GENERALIZATIONS OF HOLOGRAPHIC RENORMALIZATION GROUP FLOWS

by

Minwoo Suh

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Table of Contents

Acknowledgements ii

Abstract vi

Chapter 1: Introduction 1
- 1.1 The holographic principle 1
- 1.2 Holographic renormalization group flows 2
- 1.3 Example: The $\mathcal{N} = 1$ supersymmetric RG flow 7
- 1.4 Generalizations of holographic RG flows 11

Chapter 2: Holographic RG flows with dilaton and axion fields 14
- 2.1 Introduction 14
- 2.2 The Janus solutions in supergravity 18
  - 2.2.1 The original Janus solution 18
  - 2.2.2 The super Janus 19
  - 2.2.3 The Janus solutions in type IIB supergravity 21
- 2.3 Truncation of $\mathcal{N} = 8$ gauged supergravity in five dimensions 24
  - 2.3.1 The $SU(3)$-invariant truncation 24
  - 2.3.2 The supersymmetry equations 28
  - 2.3.3 The numerical solutions 37
- 2.4 Super Janus in $\mathcal{N} = 2$ gauged supergravity in five dimensions [23] 39
- 2.5 Lift of the $SU(3)$-invariant truncation to type IIB supergravity 40
  - 2.5.1 The metric 40
  - 2.5.2 The dilaton/axion fields 42
- 2.6 Supersymmetric Janus solution in type IIB supergravity [20] 44
- 2.7 Lift of the $SU(3)$-invariant truncation to type IIB supergravity (continued) 46
  - 2.7.1 The three-form flux 46
  - 2.7.2 The five-form flux 48
- 2.8 Type IIB supergravity on Sasaki-Einstein manifolds [50, 51] 51
- 2.9 The $SU(2)\times U(1)$-invariant flows with dilaton and axion fields 53
  - 2.9.1 The supersymmetry variations (I) 56
  - 2.9.2 The supersymmetry variations (II) 57

Chapter 3: Holographic RG flows with gauge fields 60
- 3.1 The $SU(3)$-invariant flow with electric potentials 60
  - 3.1.1 The supersymmetry variations 60
  - 3.1.2 The flow equations without dilaton and axion fields 64

3.1.3 The flow equations with dilaton and axion fields 67
3.2 The $SU(3)$-invariant flow with magnetic fields 68
3.2.1 The magnetic brane solutions 68
3.2.2 Configurations of magnetic fields 71
3.2.3 The $SU(3)$-invariant flow with magnetic fields (case I) . . . 72
3.2.4 The $SU(3)$-invariant flow with magnetic fields (case II) . . . 76

Chapter 4: Conclusions 78

Appendix A: $\mathcal{N}$ = 8 gauged supergravity in five dimensions 79

Appendix B: $SU(2,1)$ algebra 82

Appendix C: The supersymmetry variations for spin-1/2 fields 84

Appendix D: The supersymmetry variations for spin-3/2 fields 86

Appendix E: The parametrizations of the scalar manifold 88

Appendix F: The field equations of the $SU(3)$-invariant truncation 91

Bibliography 93

Abstract

The AdS/CFT correspondence conjectures the duality between type IIB supergravity on $AdS_5 \times S^5$ and $\mathcal{N}$ = 4 super Yang-Mills theory. Mass deformations of $\mathcal{N}$ = 4 super Yang-Mills theory drive renormalization group (RG) flows. Holographic RG flows are described by domain wall solutions interpolating between AdS_5 geometries at critical points of $\mathcal{N}$ = 8 gauged supergravity in five dimensions. In this thesis we study two directions of generalizations of holographic RG flows.

First, motivated by the Janus solutions, we study holographic RG flows with dilaton and axion fields. To be specific, we consider the $SU(3)$-invariant flow with dilaton and axion fields, and discover the known supersymmetric Janus solution in five dimensions. Then, by employing the lift ansatz, we uplift the supersymmetric Janus solution of the $SU(3)$-invariant truncation with dilaton and axion fields to a solution of type IIB supergravity. We identify the uplifted solution to be one of the known supersymmetric Janus solution in type IIB supergravity. Furthermore, we consider the $SU(2)\times U(1)$-invariant $\mathcal{N}$ = 2 and $\mathcal{N}$ = 1 supersymmetric flows with dilaton and axion fields.

Second, motivated by the development in AdS/CMT, we study holographic RG flows with gauge fields. We consider the $SU(3)$-invariant flow with electric potentials or magnetic fields, and find first-order systems of flow equations for each case.

Chapter 1

Introduction

1.1 The holographic principle

With the great success of 20th century physics, quantum field theory and general relativity, it is one of the final goals of physics: the unification of quantum field theory and general relativity toward theory of quantum gravity. Around the huge success of quantum field theory in the standard model of particle physics in 1970s, some crucial elements for quantum gravity were discovered: supersymmetry, extra dimensions and string theory. Based upon them, in 1980s, it turned out that there are five kinds of string theories: type I, type IIA, type IIB, heterotic O and heterotic E string theories. However, in 1995, it was shown that the five string theories are merely low energy effective theories of more fundamental, but so far unknown, theory named M-theory.

Another important element toward quantum gravity was found in the study of black holes by Hawking and Bekenstein in 1980s: The entropy of a black hole is proportional to its surface area. This suggests that the physics of (d+1)-dimensional bulk of a black hole is governed by the physics of the d-dimensional boundary of the black hole. In early 1990s, this idea was expanded to the holographic principle: Quantum field theory in d-dimensions is related to quantum gravity in (d+1)-dimensions. This principle was qualitative initially, however, in 1997, Juan Maldacena suggested the first concrete example of the holographic principle: the AdS/CFT correspondence.

The AdS/CFT correspondence [1, 2, 3] conjectures the duality between specific kinds of quantum field theory and quantum gravity. The quantum field theory in this case is $\mathcal{N}$ = 4 super Yang-Mills theory: This is the unique quantum field theory with maximal supersymmetry and conformal symmetry in four dimensions, hence it is a conformal quantum field theory (CFT). The gravity theory in this case is type IIB string theory. The low energy effective theory of type IIB string theory is type IIB supergravity which is the unique chiral maximally supersymmetric supergravity in ten dimensions. Type IIB supergravity has vacua involving five-dimensional anti-de Sitter (AdS) spacetime. The AdS/CFT correspondence means, even though $\mathcal{N}$ = 4 super Yang-Mills theory and type IIB supergravity are very different theories, there is duality between them, therefore, when we calculate physical quantities in one theory, we would get identical answers from the calculations in the other theory. To be specific, via the AdS/CFT correspondence, the strongly coupled regime of one theory corresponds to the weakly coupled regime of the other theory. Hence, the AdS/CFT correspondence is a useful tool to consider the strongly coupled regime of a theory which is usually hard to study. The AdS/CFT correspondence originally involves theories with supersymmetry and conformal symmetry, however, later, it was generalized to theories without them, hence, called the gauge/gravity dualtiy.

1.2 Holographic renormalization group flows

The AdS/CFT correspondence [1, 2, 3] conjectures a duality between type IIB string theory on $AdS_5 \times S^5$ and $\mathcal{N}$ = 4 super Yang-Mills theory (SYM). In this section, we consider the RG flows from $\mathcal{N}$ = 4 SYM, and discuss how they can be studied by solutions of type IIB supergravity via the AdS/CFT correspondence.

First, we consider the RG flow from the field theory side [4, 5]. We can deform $\mathcal{N}$ = 4 SYM by introducing mass terms to some of the chiral adjoint superfields. The mass deformation breaks the conformal invariance of $\mathcal{N}$ = 4 SYM and drives an RG flow. The

RG flow leads to a deformed theory where the conformal invariance is recovered. $\mathcal{N}$ = 4 SYM theory and the deformed theory correspond to the ultraviolet (UV) and infrared (IR) fixed points, respectively, and are both conformal field theories. However, along the RG flow, the conformal invariance is broken.

Via the AdS/CFT correspondence, RG flows in field theory correspond to certain solutions in gravity theory, the gravity duals, which have identical physics of RG flows. Regarding the RG flows from $\mathcal{N}$ = 4 SYM, the UV and IR fixed points correspond to AdS_5 solutions of type IIB supergravity. There are many ways to check this correspondence, and one of the simplest is to compare their symmetries: CFT_d has the same symmetry as AdS_{d+1}, $SO(2, d)$. Hence, the gravity duals of RG flows $i.e.$ the holographic RG flows, [6, 7, 8] can be described by domain wall solutions interpolating between two AdS_5 geometries.

