta} . \tag{6.62}$$

A somewhat different estimation can be made which will generally be lower for small α and greater for large α than that given by (6.61). From the equations (6.1) and (6.2) we know that

$$\frac{d\gamma}{\beta} - d(\gamma\dot{\xi}) = (\alpha \cos 2\pi\Delta + \pi\alpha\eta\dot{\eta}\, \frac{1}{\beta}\, \frac{1 - \beta^2}{\dot{\xi} - \beta} \sin 2\pi\Delta)d\Delta$$

$$\cong \alpha \cos 2\pi\Delta \;\; d\Delta$$

and from this, as can easily be shown by integration and limiting processes, follows the following inequality:

$$\frac{\gamma}{\beta} - \gamma\dot{\xi} \leq \left(\frac{\gamma_o}{\beta_o} - \sqrt{\gamma_o^2 - 1}\right) + \frac{3\alpha - \alpha_o}{2\pi} + \left(\frac{1}{\beta} - \frac{1}{\beta_o}\right) .$$

The second estimate is

$$\left(\frac{\omega_L}{v}\right)^4 \geq (\pi\alpha)^2 \left[\frac{\gamma_o}{\beta_o} - \sqrt{\gamma_o^2 - 1} + \frac{3\alpha - \alpha_o}{2\pi} + \frac{1}{\beta} - \frac{1}{\beta_o}\right]^2$$

$$+ \left[\frac{1}{2}\frac{\partial\alpha}{\partial\xi}\left(\gamma_o + \int_{\xi_o}^{\xi}\alpha\, d\xi\right)\right]^2 . \tag{6.63}$$

In particular, if α and β are constants, the above relation reduces to

$$(\frac{\omega_L}{\upsilon})^2 \geqq \pi\alpha \left[\frac{\gamma_o}{\beta} - \sqrt{\gamma_o^2 - 1} + \frac{\alpha}{\pi} \right] . \quad (6.64)$$

For any case it is good enough to use the lower estimate.

Now let us consider the following typical examples for purposes of illustration. $\upsilon = 2860 \times 10^6$ cycles in all cases.

Case (1): $\alpha = \pi/5$ (.93 Mev./ft.), $\beta = 1$, $\gamma_o = 5$.
From (6.62) we estimate

$$\frac{\omega_L}{\upsilon} \geqq \frac{\pi}{\sqrt{5}} , \quad \text{i.e.,} \quad H_z \geqq 457 \text{ gauss}$$

and from (6.64)

$$\frac{\omega_L}{\upsilon} \geqq .549 \frac{\pi}{\sqrt{5}} , \quad \text{i.e.,} \quad H_z \geqq 251 \text{ gauss}$$

Case (2): $\alpha = 8$ (11.8 Mev./ft.), $\beta = 1$ $\gamma_o = 2$.
The two estimates from (6.62) and (6.64) are

$$\frac{\omega_L}{\upsilon} \geqq 5.02 , \quad \text{i.e.} \quad H_z \geqq 1630 \text{ gauss}$$

and

$$\frac{\omega_L}{\upsilon} \geqq 8.41, \quad \text{i.e.} \quad H_z \geqq 2730 \text{ gauss}$$

respectively.

Case (3): $\alpha = .40$ (.59 Mev./ft.),

$$\frac{1}{\sqrt{1 - \beta^2}} - \frac{1}{\sqrt{1 - \beta_o^2}} = \alpha\xi , \quad \beta_o = .5 , \quad \gamma_o = \frac{1}{\sqrt{1 - \beta_o^2}} = 1.155 .$$

From (6.62) $\frac{\omega_L}{\upsilon} \geqq \sqrt{\frac{\pi\alpha}{\beta_o}} = 1.58,$ i.e. $H_z \geqq 514$ gauss.

Since β increases with ξ, H_z decreases with ξ.

From (6.63)

$$\frac{\omega_L}{v} \geq \sqrt{\pi\alpha}\left[\frac{\sqrt{1-\beta_o^2}}{\beta_o}+\frac{\alpha}{\pi}+\frac{1}{\beta}-\frac{1}{\beta_o}\right]^{1/2} ,$$

so we estimate

$$\frac{\omega_L}{v} \geq 1.53, \quad \text{i.e.,} \quad H_z \geq 497 \text{ gauss.}$$

As β increases $\frac{1}{\beta}-\frac{1}{\beta_o}$ decreases so H_z also decreases.

Case(4): A tapered bunching section discussed in section 5.6. $\alpha(\xi)$ and $\beta(\xi)$ are shown in Fig. 5.8. $\gamma_o = (1-\beta_o^2)^{-1/2} = 1.155$. The estimated values of H_z as calculated from (6.61) and (6.63) are plotted versus ξ in Fig. 6.13. H_z increases steadily as ξ increases and needs nowhere to be greater than some 1150 gauss.

It is true that the sure estimate is quite rough. Referring to Fig. 5.9 we see that in the latter part of the buncher all electrons move towards the wave crest steadily with $\sin 2\pi\Delta$ ever decreasing. The required strength of the focusing field should actually decrease there rather than increase as ξ increases. It would not be surprising if a detailed calculation should yield values considerably lower than what we have estimated. In fact, such calculation can easily be made by using (6.60) directly, once we have obtained the longitudinal part of the solution. This question, however, will not be pursued here.

(vi) Beam Spreading Caused by the Conserved Angular Momentum

We have mentioned before that if the optimum condition is not satisfied, i.e., if the cathode is penetrated by the flux lines of the focusing field, and if the focusing field is cut off somewhere before the output end of the accelerator, the electron will experience a defocusing force proportional to $\frac{1}{\gamma\eta^3}$, arising from its angular momentum which is conserved along the remaining path of its travel. In fact this defocusing force exists everywhere along the path, though its effect may be wholly or partly cancelled by the action of the focusing field. Now we proceed to discuss this defocusing effect. For the sake of generality, we will consider both cases, with and without the focusing field.

It is evident that for the region where γ is large we may appropriately take α = constant, $\beta = 1$ and $2\pi\Delta \rightarrow 2\pi\Delta_\infty = -\sin^{-1}A$. From equation (6.59b) we have presently

$$\frac{d}{d\tau}(\gamma\dot{\eta}) = \pi\alpha\eta(1 - \dot{\xi}) \sin 2\pi\Delta - \frac{\eta}{\gamma}\left(\frac{\omega_L}{\upsilon}\right)^2 + \frac{\mu^2}{\gamma\eta^3} \tag{6.65}$$

where

$$\mu = \frac{\upsilon}{m_r c^2} p_\phi = \left(\frac{\omega_{L,c}}{\upsilon} + \gamma_c\dot{\phi}_c\right) \eta_c^2 \tag{6.66}$$

represents the generalized angular momentum in dimensionless units which is conserved everywhere along the electron path. By the same transformations as used in deriving the equation (6.14) we obtain the following non-linear equation:

$$\left.\begin{aligned} &\frac{d^2y}{d\Delta^2} + \frac{(\pi \cos 2\pi\Delta)^2 + (\frac{2\pi}{\alpha}\frac{\omega_L}{\upsilon})^2}{(A + \sin 2\pi\Delta)^2} y - (\frac{2\pi}{\alpha})^2 \frac{\mu^2}{y^3} = 0 \\ &y = \eta\ (A + \sin 2\pi\Delta)^{1/2} \end{aligned}\right\} \quad (6.67)$$

When the independent variable is changed from Δ to ϵ $(2\pi\epsilon \ll 1)$ by $2\pi\Delta = 2\pi\Delta_\infty + 2\pi\epsilon$, the above equation is reduced to

$$\frac{d^2y}{d\epsilon^2} + \frac{1}{4\epsilon^2}\left[1 + 4b^2 + 0(\epsilon)\right] y - (\frac{2\pi}{\alpha})^2 \frac{\mu^2}{y^3} = 0 \qquad (6.68)$$

Here

$$b \equiv \frac{\omega_L}{\upsilon} \frac{1}{\alpha \cos(2\pi\Delta_\infty)} \qquad (6.69)$$

and ω_L are to be considered as constants. By neglecting the small term $0(\epsilon)$ in the coefficient of the y-term and using the substitution

$$u = \frac{y^4}{(2\pi\epsilon)^2}, \quad x = \log\epsilon \qquad (6.70)$$

equation (6.68) may further be transformed into

$$\frac{d^2u}{dx^2} - \frac{3}{4}\frac{1}{u}(\frac{du}{dx})^2 + (2b)^2u - (\frac{2\mu}{\alpha})^2 = 0 \quad . \qquad (6.71)$$

Fortunately, this non-linear equation can be integrated by multiplying with an integration factor, i.e., $2u^{-3/2}\frac{du}{dx}$. When the integration is carried out and the resulting equation is transformed by $Y = u^{1/2}$, we obtain

$$\frac{dY}{dx} = \pm\sqrt{-(\frac{2\mu}{\alpha})^2 + CY - (2b)^2 \cdot Y^2} \ .$$

Hence by integrating again

$$\left.\begin{aligned} \frac{C - 2(\frac{2\mu}{\alpha})^2 (2b)^2 Y}{\sqrt{C^2 - 4(\frac{2u}{\alpha})^2 (2b)^2}} &= \pm \sin(2b \log \frac{\epsilon_1}{\epsilon} + D) \\ Y = \sqrt{u} = \eta^2 &\cos 2\pi\Delta_\infty \ , \end{aligned}\right\} \quad (6.72)$$

C and D being integration constants. We then determine these constants from the initial values γ_1, $\dot{\xi}_1$, η_1 and $(\frac{d\eta}{d\xi})_1$ which are supposed to be known. The final result is obtained as follows:

$$\eta^2 = \left[\eta_1 \cos(b \log \frac{\epsilon_1}{\epsilon}) + \frac{\gamma_1 \dot{\xi}_1}{\omega_L/v} (\frac{d\eta}{d\xi})_1 \sin(b \log \frac{\epsilon_1}{\epsilon})\right]^2 + \left[\frac{\mu}{\eta_1} \frac{1}{\omega_L/v} \sin(b \log \frac{\epsilon_1}{\epsilon})\right]^2 \quad (6.73)$$

In passing, it may be noted that the solution (6.72) would satisfy the equation

$$\frac{d}{d\tau} (\gamma\dot{\eta}) = - \frac{\eta}{\gamma} (\frac{\omega_L}{v})^2 + \frac{\mu^2}{\gamma\eta^3}$$

if the relation $A + \sin 2\pi\Delta = 2\pi\epsilon \cos 2\pi\Delta_\infty$ were exact. This is to be expected because as ϵ decreases, the radial force arising from the accelerating field will eventually become negligible in comparison with the forces arising from ω_L and μ.

Although the solution (6.73) has the form of an oscillating function, η can hardly change when γ is large compared to γ_1; and this is due to the logarithmic factor in the argument of the circular functions. In order to increase the argument by an amount n times as large as a given interval, γ/γ_1 should be increased to the n-th power of the value reached in the first interval. For example, if $b \log \frac{\epsilon_1}{\epsilon} \cong b \log \frac{\gamma}{\gamma_1}$ should change from 0 to π, γ/γ_1 should be increased to $\exp(\frac{\pi\alpha \cos 2\pi\Delta_\infty}{\omega_L/v})$. Assuming $\alpha = 8$, $\gamma_1 = 10$, $\cos 2\pi\Delta_\infty = 1$ and a large focusing field $\omega_L/v = \sqrt{\pi\alpha}$ ($H_z = 1600$ gauss), we find $\gamma/\gamma_1 = e^{\sqrt{\pi\alpha}} \cong e^5 \cong 150$. This corresponds to an increase of energy from about 5 Mev. to 750 Mev, and a distance of travel of about 190 wavelengths. If the argument should increase from 0 to 2π, γ/γ_1 should be increased to $(150)^2$, i.e., $\gamma \cong 225{,}000$! On the other hand, if γ is only increased to e times γ_1, the argument has already been changed by $\pi/5$. η can only change appreciably when γ is relatively small. And the change of η is due primarily to the initial slope $\frac{d\eta}{d\xi}$ or the generalized angular momentum μ but not the focusing field, because for small arguments η may be approximated by

$$\eta^2 = \left[\eta_1 + \frac{\gamma_1 \dot{\eta}_1}{\alpha \cos 2\pi\Delta_\infty} \log \frac{\epsilon_1}{\epsilon}\right]^2 + \left[\frac{\mu}{\eta_1} \frac{1}{\alpha \cos 2\pi\Delta_\infty} \log \frac{\epsilon_1}{\epsilon}\right]^2 , \qquad (6.74)$$

which is independent of the focusing field. However, the application of a focusing field over the entire electron path does prevent η from increasing beyond a certain limit; in fact

$$\eta^2 \leqq \eta_1^2 + \left(\frac{\gamma_1 \dot{\eta}_1}{\omega_L/v}\right)^2 + \left(\frac{\mu}{\eta_1} \frac{1}{\omega_L/v}\right)^2 .$$

But this advantage can only be realized with a great sacrifice of power.

Now we consider the more practical case with no focusing field ($\omega_L = 0$) for $\xi \geq \xi_1$. The solution is the same as given by (6.74). It is interesting to note that the spreading of the electron beam caused by μ is also logarithmic, not according to some small finite power of γ as one might suspect. The relative magnitudes of the two logarithmic terms are in the ratio of μ/η_1 to $\gamma_1 \dot{\eta}_1$. If the cathode is perfectly shielded, i.e., $\omega_{L,c} = 0$, $\mu = \gamma_c \eta_c^2 \dot{\phi}_c$ reaches its inherent limit set by the random distribution of thermal velocities. If the cathode is not shielded and $\omega_{L,c}/\upsilon \gg \gamma_c \dot{\phi}_c$, $\mu = \frac{\omega_{L,c}}{\upsilon} \eta_c^2$ can be very large compared to $\gamma_1 \eta_1 \dot{\eta}_1$.

In the latter case, it is evident that one cannot use indiscriminately large field intensities for focusing without causing excessive beam spreading. There will be a definite limit in the degree of focusing one can achieve. When the focusing field is too weak, excessive spreading will take place at relatively short distances from the cathode. As the field is increased, the spreading will become less and will reach a minimum at a certain field intensity. After that, if the field is further increased, excessive spreading will set in again but at farther distances.

In the former case ($\omega_{L,c} = 0$), the spreading is mainly determined by the initial radial momentum $\gamma_1 \dot{\eta}_1$ because μ is at its inherent limit. If $\gamma_1 \eta_1 \dot{\eta}_1$ can be made to approach $\mu = \gamma_c \eta_c^2 \dot{\phi}_c$ by increasing the focusing field, we would be able to attain the theoretical limit of the focusing ability. We will show in the next section that while this is not strictly true, $|\gamma_1 \dot{\eta}_1|$ can under proper conditions be reduced to a value much smaller than $\left|\frac{\omega_L}{\upsilon}\right|\eta_c$ and $|\gamma_1 \eta_1 \dot{\eta}_1|$ not much larger than μ . With a shielded cathode the electron beam can be focused to a far better degree than is possible with an unshielded cathode.

(vii) Essential Requirements for Effective Focusing

With α, γ_1 and γ fixed, η depends mainly on three quantities η_1, $\gamma_1 \dot{\eta}_1$ and μ/η_1. What we should strive for by using a focusing field is, evidently, to make these three quantities as small as possible. Negative values of $\gamma_1 \dot{\eta}_1$ are always preferred, because η can only be reduced from η_1 by having a negative momentum. But it is difficult to arrange such a situation for all electrons entering at widely different phase angles. Some electrons may be acted upon by too large an inward force so that they may either go across the axis or enter into a strongly defocusing region and be repulsed from the axis. In either case they will again have positive momentum. It is important to know how these three quantities vary with ω_L and under what conditions they can be minimized.

Before proceeding with the discussion let us define a radial distance $\bar{\eta}$ by

$$\bar{\eta}^2 = \left|\frac{\mu}{\omega_L/\upsilon}\right| , \qquad (6.75)$$

which will play an important role in the following discussion. At $\eta = \bar{\eta}$, the defocusing force $\mu^2/\gamma\eta^3$ is just balanced by the focusing force $\frac{\eta}{\gamma}(\frac{\omega_L}{\upsilon})^2$. $\bar{\eta}^2$ depends on $\omega_{L,c}$.

If $\omega_{L,c} = \omega_L$, $\bar{\eta}^2 \cong \eta_c^2$. If $\omega_{L,c} = 0$, $\langle u^2 \rangle = \frac{1}{2}\frac{kT}{m_r c^2}(\eta_c)^2_{max}$

$\cong 5.7 \times 10^{-8}(\eta_c)^2_{max}$ for $T \cong 2000°C$.

and $\bar{\eta}^2 = 2.4 \times 10^{-6}\frac{(\eta_c)_{max}}{\omega_L/\upsilon}$. For example, if we take $H_z \cong 1000$ gauss, i.e. $\omega_L/\upsilon \cong 3.1$ and $(\eta_c)_{max} \cong .07$, then $\bar{\eta}^2 \cong 5.4 \times 10^{-6}$, i.e., $\bar{\eta} \cong 2.3 \times 10^{-3}$ which is a very small distance corresponding to .23 mm. for $\lambda = 10$ cm.

Equation (6.59b) can be written as

$$\frac{d(\gamma\dot{\eta})^2}{d(\eta^2)} = G(\xi, \dot{\xi}, \Delta) - (\frac{\omega_L}{\upsilon})^2 + \frac{\mu^2}{\eta^4}$$

with

$$G(\xi, \dot{\xi}, \Delta) \equiv \pi\alpha\gamma(\frac{1}{\beta} - \dot{\xi}) \sin 2\pi\Delta - \frac{\gamma}{2}\frac{\partial\alpha}{\partial\xi} \cos 2\pi\Delta \; . \qquad (6.76)$$

At $\eta = \bar{\eta}$, $-(\frac{\omega_L}{\upsilon})^2 + \frac{\mu^2}{\eta^4} = 0$, so $\frac{d(\gamma\dot{\eta})^2}{d(\eta^2)} = G$.

$$\frac{d(\gamma\dot{\eta})^2}{d(\eta^2)} > 0 \quad \text{if } (\frac{\omega_L}{\upsilon})^2 \lesseqgtr G \text{ or if } (\frac{\omega_L}{\upsilon})^2 > G$$

$$\text{but } \eta^2 < \frac{\bar{\eta}^2}{\sqrt{1 - G(\frac{\upsilon}{\omega_L})^2}} \; ; \qquad (6.77a)$$

$$\frac{d(\gamma\dot{\eta})^2}{d(\eta^2)} < 0 \quad \text{if both } (\frac{\omega_L}{\upsilon})^2 > G \text{ and } \eta^2 > \frac{\bar{\eta}^2}{\sqrt{1 - G(\frac{\upsilon}{\omega_L})^2}} \; . \qquad (6.77b)$$

Thus $\eta_{max}^2 > \dfrac{\bar{\eta}^2}{\sqrt{1 - G(\frac{v}{\omega_L})^2}}$. In order to prevent the electron from going radially far outward from the axis, $(\frac{\omega_L}{v})^2$ should be greater than G by a finite amount for at least part of the distance between $\xi = 0$ and $\xi = \xi_1$. $\xi = 0$ denotes the plane at which the electron beam enters the accelerator from the cathode-gun system; and $\xi = \xi_1$ denotes the plane at which the focusing field is to be cut off. For the sake of simplicity we shall assume ω_L = constant for $0 \leqq \xi \leqq \xi_1$.

A similar equation may be written for the region inside the cathode-gun system, i.e.,

$$\frac{d(\gamma\dot{\eta})^2}{d(\eta^2)} = \gamma f(\xi, \dot{\xi}) + \frac{\mu^2}{\eta^4} , \tag{6.78}$$

where $f(\xi, \dot{\xi})$ denotes the function of the radial force constant of the electron gun which may have any desired form. The two equations may be combined into one. Thus by defining

$$F = \begin{cases} G(\xi, \dot{\xi}, \Delta) - (\frac{\omega_L}{v})^2 & 0 \leqq \xi \leqq \xi_1 \\ \gamma f(\xi, \dot{\xi}) & \xi_c \leqq \xi \leqq 0, \end{cases} \tag{6.79}$$

we may write the equation for the combined structure as

$$\frac{d(\gamma\dot{\eta})^2}{d(\eta^2)} = F + \frac{\mu^2}{\eta^4} . \tag{6.80}$$

Now let us integrate this equation from the cathode plane $\xi = \xi_c$ through $\xi = 0$ to $\xi \leqq \xi_1$, i.e., from $\eta = \eta_c$ to η.

We obtain

$$(\gamma\dot{\eta})^2 = -\left(\frac{\omega_L}{\upsilon}\right)^2(\eta^2 - \eta_o^2) - \mu^2\left(\frac{1}{\eta^2} - \frac{1}{\eta_c^2}\right) + P(\eta) \qquad (6.81)$$

$$P(\eta) \equiv (\gamma_c\dot{\eta}_c)^2 + \int_{\eta_c}^{\eta_o} F d\eta^2 + \int_{\eta_o}^{\eta} G d\eta^2 \quad . \qquad (6.82)$$

The equation (6.81) is a quadratic equation in η^2 and may be written in its conventional form, i.e.,

$$\eta^4 - \left[\eta_o^2 + \frac{\overline{\eta}^4}{\eta_c^2} + \frac{P(\eta) - (\gamma\dot{\eta})^2}{(\omega_L/\upsilon)^2}\right]\eta^2 + \overline{\eta}^4 = 0 \; . \qquad (6.83)$$

Since η^2 must be real and positive, we note at once that the following inequalities must hold true:

$$Q(\eta) \equiv \eta_o^2 + \frac{\overline{\eta}^4}{\eta_c^2} + \frac{P(\eta) - (\gamma\dot{\eta})^2}{(\omega_L/\upsilon)^2} \geq 2\overline{\eta}^2 \qquad (6.84a)$$

$$\frac{\overline{\eta}^4}{Q(\eta)} \leq \eta^2 \leq Q(\eta) \quad . \qquad (6.84b)$$

Referring to equation (6.82) it is evident that $P(\eta)$ can only change with ω_L through the term $\int_{\eta_c}^{\eta_o} F d\eta^2$.

As $\left(\frac{\omega_L}{\upsilon}\right)^2 \rightarrow \infty$,

$$\left(\frac{\upsilon}{\omega_L}\right)^2 P(\eta) \rightarrow \left(\frac{\upsilon}{\omega_L}\right)^2 \int_{\eta_c}^{\eta_o} F d\eta^2 = \left(\frac{\upsilon}{\omega_L}\right)^2 \cdot |F|_{max} \cdot |\eta_o^2 - \eta_c^2| \; .$$

If the electron gun is effectively shielded from the magnetic focusing field (ω_L) of the accelerator, then F will be more or less independent of ω_L. Thus $\left(\frac{\upsilon}{\omega_L}\right)^2 P(\eta)$ may be reduced to a very small fraction of $|\eta_o^2 - \eta_c^2|$ by increasing ω_L. Hence we may decrease

$$Q(\eta) - \eta_o^2 \leqq \frac{\overline{\eta}^4}{\eta_c^2} + \frac{P(\eta)}{(\omega_L/\upsilon)^2}$$

$$\leqq \frac{\eta^4}{\eta_c^2} + \left(\frac{\upsilon}{\omega_L}\right)^2 \left[P(\eta_o) + |G|_{max}|\eta^2 - \eta_o^2|\right]$$

to the same small value. Whenever

$$\eta_o^2 + \frac{\overline{\eta}^4}{\eta_c^2} + \left(\frac{\upsilon}{\omega_L}\right)^2 \left[P(\eta_o) + |G|_{max} (\eta_M^2 - \eta_o^2)\right] \leqq \eta_M^2 ,$$

i.e., whenever

$$\left(\frac{\omega_L}{\upsilon}\right)^2 \geqq |G|_{max} + \frac{P(\eta_o) + \frac{\mu^2}{\eta_c^2}}{\eta_M^2 - \eta_o^2} ,$$

we have

$$\overline{\eta}^4/\eta_M^2 \leqq \eta^2 \leqq \eta_M^2 . \qquad (6.85)$$

As $\left(\frac{\omega_L}{\upsilon}\right)^2 \rightarrow \infty$, $\eta_{max} \rightarrow \eta_o$ and $\eta_{min} \rightarrow \overline{\eta}^2/\eta_o$.

If we integrate equation (6.80) from η_o to η we obtain

$$(\gamma\dot{\eta})^2 - (\gamma_o\dot{\eta}_o)^2 = - \left(\frac{\omega_L}{\upsilon}\right)^2 (\eta^2 - \eta_o^2)$$

$$- \mu^2\left(\frac{1}{\eta^2} - \frac{1}{\eta_o^2}\right) + \int_{\eta_o}^{\eta} G \, d\eta^2 .$$

Since

$$-\left(\frac{\omega_L}{\upsilon}\right)^2(\eta^2 - \eta_o^2) - \mu^2\left(\frac{1}{\eta^2} - \frac{1}{\eta_o^2}\right)$$

$$= -\left(\frac{\omega_L}{\upsilon}\right)^2\eta^2\left(1 - \frac{\bar{\eta}^2}{\eta^2}\right)^2 + \left(\frac{\omega_L}{\upsilon}\right)^2\eta_o^2\left(1 - \frac{\bar{\eta}^2}{\eta_o^2}\right)^2 ,$$

$$(\gamma\dot{\eta})^2 \leqq (\gamma_o\dot{\eta}_o)^2 + \left(\frac{\omega_L}{\upsilon}\right)^2\eta_o^2\left(1 - \frac{\bar{\eta}^2}{\eta_o^2}\right)^2 + \int_{\eta_o}^{\eta} G\, d\eta^2. \qquad (6.86)$$

The equality sign holds true for $\eta^2 = \bar{\eta}^2$. As $\left(\frac{\omega_L}{\upsilon}\right)^2 \to \infty$, $(\gamma\dot{\eta})^2 \to \left(\frac{\omega_L}{\upsilon}\right)^2\eta_o^2\left(1 - \frac{\bar{\eta}^2}{\eta_o^2}\right)^2$. Thus $(\gamma\dot{\eta})^2$ can reach very large values when $\left(\frac{\omega_L}{\upsilon}\right)^2$

But according to (6.85), η can only change within bounds if $\left(\frac{\omega_L}{\upsilon}\right)^2$ is sufficiently large, no matter how great $(\gamma\dot{\eta})^2$ may be.

By referring to the maximum or minimum points, η_m instead of η_c or η_o, the expression for $(\gamma\dot{\eta})^2$ is found to be even simpler, thus

$$(\gamma\dot{\eta})^2 = -\left(\frac{\omega_L}{\upsilon}\right)^2(\eta^2 - \eta_m^2) - \mu^2\left(\frac{1}{\eta^2} - \frac{1}{\eta_m^2}\right) + \int_{\eta_m}^{\eta} G\, d\eta^2.$$

Hence

$$(\gamma\dot{\eta})^2 \leqq \left(\frac{\omega_L}{\upsilon}\right)^2\eta_m^2\left(1 - \frac{\bar{\eta}^2}{\eta_m^2}\right)^2 + \int_{\eta_m}^{\eta} G\, d\eta^2 . \qquad (6.87)$$

The equality sign again holds true for $\eta^2 = \bar{\eta}^2$. As $\left(\frac{\omega_L}{\upsilon}\right)^2 \to \infty$,

$$(\gamma\dot{\eta})^2_{max} \to \left(\frac{\omega_L}{\upsilon}\right)^2\eta_{max}^2\left(1 - \frac{\bar{\eta}^2}{\eta_{max}^2}\right)^2 \to \left(\frac{\omega_L}{\upsilon}\right)^2\eta_o^2\left(1 - \frac{\bar{\eta}^2}{\eta_o^2}\right)^2 , \quad \text{or}$$

$$(\gamma\dot{\eta})^2_{max} \to (\frac{\omega_L}{\upsilon})^2 \eta^2_{min}(1 - \frac{\overline{\eta}^2}{\eta^2_{min}})^2 \to (\frac{\omega_L}{\upsilon})^2 \eta^2_o (1 - \frac{\overline{\eta}^2}{\eta^2_o})^2$$

as found before.

Now let us consider η_m as the maximum or minimum point nearest to $\eta_1(\xi_1)$. Near ξ_1 G will be small, i.e., $G \ll (\frac{\omega_L}{\upsilon})^2$ and $\left|\int_{\eta_m}^{\eta} G\, d\eta^2\right|$ will be negligible in comparison with the $(\frac{\omega_L}{\upsilon})^2$ term in equation (6.87). Hence we have at least approximately

$$(\gamma_1 \dot{\eta}_1)^2 \leqq (\frac{\omega_L}{\upsilon})^2 \eta^2_m (1 - \frac{\overline{\eta}^2}{\eta^2_m})^2 \quad . \tag{6.88}$$

While the maximum value of $(\gamma\dot{\eta})^2$ is reached at

$\eta^2 = \overline{\eta}^2 \Big/ \sqrt{1 - G(\frac{\upsilon}{\omega_L})^2} \cong \overline{\eta}^2$, $(\gamma\eta\dot{\eta})^2_{max}$ occurs at

$$(\frac{\omega_L}{\upsilon})^2 - \frac{u^2}{\eta^4} - G = \frac{(\gamma\dot{\eta})^2}{\eta^2}$$

i.e.,

$$\eta^2 \cong \frac{1}{2}\, \eta^2_m (1 + \frac{\overline{\eta}^4}{\eta^4_m}) \quad .$$

Hence

$$(\gamma_1 \eta_1 \dot{\eta}_1)^2 \leqq \frac{1}{4}\, (\frac{\omega_L}{\upsilon})^2 \eta^4_m \; (1 - \frac{\overline{\eta}^4}{\eta^4_m})^2 \quad . \tag{6.89}$$

As $(\frac{\omega_L}{\upsilon})^2 \to \infty$, $(\gamma_1 \eta_1 \dot{\eta}_1)^2_{max} \to \frac{1}{4}\, (\frac{\omega_L}{\upsilon})^2 \eta^4_o (1 - \frac{\overline{\eta}^4}{\eta^4_o})^2$ irrespective of whether $\eta_o > \overline{\eta}$ or $\eta_o < \overline{\eta}$.

A large focusing field can always make η_{max} small but not $|\gamma\dot{\eta}|_{max}$. To make $|\gamma\dot{\eta}|_{max}$ also small we should have, according to equation (6.86), $\dot{\eta}_o$ small and $\eta_o \cong \bar{\eta}$. If the relation $\eta_o = \bar{\eta}$ can be maintained by decreasing η_o as $\left|\frac{\omega_L}{v}\right|$ is increased, $(\gamma\dot{\eta})^2_{max}$ will have the same order of magnitude as $(\gamma_o\dot{\eta}_o)^2$ and $(\gamma\eta\dot{\eta}_1)^2_{max}$ as μ^2. But, actually, any electron gun has its inherent limitations and the size of the electron beam which it supplies cannot be decreased indefinitely in proportion to $\bar{\eta}$. Both $(\gamma\dot{\eta})^2_{max}$ and $(\gamma\eta\dot{\eta})^2_{max}$ will increase indefinitely as $(\frac{\omega_L}{v})^2 \rightarrow \infty$. There is always the possibility of over-focusing (excepting the case where the focusing field covers the entire length of the accelerator), provided that the field can be increased to very large values.

If $\omega_{L,c} = \omega_L$ and is sufficiently large, $\bar{\eta} \cong \eta_c$, $\eta_{max} \cong \eta_c \cong \eta_{min}$ and by the relation (6.87) $\gamma\dot{\eta} \cong 0$. The defocusing effect due to $\mu/\eta_1 \cong \left|\frac{\omega_L}{v}\right| \cdot \eta_c$ will be large but that due to $\gamma_1\dot{\eta}_1$ is negligible. Thus if the cathode is so small that the μ-spreading term, i.e., $\frac{\left|\frac{\omega_L}{v}\right|\eta_c}{\alpha \cos 2\pi\Delta_\infty} \log \frac{\gamma}{\gamma_1}$ can be tolerated, then the cathode shielding may not be needed. If $\omega_{L,c} = 0$ and $(\eta_o)_{max}$ is not quite small, the μ-effect will be small but $|\gamma\dot{\eta}|$ can be as large as $\left|\frac{\omega_L}{v}\right|\eta_c$. But here the large $|\gamma\dot{\eta}|$ may be tolerable because it occurs at small η, and the large η may be tolerable because it occurs with small $|\gamma\dot{\eta}|$. The shielded cathode offers appreciable advantages over the immersed cathode even if they are small. In case a large cathode is to be used and the aforementioned spreading is not tolerable, then the magnetic shielding of the cathode-gun system becomes essential. But shielding alone does not ensure effective focusing; it is also essential to make $(\eta_o)_{max}$ sufficiently small so that $|\gamma\dot{\eta}|$ can never be large.

Since the required focusing field strength for linear electron accelerators is relatively large, the requirement $(\eta_o)_{max} \cong \overline{\eta}$ is rather too severe. However, it is entirely practicable to reduce $(\eta_o)_{max}$ to a small fraction of $(\eta_c)_{max}$ and $|\gamma\eta\dot{\eta}|_{max}$ to a value not much greater than $|\mu|$. For example, let us take as before $(\eta_c)_{max} = .07$ (7 mm. for $\lambda = 10$ cm.), $\mu^2 = \langle\mu^2\rangle = 5.7 \times 10^{-8}(\eta_c)^2_{max}$, i.e. $|\mu| \cong 1.7 \times 10^{-5}$. We assume a large focusing field, $H_z \cong 2000$ gauss, i.e. $\left|\frac{\omega_L}{v}\right| \cong 6.2$; $\left(\frac{\omega_L}{v}\right)^2 = 2\,|G|_{max}$. Thus $\overline{\eta} \cong 1.7 \times 10^{-3}$(.17 mm. for $\lambda = 10$ cm.). It is certainly difficult for an electron gun to reduce the beam diameter by such a large factor as 41. So we assume a practical figure $(\eta_o)_{max} \cong .01$, thus $\frac{1}{49}(\eta_c)^2_{max} = (\eta_o)^2_{max} \cong 35\overline{\eta}^2$. We easily find $\eta_M^2 \cong (\eta_o)^2_{max}$, $|\gamma\dot{\eta}|_{max} \cong .062$ and $|\gamma\eta\dot{\eta}|_{max} \cong 17.5|\mu|$. The spreading caused by this maximum value of $|\gamma\dot{\eta}|$ is still quite small up to several billion volt energy. Most electrons will have $|\gamma\dot{\eta}|$ much less than this maximum value.

From the foregoing discussion we may summarize the essential requirements for effective focusing as follows. The cathode should be effectively shielded from the magnetic focusing field. The beam at the injection plane should have a small cross section and should be well collimated as to direction. $(\eta_o)_{max} \cong \overline{\eta}$ is an ideal condition, but for large focusing fields is too severe to be realized. $(\eta_o)_{max}$ is usually many times larger than $\overline{\eta}$ but should be as small as possible. $\dot{\eta}_o^2$ should also be as small as possible, though a small negative value of $\dot{\eta}_o$ is preferable if the focusing field strength is somewhat less than sufficient. The field strength should be large enough so that the maximum defocusing force may be balanced out. It should be sufficient to satisfy $(\frac{\omega_L}{\upsilon})^2 \cong G$ if $\dot{\eta}_o$ has negligible positive values. With a fixed $(\eta_o)_{max}$, increasing $\left|\frac{\omega_L}{\upsilon}\right|$ beyond its sufficient value will increase $(\gamma\dot{\eta})^2_{max}$, and so increase spreading. Since $\dot{\eta}$ refers to the direction of the ξ -axis, it is very important to align the accelerator tube accurately.

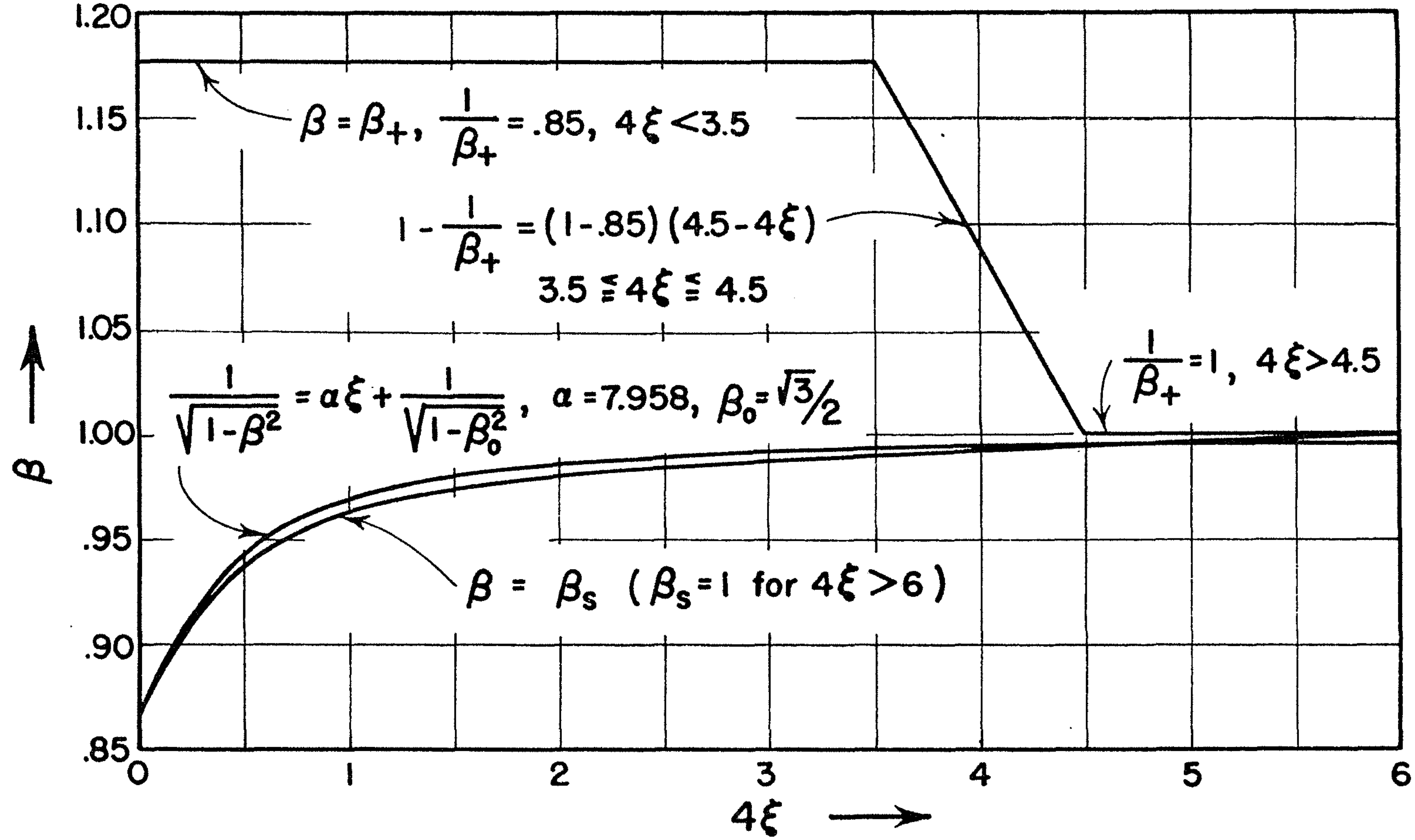

Fig. 6.1 - Shows functions of $\beta(\xi)$ for two types of short starting sections designated as β_s and β_+. $\beta_s c$ varies almost in synchronism with the velocity of an electron at zero phase angle; $\beta_+ c$ is essentially a constant greater than c. Also shown is the exact synchronous function $\frac{d}{d\xi}\left(\frac{1}{\sqrt{1-\beta^2}}\right) = \alpha$.

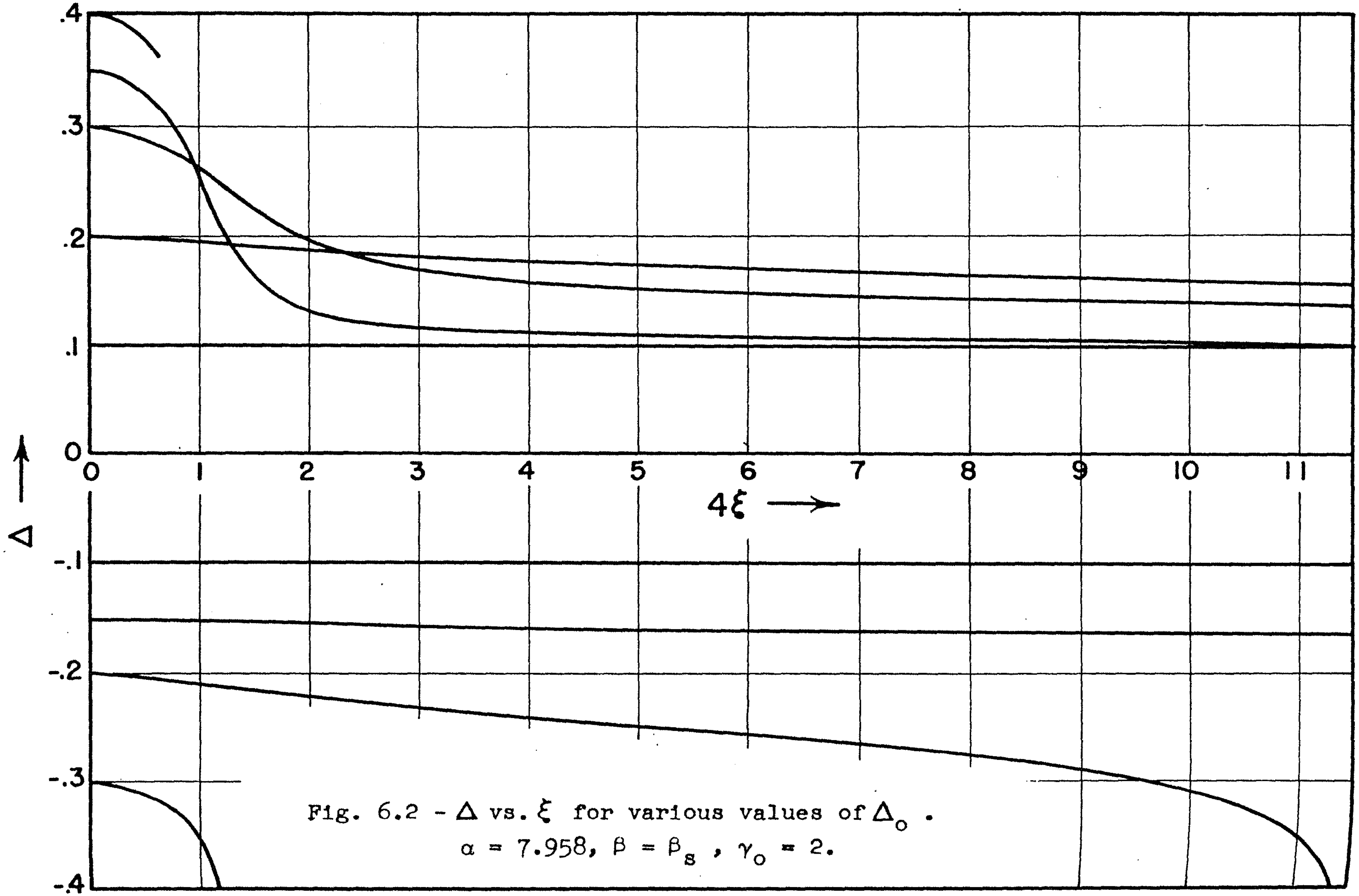

Fig. 6.2 - Δ vs. ξ for various values of Δ_o .
$\alpha = 7.958$, $\beta = \beta_s$, $\gamma_o = 2$.