It has not been proved, but with abundant evidence, it is believed that type IIB supergravity compactified on S^5 gives $\mathcal{N}$ = 8 gauged supergravity in five dimensions.[1] Hence, studying $\mathcal{N}$ = 8 gauged supergravity in five dimensions should give the equivalent physics from studying type IIB supergravity. The $SO(6)$ gauged $\mathcal{N}$ = 8 supergravity [12, 13, 14] is a maximally supersymmetric gauged supergravity in five dimensions. This theory has a scalar potential from 42 scalar fields living on the scalar manifold, $E_{6(6)}/USp(8)$. There are vacua, $i.e.$ AdS_5 solutions, at each critical point of the scalar potential. The known critical points of $\mathcal{N}$ = 8 gauged supergravity that are invariant at least under $SU(2) \times U(1)$ are listed in table 1.1 [16].

Via the AdS/CFT correspondence, holographic RG flows are described by domain wall solutions interpolating between the critical points of $\mathcal{N}$ = 8 gauged supergravity in

[1] First, we can compare symmetries and spectrum of type IIB supergravity on S^5 with those of $\mathcal{N}$ = 8 gauged supergravity in five dimensions. Second, there are solutions of type IIB supergravity uplifted from solutions of $\mathcal{N}$ = 8 gauged supergravity in five dimensions. For instance, the holographic RG flows in [9, 10, 11] are the examples. However, it should be noted that not all solutions of type IIB supergravity have their origin in $\mathcal{N}$ = 8 gauged supergravity in five dimensions.

five dimensions [6, 7, 8]. In this picture, the UV fixed point is the maximally supersymmetric $SO(6)$-invariant vacuum of $\mathcal{N}$ = 8 gauged supergravityin five dimensions, and IR fixed points are vacua with less supersymmetry and global symmetry. Along the RG flows, some supersymmetry and global symmetry are preserved.

Points	Gauge symmetry	Cosmological constant	Supersymmetry	c_{IR}/c_{UV}
(i)	$SO(6)$	$-\frac{3}{4}g^2$	$\mathcal{N} = 8$	1
(ii)	$SO(5)$	$-\frac{3^{5/3}}{8}g^2$	$\mathcal{N} = 0$	$\frac{2\sqrt{2}}{3}$
(iii)	$SU(3)$	$-\frac{27}{32}g^2$	$\mathcal{N} = 0$	$\frac{16\sqrt{2}}{27}$
(iv)	$SU(2)\times U(1)\times U(1)$	$-\frac{3}{8}\left(\frac{25}{2}\right)^{1/3}g^2$	$\mathcal{N} = 0$	$\frac{4}{5}$
(v)	$SU(2)\times U(1)$	$-\frac{2^{4/3}}{3}g^2$	$\mathcal{N} = 2$	$\frac{27}{32}$

Table 1.1 Known critical points of $\mathcal{N}$ = 8 gauged supergravity in five dimensions [16].

In the $SO(6)$ representation, the 42 scalar fields branch as

$$\mathbf{1} + \mathbf{1} + \mathbf{10} + \overline{\mathbf{10}} + \mathbf{20}' , \tag{1.1}$$

where the two singlets are the dilaton and axion fields. Via the AdS/CFT correspondence, the representations in (1.1) correspond to the gauge coupling, the θ-angle, the fermion bilinear operators, and the scalar bilinear operator of $\mathcal{N}$ = 4 SYM, respectively [6, 7]. The latter have the form

$$\mathrm{Tr}\,(\lambda^i\,\lambda^j)\,, \qquad \mathrm{Tr}\,(\overline{\lambda}^i\,\overline{\lambda}^j)\,, \qquad \mathrm{Tr}\,(X^a\,X^b) - \frac{1}{6}\,\delta^{ab}\,\mathrm{Tr}\,(X^c\,X^c)\,, \tag{1.2}$$

where i, j = 1, . . . , 6 and a, b = 1, . . . , 4. Holographic RG flows are, hence, obtained by turning on the scalar fields which are dual to the mass deformation operators in $\mathcal{N}$ = 4 SYM.

The RG flows to the nonsupersymmetric $SU(3)$ and $SO(5)$ critical points were the first examples by Girardello, Petrini, Porrati and Zaffaroni [6] and by Distler and Zamora [7]. We list some of the known RG flows by the number of massive fermion bilinears.

- One massive fermion bilinear: $\mathcal{N}$ = 1 flow

  It corresponds to the $\mathcal{N}$ = 1 supersymmetric RG flow to the $\mathcal{N}$ = 2 supersymmetric $SU(2) \times U(1)$ critical point. It involves two scalar fields, χ and α, dual to fermion bilinear and scalar bilinear, respectively,

$$\begin{aligned} \mathrm{Tr}\,(\lambda^4\,\lambda^4) \qquad &\longleftrightarrow \qquad \chi\,, \\ \sum_{j=1}^{4} \mathrm{Tr}\,(X^j\,X^j) \;-\; 2\sum_{j=5}^{6} \mathrm{Tr}\,(X^j\,X^j) \qquad &\longleftrightarrow \qquad \alpha\,. \end{aligned} \tag{1.3}$$

  The $\mathcal{N}$ = 1 flow corresponds to the phase discovered by Leigh and Strassler [4], and, is known as the LS flow. The $SU(2) \times U(1)$ critical point in supergravity was discovered by Khavaev, Pilch and Warner [16]. The holographic RG flow was studied by Freedman, Gubser, Pilch and Warner, hence, is also known as the FGPW flow from the supergravity aspect [8]. Later, the flow was uplifted to type IIB supergravity by Pilch and Warner [10].

We are not always lead to an IR critical point by RG flows. There are flows which lead the scalar fields to divergences, *flows to Hades*. However, there are examples that these five-dimensional singularities are overcome when the flow solutions are uplifted to type IIB supergravity [9, 10]. Below are the examples.

- Two massive fermion bilinears: $\mathcal{N} = 2^*$ flow

  It involves two scalar fields, χ and α, dual to a fermion bilinear and a scalar bilinear, respectively,

$$\begin{aligned} \mathrm{Tr}\,(\lambda^3\,\lambda^3 \,+\, \lambda^4\,\lambda^4) \qquad &\longleftrightarrow \qquad \chi\,, \\ \sum_{j=1}^{4} \mathrm{Tr}\,(X^j\,X^j) \,-\, 2\sum_{j=5}^{6} \mathrm{Tr}\,(X^j\,X^j) \qquad &\longleftrightarrow \qquad \alpha\,. \end{aligned} \tag{1.4}$$

  The $\mathcal{N} = 2^*$ flow was studied and then uplifted to type IIB supergravity [9]. It describes the Coulomb branch of $\mathcal{N} = 4$ SYM.

- Three massive fermion bilinears: $\mathcal{N} = 1^*$ flow

  Minimally, it involves two scalar fields, m and σ, dual to a fermion bilinear and a gaugino condensate, respectively,

$$\begin{aligned} \sum_{a=1}^{3} \mathrm{Tr}\,(\,\lambda^a\,\lambda^a) \qquad &\longleftrightarrow \qquad m\,, \\ \mathrm{Tr}\,(\,\lambda^4\,\lambda^4) \qquad &\longleftrightarrow \qquad \sigma\,. \end{aligned} \tag{1.5}$$

  The vacua of $\mathcal{N} = 1^*$ theories were extensively studied from the field theory aspect $e.g.$ references in [10]. The holographic $\mathcal{N} = 1^*$ flow was first studied by Girardello, Petrini, Porrati and Zaffaroni, and known as GPPZ flow [17]. Later, it was revisited by Pilch and Warner with more general scalar fields, and then uplifted to type IIB supergravity [10], however, the full uplift of this flow is not known.

1.3 Example: The $\mathcal{N}$ = 1 supersymmetric RG flow

One of the main technical issues in the study of holographic RG flows is to manage the complexity from the 42 noncompact scalar fields of $\mathcal{N}$ = 8 gauged supergravity in five dimensions. Each vacuum corresponds to a critical point of the scalar potential, however, handling the scalar potential with all 42 scalar fields is not practical. So it turned out to be convenient to truncate $\mathcal{N}$ = 8 gauged supergravity to its subsector with global symmetry smaller than $SO(6)$ [16]. In this manner, the RG flows with $SU(3)$ and $SU(2) \times U(1)$ invariance have been studied [6, 7, 8, 17, 9, 10, 18].