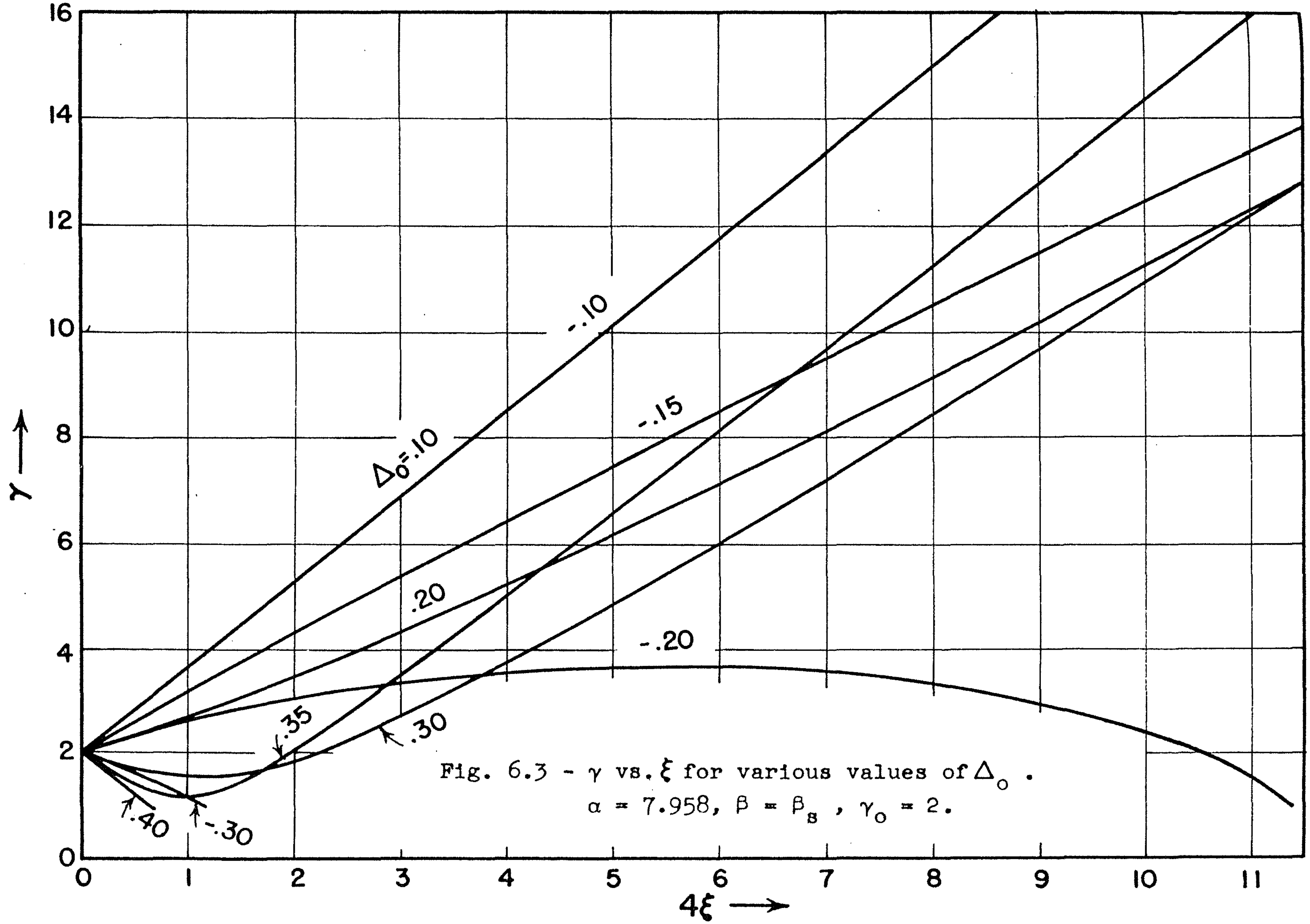

Fig. 6.3 - γ vs. ξ for various values of Δ_o.
$\alpha = 7.958$, $\beta = \beta_s$, $\gamma_o = 2$.

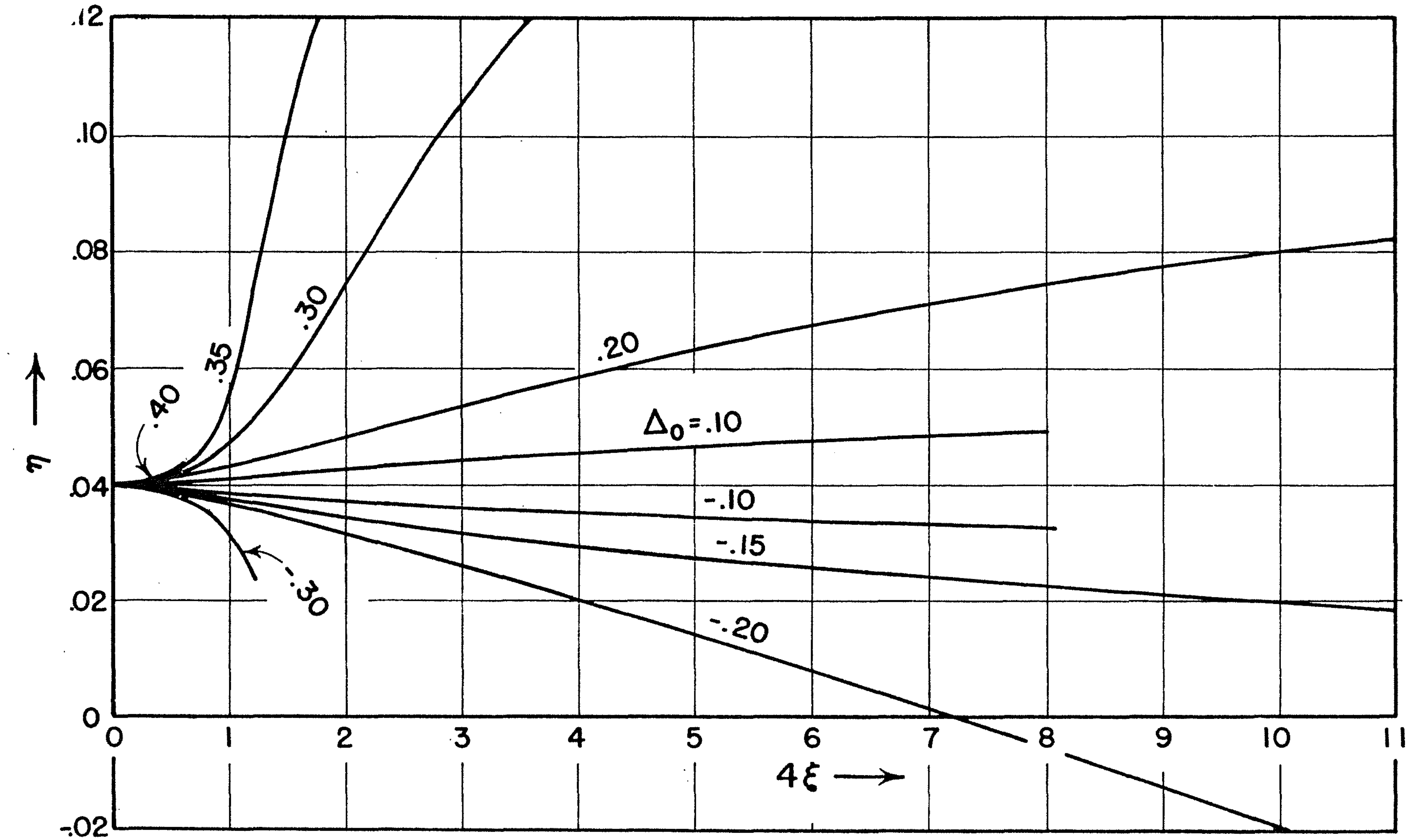

Fig. 6.4 - η vs. ξ for various values of Δ_o .

$\alpha = 7.958$, $\beta = \beta_s$, $\gamma_o = 2$, $\eta_o = .040$, $(d\eta/d\xi)_o = 0$.

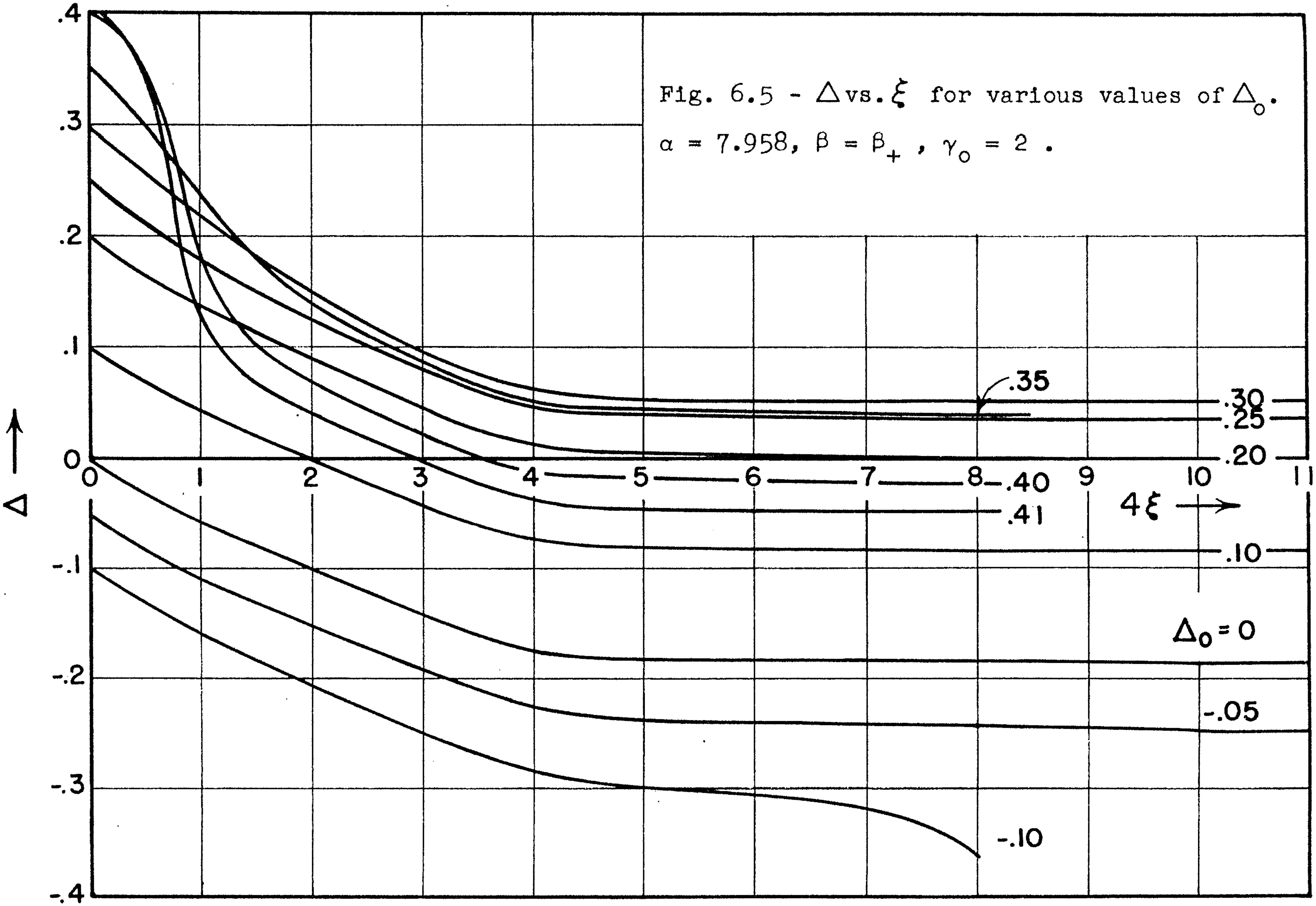
Fig. 6.5 - Δ vs. ξ for various values of Δ_o.
α = 7.958, $\beta = \beta_+$, γ_o = 2 .
.4
.3
.2
.1
0
-.1
-.2
-.3
-.4
Δ
0
1
2
3
4
5
6
7
8
9
10
11
4ξ
.35
.30
.25
.20
.40
.41
.10
Δ_o = 0
-.05
-.10

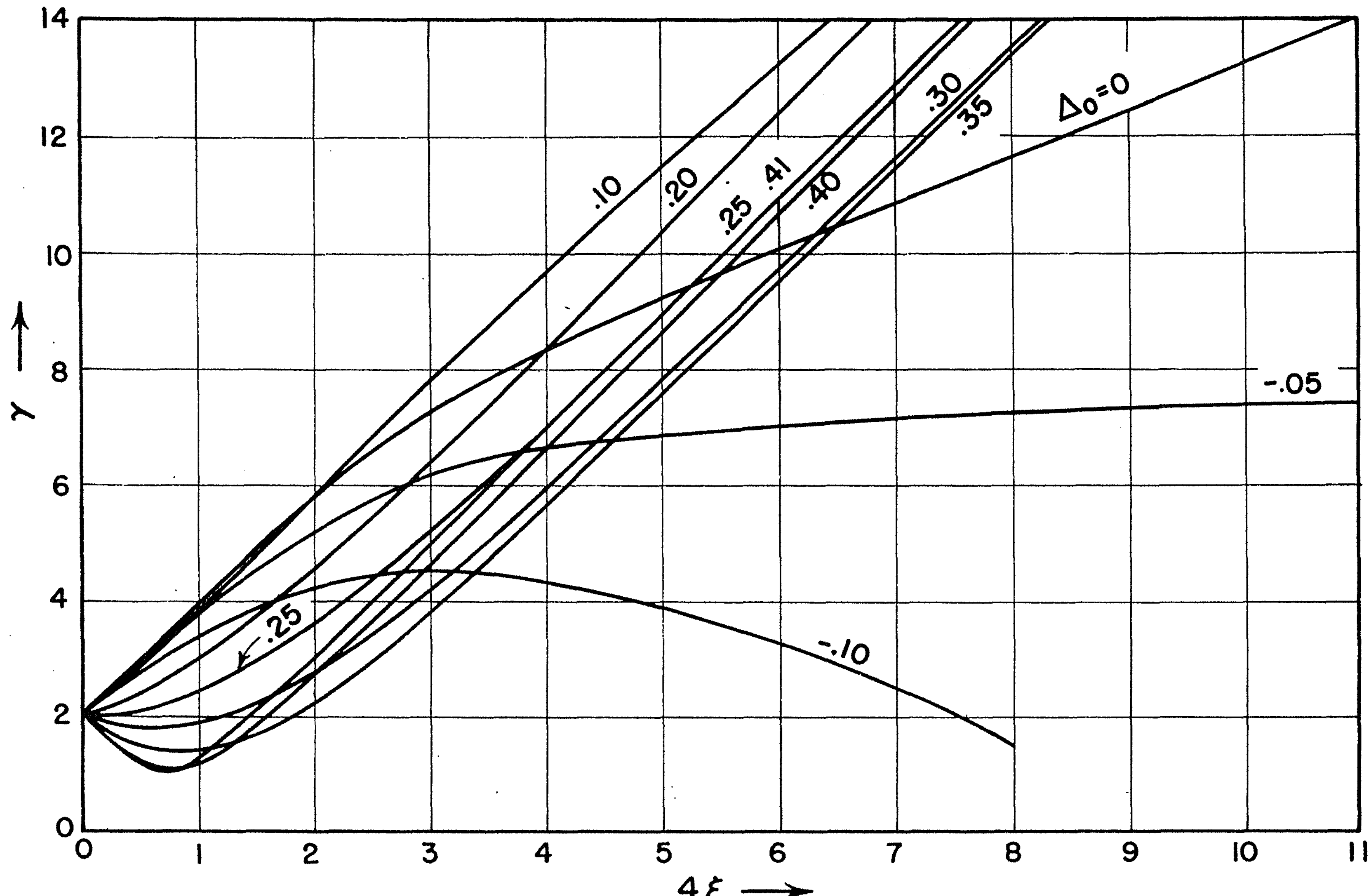

Fig. 6.6 - γ vs. ξ for various values of Δ_o .

$\alpha = 7.958$, $\beta = \beta_+$, $\gamma_o = 2$.

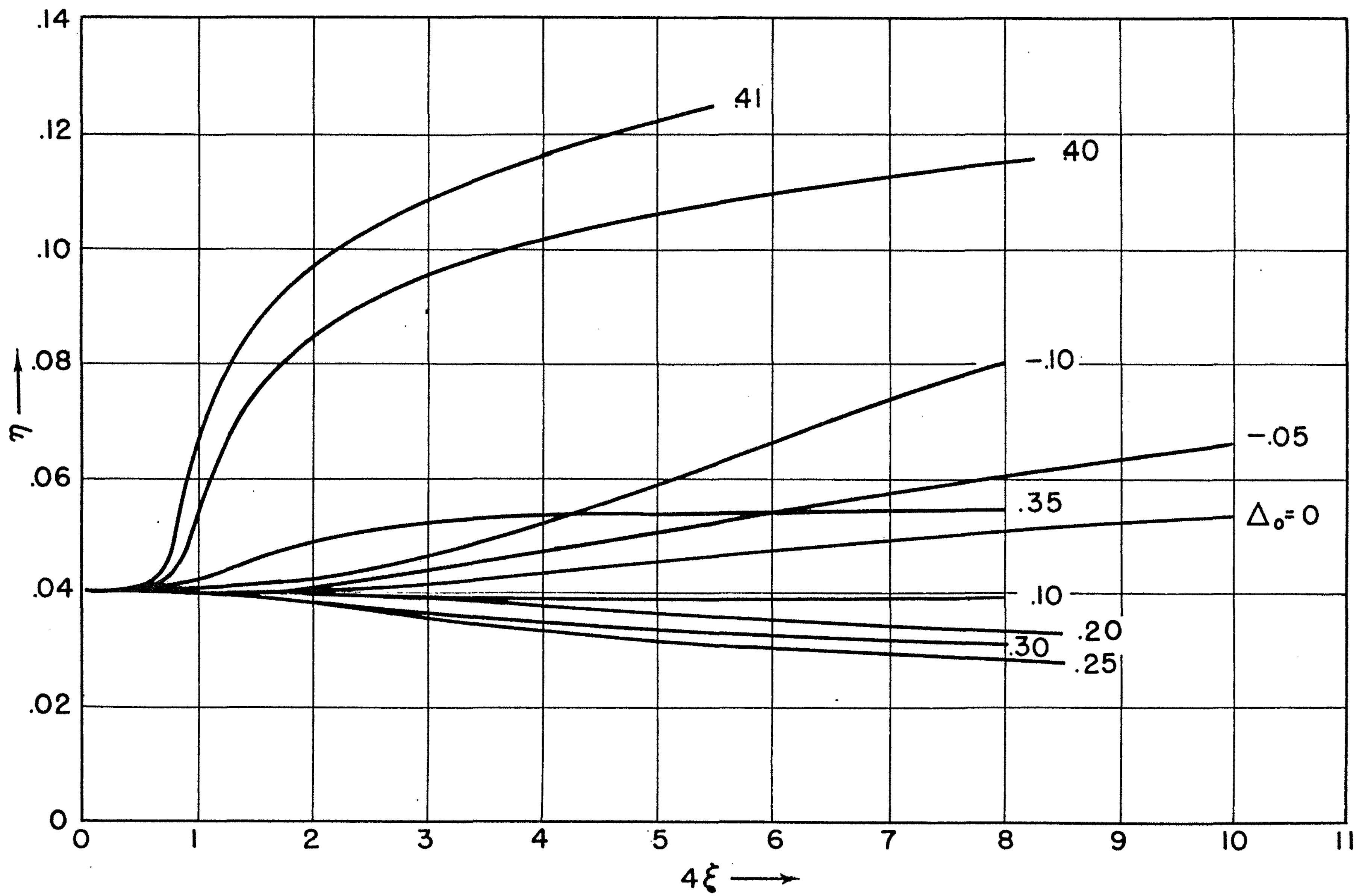

Fig. 6.7 - η vs. ξ for various values of Δ_o .

$\alpha = 7.958$, $\beta = \beta_+$, $\gamma_o = 2$, $\eta_o = .040$, $(d\eta/d\xi)_o = 0$.

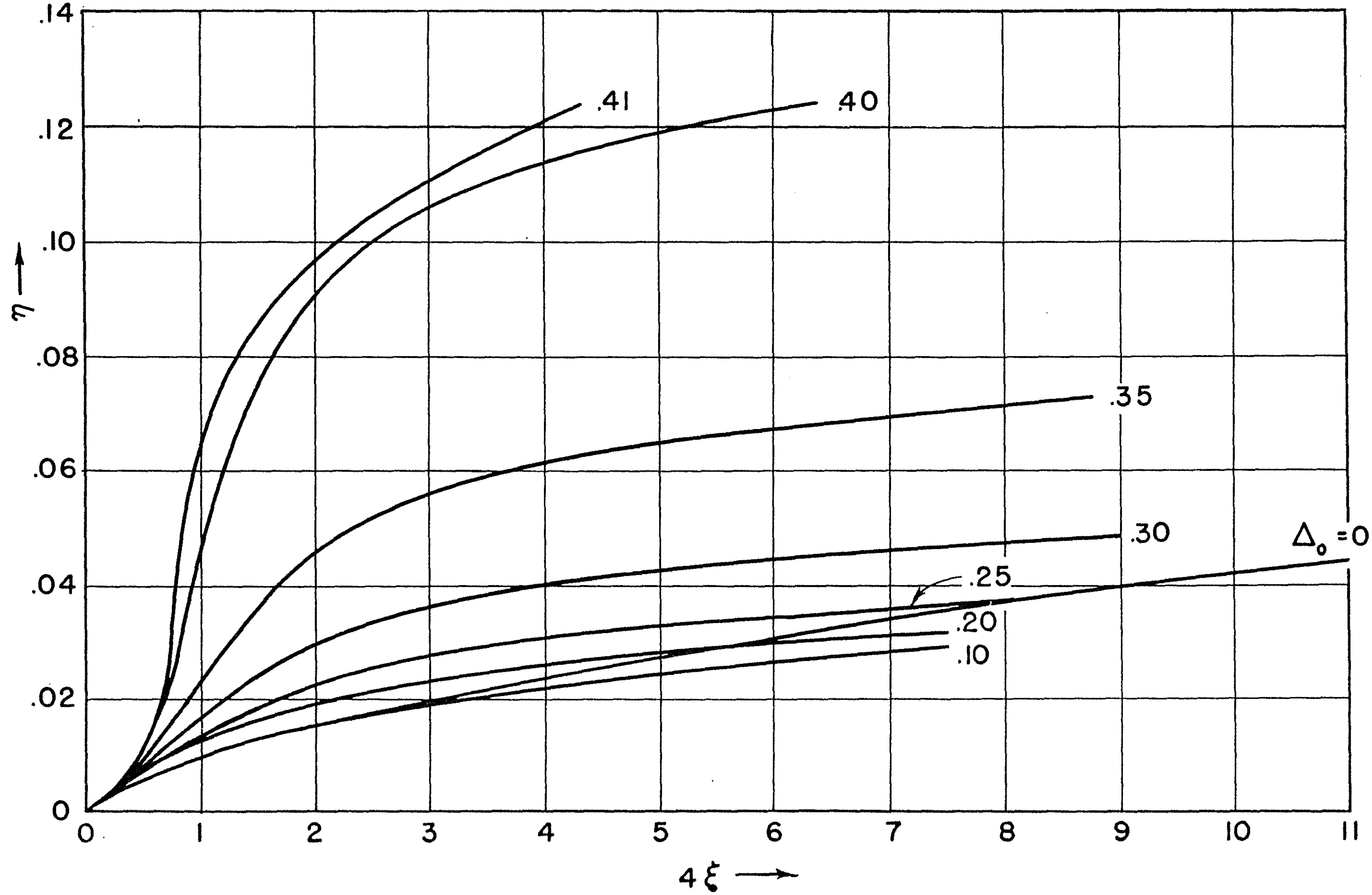

Fig. 6.8 - η vs. ξ for various values of Δ_o .