As a specific example, let us consider the $\mathcal{N}$ = 1 supersymmetric $SU(2)\times U(1)$-invariant flow [8] in (1.2). We set the gauge fields to vanish. The bosonic part of the Lagrangian of the $SU(2)\times U(1)$-invariant truncation is

$$e^{-1}\,\mathcal{L} \,=\, -\frac{1}{4}\,R \,+\, 3\,\partial_\mu\alpha\,\partial^\mu\alpha \,+\, \frac{1}{2}\,\partial_\mu\chi\,\partial^\mu\chi \,+\, \mathcal{P} \,-\, \frac{1}{4}\,e^{4\alpha}\,F_{\mu\nu}\,F^{\mu\nu} \,+\, \mathcal{L}_{CS}\,. \quad (1.6)$$

The superpotential is

$$W \,=\, \frac{1}{4\rho^2}\Big[\cosh(2\chi)\,(\rho^6 \,-\, 2) \,-\, (3\rho^6 \,+\, 2)\Big]\,, \quad (1.7)$$

and the scalar potential is obtained by

$$\mathcal{P} \,=\, \frac{g^2}{8}\,\left|\frac{\partial W}{\partial\varphi_j}\right|^2 \,-\, \frac{g^2}{3}\,|W|^2\,, \quad (1.8)$$

where $\rho = e^\alpha$, and φ_j are properly normalized fields, $\varphi_1 = \chi$, $\varphi_2 = \sqrt{6}\,\alpha$. The scalar potential has three critical points: the maximally supersymmetric $SO(6)$-invariant point, the $\mathcal{N}$ = 2 supersymmetric $SU(2)\times U(1)$-invariant point, and the nonsupersymmetric $SU(3)$-invariant point. In figure 1.1, points 2 and 3 are $\mathbb{Z}_2$ equivalent SU(3)-invariant points and points 4 and 5 are $\mathbb{Z}_2$ equivalent $\mathcal{N}$ = 2 supersymmetric points.

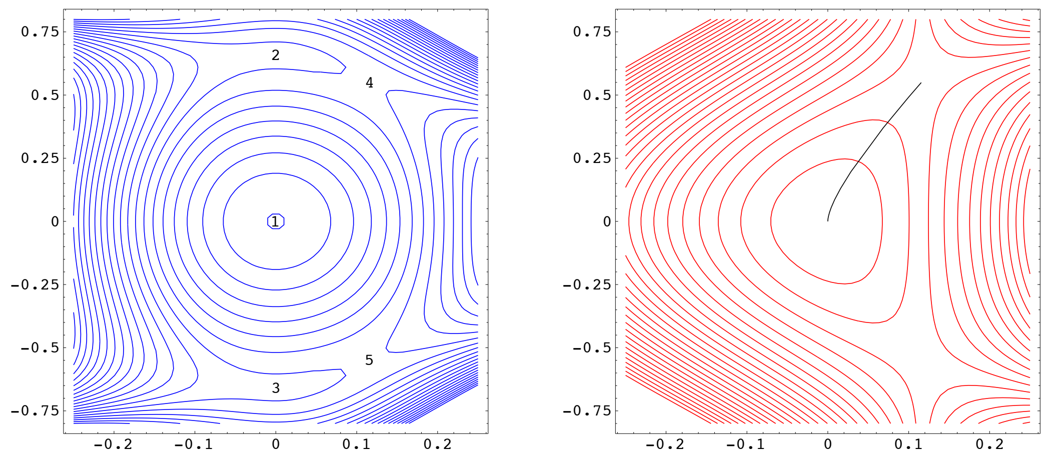

Figure 1.1: *The contour map of the scalar potential, P, (left) and the superpotential, W, (right), with χ on the vertical axis and α on the horizontal axis [8].*

Now we consider the domain wall solution which preserves the Poincaré invariance in four dimensions [8],

$$ds^2 \;=\; e^{2U(r)}\,\eta_{\mu\nu}\,dx^\mu\,dx^\nu \;-\; dr^2\,, \tag{1.9}$$

where $\eta_{\mu\nu}$ is a Minkowski metric and r is a radial direction corresponding to the energy scale in dual field theory. By having the supersymmetry variations of fermionic fields, $i.e.$ the spin-3/2 and spin-1/2 fields, vanish, we obtain the RG flow equations,

$$\frac{d\varphi_j}{dr} \;=\; \frac{g}{2}\,\frac{\partial W}{\partial\,\varphi_j}\,, \tag{1.10}$$

$$\frac{dU}{dr} \;=\; -\,\frac{g}{3}\,W\,, \tag{1.11}$$

whose solution interpolates between the critical point with maximal supersymmetry and the $\mathcal{N} = 2$ supersymmetric critical point. A numerical solution of the steepest descent equations is shown on the contour plot of W in figure 1.1. Along the flow $\mathcal{N} = 1$ supersymmetry is preserved.

$\mathcal{N}$ = 8 gauged supergravity in five dimensions is believed to be a consistent truncation of type IIB supergravity on S^5, $i.e.$ solution of $\mathcal{N}$ = 8 gauged supergravity in five dimensions can be uplifted to a solution of type IIB supergravity. There are proposed consistent truncation ansätse for type IIB supergravity fields, $i.e.$ metric, dilaton/axion fields, three- and five-form fluxes. A consistent truncation ansatz for metric was proposed in [16], for dilaton/axion fields in [9], and for three- and five-form fluxes in [18]. Employing those ansätse, the $\mathcal{N}$ = 1 and $\mathcal{N}$ = 2 supersymmetric $SU(2){\times}U(1)$-invariant flows were uplifted to type IIB supergravity in [10] and [9], respectively.

Now we briefly present the uplift of the $\mathcal{N}$ = 1 supersymmetric $SU(2){\times}U(1)$-invariant flow [10]. The IIB dilaton and axion fields are trivial for this flow. The IIB metric is

$$ds^2 \,=\, \Omega^2\, ds^2_{1,4} \,+\, ds^2_5\,, \tag{1.12}$$

where $ds^2_{1,4}$ is an arbitrary solution of $\mathcal{N}$ = 8 gauged supergravity in five dimensions. The internal space metric is

$$ds^2_5 \,=\, \frac{a^2}{2}\,\frac{\text{sech}\chi}{\xi}\,(dx^I\,Q^{-1}_{IJ}\,dx^J) \,+\, \frac{a^2}{2}\,\frac{\sinh\chi\,\tanh\chi}{\xi^3}\,(x^I\,J_{IJ}\,dx^J)^2\,, \tag{1.13}$$

where Q_{IJ} is a diagonal matrix with $Q_{11} = \ldots = Q_{44} = e^{-2\alpha}$, $Q_{55} = Q_{66} = e^{4\alpha}$, J_{IJ} is an antisymmetric matrix with $J_{14} = J_{23} = J_{65} = 1$, and $\xi^2 = x^I\,Q_{IJ}\,x^J$. The warp factor is

$$\Omega^2 \,=\, \xi\,\cosh\chi\,. \tag{1.14}$$

We define complex coordinates corresponding to J_{IJ},

$$u^1 \,=\, x^1 \,+\, i\,x^4\,, \qquad u^2 \,=\, x^2 \,+\, i\,x^3\,, \qquad u^3 \,=\, x^5 \,-\, i\,x^6\,, \tag{1.15}$$

and then

$$\begin{pmatrix} u^1 \\ u^2 \end{pmatrix} = \cos\theta \; g\,(\alpha_1,\,\alpha_2,\,\alpha_3) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad u^3 \,=\, e^{-\phi}\,\sin\theta\,, \tag{1.16}$$

where g (α_1,α_2,α_3) is an $SU(2)$ invariant matrix in terms of Euler angles.

The three-form flux is given by [10]

$$F_{(3)} \,=\, d\,A_{(2)}\,, \tag{1.17}$$

where the two-form gauge potential is

$$A_{(2)} \,=\, C_{(2)} \,-\, i\,B_{(2)}\,, \tag{1.18}$$

and $C_{(2)}$ and $B_{(2)}$ are RR and NSNS two-form gauge potentials, respectively. We have

$$A_{(2)} \,=\, e^{-i\,\phi}\,(a_1\,d\theta \,-\, a_2\,\sigma_3 \,-\, a_3\,d\phi) \,\wedge\, (\sigma_1 \,-\, i\,\sigma_2)\,, \tag{1.19}$$

where

$$a_1 \,=\, \frac{2}{g^2}\,\tanh\chi\,\cos\theta\,, \tag{1.20}$$

$$a_2 \,=\, \frac{1}{g^2}\,\frac{\rho^6\,\tanh\chi}{X_1}\,\cos^2\theta\,\sin\theta\,, \tag{1.21}$$

$$a_3 \,=\, -\,\frac{2}{g^2}\,\frac{\tanh\chi}{X_1}\,\cos^2\theta\,\sin\theta\,, \tag{1.22}$$

with

$$X_1 \,=\, \cos^2\theta \,+\rho^6\,\sin^2\theta\,, \tag{1.23}$$

and σ_i, i = 1, 2, 3, are the $SU(2)$-invariant one-forms.