$\alpha = 7.958$, $\beta = \beta_+$, $\gamma_o = 2$, $\eta_o = 0$, $(d\eta/d\xi)_o = .060$.

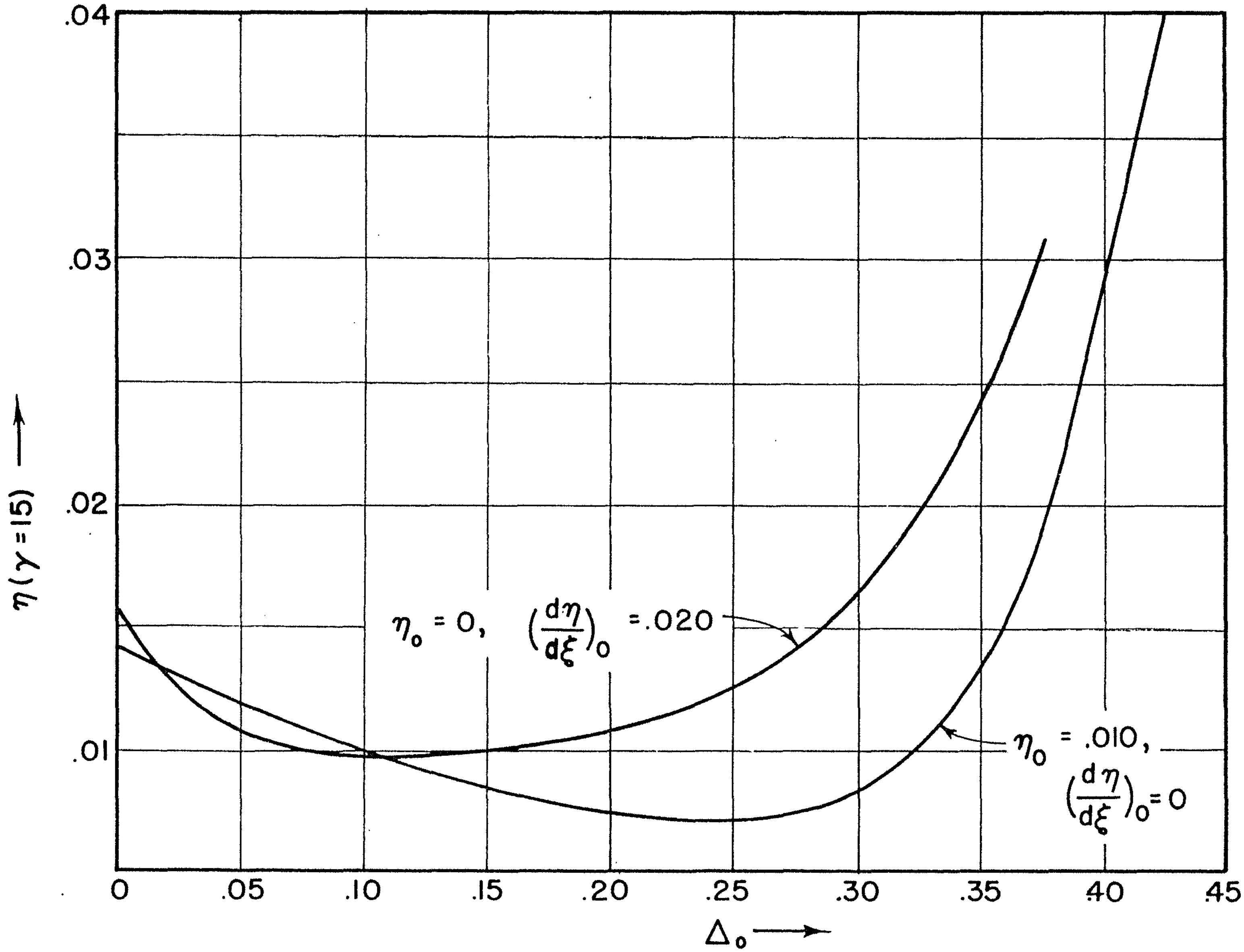

Fig. 6.9 - $\eta(\gamma = 15)$ vs. Δ_o for two different sets of initial values. $\alpha = 7.958$, $\beta = \beta_+$, $\gamma_o = 2$.

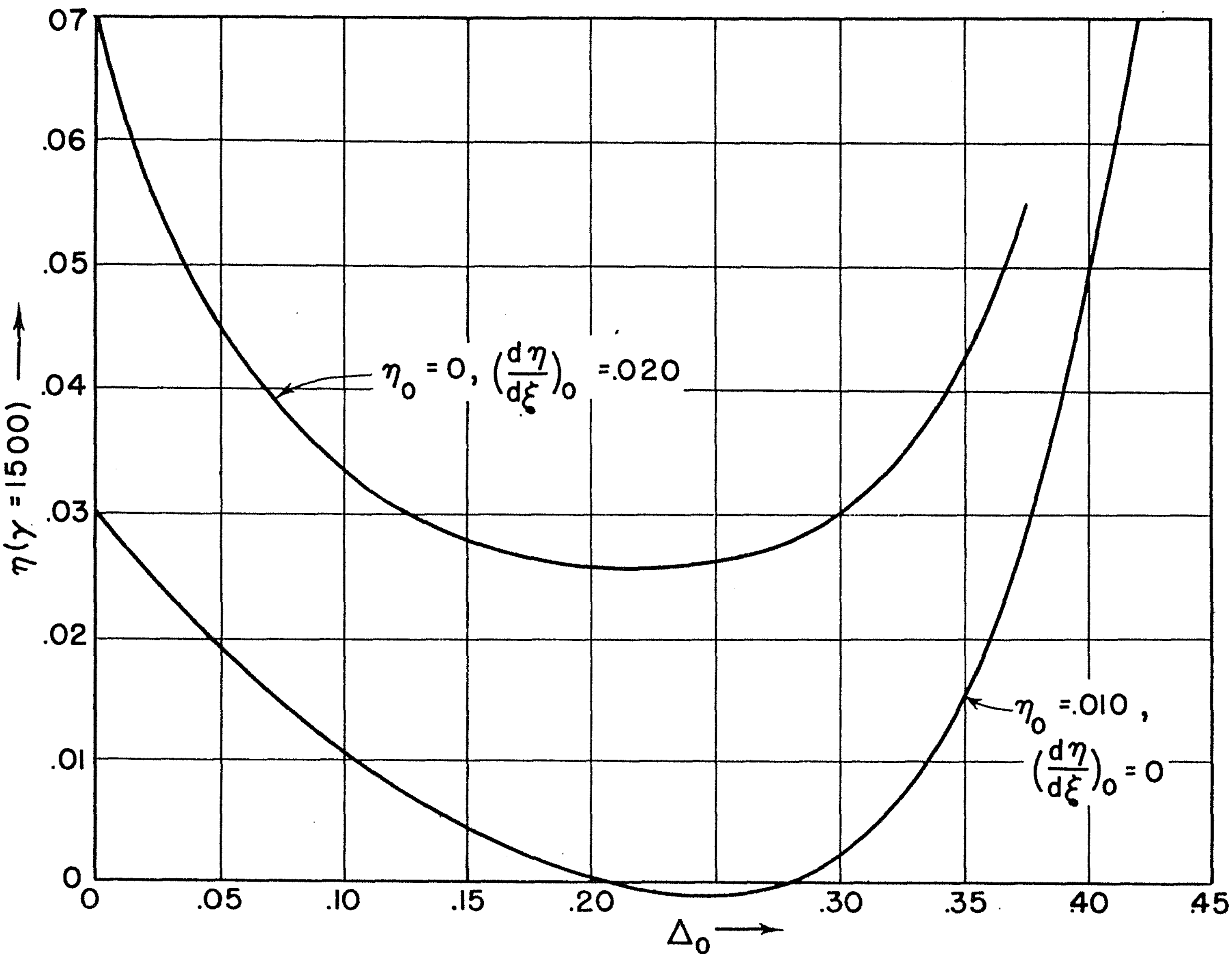

Fig. 6.10 - $\eta(\gamma = 1500)$ vs. Δ_0 for two different sets of initial values. $\alpha = 7.958$, $\beta = \beta_+$, $\gamma_0 = 2$.

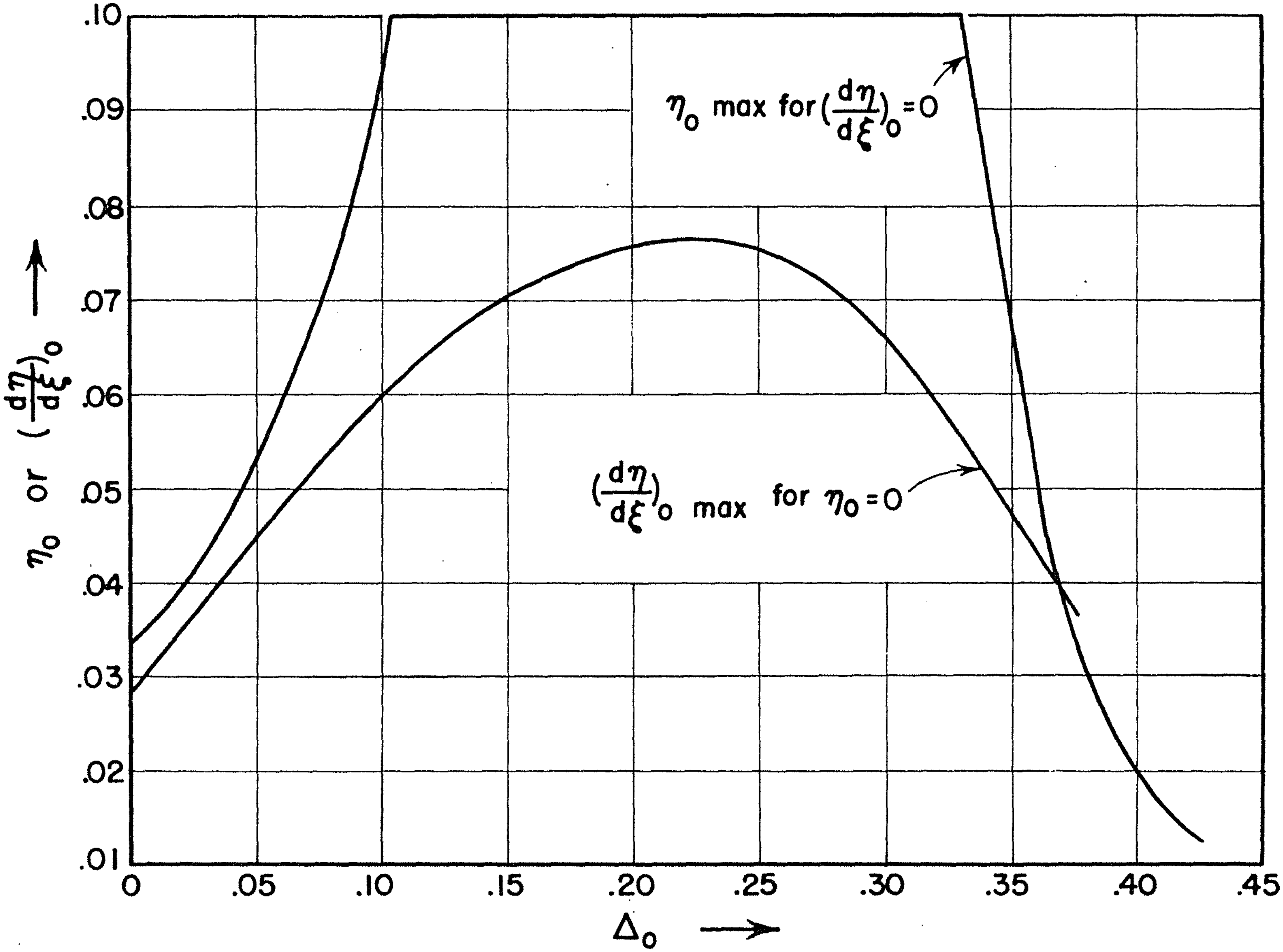

Fig. 6.11 - Maximum allowable values of η_o and $(d\eta/d\xi)_o$ vs. Δ_o .
$\eta(\gamma = 1500) \leq .100$. $\alpha = 7.958$, $\beta = \beta_+$, $\gamma_o = 2$.

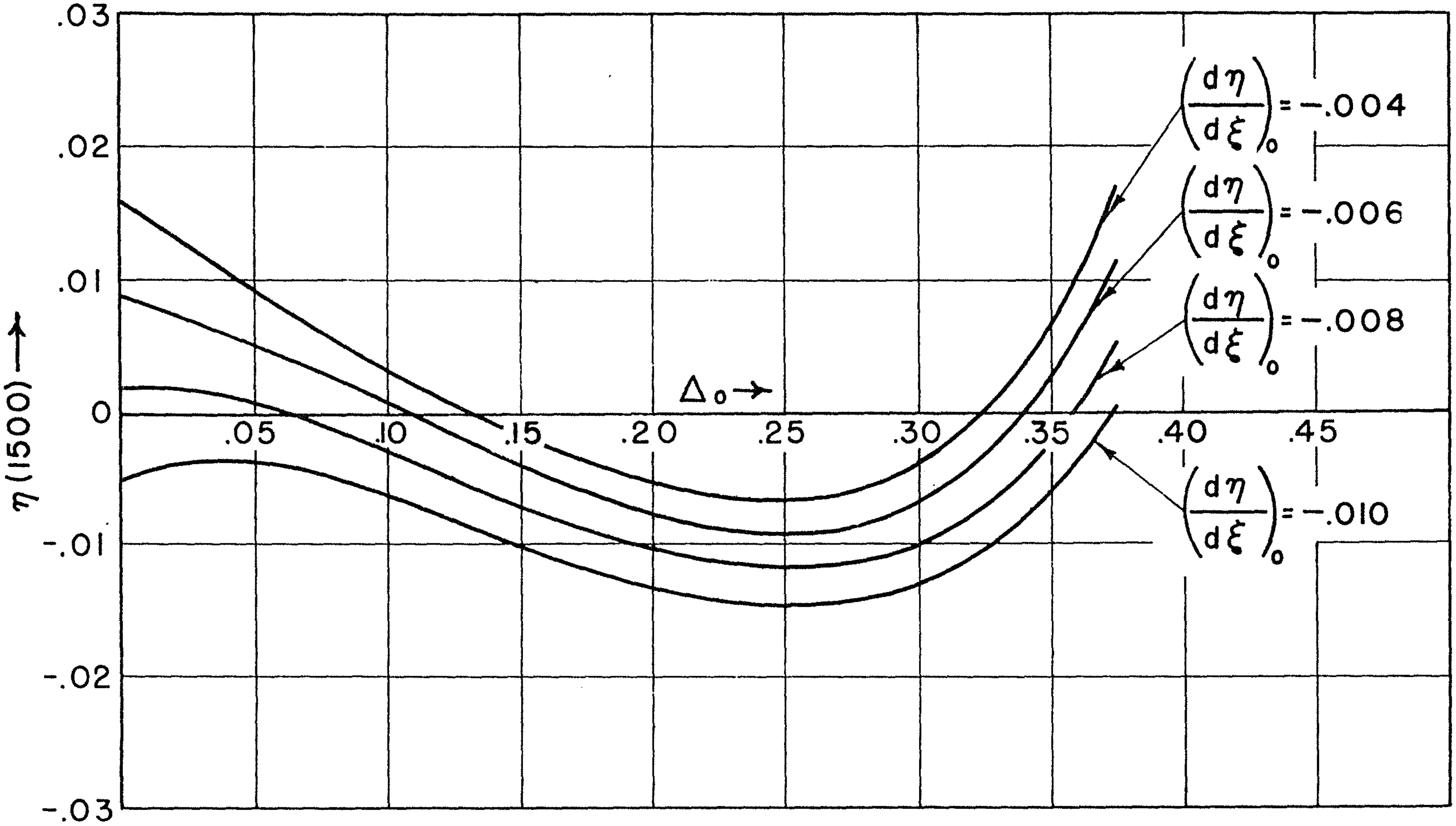

Fig. 6.12 - Curves showing $\eta(\gamma = 1500)$ vs. Δ_o for initially converging beams. η_o = .010 for all cases; $(d\eta/d\xi)_o$ has different values. $\alpha = 7.958$, $b = \beta_+$, $\gamma_o = 2$.

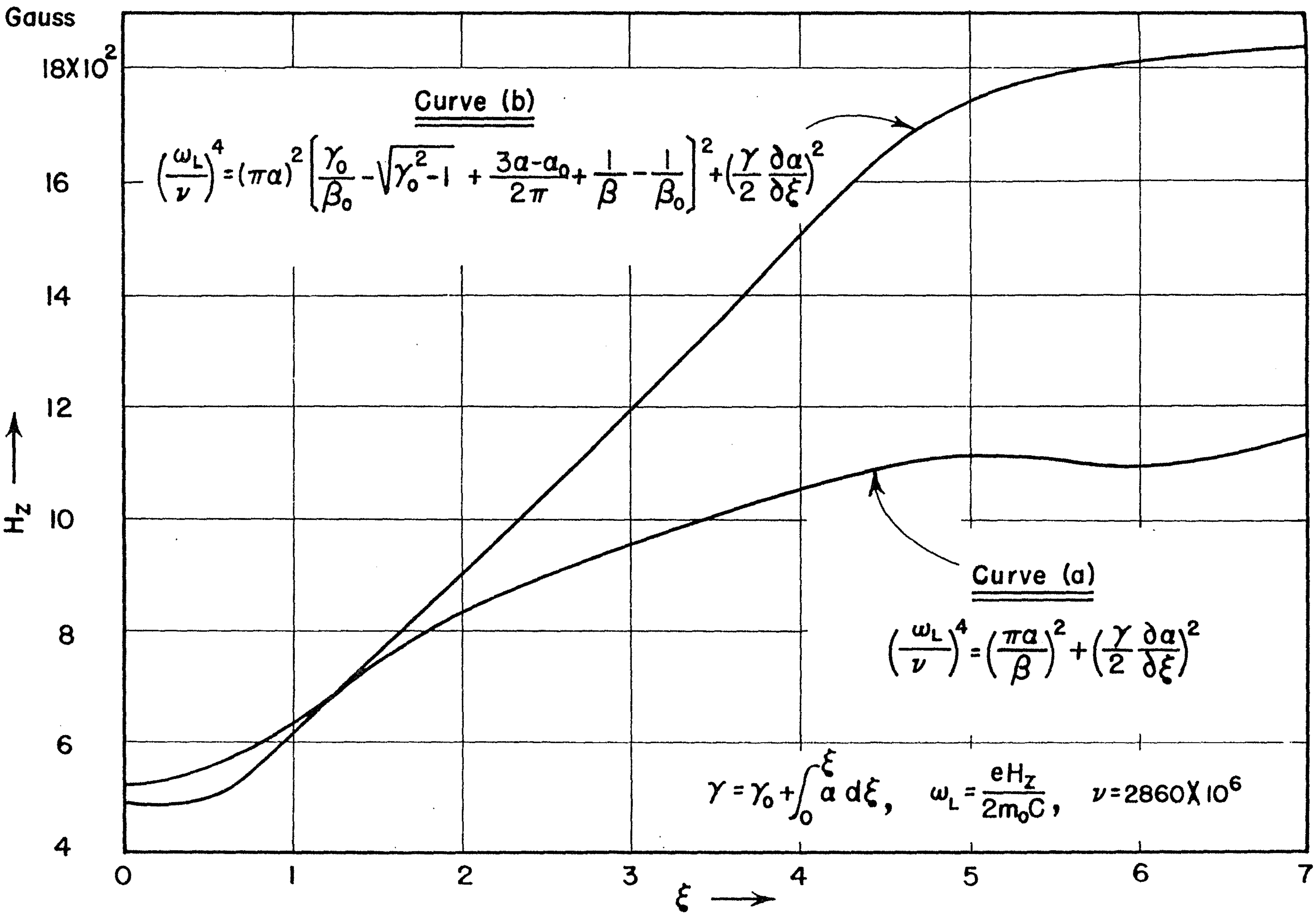

Fig. 6.13 - The estimated values of the focusing field strength for a buncher having parameters shown in Fig. 5.8 and characteristics shown in Fig. 5.9 are plotted against ξ.

CHAPTER VII

FIELD ENERGY AND RELATED CHARACTERISTICS

In the previous discussion of the accelerating field, our attention has been directed almost entirely to the field intensities and, in particular, to the two parameters α and β which represent respectively the amplitude and the phase velocity of the field. The field problem is completely solved once the field intensities are determined. From the intensities other field quantities may be derived by the well-known relation of electromagnetism. Characteristic quantities related to the energy concept, such as the energy flow, energy density, group and energy velocities, are of fundamental importance in both theory and practice. In this chapter we shall devote ourselves to the discussion of these quantities and their inter-relations together with the attenuation loss, which so far has been neglected, and the related parameters such as Q and the shunt impedance. The complication introduced by the loading of the waveguide and the consequent presence of a manifold of Fourier components appears here in having non-vanishing cross products of the component intensities. This greatly increases the numerical labor required to evaluate these quantities especially when high accuracy is desired, but does not add any real difficulty to the discussion of principles. Consequently, we shall discuss these quantities in a general manner and only use the approximate forms of field expressions for giving specific results.

7.1. Energy Flow

The flow of energy per unit area per unit time at a certain point is given by the Poynting vector $\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{H}$. Here $\vec{E} \times \vec{H}$ has only two components: $E_r H_\phi$ in the $\vec{z}$ direction and $-E_z H_\phi$ in the $\vec{r}$ direction. The time-averaged radial flow of energy across the surface r = a supplies the ohmic loss to the outer metallic boundary. For any one cell or cavity between two neighboring loading disks, the average radial flow of energy across r = a must be equal to the difference between the average rate of the axial energy flow at one disk hole and that at the next hole. The energy loss on the boundary may most conveniently be evaluated from the Poynting vector right at the metallic surfaces and will be discussed in a later section. At present we shall only be concerned with the axial flow of energy which in a traveling-wave system supplies the power dissipated in the terminating load in addition to the loss in the cavities.

From $\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{H}$ we have

$$\overline{S}_z = \frac{c}{8\pi} \text{R.P.} \left[E_r^* \; (r,z;t) \; H_\phi(r,z;t) \right] , \tag{7.1}$$

where E_r and H_ϕ are given in equations (4.3) or (4.3'). Let us write $\overline{S}_z$ in its series form

$$\overline{S}_z = \sum_{m,n} (\overline{S}_z)_{mn} \tag{7.2}$$

with

$$(\overline{S}_z)_{mn} = \frac{c}{8\pi} E_m E_n \frac{k_{zm}a \; J_1(k_{rm}a)}{k_{rm}a} \; \frac{ka \; J_1(k_{rn}a)}{k_{rn}a}$$

$$\cdot \cos\left[(k_{zm}z + \phi_m) - (k_{zn}z + \phi_n)\right] , \qquad (7.3)$$

and denote

$$P_{mn} \equiv \int_0^a (\overline{S}_z)_{mn} \, 2\pi r \, dr , \qquad (7.4)$$

then

$$P = \int_0^a \overline{S}_z \, 2\pi r \, dr = \sum_{m,n} P_{mn} \qquad (7.5)$$

is the time-averaged energy flow in the axial direction across a disk hole or any transverse plane with $r \lesseqgtr a$. The integration can easily be carried out, thus

$$P_{mm} = \frac{c}{8} E_m^2 \, a^2 \frac{k_{zm}a \; ka}{(k_{rm}a)^2} \left[J_1^2(k_{rm}a) - J_o(k_{rm}a) \; J_2(k_{rm}a)\right] , \qquad (7.6a)$$

$$P_{mn} = \frac{c}{4} E_m E_n \; a^2 \frac{k_{zm}a \; ka}{(k_{rm}a)^2 - (k_{rn}a)^2} \left[\frac{J_o(k_{rm}a) \; J_1(k_{rn}a)}{k_{rn}a}\right.$$

$$\left. - \frac{J_o(k_{rn}a) \; J_1(k_{rm}a)}{k_{rm}a}\right] \cdot \cos\left[(k_{zm}z + \phi_m) - (k_{zn}z + \phi_n)\right]$$

$$m \neq n . \qquad (7.6b)$$

P_{mm} is independent of z but P_{mn} $(m \neq n)$ depends on z. The origin of the coordinate axes may be so chosen that at a disk hole z = Nd, N being an integer, then for such values of z

$$\cos\left[(k_{zm}z + \phi_m) - (k_{zn}z + \phi_n)\right] = \cos(\phi_m - \phi_n).$$

From equations (7.6) we note that P_{mm} is positive if $k_{zm} > 0$, i.e., if $m > 0$, and is negative if $m < 0$. Naturally P_{mm} is interpreted as the flow of energy carried by the m-th Fourier component while P_{mn} is the flow of energy arising from the interaction between the m-th and the n-th components. $P_{mn} \cdot k_{zn}a = P_{nm} \cdot k_{zm}a$. P_{mn} and P_{nm} have the same or opposite signs like k_{zm} and k_{zn}, so like m and n. P_{oo}/P is the fraction of energy flow or power due exclusively to the fundamental component which only is responsible for the acceleration of electrons. $E_o^2 c/P$ has the dimension of the reciprocal of an area and is proportional to the square of the electron energy gain per unit length divided by the total energy flow, so it measures the overall effectiveness of acceleration.

If the Fourier amplitudes are determined, the numerical evaluation of P_{mn} is straightforward and easy. While the technique of using simple trial functions has met with great success in the solution of eigenvalue problems, it nevertheless is not quite reliable for determining the first few Fourier amplitudes. In contrast to the eigenvalues, these amplitudes are rather sensitive to the choice of trial functions. If accurate values are desired, we must use more elaborate trial functions such as suggested at the end of Chapter III and apply variations to the unknown coefficients, or use some standard method to obtain an approximate solution of the infinite set of simultaneous equations which constitute the formal analytic solution of the field problem (equations

(3.14)). In this connection it may be pointed out that Slater's method[45] of reducing an infinite set to a finite number of equations by using asymptotic expansions is equivalent to the variational method using a trial function with one or more variable coefficients, e.g.,

$$E_z(r = a) = \frac{1}{\sqrt{d^2 - z^2}} + \alpha \cos k_{zo}z + \ldots \quad (\alpha \text{ variable}).$$

Regarding the magnitude of the energy flow due to the higher Fourier components, Walkinshaw[46] found by a rough calculation that approximately 98 per cent of the power is carried by the fundamental component in a waveguide with $\beta = .4$, $a = .2\lambda$, $k_{zo}d = \pi/2$ and disk thickness $= d/4$, but we have obtained figures as low as 75 per cent for similar dimensions. And our experience indicates that certain simple trial functions do give optimistic values. We believe that quite an appreciable fraction of the total power may be carried by the higher components including their interaction with the fundamental wave. However, for a qualitative discussion or for the calculation of other quantities, such as attenuation and Q, which are not so sensitive to the distribution of the field components, the omission of the higher components greatly simplifies the problem and therefore is highly recommended.

Thus instead of considering $E_o^2 c/P$ we consider the relation between E_o and P_{oo}. In practical units ($\mathcal{P}$ in watts

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45. J.C. Slater, loc.cit., footnote 13.

46. W.Walkinshaw, Proc. Phys. Soc., 61, 246-254 (1948).

and $\mathcal{E}$ in volts per cm.) the relation is

$$\mathcal{P}_{oo} = \frac{(\mathcal{E}_o\lambda)^2}{30} \frac{1}{32\pi^2} \frac{1}{\beta} \frac{(ka)^4}{(k_{ro}a)^2} \left[J_1^2(k_{ro}a) - J_o(k_{ro}a)J_2(k_{ro}a)\right],$$

i.e.,

$$\mathcal{E}_o\lambda = \left[\frac{\lambda^2}{\pi a^2} \frac{(k_{ro}a)^2}{(ka)^2} \frac{2\,\mathcal{P}_{oo}\,\beta\,Z_o}{J_1^2(k_{ro}a) - J_o(k_{ro}a)J_2(k_{ro}a)}\right]^{1/2}, \tag{7.7}$$

where $Z_o = 120\pi$ is the so-called intrinsic impedance of free space. The case of $\beta = 1$ is most important. In this case $k_{ro} = 0$, so

$$\mathcal{E}_o\lambda = \frac{4}{\pi}\left(\frac{\lambda}{a}\right)^2 \sqrt{30\mathcal{P}}\ , \qquad (\beta = 1)\ . \tag{7.8}$$

We may also consider the relation between $\mathcal{P}_{oo}$ and the peak transverse voltage $\int_o^a \text{Amp.}\ \mathcal{E}_{r,o}\ dr$. From the field expression (4.3') we easily find

$$\int_o^a \text{Amp.}\ \mathcal{E}_{r,o}\ dr = \mathcal{E}_o\lambda \frac{kk_{zo}}{2\pi k_{ro}^2}\left[1 - J_o(k_{ro}a)\right]\ . \tag{7.9}$$

Needless to say, all these relations refer to a single transverse plane or to short distances within which the attenuation loss can be neglected. The question about the energy flow over long distances must wait till the subject of attenuation has been taken up.

7.2. Group Velocity

As is usual, the group velocity v_g is defined by

$$\frac{v_g}{c} = \frac{1}{c}\frac{d\upsilon}{d\tau_z} = \frac{dk}{dk_z} = \beta^2 \frac{d\lambda}{d\lambda_z} \tag{7.10}$$

and may be conceived as the velocity with which the envelope of a finite pulse of waves travels through the guide. v_g/c is given by the slope of the tangent to the υ/c vs. τ_{zo} curve shown in Fig. 2.2. Since υ is a periodic function of τ_z with a period 1/d, the group velocity is the same whether $\tau_z = \tau_{zo}$ or $\tau_z = \tau_{zn} = \tau_{zo} + n/d$. In other words all different Fourier components have the same group velocity, though those with negative n propagate in the negative direction.

By considering a closed section of the loaded waveguide such as shown in Fig. 2.4a we can easily show by perturbation arguments (see equation (3.44)) that

$$dk = -\int_{\gamma} (\vec{E} \times \vec{H})_n \, d\sigma \Big/ \int_{R} (E^2 + H^2) \, d\tau$$

where γ denotes the transverse plane at a distance $dz = (q/2)|d\lambda_{zo}|$ from one end plate (z = 0), R the total volume of the closed section and $\vec{E}$, $\vec{H}$ are real vector functions of space coordinates. Since

$$-\int_{\gamma} (\vec{E} \times \vec{H})_n \, d\sigma = \int_{\delta} \mathrm{div}\, (\vec{E} \times \vec{H}) \, d\tau ,$$

δ being the small volume of thickness dz bounded by the plane γ and the end plate, we obtain

$$\frac{dk}{d|\lambda_{zo}|} = -k \frac{\int_{\gamma} (H^2 - E^2) \, d\sigma}{\int_{V} (H^2 + E^2) \, d\tau} \tag{7.11}$$

where V is the volume of the guide of length λ_{zo}. The same relation can also be obtained by considering the radiation pressure on the end plates. To move one end plate by a distance dz a certain amount of work must be done. This increases the energy W of the system. Since W/υ is an adiabatic invariant, υ is increased by increasing W.[47] From (7.11) the group velocity follows directly:

$$\frac{\beta c}{v_g} = \frac{1}{\beta\lambda} \frac{\int_V (H^2 + E^2)\, d\tau}{\int_\gamma (H^2 - E^2)\, d\sigma} . \tag{7.12}$$

This relation states that the ratio of phase velocity to group velocity is equal to the ratio of the time-averaged electromagnetic energy per unit length of the resonating guide to the total time-averaged radiative force acting on one end plate.

From the defining relations (7.10) we also have

$$\frac{\beta c}{v_g} = 1 + \frac{\lambda}{\beta}\frac{d\beta}{d\lambda} = 1 - \frac{\upsilon}{\beta}\frac{d\beta}{d\upsilon} . \tag{7.10'}$$

The group velocity can only be equal to the phase velocity in non-dispersive media in which $d\beta/d\upsilon = 0$. For example, in free space $\beta = 1$ for all frequencies and both velocities are equal to c. In the loaded waveguide $\beta c = v_g$ when and only when

$$\frac{1}{\beta\lambda} \int_V (H^2 + E^2)\, d\tau = \int_\gamma (H^2 - E^2)\, d\sigma .$$

This relation, perhaps, can be satisfied for a single frequency for $\beta < 1$ by a proper choice of physical dimensions. Disregard-

47. See e.g. E.T. Jaynes, "Lecture Notes on Advanced Microwave Theory," Stanford University; R.B.R. Shersby-Harvie, Proc. Phys. Soc., 61, 267 (1948).

ing this exceptional case the disk-loaded waveguide acts like a dispersive medium, in which $d\beta/d\upsilon \neq 0$ and distortion of wave form takes place.

The concept of group velocity is very important in the discussion of energy- or phase-frequency relations. For effective acceleration the electrons must be maintained near the peak of the traveling wave. If the energy should not fall below the maximum obtainable value by a certain amount, the phase angle $2\pi\Delta$ must be controlled with a certain precision. Since

$$2\pi\Delta = \int k_{zo}\, dz - \omega \int dt \ ,$$

we obtain by differentiation with respect to υ

$$\frac{d\Delta}{d\upsilon} = - \int \frac{1}{\lambda_{zo}^2} \frac{d\lambda_{zo}}{d\upsilon}\, dz - \upsilon \frac{d}{d\upsilon} \int dt \ .$$

Let us consider a long accelerator of length ℓ. If ℓ is large compared to the length of the buncher in which the electron velocity may differ greatly from c, then $\int dt \cong \ell/c$ and

$$d\Delta = \frac{d\upsilon}{\upsilon} \left(\frac{c}{v_g} - 1\right) \frac{\ell}{\lambda} \qquad (7.13)$$

or

$$d\Delta = - \frac{d\lambda}{\lambda} \left(\frac{c}{v_g} - 1\right) \frac{\ell}{\lambda} \ . \qquad (7.13')$$

To avoid appreciable phase shift we must keep the frequency constant. An increase in length of the accelerator guide due to an increase in temperature has the same effect on Δ as the same proportional decrease in wavelength; so we must also control the temperature. It will be seen later that c/v_g is usually large compared to unity, the phase shift per unit fractional change of frequency or wavelength is directly proportional to ℓ/λ and approximately to the reciprocal of the group velocity. This kind of instability, if not minimized, can prevent the accelerator from working satisfactorily, especially if ℓ is very large. Shersby-Harvie[48] has given an excellent account of the effect of wavelength and temperature variations; reference is to be made to his paper for further details.

With wave functions given by equations (3.1), (3.2), (3.6) and (3.7), we easily find that the volume integral in (7.12) contains no cross products but the surface integral does. By neglecting the cross-product and the higher order terms, (7.12) may be written as[49]

$$\frac{\beta c}{v_g} = \beta^2 \, \frac{\int_0^a \left[H_\phi^{I}(o)\right]^2 2\pi r \, dr + 2\int_a^b (H_\phi^{II})^2 \, 2\pi r \, dr}{\int_0^a \left[H_\phi^{I}(o)\right]^2 \, 2\pi r \, dr - \beta^2 \cdot \int_a^b \left[(H_\phi^{II})^2 - (E_z^{II})^2\right] 2\pi r \, dr}, \tag{7.14}$$

where $H_\phi^I(o)$ denotes the value of H_ϕ^I at $z = 0$. We may note from this relation that as $a \rightarrow b$, $\beta c/v_g \rightarrow \beta^2$, i.e., $(\beta c)v_g \rightarrow c^2$ as is to be expected for an unloaded waveguide.

The approximate expression for group velocity may also be obtained directly by differentiating the approximate solution (equation (3.36)), which we discussed in section 3.2(i)

48. R.B.R. Shersby-Harvie, loc. cit., footnote 47, p.260.

49. It may be pointed out that the expression given by Shersby-Harvie is energy velocity instead of group velocity. See later discussion.

Thus

$$\frac{d\phi(k_{ro}a)}{d(k_{ro}a)} \frac{dk_{ro}}{dk} = \frac{d\alpha(ka,kb)}{d(ka)} \frac{1}{1-\eta} \quad .$$

Since $k^2 = k_{ro}^2 + k_{zo}^2$, i.e., $\frac{dk_{zo}}{dk} = \frac{k}{k_{zo}} - \frac{k_{ro}}{k_{zo}} \frac{dk_{ro}}{dk}$,

we obtain

$$\frac{dk_{zo}}{dk} = \frac{k}{k_{zo}} - \frac{k_{ro}}{k_{zo}} \frac{d\alpha/d(ka)}{d\phi/d(k_{ro}a)} \quad , \qquad (k_{ro}a \neq 0), \qquad (7.15)$$

this being equal to c/v_g. For the special case $\beta = 1$, i.e., $k_{ro} = 0$,

$$\frac{dk_{zo}}{dk} = 1 - \frac{1}{1-\eta} \frac{8}{ka} \frac{d\alpha}{d(ka)} \qquad (k_{ro}a = 0) \ . \qquad (7.15a)$$

In Fig. 7.1 we plot $v_g/c = dk/dk_{zo}$ against ka for kb = 2.66, η = .240 and β = 1. Another approximate expression, shown dotted in the same figure, is good for a $\ll$ d, and is obtained simply by differentiating the relation (3.46). Two other dotted curves are to be explained in the following section.

7.3. Energy Density and Energy Velocity

The group velocity is not to be confused with the energy velocity. They are not the same if there is attenuation and not necessarily the same if there is no attenuation. The energy velocity is defined as the ratio of the energy flow to the energy density per unit length. Let W denote the energy density and v_e the energy velocity. Since

$$W = \frac{1}{\beta\lambda}\frac{1}{16\pi}\int_V (H^* \cdot H + E^* \cdot E)\, d\tau = \frac{1}{\beta\lambda}\frac{1}{8\pi}\int_V H^* \cdot H\, d\tau \,, \tag{7.16}$$

we easily find

$$\frac{\beta c}{v_e} = \frac{1/\lambda \int_v H^* \cdot H\, d\tau}{\mathrm{R.P.} \int_0^a E_r^* \cdot H_\phi \, 2\pi r\, dr} \tag{7.17}$$

A comparison between the two equations (7.12) and (7.17) reveals that $v_g \neq v_e$. The loaded waveguide presents an interesting example of a dispersive medium in which the two velocities would not be equal even if there were no attenuation.

Thus we have three different wave velocities (phase, group and energy), all having important physical significance. It would be very interesting to perform a Lorentz transformation of the wave functions to determine what would appear to a moving observer with each of these velocities. Hershberger[50] has made such transformations for the field in an ordinary waveguide. He found that the field is reduced to the cut-off solution if transformed with the group or energy velocity and to a magnetostatic field if transformed with the phase velocity greater than c (an impossible velocity for a moving observer). The loaded waveguides have simple cut-off solutions, different group and energy velocities and can have any phase velocity greater or less than c; they would indeed offer many interesting aspects to the applied mathematicians. The essential features of a single and many successive Lorentz

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50. W.D. Hershberger, J. App. Phys., 16, p.465 (1945).

transformations with respect to the phase velocity have already been discussed by Ginzton, Hansen and Kennedy.[51]

The energy velocity may similarly be evaluated by neglecting the higher order terms. Thus we substitute the traveling-wave field expressions[52] in (7.17) and obtain

$$\frac{\beta c}{v_e} = \beta^2 \left[1 + \frac{\int_a^b H^* \cdot H \; 2\pi r \; dr}{\int_0^a H^* \cdot H \; 2\pi r \; dr} \right] . \tag{7.18}$$

Here again $(\beta c)\, v_e \rightarrow c^2$ as $a \rightarrow b$. For an unloaded waveguide the energy velocity equals the group velocity. It may be noted that $\int_a^b H^* \cdot H \; 2\pi r \; dr$ is proportional to the loss on the surface of a loading disk and $\int_0^a H^* \cdot H \; 2\pi r \; dr$ is proportional to the energy flow down the guide. Since $v_e < c$, the above relation shows most distinctly the fact that $\beta < 1$ is achieved at the expense of energy. The smaller β is, the greater the loss on the loading disks per unit energy flow.

Two useful approximations to (7.17) are exceedingly simple. First, if $ka \rightarrow 0$ we can approximate H_ϕ quite well as $\frac{2.405 \, E_0}{(1 - \eta) kb} \cdot J_1 (2.405 \, \frac{r}{b})$ in which case the energy density, as computed by doubling the magnetic energy, comes out as

$$W \cong 0.195 \, E_0^2 \, / \, (1 - \eta) \, k^2 , \qquad ka \ll 1 \; . \tag{7.19}$$

Combining this with the expression for energy flow we get one

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51. E.L.Ginzton, W.W.Hansen, W.R. Kennedy, loc. cit. footnote 7.

52. E.L.Chu, W.W. Hansen, loc. cit., footnote 1.

of the approximations for the energy velocity used in Fig. 7.1.

Next, consider the special case $\beta = 1$ and let $kb \to \infty$, so that $kb \cong ka + \frac{2}{(1 - \eta)\, ka}$. Then we have $H_\phi = E_o \frac{kr}{2}$ for $r < a$ and we can take $H_\phi \cong \frac{E_o}{1 - \eta} \frac{1}{kb - ka}$ = constant for $a < r < b$. This approximation is obviously good for large ka and is surprisingly close for small ka, being only 1.55 times too small for ka = 0. Using these fields, we find

$$W \cong \frac{E_o^2}{8k^2} \left(\frac{k^4 a^4}{8} + \frac{1}{1 - \eta} \frac{kb + ka}{kb - ka}\right) ,$$

$$\beta = 1, \quad ka \gg 1 , \qquad (7.20)$$

and this may be further simplified by using (3.42) to relate kb and ka. Doing so, we get the remaining approximate formula of Fig. 7.1.

From this figure we see that the group and energy velocities, though different in principle, are not far from each other numerically. For practical purposes either one may be used in place of the other.

Knowing the velocity of the transport of energy we obtain the time required for the traveling field to fill up the accelerator guide by dividing the length of the guide by the energy velocity, i.e. ℓ/v_e. For a long accelerator the power is invariably fed at many different points, thus dividing the whole guide into many sections. Let the length of each section be $\Delta\ell$, then the filling time of the accelerator is $\Delta\ell/v_e$. The duration of the power pulse must in no circumstances be shorter than the sum of the filling time and the acceleration time in order that the waveguide may be completely filled up with power during the period of active acceleration. While the filling time is easily understood, the transient phenomena in the loaded guide are very complicated. In fact, the corresponding problem for ordinary waveguides is already formidable enough and has only recently been investigated.[53]

7.4. Attenuation: Attenuation Constant, Attenuation Length, Q, and Shunt Impedance.

When actual power requirements are considered, the attenuation loss must not be neglected. If there were no loss of power accompanying the transmission of waves down the waveguide, it would be possible to increase the total electron energy indefinitely by simply increasing the length of the accelerator without feeding additional power. We shall assume that the attenuation is caused entirely by the ohmic loss on the metallic walls; the absorption of power by the electrons themselves is usually negligible or can readily be accounted for. The calculation of the power loss on the walls is simple enough in principle; the time-averaged energy loss per unit length of guide is equal to

$$\left|\frac{dP}{dz}\right| = \frac{\delta \upsilon}{8} \frac{1}{\beta\lambda} \int_{\Gamma} H^{*} \cdot H \, d\sigma \tag{7.21}$$

where $\delta = \frac{1}{2\pi}\sqrt{\frac{\rho\lambda}{\mu}}$ is the skin depth, μ the permeability and

53. Manuel Cerrillo, M.I.T.Research Laboratory of Electronics, Technical Report No. 33, Jan. 3, 1948.

ρ the resistivity of the metallic wall (at $\lambda = 10$ cm., $\delta \cong 1.20 \times 10^{-4}$ cm. for Cu and $\delta \cong 1.17 \times 10^{-4}$ cm. for Ag.), and Γ denotes the surface enclosing V of length $\beta\lambda$. Hence we shall not describe the detailed numerical work; we shall only give some specific results obtained from a typical calculation based on the approximate theory of disk-loaded waveguides. Before doing this we shall first consider how the results are to be expressed in various useful forms.