The five-form flux is given by [10]

$$F_{(5)} \,=\, \mathcal{F} \,+\, *\mathcal{F}\,, \tag{1.24}$$

where

$$\mathcal{F} \,=\, \omega_r\, dr \,\wedge\, dx^0 \,\wedge\, dx^1 \,\wedge\, dx^2 \,\wedge\, dx^3 \,+\, \omega_\theta\, d\theta \,\wedge\, dx^0 \,\wedge\, dx^1 \,\wedge\, dx^2 \,\wedge\, dx^3\,, \tag{1.25}$$

and

$$\omega_r = \frac{g}{8} e^{4U} \frac{\cosh^2\chi}{\rho^4} \big((\cosh(2\chi)-3)\cos^2\theta + \rho^6(2\rho^6 \sinh^2\chi \sin^2\theta \,+ \cos(2\theta)-3)\big)\,, \tag{1.26}$$

$$\omega_\theta \,= \frac{1}{8}\, e^{4U}\, \frac{1}{\rho^2} \big(2\cosh^2\chi \,+\, \rho^6\,(\cosh(2\chi) \,-\, 3\big)\, \sin(2\theta)\,. \tag{1.27}$$

1.4 Generalizations of holographic RG flows

In this introduction we briefly considered the RG flows from $\mathcal{N}$ = 4 SYM and their holographic description from $\mathcal{N}$ = 8 gauged supergravity in five dimensions. In this section, we consider some generalizations of holographic RG flows.

To understand the first generalization, we consider a class of solutions in type IIB supergravity called the Janus solutions. Via the AdS/CFT correspondence, the only two scalar fields in type IIB supergravity, the dilaton and axion fields, Φ and $C_{(0)}$, are dual to the gauge coupling and θ-angle in $\mathcal{N}$ = 4 SYM, respectively. Unlike other solutions of type IIB supergravity, the Janus solutions have nontrivial profile of the dilaton field. To be specific, the Janus solutions are characterized by two main features: (i) they are AdS-domain wall solutions with an interface, (ii) the dilaton field takes constant values on both sides of the interface, but it jumps across the interface. As the dilaton field varies, the gauge coupling of the dual gauge theory varies across the interface, *i.e.* the dual gauge theories are defect conformal field theories. The dual gauge theory is $\mathcal{N}$ = 4 SYM in 3+1 dimensions with a 2+1 dimensional interface.

The holographic RG flows discussed so far have only involved the scalar fields dual to the fermion or scalar bilinear operators, but not the singlets in (1.1) which are dual to the five-dimensional dilaton and axion fields. Motivated by the Janus solutions we study the holographic RG flows with dilaton and axion fields. Specifically we will concentrate on the $SU(3)$-invariant flow [11], and will discover that this flow solution involving the dilaton and axion fields indeed reproduces the known Janus solutions with $SU(3)$-invariance in $\mathcal{N}$ = 2 gauged supergravity [23] and in type IIB supergravity [20]. Furthermore, we will consider the $SU(2)\times U(1)$-invariant $\mathcal{N}$ = 1 and $\mathcal{N}$ = 2 supersymmetric RG flows with dilaton and axion fields, however, unlike the $SU(3)$-invariant flow, it appears that they cannot involve nontrivial dilaton and axion fields.

To consider the second kind of generalization, we briefly discuss the recent development in applying the AdS/CFT correspondence to condensed matter physics: AdS/CMT. One of the obstacles in condensed matter physics is that the interesting condensed matter systems are usually described by strongly coupled field theories. On the other hand, the AdS/CFT correspondence provides an effective tool to study strongly coupled field theories through weakly coupled gravity theories. This AdS/CMT was initiated by phenomenological models in gravity theories which exhibit some properties of interesting condensed matter systems, $e.g.$ superconductors, Fermi liquids, and magnetism. One of the popular phenomenological models is the Abelian Higgs model which involves a metric, scalar fields with nontrivial scalar potential, and gauge fields [24, 25, 26]. Holographic superconductors were constructed as electrically charged black hole solutions of this model that develop scalar hair below a critical temperature. On the other hand, there are also top-down models of AdS/CMT from supergravity theories. Unlike the phenomenological models, they provide the precise dual field theories. For instance, holographic superconductors were constructed in type IIB supergravity [27, 28] and in d = 11 supergravity [29, 30]. Also magnetically charged brane solutions were studied in various supergravity theories [31, 32, 33, 34, 35, 36]. Due to the top-down models of AdS/CMT, consistent truncation involving gauge fields has become an interesting topic.

The holographic RG flows discussed so far have involved only the scalar fields, and not the gauge fields. Recently, however, motivated by AdS/CMT models with electric potentials, the $SU(2)\times U(1)$-invariant $\mathcal{N}$ = 1 supersymmetric RG flow [8, 10] in (1.2) was generalized to involve electric potentials, and a flow interpolating between two global AdS_5 was discovered [37]. In the same spirit, we will study electrically charged $SU(3)$-invariant flow. Furthermore, we will also study magnetically charged $SU(3)$-invariant flows.

The plan for this thesis is as follows. In chapter 2 we study the generalization of holographic RG flows to involve the dilaton and axion fields. From section 2.1 to section 2.8 we concentrate on the $SU(3)$-invariant truncation with dilaton and axion fields based on the paper [11]. In section 2.9 we, further, consider the $SU(2)\times U(1)$-invariant flows with dilaton and axion fields. In chapter 3 we study holographic RG flows involving electric potentials or magnetic fields in section 3.1 and section 3.2, respectively. Conclusions are presented in chapter 4. Technical details are collected in appendices.

Chapter 2

Holographic RG flows with dilaton and axion fields

2.1 Introduction

The Janus solutions provide a class of examples for the AdS/CFT correspondence [1]. The Janus solutions are characterized by two main features: (i) they are AdS-domain wall solutions with an interface, (ii) the dilaton field takes constant values on both sides of the interface, but it jumps across the interface. As the dilaton field is not constant, the coupling constant of the dual gauge theory varies across the interface, *i.e.* the dual gauge theories are defect conformal field theories. The first example of Janus solutions was discovered in type IIB supergravity with no supersymmetries by Bak, Gutperle and Hirano in [19]. The dual gauge theory is $\mathcal{N}$ = 4 super Yang-Mills theory in 3+1 dimensions with a 2+1 dimensional interface. Even though this solution breaks all the supersymmetries, the stability against a large class of perturbations was proved in [19, 38].

After the discovery of the original Janus solution, the dual gauge theory was studied in [39]. It was observed that by reducing $SO(6)$ R-symmetry of the dual gauge theory down to at least $SU(3)$, some supersymmetries were restored. Motivated by this observation, Clark and Karch constructed a supersymmetric Janus solution with

$SU(3)$ isometry, super Janus [23], based on the studies of curved domain wall solutions [40, 41, 42, 43, 44] in $\mathcal{N}$ = 2 gauged supergravity with one hypermultiplet in five dimensions [45, 46].

Later, Janus gauge theories were constructed more systematically in [47]. It gives the complete classification of all possible Janus solutions in type IIB supergravity. According to the classification, there are four kinds of solutions with $SO(6)$, $SU(3)$, $SU(2)\times U(1)$ and $SO(3)\times SO(3)$ isometries, and each of them has zero, four, eight, and sixteen Poincaré supersymmetries, respectively. Among these, the Janus solution with no supersymmetry is the original Janus solution [19]. By D'Hoker, Estes and Gutperle, the Janus solutions with four and sixteen supersymmetries were constructed in type IIB supergravity in [20] and [21, 22], respectively. Later, the Janus field theories in [47] were generalized to allow the theta-angle to vary which is holographicallly dual to the axion field, and were also applied to construct three-dimensional Chern-Simons theories with $\mathcal{N}$ = 4 supersymmetries in [48].