At least three forms are useful. First, we may consider the attenuation constant, i.e., the voltage attenuation per unit length

$$I = \frac{1}{2P}\left|\frac{dP}{dz}\right| , \qquad \text{i.e., } P = P(o)\ e^{-2Iz} \quad , \tag{7.22}$$

where P(o) is the energy flow at z = 0. It can readily be shown that for any given guide

$$I \sim \frac{\delta}{\lambda^2} \sim \lambda^{-3/2} \qquad \text{or} \qquad I = I_o \lambda^{-3/2} \quad , \tag{7.23}$$

I_o being a proportionality constant which depends on the geometry and the material of the waveguide. I and I_o may further be separated into two parts, one pertaining to the cylindrical wall or tube and the other to the loading disks. The loss on the disks is proportional to the number of disks per wavelength, $n = \lambda_{zo}/d$, and constitutes by far the major part of the attenuation; consequently, I and I_o are approximately proportional to n. Since I_o determines the phase velocity, I is certainly a function of k_z or λ_z.

Second, we can determine

$$Q = \frac{\omega \text{ x energy density}}{\text{power loss per unit length}} = \omega W / \left| \frac{dW}{dt} \right| \qquad (7.24)$$

irrespective of whether the flow is steady or not. Q plays an important role in the discussion of the standing wave accelerator.[54] If the accelerator tube is closed by two lossless conducting end plates to form a resonant cavity, the Q measured should be the same as given by (7.24). From this relation we obtain immediately for a resonant cavity

$$W = W_o \, e^{-\frac{\omega}{Q} t} \quad . \qquad (7.25)$$

Here $Q/\omega \equiv T_o$ may be called the decay time in which the energy density falls to 1/e of its initial value. By an analogy of the lumped circuit theory, the time required for the energy density in a resonant cavity to build up to (1 - 1/e) of its steady-state value will also be T_o.

To consider the traveling waves we make use of the relation

$$\frac{dW}{dt} = \frac{\partial W}{\partial t} + \frac{\partial W}{\partial z}\frac{dz}{dt} = \frac{\partial W}{\partial t} + v_e \frac{\partial W}{\partial z} \quad ,$$

i.e.,

$$\frac{\partial W}{\partial t} + v_e \frac{\partial W}{\partial z} + \frac{\omega}{Q} W = 0 \quad .$$

Here, a strip of waves is treated as one of particles moving with the velocity dz/dt which is identified as equal to v_e. Since $P = Wv_e$ we obtain

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54. J.C. Slater, loc. cit., footnote 8.

$$\frac{\partial P}{\partial t} + v_e \frac{\partial P}{\partial z} + \frac{\omega}{Q} P = 0 \quad . \tag{7.26}$$

For a steady flow $\partial W/\partial t = 0$, $\partial P/\partial t = 0$ and $\frac{dW}{dt} = \frac{\partial P}{\partial z} = \frac{dP}{dz}$.

Hence

$$Q = \omega W / \left| \frac{dP}{dz} \right| \tag{7.27}$$

and

$$P(z) = P(o) \, e^{-\frac{\omega}{v_e Q} z} \quad . \tag{7.28}$$

Comparing this relation with (7.22) we find $I = \frac{1}{2} \frac{\omega}{v_e Q}$.

$$\frac{1}{2I} = \frac{v_e Q}{\omega} \equiv \ell_o \tag{7.29}$$

is the so-called attenuation length. Stating in words, the attenuation length is the reciprocal of the power attenuation and is equal to the decay time multiplied by the energy velocity; it is the distance in which the energy flow falls to 1/e of its initial value. The attenuation length, the decay and the filling time have important bearing on the design of the power-feed system, and have been discussed in detail in at least two well-known articles.[55]

Both I and Q are criterions for judging the performance of the accelerator tube as a waveguide or as a resonator. The third form is the so-called shunt impedance per unit length which is defined as the ratio of the square of the electron voltage gain per unit length to the power dissipation per unit length, i.e., $E_o^2 / \left| \frac{dP}{dz} \right| = E_o^2 / \, 2IP$. This quantity is useful because it measures the effectiveness or

55. J.C. Slater, loc. cit. footnote 8; E.L.Ginzton, W.W. Hansen, W.R.Kennedy, loc. cit., footnote 7.

power economy of the accelerator. Let r denote the shunt impedance per unit length in practical units (ohms/cm.), $r/30 = c \frac{E_o^2}{\left|\frac{dP}{dz}\right|} = c \frac{E_o^2}{2IP}$ where E_o, P and $\frac{dP}{dz}$ are as usual in Gaussian units.

The three quantities I, Q and r are inter-connected with one another, all being functions of one or more of the fundamental quantities, namely, the energy flow, energy density and the power loss per unit length. It is convenient to list them together in explicit integral forms:

$$\frac{I\lambda^2}{\delta} = \frac{\pi}{2} \frac{\lambda \int_\Gamma H^* \cdot H \, d\sigma}{\int_V H^* \cdot H \, d\tau} \cdot \frac{c}{v_e} \quad , \tag{7.30}$$

$$\frac{Q\delta}{\lambda} = \frac{2}{\lambda} \frac{\int_V H^* \cdot H \, d\tau}{\int_\Gamma H^* \cdot H \, d\sigma} \quad , \tag{7.31}$$

$$\frac{r\delta}{30} = 8\beta \frac{(E_o\lambda)^2}{\int_\Gamma H^* \cdot H \, d\sigma} \, , \tag{7.32}$$

$$\frac{c}{v_e} = \frac{1}{\beta\lambda} \frac{\int_V (H^* \cdot H) \, d\tau}{\text{R.P.} \int_{\substack{\text{disk}\\ \text{hole}}} E_r^* \cdot H_\phi \, d\sigma} \quad . \tag{7.17}$$

In passing we may point out that all these quantities can directly or indirectly be measured but none too accurately.[56]

56. E.L.Ginzton, W.W.Hansen, W.R.Kennedy, loc. cit. footnote 7. W.W.Hansen, R.F.Post, J.App.Phys., 19, 1059-1061, (1948).

The only quantity which can be measured with ease and precision is the group velocity; and this, as we have pointed out before, is not the same as the energy velocity. Until a rigorous calculation of the field amplitudes has been made, it is not even simple to judge the accuracy of various measurements.

The results of a typical calculation are given in Figs. 7.2, 7.3 and 7.4, which give $\frac{I}{k_{zo}}\frac{\lambda}{\delta}$, $\frac{Q\delta}{\lambda}$, and $\frac{1}{r\delta}$ as functions of ka, when kb is such that $\beta = 1$ and there are four loading disks per wavelength.

While the general expressions are quite complex, some approximations will now be given which are usefully simple. All of these are for the special case $\beta = 1$.

Two sub-cases are to be considered: $ka \ll 1$ and $ka \gg 1$. When $ka \ll 1$ we find by using the same field as that used for (7.19)

$$\frac{I\lambda^2}{\delta} \cong 2\pi \frac{6.25}{(ka)^4(1-\eta)^2}\left[n + 2.61(1-\eta)\right], \qquad (7.33)$$

$$\frac{Q\delta}{\lambda} \cong \frac{1-\eta}{n + 2.61(1-\eta)}, \qquad (7.34)$$

$$\frac{30}{r\delta} \cong \frac{.0308}{(1-\eta)^2}\left[n + 2.61(1-\eta)\right] \qquad (7.35)$$

$$(ka \ll 1, \quad \beta = 1).$$

When $ka \gg 1$ we find by using the approximation of (7.20)

$$\frac{I\lambda^2}{\delta} \cong \frac{(4\pi)^2}{(ka)^4}\left[\frac{kb}{(1-\eta)(kb-ka)^2} + \frac{n}{2\pi(1-\eta)^2}\cdot\frac{kb+ka}{kb-ka}\right.$$

$$\left. + \frac{n}{2\pi}\frac{\eta k^2 ad}{(1-\eta)^2(kb-ka)^2}\right] , \qquad (7.36)$$

$$\frac{Q\delta}{\lambda} \cong \left[\frac{(ka)^4}{8} + \frac{1}{1-\eta}\,\frac{kb+ka}{kb-ka}\right] \Bigg/ \left[\frac{2\pi kb}{(1-\eta)^2(kb-ka)^2}\right.$$

$$\left. + \frac{n}{(1-\eta)^2}\frac{kb+ka}{kb-ka} + \frac{n\eta k^2 ad}{(1-\eta)^2(kb-ka)^2}\right] , \qquad (7.37)$$

$$\frac{30}{r\delta} \cong \frac{1}{8}\left[\frac{kb}{(1-\eta)(kb-ka)} + \frac{n}{2\pi(1-\eta)^2}\;\frac{kb+ka}{kb-ka}\right.$$

$$\left. + \frac{n\eta k^2 ad}{(1-\eta)^2(kb-ka)^2}\right] \qquad (7.38)$$

$$(ka \gg 1, \quad \beta = 1)$$

When ka is sufficiently large, these may be simplified by using the approximate relation (3.42) with results

$$\frac{I\lambda^2}{\delta} \cong \frac{2\pi\lambda}{b} , \qquad (7.39)$$

$$\frac{Q\delta}{\lambda} \cong \frac{b}{2\lambda} , \qquad (7.40)$$

$$\frac{30}{r\delta} \cong \frac{1}{32}(kb)^3 . \qquad (7.41)$$

A qualitative understanding of the above relations is useful and simple. For $\beta = 1$ we have $E_r = H_\phi = E_o\, kr/2$ for $r < a$, and we will not do too badly by continuing H_ϕ in a linear manner to $r = b$. Thus the energy flow will be proportional to $(ka)^4$, the energy density to $(kb)^4$ and the energy loss on the walls will vary like $(kb)^3$, all for constant axial field. Thus the energy velocity, which is the ratio of energy flow to energy density, at first rises rapidly as ka increases and then levels off as ka approaches kb. Likewise, the attenuation constant, which depends on the ratio of power loss per unit length to energy flow, starts by decreasing rapidly as ka increases. When ka is large, the decrease is slower, being simply due to greater cross-sectional area for the wave to carry power as compared to the perimeter bounding the wave and introducing losses. On the other hand, Q which depends on the ratio of energy stored to energy loss, depends mainly on kb which at first increases only slowly with rising ka. Thus 1/Q at first drops slowly, though finally varying like 1/ka, for essentially the same reason as the attenuation.

By reducing a/b, the energy velocity can be decreased to any desired extent while $Q \sim W/\left|\frac{dP}{dz}\right|$ and $r \sim E_o^2/\left|\frac{dP}{dz}\right| \sim W/\left|\frac{dP}{dz}\right|$ are not much affected. Consequently, the attenuation length can similarly be decreased; on the other hand the attenuation constant and the filling time can similarly be increased. It will be seen in the next section that it is this reduction of the attenuation length which enables the short traveling wave accelerators to have good power economy.

7.5. Power Input and Power Economy: Traveling Waves vs. Standing Waves; Single, Multiple, and Distributed Feeds.

The power input which we shall denote $\mathcal{P}_{in}$ is equal to the total energy flow from the power sources to the accelerator guide, i.e., $\mathcal{P}_{in} = \sum_m \mathcal{P}_m$. In standing-wave guides all power is dissipated on the metallic walls; in traveling-wave guides part of the power, $\mathcal{P}(\ell)$, flows into a matched termination on the output end of the guide of length ℓ. In the particular case where there is only one source feeding the traveling-wave guide, $\mathcal{P}_{in}$ is evidently equal to $\mathcal{P}(o)$. Since our end objective is electron energy or voltage, $V(\text{volts}) = \int_o^\ell \mathcal{E}_o(z)\, dz$, a convenient factor of power economy may be defined as $V^2/\mathcal{P}_{in}$. This factor differs somewhat from the total shunt impedance, which according to our previous definition should be $\int_o^\ell r dz = \dfrac{V^2}{\mathcal{P}_{in} - \mathcal{P}(\ell)}$, provided that $\mathcal{P}(\ell) \neq 0$.

Let us consider the single feed first. In this case

$$\left.\begin{aligned} &\mathcal{E}_o(z) = \mathcal{E}_o(o)\, e^{-Iz}, \qquad V = \frac{\mathcal{E}_o(o)}{I}(1 - e^{-I\ell}), \\ &\frac{V^2}{\mathcal{P}_{in}} = \frac{[\mathcal{E}_o(o)]^2}{\mathcal{P}(o)} \cdot \frac{(1 - e^{-I\ell})^2}{I^2} = \frac{2r}{I}(1 - e^{-I\ell})^2 \end{aligned}\right\} \quad (7.42)$$

for traveling waves;

$$\left.\begin{aligned} &\mathcal{E}_{o,c}(z) = \mathcal{E}_{o,c} = \text{const.}, \qquad V_c = \mathcal{E}_{o,c}\,\ell , \\ &\frac{V_c^2}{P_{in}} = \frac{(\mathcal{E}_{o,c}\,\ell)^2}{P_{in}} = \frac{r}{2}\,\ell \end{aligned}\right\} \qquad (7.43)$$

for standing waves. Thus for single feed

$$\frac{V}{V_c} = 2\,\frac{1 - e^{-I\ell}}{\sqrt{I\ell}} = 2\,\frac{1 - e^{-\ell/2\ell_o}}{\sqrt{\ell/2\ell_o}} \qquad (7.44)$$

This expression can be maximized by differentiating with respect to I or ℓ_o. The maximum value occurs at $\frac{I\ell}{2}\,e^{-I\ell/2} = \frac{1}{2}\sinh\frac{I\ell}{2}$, i.e., $I\ell = \ell/2\ell_o \cong 1.24$ and is equal to

$$\left(\frac{V}{V_c}\right)_{max.} = 4\sqrt{I\ell}\;e^{-I\ell} \cong 1.28 \quad . \qquad (7.45)$$

If $\ell \ll \ell_o$, $V/V_c \cong \sqrt{2\ell/\ell_o}$; if $\ell_o \ll \ell$, $V/V_c \cong \sqrt{8\ell_o/\ell}$. $V = V_c$ for $I\ell = \ell/2\ell_o \cong 0.35$ or 3.85.

From the viewpoint of power economy the traveling wave accelerators are superior to the standing wave accelerators for $.70\,\ell_o \lesseqgtr \ell \lesseqgtr 7.70\,\ell_o$ and inferior if ℓ/ℓ_o is either too small or too large. As pointed out before, r and Q do not change greatly with geometric factors, thus with a guide of a given length and operated with standing waves there is not much to be gained by maximizing r. On the other hand, if the guide is operated with traveling waves we can always choose a suitable value of a/b such that $\ell \cong 2.48\,\ell_o$ and get a 28 per cent increase in voltage for the same input power and save 39 per cent of power for the same total voltage as compared to the case of standing waves. For given $\mathcal{P}_{in}$ and r, V always increases with ℓ. In the standing-wave case $V^2 \sim \ell$, while in the traveling-wave case $V^2 \rightarrow \frac{2r}{I} \mathcal{P}_{in}$ as $\ell \rightarrow \infty$. But it must be noted that unless the pulse time is infinite, ℓ cannot be increased indefinitely, because with a single feed ℓ has an upper limit for any given pulse time beyond which the accelerating field cannot reach the far end of the guide.

Next we consider the other extreme case of distributed feeds. To simplify the discussion the feeds are so distributed that a uniform energy density W is maintained throughout the guide. Thus in the standing-wave case the power feed per unit length is equal to the power loss per unit length, $P_{in} = \omega W_c/Q$, while in the traveling-wave case the feed system should be so arranged that W and $v_e W$ are constants. In the latter case $P_{in} = (\frac{\omega \ell}{Q} + v_e)\, W = \frac{\omega W}{Q}\, (\ell + \ell_o)$, $v_e W$ being finally absorbed by a matched termination. Since by definition the energy is proportional to $E_o^2 \lambda^2$, i.e., $W = AE_o^2 \lambda^2$ and $W_c = 2AE_{o,c}^2 \lambda^2$ (A being a constant determined by the geometry), we obtain

$$\frac{V^2}{\mathcal{P}_{in}} = \frac{30}{2\pi A} \frac{Q\ell}{\lambda} \frac{\ell}{\ell + \ell_o} \quad , \tag{7.46}$$

$$\frac{V_c^2}{\mathcal{P}_{in}} = \frac{30}{2\pi A} \frac{Q\ell}{2\lambda} = \frac{r}{2} \ell \quad . \tag{7.47}$$

The relation between Q and r is obvious because $V_c^2/\mathcal{P}_{in}$ cannot depend on the method of feeding. Hence

$$\frac{V}{V_c} = \sqrt{\frac{2\ell}{\ell + \ell_o}} \tag{7.48}$$

for distributed feeds with uniform W. $V/V_c \cong \sqrt{2\ell/\ell_o}$ if $\ell \ll \ell_o$ and $V/V_c \cong \sqrt{2}$ if $\ell_o \ll \ell$. With distributed feeds, the traveling-wave tubes having $\ell > \ell_o$ are even more effective than with a single feed. For $\ell \gg \ell_o$ the power ratio between the two types of tubes is approximately two to one.

The above discussion, however, applies only to the same guide operated with the two types of waves. If the standing-wave tube operates on the π-mode, i.e. $d = \lambda_{z,o}/2$, the same guide cannot be operated as a traveling-wave tube. The latter type of tube must have $d < \lambda_{z,o}/2$. In other words the standing-wave tube can have fewer loading disks per wavelength and so a higher Q and a greater r as compared to the traveling-wave tube. Let us take $d = \lambda_{z,o}/4$ for the latter tube and compare its r with that of the other tube with $d = \lambda_{z,o}/2$. From (7.35) we find that the ratio of the disk loss to the cylindrical wall loss is roughly n/2.61; hence the ratio of Q's or the ratio of r's of the two cases is about $\frac{2 + 2.61}{4 + 2.61} \cong .70$. When this factor is taken into account, $(V/V_c)_{max} \cong 1.07$ for single feed and $(V/V_c)_{max} \cong 1.18$ for distributed feeds.

Lastly we consider the case of multiple feeds. Let each feed supply the same amount of power $\mathcal{P}_o$ and the spacing between any two neighbouring feeds be $\Delta \ell$. Also let $\mathcal{P}(m)$ and V(m) be the energy flow and the voltage gain in the m-th section ($(m - 1)\Delta\ell \leqq z \leqq m\Delta\ell$) respectively, then

$$\mathcal{P}(m) = \mathcal{P}_o \sum_{q=0}^{m-1} e^{-2I(z - q\Delta\ell)}$$

$$= \mathcal{P}_o \, e^{-2I(z-(m-1)\Delta\ell)} \cdot \sum_{q=0}^{m-1} e^{-2Iq\Delta\ell} \quad .$$

If the energy flow in the m-th section were equal to $\mathcal{P}_o e^{-2I(z-(m-1)\Delta\ell)}$, the voltage gain in this section would be

$\frac{\mathcal{E}_o}{I}(1 - e^{-I\Delta\ell}) = \sqrt{\frac{2r}{I}\mathcal{P}_o} \cdot (1 - e^{-I\Delta\ell})$. Hence

$$V(m) = \sqrt{\frac{2r}{I}\mathcal{P}_o} \cdot (1 - e^{-I\Delta\ell}) \cdot \left[\sum_{q=0}^{m-1} e^{-2Iq\Delta\ell} \right]^{1/2} \quad .$$

Since $V = \sum_{m=1}^{N=\ell/\Delta\ell} V(m)$ and $\mathcal{P}_{in} = \mathcal{P}_o \frac{\ell}{\Delta\ell}$, we easily obtain

$$\frac{V^2}{\mathcal{P}_{in}} = r\ell \, \frac{2}{I\Delta\ell} \tanh \frac{I\Delta\ell}{2} \cdot \left[\frac{1}{N} \sum_{m=1}^{N} (1 - e^{-2Im\Delta\ell})^{1/2} \right]^2 \quad . (7.49)$$

If $\Delta\ell \cong \ell_o$, i.e., $2I\Delta\ell \cong 1$ and N is large, then the expression inside the bracket is approximately unity. Under these conditions we have simply

$$\frac{V^2}{\mathcal{P}_{in}} \cong r\ell \, \frac{2}{I\Delta\ell} \tanh \frac{I\Delta\ell}{2} \cong r\ell \; .$$

Here again the voltage is about $\sqrt{2}$ times that obtainable in the corresponding case of standing waves.

To obtain the same amount of output voltage one can either use a larger input power and a shorter tube length or a smaller power and a larger length. The cost of the tube increases with length and the cost of power decreases with length. Optimum values should be used so that the total cost is a minimum. For any given length and power, the output voltage is a function of r and I and depends on the geometry of the guide and the wall material. There are many relevant factors but most of them can be dictated by practical considerations.[57] As shown by Ginzton, Hansen and Kennedy[58] the design problem can actually be boiled down to the determination of a single parameter (a/b) to give maximum total voltage. The procedure, though straight-forward, is rather tedious. A very detailed account of this procedure, including a set of graphs showing optimum results, is given in their paper.

57, 58. E.L.Ginzton, W.W.Hansen, W.R.Kennedy, loc. cit. footnote 7.

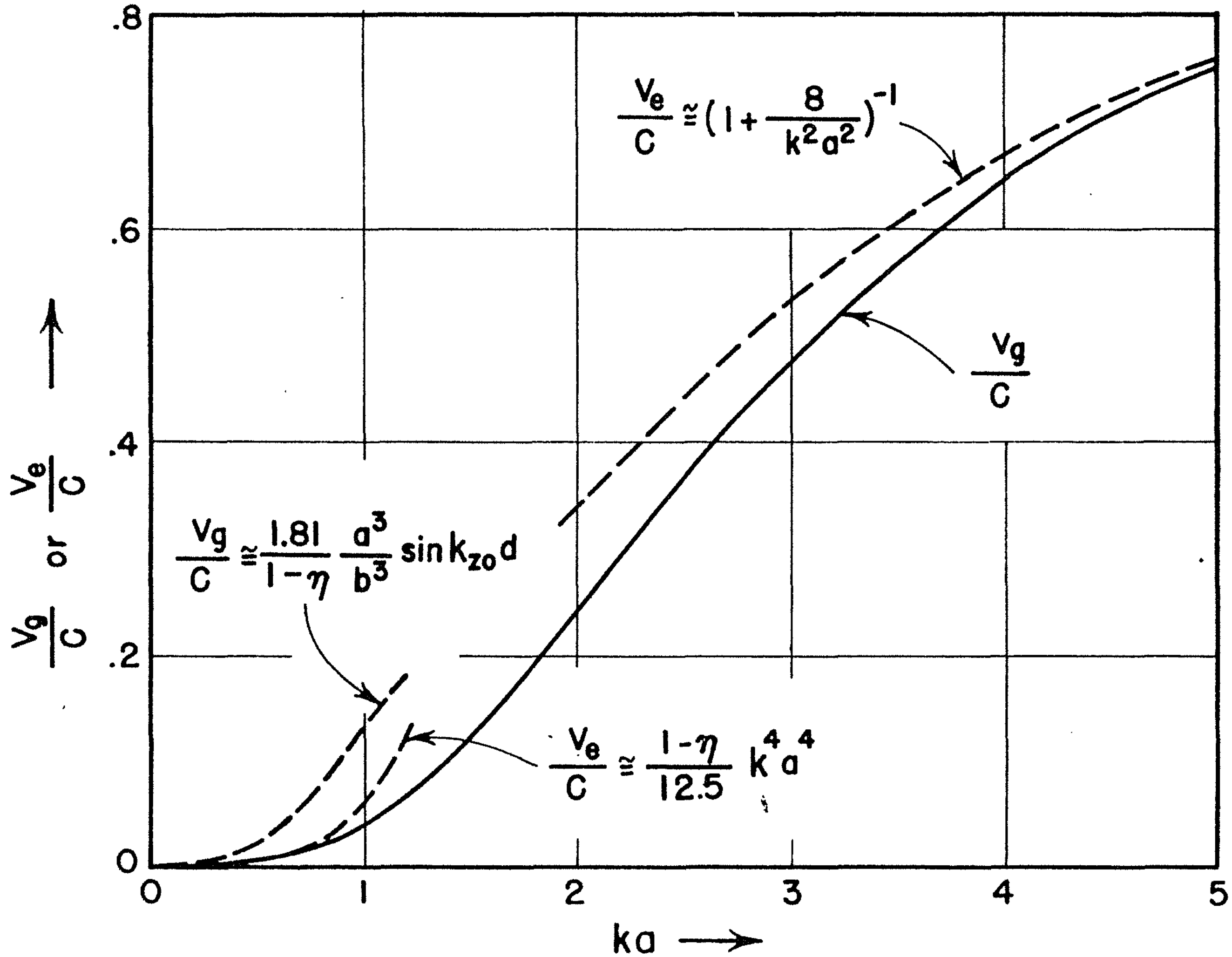

Fig. 7.1 - Curves showing the group velocity v_g/c and the energy velocity v_e/c as functions of ka. $\beta = 1$, $kb = 2.66$, $\eta = .240$. All except the dotted v_g/c curve are computed from a theory valid when $d \ll a$, $d \ll b-a$. The other curve is good for $a \ll d$, $a \ll b$.

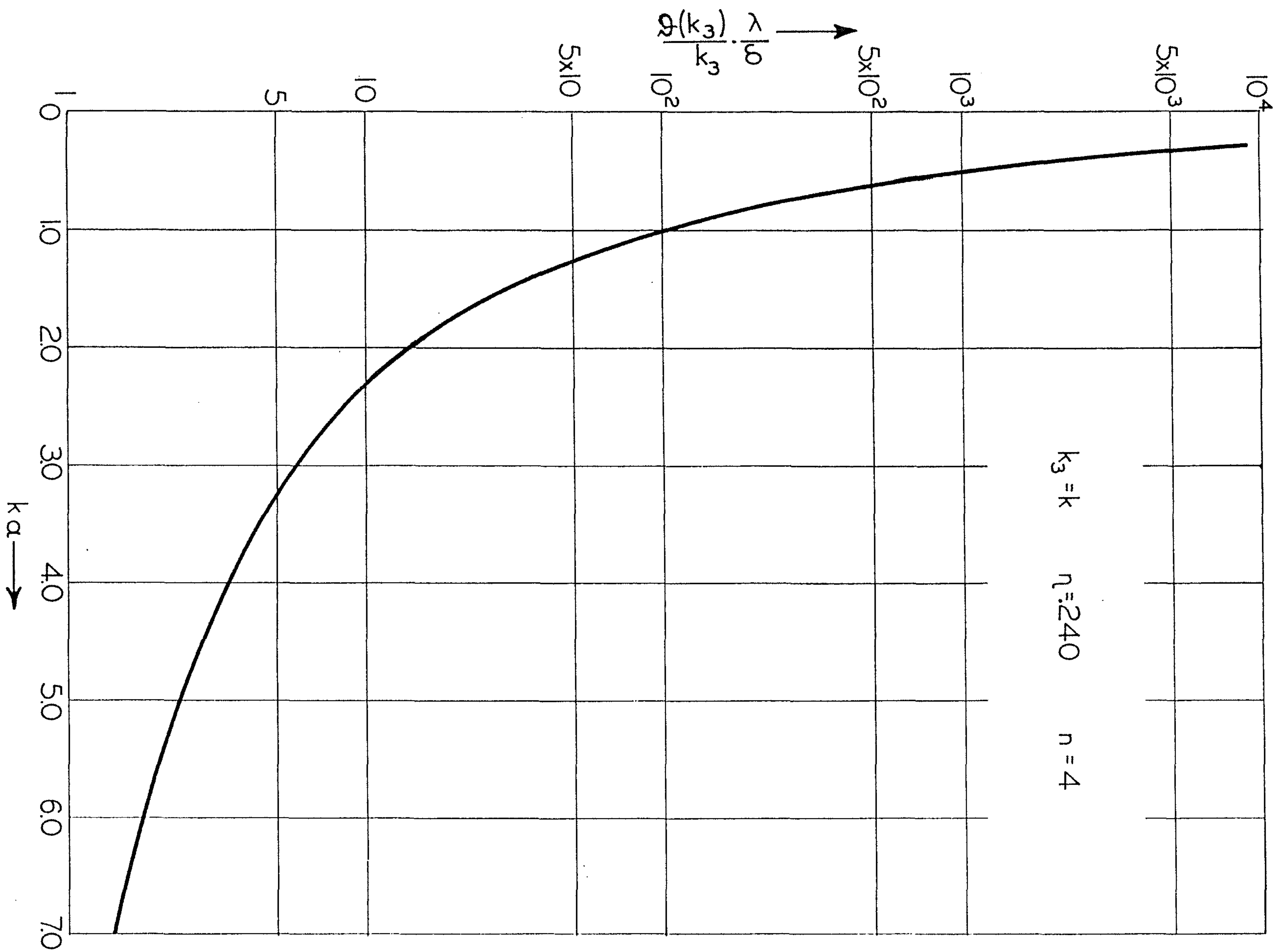

Fig. 7.2 - The imaginary part of the propagation constant, multiplied by $\lambda/k_3\delta$ ($k_3 \equiv k_{zo}$) is plotted against ka. The radius b is such as to give $k_3 = k$, there are 4 loading disks per wavelength, and the ... derived from Eqs. (3.32) and (3.33).

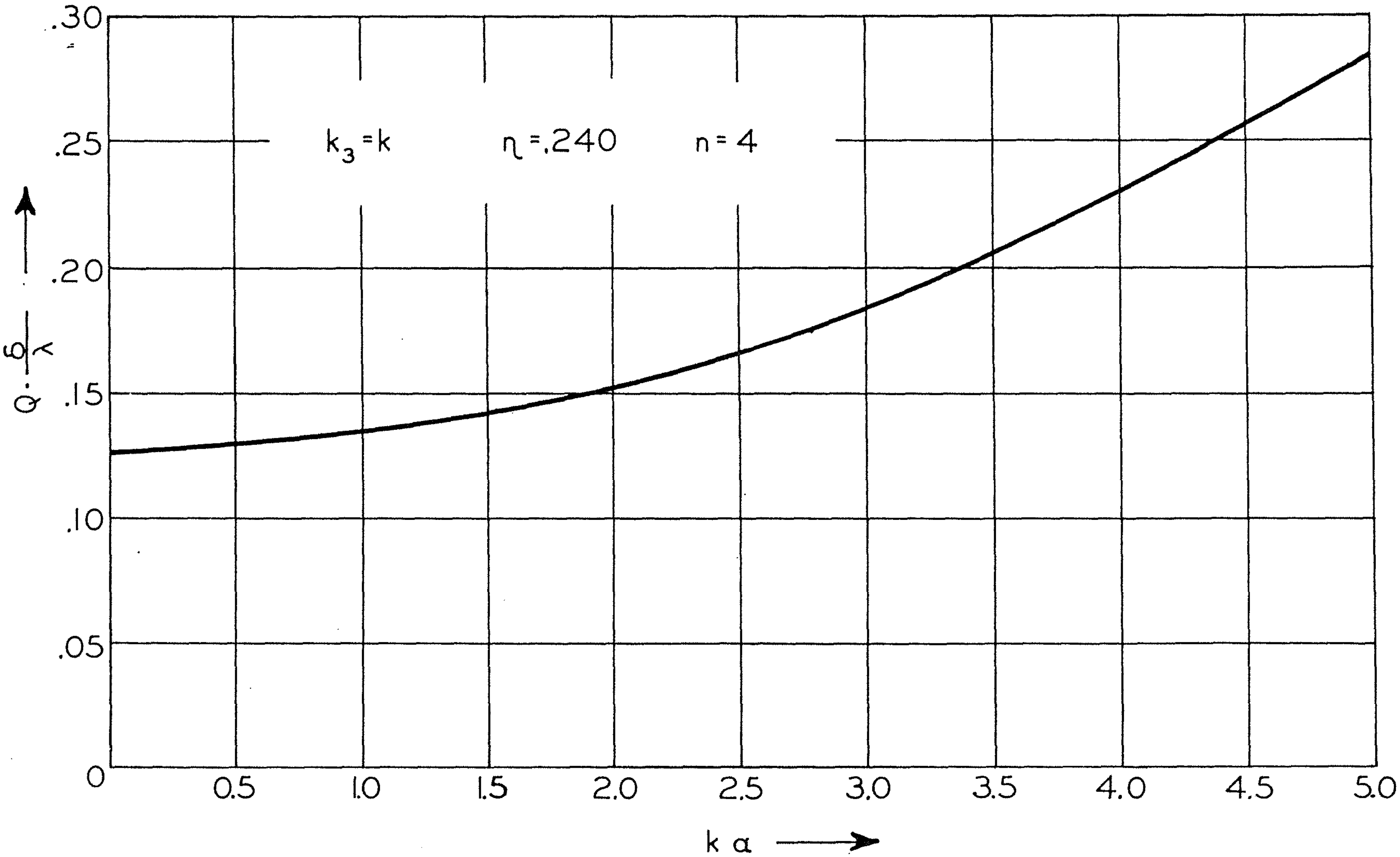

Fig. 7.3 - The quantity $Q\delta/\lambda$ is plotted against ka under the same assumptions as in Fig. 7.2.

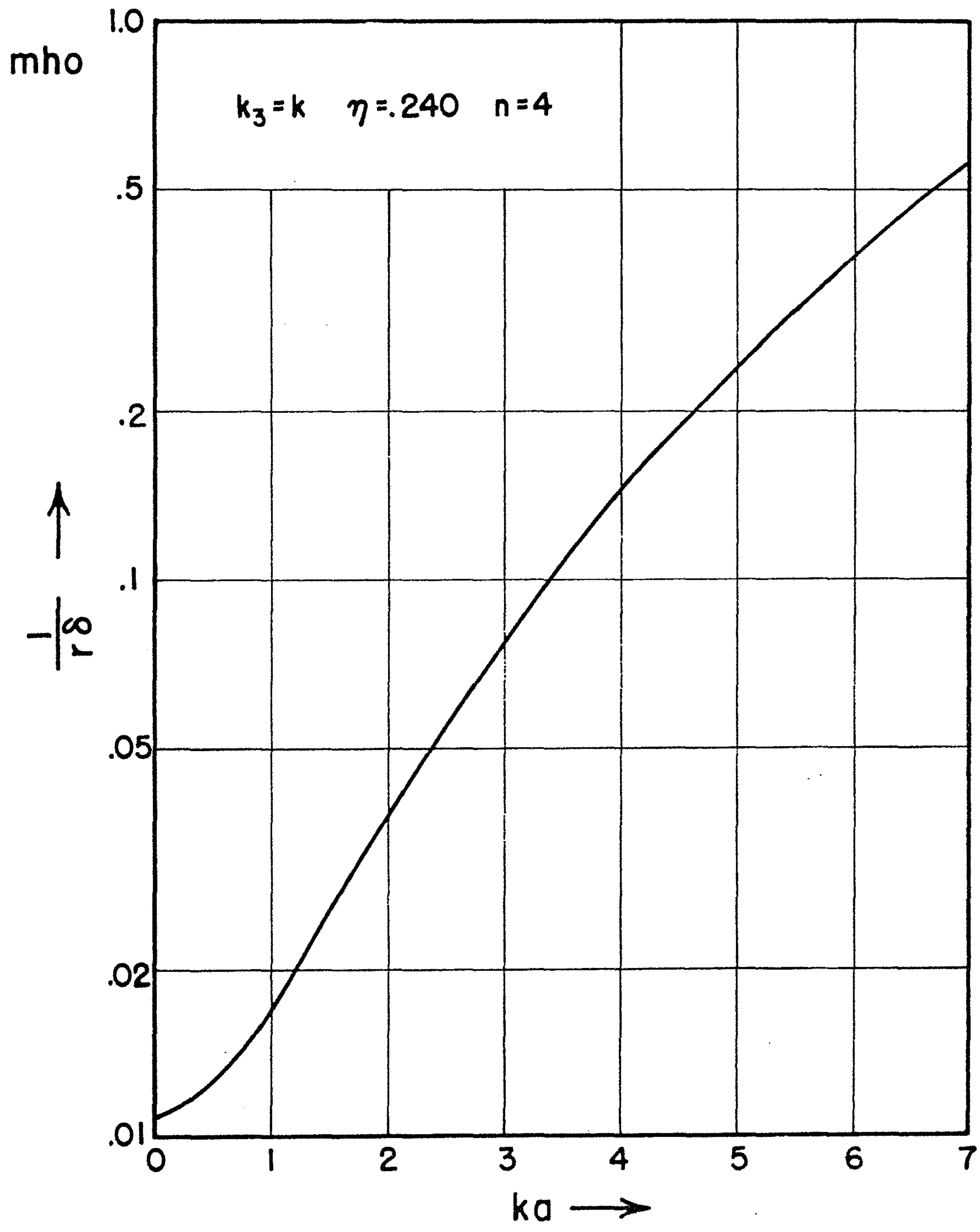

Fig. 7.4 - The quantity $1/r\delta$, measured in mho, is plotted against ka, the conditions being as described for Fig. 7.2.

CHAPTER VIII

ELECTRON ENERGY-LOSS DUE TO RANDOM CONSTRUCTIONAL ERRORS

Effective acceleration depends on the maintenance of a precise phase relation between the field and the electrons. In a disk-loaded waveguide the phase velocity of the field, characterized by β or k_{zo} is rather sensitive to the variation of geometric dimensions. If the constructional accuracy is not sufficiently good, both the field amplitude α and the phase angle $k_{zo}d$ will change appreciably from their respective desired values from cell to cell along the accelerator tube. Although the positive and negative errors tend to compensate each other, the magnitude of the accumulated phase error builds up in proportion to the square root of the distance, as in the "random walk" problem, while the field amplitude suffers an attenuation which is proportional to the mean square of the phase error. The consequent loss of electron energy can be quite serious if not properly compensated, especially if the accelerator length is great. In this chapter the random loss problem will be discussed by considering a simple equivalent circuit. The phase error will be analyzed into its constituent parts which depend on errors in different dimensional quantities. They are connected by a simple relation derived from the principle of similitude. Numerical values will be given to show the relative importance of different dimensional errors. The probable magnitude of the energy loss for a practical set of tolerance values will be calculated.

8.1. Scattering of Waves in an Almost Uniform Transmission Line.

We consider an equivalent problem dealing with the scattering loss in an almost uniform transmission line. As far as the k_{zo} field component is concerned, an exactly periodic loaded-guide may be represented by a uniform transmission line in which the phase velocity is a constant and no reflected waves are present. The effect of non-uniformity from cell to cell in an actual accelerator guide due to constructional errors may be simulated by loading the uniform line with shunt susceptances such as shown schematically in Fig. 8.1. The loading susceptances x_n's are measured in units of the characteristic admittance of the uniform line; x_n's have random sign and small random magnitudes but are equally spaced at a distance d. It is assumed that the error in d may be taken care of by a corresponding error in x.

Let us first consider the scattering process at x_n. Let $E_+(\rightarrow n)$ and $I_+(\rightarrow n)$ denote respectively the incident voltage wave and the incident current wave moving towards x_n, $E_r(\leftarrow n)$, $I_r(\leftarrow n)$ the corresponding reflected waves; $E(\leftrightarrow n) \equiv E_+(\rightarrow n) + E_r(\leftarrow n)$, $I(\leftrightarrow n) \equiv I_+(\rightarrow n) + I_r(\leftarrow n)$; $E_t(n \rightarrow)$, $I_t(n \rightarrow)$ denote the transmitted voltage and current waves moving away frmm x_n towards the right. If the characteristic admittance is normalized to unity, $E_+(\rightarrow n) = e^{jk_{zo}(z-z_n)}$,

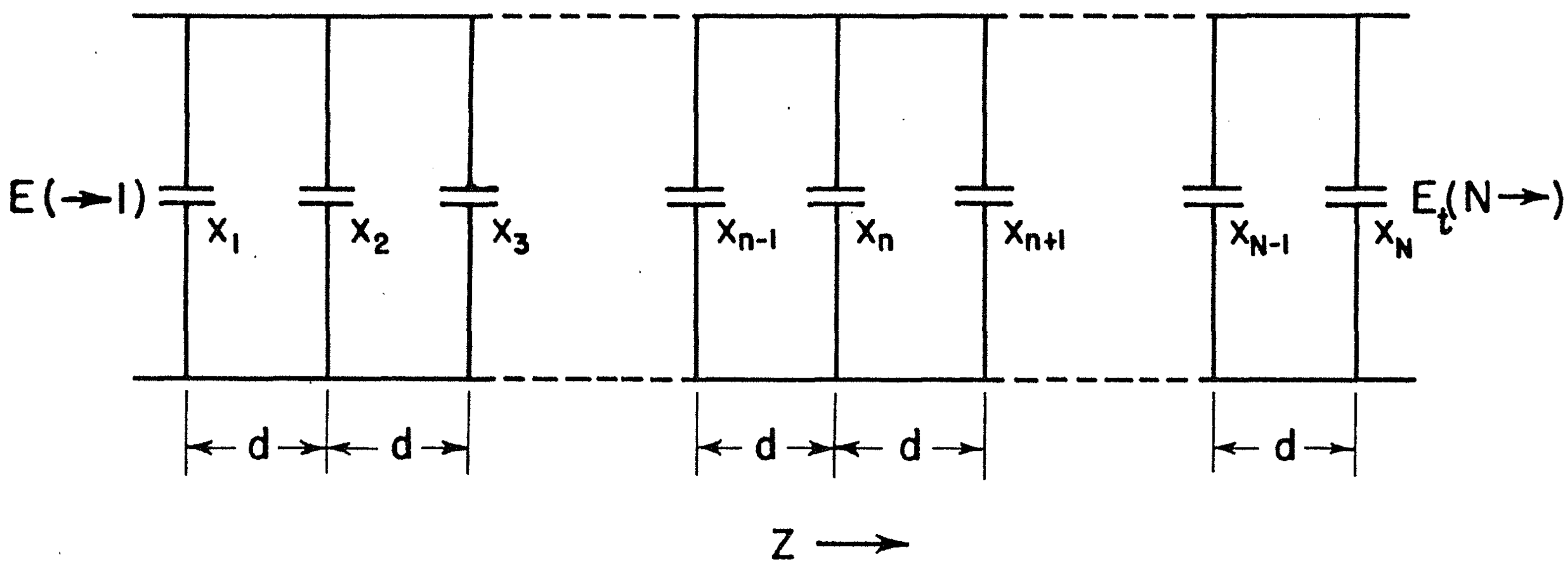

Fig. 8.1 - Equivalent circuit of an almost uniform transmission line. Susceptances x_n's are measured in units of the characteristic admittance of the unloaded uniform line. $E_+(\rightarrow 1) \sim e^{jk_{zo}z}$, $E_t(N \rightarrow) \sim e^{jk_{zo}z}$ are the incident and the transmitted voltage waves respectively.

and a reflection coefficient Γ_n of x_n is defined such that $E_r(\leftarrow n) = \Gamma_n e^{-jk_{zo}(z-z_n)}$, then we have from the usual circuit theory the following set of relations:

$$\left.\begin{aligned} I_+(\rightarrow n) &= E_+(\rightarrow n) \\ I_r(\leftarrow n) &= -E_r(\leftarrow n) \\ E(\leftrightarrow n) &= e^{jk_{zo}(z-z_n)} + \Gamma_n e^{-jk_{zo}(z-z_n)} \\ I(\leftrightarrow n) &= e^{jk_{zo}(z-z_n)} - \Gamma_n e^{-jk_{zo}(z-z_n)} \\ E_t(n\rightarrow) &= I_t(n\rightarrow) = (1+\Gamma_n)e^{jk_{zo}(z-z_n)} \\ I(\leftrightarrow n) &= I_t(n\rightarrow) - jx_n\, E(z=z_n)\ . \end{aligned}\right\} \quad (8.1)$$

Then we easily find

$$\Gamma_n = \frac{jx_n}{2-jx_n} \quad (8.2)$$

and

$$\frac{E_t(n\rightarrow)}{E_+(\rightarrow n)} = \frac{2}{2-jx_n} \cong \left(1-\frac{x_n^2}{8}\right) e^{jx_n/2} \quad . \quad (8.3)$$

By scattering once at x_n, the transmitted wave experiences a phase shift $\theta_n = x_n/2$ and suffers a reduction in amplitude by a fraction $\frac{1}{2}\theta_n^2$. It may be pointed out that this relation between the phase shift and the reduction in amplitude remains true if the shunt element is replaced by a T-network.