Despite of all these developments in Janus geometries, as the five- and ten-dimensional solutions were constructed independently, the relation between those solutions are far from obvious. However, as $\mathcal{N}$ = 2 gauged supergravity with one hypermultiplet is a truncation of $\mathcal{N}$ = 8 gauged supergravity in five dimensions [12, 13, 14], it was conjectured by Clark and Karch in [39] that the super Janus in $\mathcal{N}$ = 2 gauged supergravity could be embedded in $\mathcal{N}$ = 8 gauged supergravity in five dimensions. If this embedding could be achieved, as partial results of lift for embedding $\mathcal{N}$ = 8 gauged supergravity to type IIB supergravity on S^5 are readily known [16, 9, 10], one should be able to uplift the supersymmetric Janus solution in five dimensions to the one in type IIB supergravity. This will provide us with the bridge between the known supersymmetric Janus solutions in five and ten dimensions. In this section, we indeed show that the super

Janus can be embedded in $\mathcal{N} = 8$ gauged supergravity in five dimensions and its uplift gives the supersymmetric Janus solution in type IIB supergravity [20].

In order to address these questions, we will revisit the $SU(3)$-invariant truncation of $\mathcal{N} = 8$ gauged supergravity in five dimensions which was studied in [14] and [6, 7, 17]. Later it was uplifted to type IIB supergravity in [10]. However, in these studies, there was only one real scalar field in the flat domain wall, and the dilaton/axion fields were suppressed. In order to construct Janus solutions, we will generalize the previous studies in two aspects: (i) we extend the field content to include the dilaton/axion fields, so we will have two complex or four real scalar fields, (ii) we consider the AdS-domain wall instead of the flat domain wall. However, as it was known in $\mathcal{N} = 2$ gauged supergravity in five dimensions in [40, 43], we will find that the two directions of generalization are in fact equivalent, *i.e. one can turn on the dilaton/axion fields only in the curved background, and vice versa*. Finally we will show that the $SU(3)$-invariant truncation with the dilaton/axion fields indeed has a solution identical to the super Janus in [23].

Then we will uplift the solution of the $SU(3)$-invariant truncation to type IIB supergravity by employing the consistent truncation ansatz for metric and dilaton/axion fields in [16, 9, 10]. Though there are the lift formulae for three- and five-form fluxes proposed in [18], we find that they do not work for the curved domain walls. We propose modified lift formulae similar to those of [18] for three- and five-form fluxes, and check that they generate correct fluxes for the cases we are considering. Finally we will show that the lift of the $SU(3)$-invariant truncation indeed falls into a special case of the supersymmetric Janus solution in type IIB supergravity in [20].

Of independent interest from the Janus solutions, there has been notable development in consistent truncation of type IIB supergravity on Sasaki-Einstein manifolds recently [50, 51, 52, 53]. We will show that the lift of the $SU(3)$-invariant truncation to type IIB supergravity provides a particular example of the truncation in [50, 51].

Furthermore, we study the $SU(2)\times U(1)$-invariant flows with dilaton and axion fields. We will find that, unlike the $SU(3)$-invariant truncation, the dilaton and axion fields are trivial in the $\mathcal{N}$ = 2 supersymmtric flow.

In section 2.2 we review the Janus solutions in supergravity. In section 2.3 we begin by studying the $SU(3)$-invariant truncation of $\mathcal{N}$ = 8 gauged supergravity in five dimensions with dilaton and axion fields. In section 2.4 we show that a solution of the $SU(3)$-invariant truncation is identical to the super Janus in $\mathcal{N}$ = 2 supergravity in five dimensions. In section 2.5 we lift the solution of the $SU(3)$-invariant truncation to type IIB supergravity by employing consistent truncation ansatz for metric and dilaton/axion fields. In section 2.6 we show that the lifted metric and dilaton/axion fields completely fix the supersymmetric Janus solution with $SU(3)$ isometry in type IIB supergravity. In section 2.7 we continue the lift of the $SU(3)$-invariant truncation for three- and five-form fluxes. In section 2.8 we consider the consistent truncation of type IIB supergravity on Sasaki-Einstein manifolds in relation with the $SU(3)$-invariant truncation. In section 2.9 we study the $SU(2)\times U(1)$-invariant flows with dilaton and axion fields. In appendix A we briefly review $\mathcal{N}$ = 8 gauged supergravity in five dimensions. In appendix B the $SU(2,1)$ algebra is presented. In appendix C details of the supersymmetry variation for spin-3/2 fields are presented for the $SU(3)$-invariant truncation. Appendix D summarizes the different parametrizations of the scalar manifold in this paper. In appendix E we present the field equations in five dimensions.

2.2 The Janus solutions in supergravity

In this section we review the Janus solutions in supergravity.

2.2.1 The original Janus solution

The original Janus solution [19] in type IIB supergravity is an asymptotically AdS_5 space with a spatially varying dilaton. The original Janus solution has a metric, a dilaton, and a five-form flux with the other fields vanishing. The metric takes the form of AdS_4-sliced AdS_5,

$$ds^2 = f(r)ds^2_{AdS_4} - dr^2 + ds^2_{S^5}. \tag{2.1}$$

The dilaton field and the five-form flux are, respectively,

$$\Phi = \Phi(r), \tag{2.2}$$

$$F_{(5)} = 2f(r)^{1/2}\, dr \wedge \omega_{AdS_4} + 2\omega_{S^5}, \tag{2.3}$$

where ω is the unit volume form for the respective space. When one solves the equations of motion, one finds that the dilaton field takes constant values at the boundaries, but it jumps across an interface on the coordinate r. Due to this nontrivial profile of the dilaton field, this solution is named as Janus solution. This solution breaks all supersymmetries, but the stability against a large class of perturbations has been proved in [19, 38].

The dual gauge theory is a 3+1 dimensional gauge theory with a 2+1 dimensional planar interface. The gauge theory on each side of the planar interface is $\mathcal{N}$ = 4 super Yang-Mills theory, and the gauge coupling varies discontinuously across the interface. Via the AdS/CFT correspondence, $e^{\Phi} = \frac{g_{YM}^2}{4\pi}$, where Φ is the dilaton field of type IIB supergravity and g_{YM} is the coupling constant of $\mathcal{N}$ = 4 super Yang-Mills theory. Hence, as the dilaton field varies, the gauge coupling in dual field theory varies.

2.2.2 The super Janus

We briefly review $\mathcal{N} = 2$ gauged supergravity with one hypermultiplet in five dimensions [45, 46]. The bosonic sector of the theory has a graviton $e_\mu{}^a$, a vector field A_μ, and four scalar fields q^X. The scalar fields parametrize the coset manifold $\frac{SU(2,1)}{SU(2)\times U(1)}$. The bosonic part of the Lagrangian is

$$e^{-1}\mathcal{L} \,=\, -\frac{1}{2}R \,-\, \frac{1}{2}g_{XY}\,D_\mu\,q^X\,D^\mu\,q^Y \,-\, \mathcal{P}(q) \,-\, \frac{1}{4}F_{\mu\nu}\,F^{\mu\nu}\,, \tag{2.4}$$

where

$$D_\mu q^X \,=\, \partial_\mu q^X \,+\, g\,A_\mu\,K^X(q)\,, \tag{2.5}$$

and K^X are the four Killing vectors of the gauged isometries on the scalar manifold. Parametrizing the scalar fields by $q^X = \{V,\,\sigma,\,R,\,\alpha\}$, [1] the scalar potential is given by

$$\mathcal{P} \,=\, g^2\left(-6 \,-\, \frac{3R^2}{V} \,+\, \frac{3R^4}{V^2}\right)\,, \tag{2.6}$$

and the superpotential is

$$W \,=\, 1 \,+\, \frac{R^2}{V}. \tag{2.7}$$

The metric g_{XY} of the scalar manifold is

$$ds^2 \,=\, \frac{1}{2V^2}\,dV^2 \,+\, \frac{1}{2V^2}\,d\sigma^2 \,-\, \frac{2R^2}{V^2}\,d\sigma\,d\alpha \,+\, \frac{2}{V}\,dR^2 \,+\, \frac{2R^2}{V}\,(1 \,+\, \frac{R^2}{V})\,d\alpha^2\,. \tag{2.8}$$

This $\mathcal{N} = 2$ gauged supergravity with one hypermultiplet in five dimensions can be obtained from the $SU(3)$-invariant truncation of $\mathcal{N} = 8$ gauged supergravity in five dimensions. They have identical field content and the scalar manifold.

[1] The scalar field, R, was denoted by r in [23]. It should not be confused with the Ricci scalar in (2.4).