The transmitted wave $E_t(n\rightarrow)$ will impinge upon x_{n+1} and give rise to a reflected wave. The reflected waves will further be reflected giving rise to + waves. And the same process proceeds indefinitely. But the + waves arising from the reflected waves have only small amplitudes random in both sign and magnitude, being at most of the order of $|x_m \cdot x_p|$; their resultant may be neglected in comparison with the directly transmitted wave. Thus we may take $E_t(n \rightarrow) = E_+(\rightarrow n+1)$, so

$$\frac{E_+(\rightarrow n+1)}{E_+(\rightarrow n)} \cong \left(1 - \frac{x_n^2}{8}\right) e^{jx_n/2} \quad .$$

By taking continued products we obtain

$$\frac{E_t(n \rightarrow)}{E_+(\rightarrow 1)} = \left(1 - \sum_{p=1}^{n} x_p^2/8\right) e^{j \sum_{p=1}^{n} x_p/2} \quad . \tag{8.4}$$

Due to the presence of the scattering susceptances x_n's, the power transmitted through the n-th cell is less than the power in the incident wave at the first cell by a fraction $\sum_{p=1}^{n} x_p^2/4$ which, for sufficiently large values of n, is equal to n/4 times the expectation value of $\overline{x^2}$, i.e. $(n/4)\langle x^2 \rangle$.

8.2. Scattering Susceptance vs. Loading-Disk Susceptance

In the accelerator tube the phase shift in one cell is $\delta(k_{zo}d)$. k_{zo} is a function of frequency and dimensions. With the same notations as used before (see Fig. 2.1),

$$k_{zo} \equiv k_{zo}(k;\ a,b,d,\ \eta d) \tag{8.5}$$

The equivalent scattering susceptance $x = 2\delta(k_{zo}d)_{k=\text{const.}}$ may be decomposed into its constituent parts, namely

$$\frac{x}{2} \equiv \frac{1}{2} \sum_{q_i} x(q_i) = \sum_{q_i} \frac{\partial(k_{zo}d)}{\partial q_i} \cdot \delta\, q_i \tag{8.6}$$

$$(q_i = a,\ b,\ d,\ \eta d)\ .$$

The concept of the scattering susceptance is not to be confused with that of the equivalent loading-disk susceptance; it is interesting to know how they are related.

In Section 2.5, the periodically loaded waveguide is represented by a loaded transmission line (instead of a uniform line). The equation of the loaded line is given by (2.29). With respect to the k_{zo} field component the equation is

$$\cos k_{zo}d = \cos k_1 d - \frac{1}{2} X \sin k_1 d\ , \tag{8.7}$$

where X is the equivalent loading-disk susceptance; $X = k_1 C$, C being the equivalent loading-disk capacity; $k_1 = 2\pi/$unloaded guide wavelength. Differentiating equation (8.7) we obtain

$$\frac{\partial(k_{zo}d)}{\partial\, q_i} = \frac{\sin k_1 d + (X/2)\cos k_1 d}{\sin k_{zo}d} \frac{\partial(k_1 d)}{\partial q_j}$$

$$+ \frac{1}{2} \frac{\sin k_1 d}{\sin k_{zo}d} \frac{\partial X}{\partial q_j} \tag{8.8}$$

$$(q_j = k, a, b, d,\ \eta d)\ .$$

$2\frac{\partial(k_1 d)}{\partial q_i}\, \delta q_i$ and $2\, \frac{\partial X}{\partial q_i}\, \delta q_i$ may similarly be considered as the scattering susceptances due to the errors in the unloaded line characterized by k_1 and in the loading-disk respectively; a proper combination of the two according to equation (8.8) gives

gives the total equivalent scattering susceptance $x(q_i)$. The relation between $x(q_i)$ and $\partial X/\partial q_i$ is not as simple as one might first suppose.

Since the loaded waveguide has narrow band widths, $k_1 \ll k$ and $k_1 d \ll 1$. Equations (8.7) and (8.8) may be reduced to

$$\frac{1}{2} X \cdot k_1 d \cong 1 - \cos k_{zo} d \tag{8.7a}$$

$$\frac{\partial(k_{zo}d)}{\partial q_j} \cong \frac{k_1 d + (X/2)}{\sin k_{zo}d} \frac{\partial(k_1 d)}{\partial q_j} + \frac{1}{2} \frac{k_1 d}{\sin k_{zo}d} \frac{\partial X}{\partial q_j} \tag{8.8a}$$

If $k_{zo}d \cong \pi/2$, the above equations can further be simplified; thus

$$\frac{1}{2} X \cong \frac{1}{k_1 d} \tag{8.9}$$

$$\frac{\partial(k_{zo}d)}{\partial q_j} \cong \frac{1}{k_1 d} \frac{\partial(k_1 d)}{\partial q_j} + \frac{1}{X} \frac{\partial X}{\partial q_j} \tag{8.10}$$

$$(k_{zo}d \cong \pi/2)$$

On the assumption that $C(= X/k_1)$ is practically independent of frequency we obtain a very simple relation from (8.10) with $q_j = k$, i.e.,

$$kC \cong \frac{kd}{2} X^2 \cong \frac{\partial k_{zo}}{\partial k} = c/v_g \qquad (k_{zo}d \cong \pi/2) .$$

C or X is a simple function of the group velocity. The other equations of (8.10) with q_j = a, b, d, ηd are somewhat more complicated due to the fact that k_1 is not independent of any of the dimensions if $\eta d \neq 0$.

8.3. Principle of Similitude

The various derivatives $\frac{\partial k_{zo}}{\partial q_i}$ are connected by a simple relation specified by the principle of similitude.[59] Referring to the present problem and assuming a perfectly conducting boundary, this principle may simply be stated as follows: If all geometrical dimensions are multiplied by the same factor, then the free-space wavelength and the guide wavelength must also be multiplied by the same factor in order that the two cases before and after the change of scale may be exactly similar. Let the factor be $(1 + \epsilon)$ with $\epsilon \ll 1$; $\frac{1}{1+\epsilon} \cong 1 - \epsilon$. Thus

$$(1 - \epsilon)\, k_{zo} \equiv k_{zo}\left[(1 - \epsilon)k;\ (1 + \epsilon)\, q_i\right] ,$$

i.e.,

$$-\epsilon\, k_{zo} = -\frac{\partial k_{zo}}{\partial k}\epsilon k + \sum_{q_i} \frac{\partial k_{zo}}{\partial q_i}\,\epsilon\, q_i \quad .$$

Hence

$$-\frac{k}{k_{zo}}\frac{\partial k_{zo}}{\partial k} + \sum_{q_i}\frac{q_i}{k_{zo}}\frac{\partial k_{zo}}{\partial q_i} = -1 \quad . \tag{8.11}$$

Since $\frac{d}{k_{zo}}\frac{\partial k_{zo}}{\partial d} + 1 = \frac{d}{k_{zo}d}\frac{\partial(k_{zo}d)}{\partial d}$, (8.11) may be written as

$$-\frac{k}{k_{zo}d}\frac{\partial(k_{zo}d)}{\partial k} + \sum_{q_i}\frac{q_i}{k_{zo}d}\frac{\partial(k_{zo}d)}{\partial q_i} = 0 \quad . \tag{8.12}$$

In actual calculation or measurements it is more convenient to keep $k_{zo}d$ = constant and observe the variation of the resonant frequency with respect to q_i. Let us denote the resonant frequency by k_o to distinguish it from the operating

59. J.A. Stratton, "Electromagnetic Theory" (McGraw-Hill, New York, 1941), p.488.

frequency k.

$$k_o \equiv k_o(a,\ b,\ d,\ \eta d) \tag{8.13}$$

$$(k_{zo}d = \text{const.})$$

By applying the principle of similitude we obtain

$$\sum_{q_i} \frac{q_i}{k_o} \frac{\partial k_o}{\partial q_i} = -1 \quad . \tag{8.14}$$

With this relation, (8.12) may be written as

$$\sum_{q_i} q_i \left[\frac{\partial(k_{zo}d)}{\partial q_i} + \frac{k}{k_o} \frac{\partial(k_{zo}d)}{\partial k} \frac{\partial k_o}{\partial q_i} \right] = 0 \quad .$$

At $k = k_o$, it can readily be verified that

$$\frac{\partial(k_{zo}d)}{\partial q_i} = -\frac{\partial(k_{zo}d)}{\partial k} \frac{\partial k_o}{\partial q_i} = -d \cdot \frac{\partial k_o}{\partial q_i} \frac{c}{v_g} \tag{8.15}$$

$$(k = k_o; \quad q_i = a, b, d, \eta d)$$

Both v_g and $\partial k_o/\partial q_i$ can be determined accurately either by calculation or by measurement. The methods of calculation have been discussed in Chapter III; some of the results are directly applicable. In particular, the perturbation method has great usefulness.

8.4. Energy Loss Due to Random Errors

The electron energy or mass is given by the integral

$$\left.\begin{aligned} &\gamma = \int \alpha \cos 2\pi\Delta \, d\xi \\ \text{with} \quad &2\pi\Delta = \int \left(k_{zo}\lambda - \frac{2\pi}{\dot{\xi}}\right) d\xi = 2\pi \int \left(\frac{1}{\beta} - \frac{1}{\dot{\xi}}\right) d\xi \end{aligned}\right\} \tag{8.16}$$

Let us consider α = constant, $\beta = 1$ as the ideal case. Due to errors we actually have

$$\left.\begin{aligned} \alpha(\xi) &= \alpha\left[1 + f(\xi)\right] \\ \frac{1}{\beta} &= 1 + h(\xi) \end{aligned}\right\} \tag{8.17}$$

f, h being random functions of ξ. $|f| \ll 1$; $|h| \ll 1$. Neglecting the first ten wavelengths or so, we may very well take

$$\frac{1}{\beta} - \frac{1}{\dot{\xi}} \cong h(\xi) \ .$$

Thus

$$2\pi\Delta \cong 2\pi \int h(\xi)\ d\xi \ ,$$

or

$$\left.2\pi\Delta(n) = \phi_o + \sum_{p=1}^{n} \theta_p = \phi_o + \phi(n)\right\} \tag{8.18}$$

where ϕ_o is a constant and $\theta_p = 2\pi \int_{\text{p-th cell}} h(\xi)\ d\xi = \left[\delta(k_{zo}d)\right]_p$ is the phase shift in the p-th cell. So

$$\gamma = \alpha \int \left[1 + f(\xi)\right]\cdot \cos\left[2\pi \int f(\xi')\ d\xi'\right]\ d\xi \ ,$$

or

$$\gamma(N) = \alpha \sum_{n=1}^{N} \left[1 + f(n)\right]\cdot \cos\left[\phi_o + \phi(n)\right]$$

$$= \sum_{n=1}^{N} V(n) \ . \tag{8.19}$$

Here V(n) corresponds to $E_t(n \rightarrow)$ in the equivalent circuit discussed in Section 8.1. From the result obtained there we may ascertain that

$$f(n) = -\sum_{p=1}^{n} \theta_p^2/2 \ . \tag{8.20}$$

Substituting this into (8.19) and calculating the expectation value $\langle\gamma(N)\rangle$ on the assumption that $\langle\theta_p\theta_{p'}\rangle = \theta^2 \cdot \delta_{pp'}$, we obtain

$$\langle\gamma(N)\rangle \cong N\alpha \cos \phi_o \left[1 + \langle \bar{f} \rangle - \frac{1}{2} \langle \overline{\phi^2} \rangle\right] ,$$

i.e.,

$$\langle\gamma(N)\rangle \cong N\alpha \cos \phi_o \left(1 - \frac{1}{2} N\theta^2\right) \quad . \tag{8.21}$$

The electron energy is reduced by a fraction $\frac{1}{2}N\theta^2$; the loss can be serious if N is great.

To illustrate, we give a set of numerical values obtained from measurements:

$$2a = .8717'' , \ 2b = 3.260'' , \ d = 1.030'' , \ \eta d = .240'' .$$

$$\upsilon = k_o c/2\pi = 2857.0 \text{ mc.}, \ k_{zo} d = \pi/2, \ \beta = 1.$$

$\frac{\partial\upsilon}{\partial a} = .520$ mc/mil	$\frac{a}{\upsilon}\frac{\partial\upsilon}{\partial a} = .0793$
$\frac{\partial\upsilon}{\partial b} = -\ 1.970$ mc/mil	$\frac{b}{\upsilon}\frac{\partial\upsilon}{\partial b} = -\ 1.123$
$\frac{\partial\upsilon}{\partial d} = -\ .104$ mc/mil	$\frac{d}{\upsilon}\frac{\partial\upsilon}{\partial d} = -\ .0375$
$\frac{\partial\upsilon}{\partial(\eta d)} = .274$ mc/mil	$\frac{\eta d}{\upsilon}\frac{\partial\upsilon}{\partial(\eta d)} = .0230$

c/Group velocity $\equiv \partial k_{zo}/\partial k = 85.6$

Voltage attenuation constant $\equiv I = 2.73 \times 10^{-3}$.

From the above data we find

$$\sum_{q_i} \frac{q_i}{\upsilon}\frac{\partial\upsilon}{\partial q_i} = -\ 1.058 \quad ,$$

being correct within 6 per cent of the value specified by the principle of similitude. The partial derivatives of $k_{zo}d$ are calculated as follows:

$$\frac{\partial(k_{zo}d)}{\partial a} = -\ 24.46 \times 10^{-3} \text{ per mil}$$

$$\frac{\partial(k_{zo}d)}{\partial b} = 92.50 \times 10^{-3} \text{ per mil}$$

$$\frac{\partial(k_{zo}d)}{\partial d} = 4.92 \times 10^{-3} \text{ per mil}$$

$$\frac{\partial(k_{zo}d)}{\partial(\eta d)} = -\ 12.90 \times 10^{-3} \text{ per mil}$$

Assuming $\delta a = 1/2$ mil, $\delta b = 1/4$ mil, $\delta d = 4$ mils, $\delta(\eta d) = 1$ mil and $N = 500$ we obtain

$$\theta_a^2 = .000150\ , \qquad \theta_b^2 = .000535\ ,$$

$$\theta_d^2 = .000387\ , \qquad \theta_{\eta d}^2 = .000166\ .$$

$$\theta^2 = \sum_{q_i} \theta_{q_i}^2 = .00124\ .$$

$$\frac{1}{2} N\theta^2 = .310\ .$$

The error in b is the most serious while that in ηd is the least. The reduction in amplitude corresponds to an increase in the voltage attenuation constant; $\delta I = \theta^2/2d = .237 \times 10^{-3}$, $\delta I/I = .0868$.

Means for correcting the errors have been discussed by Schiff and Post.[60] Assuming a constant amplitude, they have

60. L.I. Schiff and R.F. Post, Phys.Rev. 81, 655A(1951); R.F.Post, A 50-Mev Linear Electron Accelerator, (Ph.D. Dissertation, Stanford Univ.,) Oct. 1950.

shown that the energy loss due to phase errors can be reduced from $\frac{1}{4} N\theta^2$ to $\frac{1}{30} N\theta^2$ by an optimum choice of phase and phase velocity. Further details may be found in their work.

CHAPTER IX

CONCLUSION

Four main subjects have been discussed: 1. the field problem, 2. the orbit problem, 3. field energy and related characteristic quantities and 4. the electron energy-loss due to random constructional errors. The first two subjects deal with the basic theory of linear acceleration of electrons; the third is important from the practical point of view while the last concerns the effectiveness of long accelerators. Although this work is rather detailed, no attempt has been made at completeness. Perhaps it may be said that the analytical discussion is balanced against the numerical work, and in both accuracy has been striven for wherever expedient. The numerical examples intend to serve the double purposes of illustrating the theory and giving some actual or estimated characteristics of the Stanford billion-volt accelerator, now under construction.

Regarding the field problem, much emphasis has been given to the eigenvalues and less to the field amplitudes. While the eigenvalues can easily be calculated with precision by using a variational method with simple trial functions, the field amplitudes are rather sensitive to the choice of trial functions and must be determined by more elaborate processes. The finite thickness of the loading disks also complicates the problem. The trial function which has the correct asymptotic behavior is $E_z \sim \left[d^2 - (z - \frac{\pi d}{2})^2 \right]^{-1/3}$. With this function the Fourier amplitudes turn out to be Bessel functions of order 1/6 instead of order zero as in the case of zero disk thickness. To the writer's knowledge direct tables of such functions are not yet available.

Both the longitudinal and the transverse motions of electrons have been discussed in detail. For the sake of simplicity, two rather important factors have been neglected, space charge and the earth's magnetic field. The earth's field tends to deflect the moving electron beam continuously away from the axis. If not shielded or compensated, the cumulative effect can be quite serious. The shielding problem has been studied elsewhere.[61] The mutual repulsion between charges causes the beam to spread; but the repulsive force between any two moving charges having the same velocity $\vec{\beta}_e c$ decreases as[62] $1 - \beta_e^2 \cong 1 - \beta^2$ just like the electromagnetic force exerted by the field. Thus, as far as beam spreading is concerned, no qualitative error would be incurred by neglecting the space charge. If the charge density is large, the electron beam would absorb quite a fraction of the transmitted field energy. Knowing the electron current, the power absorbed can easily be calculated. On the other hand, a rigorous discussion of the

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61. R.B. Neal, Magnetic Shielding of the Linear Electron Accelerator, Stanford University Microwave Laboratory Report No. 132. Nov. 1950.

62. E.L.Ginzton, W.W.Hansen, and W.R. Kennedy, loc. cit., footnote 7; E. Madelung, Die Mathematischen Hilfsmittel des Physikers (Dover Publications, New York, 1943) p.264.

charge-field interaction is much more complicated, but hardly worth while at this time.

A set of general equations describing the motion of an electron in an axially symmetric field, which includes both the accelerating and the focusing field, applicable to the case of rapidly varying parameters, have been derived and put in convenient forms so that, if a calculating machine is available, one can at once set up the problem and obtain the orbits.

Assuming a sufficiently strong magnetic focusing field, which is no great practical problem for linear electron accelerators, electron orbits can be sufficiently stabilized. It is both possible and practicable to bunch all the electrons in one wave cycle to within a small phase angle around the crest of the wave and at the same time to focus them to a beam of a small cross section, both actions taking place in a relatively short distance in the initial stage of acceleration. However, it may be advisable from an engineering point of view to compromise some of the theoretically obtainable results for economy or convenience.

The discussion of the transverse motion and focusing is in particular detail. Analytical solutions to a number of useful cases have been obtained, mostly in a form capable of direct application. The treatment of the focusing problem, if supplemented by the space charge effect, is quite general and can be applied to electron beams of other devices with an axially symmetric field.

The small radiative loss is the most outstanding feature of linear accelerators and has been discussed by various authors, notably Schiff[63] and Schwinger.[64] It seems desirable to express the result in explicit form, using the present notation. The expression for the rate of energy-loss due to radiation is

$$P = \frac{2}{3}\frac{e^2}{\lambda}\frac{c}{\lambda}\frac{1}{1 - \beta_e^2}\left[(\dot{\vec{\rho}})^2 - (\dot{\gamma})^2\right] , \qquad (9.1)$$

where $\vec{\rho} = \gamma \vec{\beta}_e$, $\vec{\beta}_e c$ = velocity of electron. The losses due to the longitudinal and the radial component of acceleration are P_ξ and P_η respectively:

$$P_\xi = \frac{2}{3}\frac{e^2}{\lambda}\frac{c}{\lambda}(\alpha \cos 2\pi\Delta)^2 \qquad (9.2a)$$

$$P_\eta = P_\xi \cdot \left[(\gamma\dot{\eta})^2 + \pi^2(\gamma\eta)^2(1 - \dot{\xi})^2 \tan^2 2\pi\Delta\right]$$

$$\cong P_\xi \cdot (\gamma\dot{\eta})^2 . \qquad (9.2b)$$

The angular component is negligible. P_ξ is independent of γ or electron energy; P_η is usually much smaller than P_ξ because $(\gamma\dot{\eta})$ is usually much smaller than unity. They are always negligible in comparison with the rate of energy-gain unless $\frac{e^2}{\lambda}\alpha$ approaches the order of unity.

From the discussion of the constructional errors it is clear that the dimensions of the loaded guide should be held

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63. L.I. Schiff, Rev.Sci. Inst., 17, 6 (1946).

64. J. Schwinger, Phys. Rev., 75, 1912 (1949).

very accurately, especially the tube diameter and the disk-hole diameter. With a given structure the frequency of the power source, on which the phase velocity depends, must be very accurately controlled. The longer the accelerator tube, the more severe is the requirement of frequency stability. For really long accelerators, some precise method of frequency control will be needed.

Consequently, the techniques of detecting and of correcting the errors are of great practical importance in the development of long accelerators. Two methods of measurement may be mentioned. Becker and Caswell[65] have made accurate nodal measurements in testing the 6-Mev. accelerator. Post[66] has made bead measurements on the 50-Mev. accelerator and developed the method to a convenient technique. He has also corrected the errors by placing appropriate metal slugs in the different cells.

Finally, it may be mentioned that an investigation of other forms of the loaded guide might perhaps lead to noticeable improvements, e.g., to a higher value of Q, or a greater shunt impedance, or to a higher limiting value of the cold emission field-strength, thus permitting the use of a larger α.

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65. G.E. Becker and D.A. Caswell, Operation of a 6-Mev. Linear Electron Accelerator, to be published in Rev.Sci.Inst.

66. R.F. Post, loc. cit., footnote 60.