Now we briefly review the super Janus in $\mathcal{N} = 2$ gauged supergravity in five dimensions [23]. The metric is the AdS-domain wall,

$$ds^2 \,=\, e^{2U(r)}\, ds^2_{AdS_4} \,-\, dr^2\,. \tag{2.9}$$

There are also four scalar fields,

$$V \,=\, V(r)\,, \quad \sigma \,=\, \sigma(r)\,, \quad R \,=\, R(r)\,, \quad \alpha \,=\, \alpha(r)\,, \tag{2.10}$$

which depend on the r-coordinate only. We set the gauge field, A_μ, to vanish. Then by having the supersymmetry variations of fermionic fields, $i.e.$ the spin-3/2 and spin-1/2 fields, to vanish, one obtains the supersymmetry equations,

$$U' \,=\, \mp\, g\, W\, \gamma\,, \tag{2.11}$$

$$V' \,=\, 6\, g\, \left(\mp\, R^2\, \gamma \,+\, R\, \sqrt{V}\, \sqrt{1-\gamma^2}\right)\,, \tag{2.12}$$

$$R' \,=\, 3\, g\, \left(\pm\, R\, \gamma \,+\, \frac{R^2}{\sqrt{V}}\, \sqrt{1-\gamma^2}\right)\,, \tag{2.13}$$

where

$$\gamma \,=\, \sqrt{1 \,-\, \frac{\lambda^2\, e^{-2U}}{g^2\, W^2}}\,, \tag{2.14}$$

and the scalar fields σ and α are consistently set to be constant. Then, numerically plotting $V \,=\, V(r)$, we find that it exhibits the nontrivial profile of the dilaton field in Janus solutions.

2.2.3 The Janus solutions in type IIB supergravity

Later in [47] the Janus gauge theories were constructed more systematically. It also gives the complete classification of all possible Janus solutions in type IIB supergravity.

Isometry	$SO(6)$	$SU(3)$	$SU(2)\times U(1)$	$SO(3)\times SO(3)$
Supersymmetries	zero	four	eight	sixteen

Table 2.1 Classification of all possible Janus solutions in type IIB supergravity

In table 1 it shows the isometry of internal space and the number of real supersymmetry out of total 32 real supersymmetries of type IIB supergravity. The number of supersymmetry counts real supercharges with both Poincaré and conformal supercharges. The Janus solution with no supersymmetry is the original Janus solution. From these observations, Janus solutions with four and sixteen supersymmetries were constructed in [20] and [21, 22] respectively. Here we take a look at the one with four supersymmetries as this one has $SU(3)$ isometry.

We briefly review the supersymmetric Janus solution with the internal space isometry $SU(3)$ in type IIB supergravity [20]. The metric is given by

$$ds^2 = f_4^2\, ds^2_{AdS_4} - dr^2 + f_1^2\,(d\beta + A_1) + f_2^2\, ds^2_{CP_2}\,, \tag{2.15}$$

where

$$ds^2_{CP_2} = d\alpha^2 + \frac{1}{4}\sin^2\alpha\,\left(\sigma_1^2 + \sigma_2^2 + \cos^2\alpha\,\sigma_3^2\right)\,, \tag{2.16}$$

and

$$A_1 = \frac{1}{2}\sin^2\theta\,\sigma_3\,, \tag{2.17}$$

and σ_i, i = 1, 2, 3, are the $SU(2)$-invariant one-forms,

$$\begin{aligned} \sigma_1 &= -\sin\alpha_2\,\cos\alpha_3\,d\alpha_1 + \sin\alpha_3\,d\alpha_2\,, \\ \sigma_2 &= +\sin\alpha_2\,\sin\alpha_3\,d\alpha_1 + \cos\alpha_3\,d\alpha_2\,, \\ \sigma_3 &= -\cos\alpha_2\,d\alpha_1 - d\alpha_3\,, \end{aligned} \tag{2.18}$$

The five-form flux is given by

$$F_{(5)} = f_5\left(-e^0\wedge e^1\wedge e^2\wedge e^3\wedge e^4 + e^5\wedge e^6\wedge e^7\wedge e^8\wedge e^9\right), \tag{2.19}$$

where e^n, $n = 0,\,\ldots,\,9$ are the frames of the metric,

$$e^i = f_4\,\hat{e}^i\,, \qquad e^4 = dr\,, \qquad e^5 = f_1\hat{e}^5 = f_1\,(d\beta + A_1)\,, \qquad e^a = f_2\,\hat{e}^a\,, \tag{2.20}$$

where i = 0, 1, 2, 3, a = 6, 7, 8, 9, and

$$\hat{e}^6 = d\alpha\,, \qquad \hat{e}^7 = \frac{1}{4}\sin(2\alpha)\,\sigma_3\,, \qquad \hat{e}^8 = \frac{1}{2}\sin\alpha\,\sigma_1\,, \qquad \hat{e}^9 = \frac{1}{2}\sin\alpha\,\sigma_2\,. \tag{2.21}$$

The two-form gauge potential is given by

$$B^{\rm DEG}_{(2)} = C_{(2)} - i\,B_{(2)} = i\,f_3\,\Omega_2 - i\,\overline{g}_3\,\overline{\Omega}_2\,, \tag{2.22}$$

where $B^{\rm DEG}_{(2)}$ is the two-form gauge potential defined in [20], $C_{(2)}$ and $B_{(2)}$ are RR and NSNS two-form gauge potentials respectively, Ω_2 is the holomorphic (2,0)-form on $\mathbb{CP}_2$, f_3 and g_3 are complex functions, and the bar denotes complex conjugation. The dilaton/axion fields are denoted by B with its associated function f. Overall, the most general solution with the $SU(3)$ isometry of internal space is specified by the seven functions, f_1, f_2, f_3, g_3, f_4, f_5, and B, and they depend only on the r-coordinate.

In section 9 of [20], a special case is presented when

$$a^{DEG} = -\frac{3}{f_1 f_2^2} f (f_3 - B g_3) = 0 , \tag{2.23}$$

where a^{DEG} is a function defined for convenience in [20]. Furthermore, in this case,

$$f_1 f_2 = \rho , \qquad f_5 = \frac{3}{2 f_1} - \frac{1}{2} \frac{f_1}{f_2^2} , \tag{2.24}$$

where ρ is a constant. Some functions are integrated to hyper-elliptic integral as

$$f_4^2 \left(\frac{\partial \Psi}{\partial r} \right)^2 = \left(1 + \frac{C_2^2}{9\rho^8} \Psi^6 \right)^2 - \Psi^2 , \tag{2.25}$$

where $\Psi = \psi^{\mathrm{DEG}} = \frac{\rho}{f_2 f_4}$ and C_2 is a constant. Here ψ^{DEG} is a quantity defined in [20].

2.3 Truncation of $\mathcal{N}$ = 8 gauged supergravity in five dimensions

2.3.1 The $SU(3)$-invariant truncation

We study the $SU(3)$-invariant truncation of $\mathcal{N}$ = 8 gauged supergravity in five dimensions. There are a graviton $e_\mu{}^a$, a vector field A_μ, and four real scalars x_i for the bosonic field content in the $SU(3)$-invariant sector. As mentioned in the introduction, there have been studies on the $SU(3)$-invariant truncation in [14, 6, 7, 17] and [10], but consistently they did not included the dilaton and axion fields in these studies. Here we extend the field content to all four scalar fields including dilaton and axion fields.

Let us count the number of bosonic fields in the $SU(3)$-invariant truncation. In the full theory, under the gauge group, $SU(4) \simeq SO(6)$, 1 graviton $e_\mu{}^a$ transforms as $\mathbf{1}$, 15 vector fields $A_{\mu IJ}$ as $\mathbf{15}$, 12 two-form tensor fields $B_{\mu\nu}{}^{I\alpha}$ as $\mathbf{6}+\mathbf{6}$, and 42 scalar fields ϕ^{abcd} as $\mathbf{20}'+\mathbf{10}+\overline{\mathbf{10}}+\mathbf{1}+\mathbf{1}$. By breaking $SU(4)$ down to $SU(3)$ they branch as [7]

$$
\begin{array}{lrcll}
e_\mu{}^a & \mathbf{1} & \rightarrow & \mathbf{1}, & (2.26)\\
A_{\mu IJ} & \mathbf{15} & \rightarrow & \mathbf{8}+\mathbf{3}+\overline{\mathbf{3}}+\mathbf{1}, & (2.27)\\
B^{I\alpha}_{\mu\nu} & \mathbf{6}+\mathbf{6} & \rightarrow & (\mathbf{3}+\overline{\mathbf{3}})+(\mathbf{3}+\overline{\mathbf{3}}), & (2.28)\\
 & \mathbf{20}' & \rightarrow & \mathbf{8}+\mathbf{6}+\overline{\mathbf{6}}, & \\
\phi^{abcd} & \mathbf{10}+\overline{\mathbf{10}} & \rightarrow & (\mathbf{1}+\overline{\mathbf{3}}+\mathbf{6})+(\mathbf{1}+\mathbf{3}+\overline{\mathbf{6}}), & (2.29)\\
 & \mathbf{1}+\mathbf{1} & \rightarrow & \mathbf{1}+\mathbf{1}, &
\end{array}
$$

so we have a graviton $e_\mu{}^a$, a vector field A_μ, and four scalars x_i in the $SU(3)$-invariant sector.

The 42 scalar fields of $\mathcal{N}$ = 8 gauged supergravity in five dimensions live on the coset manifold $E_{6(6)}/USp(8)$. The basic structure of the coset manifold is explained in [14], and is summarized in appendix A. Fundamental representation of $E_{6(6)}$ is real and 27-dimensional. The infinitesimal $E_{6(6)}$ transformation in the $SL(6,\mathbb{R})\times SL(2,\mathbb{R})$ basis, (z_{IJ}, $z^{I\alpha}$), is [14]

$$\delta z_{IJ} = -\Lambda^K{}_I\, z_{KJ} - \Lambda^K{}_J\, z_{IK} + \Sigma_{IJK\beta}\, z^{K\beta}\,, \tag{2.30}$$

$$\delta z_{I\alpha} = \Lambda^I{}_K\, z^{K\alpha} + \Lambda^\alpha{}_\beta\, z_{I\beta} + \Sigma^{KLI\beta}\, z_{KL}\,, \tag{2.31}$$

where $\Lambda^I{}_J$ and $\Lambda^\alpha{}_\beta$ are real and traceless generators of $SL(6,\mathbb{R})$ and $SL(2,\mathbb{R})$ respectively, and the coset elements $\Sigma_{IJK\alpha}$ are real and antisymmetric in IJK.

Among the $E_{6(6)}$ generators, the $SU(3)$ generators of the gauge group $SO(6)$ are the ones that commute with the complex structure, J_{IJ}, which is an antisymmetric tensor with nonzero components, $J_{12} = J_{34} = J_{56} = 1$. Then we obtain the $SU(3)$-invariant generators by finding ones that commute with the $SU(3)$ generators. There are eight $SU(3)$-invariant generators, and they close onto an $SU(2,1)$ algebra,

$$\Sigma^{(1)}_{IJK\alpha} = +\,(\delta^{1357}_{IJK\alpha} - \delta^{2468}_{IJK\alpha}) + (\delta^{1368}_{IJK\alpha} - \delta^{2457}_{IJK\alpha}) + (\delta^{1458}_{IJK\alpha} - \delta^{2367}_{IJK\alpha}) - (\delta^{1467}_{IJK\alpha} - \delta^{2358}_{IJK\alpha}), \tag{2.32}$$

$$\begin{aligned}\Sigma^{(2)}_{IJK\alpha} = &+\,(-\delta^{1358}_{IJK\alpha} - \delta^{2467}_{IJK\alpha}) + (\delta^{1367}_{IJK\alpha} + \delta^{2458}_{IJK\alpha}) + (\delta^{1457}_{IJK\alpha} + \delta^{2368}_{IJK\alpha}) \\ &- (-\delta^{1468}_{IJK\alpha} - \delta^{2357}_{IJK\alpha}),\end{aligned} \tag{2.33}$$

$$\Sigma^{(3)}_{IJK\alpha} = +\,(\delta^{1358}_{IJK\alpha} - \delta^{2467}_{IJK\alpha}) + (\delta^{1367}_{IJK\alpha} - \delta^{2458}_{IJK\alpha}) + (\delta^{1457}_{IJK\alpha} - \delta^{2368}_{IJK\alpha}) - (\delta^{1468}_{IJK\alpha} - \delta^{2357}_{IJK\alpha}), \tag{2.34}$$

$$\begin{aligned}\Sigma^{(4)}_{IJK\alpha} = &+\,(\delta^{1357}_{IJK\alpha} + \delta^{2468}_{IJK\alpha}) + (-\delta^{1368}_{IJK\alpha} - \delta^{2457}_{IJK\alpha}) + (-\delta^{1458}_{IJK\alpha} - \delta^{2367}_{IJK\alpha}) \\ &- (\delta^{1467}_{IJK\alpha} + \delta^{2358}_{IJK\alpha}),\end{aligned} \tag{2.35}$$

$$\Lambda^{(5)\,I}{}_{J} \;=\; J_{IJ}, \tag{2.36}$$

$$\Lambda^{(6)\,\alpha}{}_{\beta} \;=\; (S_1)^{\,\alpha}{}_{\beta}, \tag{2.37}$$

$$\Lambda^{(7)\,\alpha}{}_{\beta} \;=\; (S_2)^{\,\alpha}{}_{\beta}, \tag{2.38}$$

$$\Lambda^{(8)\,\alpha}{}_{\beta} \;=\; (S_3)^{\,\alpha}{}_{\beta}, \tag{2.39}$$

where

$$S_1 \;=\; \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\,,\quad S_2 \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,,\quad S_3 \;=\; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,, \tag{2.40}$$

are three $SL(2,\mathbb{R})$ generators. We refer to appendix B for the $SU(2,1)$ algebra of these generators. The generators $\Lambda^{(6)}$, $\Lambda^{(7)}$ are symmetric, and with the self-duality defined by

$$\Sigma_{IJK\alpha} \;=\; +\,\frac{1}{6}\,\epsilon_{IJKLMNP}\,\epsilon_{\alpha\beta}\,\Sigma^{MNP\beta}\,, \tag{2.41}$$

$\Sigma^{(1)}$, $\Sigma^{(2)}$ are self-dual. By computing the Cartan-Killing form [14] these symmetric and self-dual generators turn out to be the noncompact generators of the scalar manifold [46],

$$\mathcal{M} \;=\; \frac{SU(2,1)}{SU(2)\times U(1)}\,. \tag{2.42}$$

We exponentiate the transformations by four noncompact generators,

$$\begin{aligned} T_1 \;&=\; \frac{1}{4\sqrt{2}}\,\Sigma^{(1)}\,, & T_2 \;&=\; \frac{1}{4\sqrt{2}}\,\Sigma^{(2)}\,, \\ T_3 \;&=\; \frac{1}{2\sqrt{2}}\,(\Lambda^{(7)} + \Lambda^{(6)})\,, & T_4 \;&=\; \frac{1}{2\sqrt{2}}\,(\Lambda^{(7)} - \Lambda^{(6)})\,, \end{aligned} \tag{2.43}$$

with parameters, x_1, x_2, x_3, x_4, respectively. Schematically the exponentiation of the generators is

$$z' \;=\; e^{(x_3\,T_3\,+\,x_4\,T_4)}\,e^{(x_1\,T_1\,+\,x_2\,T_2)}\,z\,. \tag{2.44}$$

From the exponentiation we can extract the coset representatives in the $SL(6,\mathbb{R})\times SL(2,\mathbb{R})$ basis, $U^{IJ}{}_{KL}$, $U^{IJK\alpha}$, $U_{I\alpha}{}^{KL}$ and $U_{I\alpha}{}^{J\beta}$, by (A.1) and (A.2). The coset representatives in the $USp(8)$ basis, $\mathcal{V}^{IJab}$, $\mathcal{V}_{I\alpha}{}^{ab}$, are obtained by (A.3) and (A.4).

Now with the coset representatives in the $USp(8)$ basis, we can reduce the bosonic part of the Lagrangian of the $SU(3)$-invariant truncation. We introduce an angular parametrization of the scalar fields,

$$\begin{aligned} x_1 &= 2\,\chi\,\cos\psi\,, \quad x_2 \,=\, 2\,\chi\,\sin\psi\,, \\ x_3 &= 2\,\phi\,\cos a\,, \quad x_4 \,=\, 2\,\phi\,\sin a\,. \end{aligned} \tag{2.45}$$

The bosonic part of the Lagrangian is

$$e^{-1}\,\mathcal{L} \,=\, -\frac{1}{4}\,R \,+\, \mathcal{L}_{kin} \,+\, \mathcal{P} \,-\, \frac{3}{4}\,F_{\mu\nu}\,F^{\mu\nu}\,, \tag{2.46}$$

where the kinetic term for the scalar fields is

$$\begin{aligned} \mathcal{L}_{kin} \,=&\, \frac{1}{2}\,\partial_\mu\chi\,\partial^\mu\chi \,+\, \frac{1}{8}\,\sinh^2(2\,\chi)\,\left(\partial_\mu\psi \,+\, \sinh^2\phi\,\partial_\mu a \,+\, g\,A_\mu\right)^2 \\ &+\, \cosh^2\chi\,\left(\frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi \,+\, \frac{1}{8}\,\sinh^2(2\,\phi)\,\partial_\mu a\,\partial^\mu a\right)\,, \end{aligned} \tag{2.47}$$

and the scalar potential is

$$\mathcal{P} \,=\, \frac{3}{32}\,g^2\left(\cosh^2(2\,\chi) \,-\, 4\,\cosh(2\,\chi) \,-\, 5\right)\,. \tag{2.48}$$

Note that the scalar potential is manifestly invariant under $SL(2,\mathbb{R})$, $i.e.$ it is independent of ϕ and a. We note that ϕ and a are dilaton and axion fields in five dimensions.

The scalar potential has two critical points which are the AdS_5 vacua in the $SU(3)$-invariant truncation [16, 8].[2] One of the critical points is the $\mathcal{N}$ = 8 supersymmetric $SO(6)$ point where $\chi = 0$ and $\mathcal{P} = -\frac{3}{4}\,g^2$. This point lifts to $AdS_5 \times S^5$ vacuum in type IIB supergravity. Another one is the nonsupersymmmtric $SU(3)$ point where $\chi = \frac{1}{2}\log(2-\sqrt{3})$ and $\mathcal{P} = -\frac{27}{32}\,g^2$. This point lifts to a solution found by Romans in type IIB supergravity in [49]. The holographic renormalization flows studied in [6, 7, 17, 9] and the domain wall solution for holographic superconductor in [27, 28] flow to this critical point.

2.3.2 The supersymmetry equations

In this section we will explicitly derive the supersymmetry equations for the $SU(3)$-invariant truncation with the dilaton and axion fields, and then solve them numerically. We set the gauge field, A_μ, to vanish. Some equivalent equations in $\mathcal{N}$ = 2 gauged supergravity were obtained in [40, 43], however, this subsection is to have equations in the parametrization of $\mathcal{N}$ = 8 gauged supergravity in five dimensions with more scalar fields.

We will consider the AdS-domain wall,

$$ds^2 \,=\, e^{2\,U(r)}\,ds^2_{AdS_4} \,-\, dr^2\,, \tag{2.49}$$

where

$$ds^2_{AdS_4} \,=\, \frac{1}{z^2}\,(dt^2 \,-\, dx^2 \,-\, dy^2 \,-\, dz^2)\,. \tag{2.50}$$

[2] The scalar field χ was denoted by $\varphi_1 = \chi$ in [8].

We begin by considering the superpotential and the spinors in five dimensions. The superpotential, W, is obtained as one of the eigenvalues of W_{ab} tensor [8],

$$W_{ab}\,\eta^b_{(k)} \,=\, W\,\eta^a_{(k)}\,, \tag{2.51}$$

where $k \,=\, 1,\,2.$ There are two eigenvalues with degeneracy of two and six, and they are, respectively,

$$W_1 \,=\, -\,\frac{3}{4}\Big(1\,+\,\cosh(2\,\chi)\Big)\,, \tag{2.52}$$
$$W_2 \,=\, -\,\frac{1}{4}\Big(5\,+\,\cosh(2\,\chi)\Big)\,, \tag{2.53}$$

but only $W \,=\, W_1$ gives the scalar potential by

$$\mathcal{P} \,=\, \frac{g^2}{8}\left|\frac{\partial W}{\partial\varphi_i}\right|^2 \,-\, \frac{g^2}{3}|W|^2\,, \tag{2.54}$$

where $\varphi_i \,=\, \chi\,,\,\phi\,,\,\psi\,,\,a\,.$ The eigenvectors, $\eta^a_{(1)}$, $\eta^a_{(2)}$, for the superpotential, W, are

$$\eta^a_{(1)} \,=\, (0,\,1,\,0,\,1,\,-1,\,0,\,1,\,0)\,, \tag{2.55}$$

$$\eta^a_{(2)} \,=\, (-1,\,0,\,1,\,0,\,0,\,-1,\,0,\,-1)\,, \tag{2.56}$$

and they are related to each other by

$$\Omega_{ab}\,\eta^b_{(1)} \,=\, -\,\eta^a_{(2)}, \qquad \Omega_{ab}\,\eta^b_{(2)} \,=\, +\,\eta^a_{(1)}\,, \tag{2.57}$$

where Ω_{ab} is the $USp(8)$ symplectic form given in *e.g.* [8]. We employed the gamma matrix conventions in [8]. Then the $SU(3)$-invariant five-dimensional spinors are given by

$$\epsilon^a \;=\; \eta^a_{(1)}\,\hat{\epsilon}_1 \,+\, \eta^a_{(2)}\,\hat{\epsilon}_2\,, \tag{2.58}$$

$$\epsilon_a \;=\; \Omega_{ab}\,\epsilon^b \;=\; -\,\eta^a_{(2)}\,\hat{\epsilon}_1 \,+\, \eta^a_{(1)}\,\hat{\epsilon}_2\,, \tag{2.59}$$

where $\hat{\epsilon}_1$ and $\hat{\epsilon}_2$ are spinors with four complex components.

The supersymmetry equations are obtained by setting the supersymmetry variations of fermionic fields, *i.e.* the spin-3/2 and spin-1/2 fields, to zero. For the supersymmetry analysis we will suppress the gauge field, A_μ, below. The purely bosonic parts of the variations are [14]

$$\delta\,\psi_{\mu a} \;=\; D_\mu\,\epsilon_a \,-\, \frac{1}{6}\,g\,W_{ab}\,\gamma_\mu\,\epsilon^b\,, \tag{2.60}$$

$$\delta\,\chi_{abc} \;=\; \sqrt{2}\left[\gamma^\mu\,P_{\mu abcd}\,\epsilon^d \,-\, \frac{1}{2}\,g\,A_{dabc}\,\epsilon^d\right]. \tag{2.61}$$

First we solve the spin-3/2 field variation. For the t-, x-, y- directions,

$$U'\,\gamma^{(4)}\,\epsilon_a \,-\, e^{-U}\,\gamma^{(3)}\,\epsilon_a \,-\, \frac{1}{3}\,g\,W_{ab}\,\epsilon^b \;=\; 0\,, \tag{2.62}$$

where the prime denotes the derivative with respect to the r-coordinate. We plug the spinors, (2.58), in (2.62) and rearrange to obtain

$$\begin{pmatrix} U'\,\gamma^{(4)} \,+\, e^{-U}\,\gamma^{(3)} & \frac{1}{3}\,g\,W \\ \frac{1}{3}\,g\,W & U'\,\gamma^{(4)} \,+\, e^{-U}\,\gamma^{(3)} \end{pmatrix} \begin{pmatrix} \hat{\epsilon}_1 \\ \hat{\epsilon}_2 \end{pmatrix} \;=\; 0\,. \tag{2.63}$$

From the integrability of the variation, $i.e.$ the determinant of the matrix in (2.63) vanishes, we obtain [40, 43]

$$U' = \mp \frac{1}{3} g W \gamma, \tag{2.64}$$

where

$$\gamma = \sqrt{1 - \frac{9 e^{-2U}}{l^2 g^2 W^2}} . \tag{2.65}$$

From here the upper and lower signs in equations are related. Note that for the flat domain wall, $l \to \infty$, we have $\gamma = 1$. By plugging (2.64) back into (2.63), we obtain a projection condition for the spinors,

$$\begin{aligned} \hat{\epsilon}_1 &= + (\mp \gamma \gamma^{(4)} + \sqrt{1 - \gamma^2} \gamma^{(3)}) \hat{\epsilon}_2 , \\ \hat{\epsilon}_2 &= - (\mp \gamma \gamma^{(4)} + \sqrt{1 - \gamma^2} \gamma^{(3)}) \hat{\epsilon}_1 . \end{aligned} \tag{2.66}$$

For the flat domain wall limit, $l \to \infty$, it reduces to the projection condition in [8]. By multiplying $\gamma^{(4)}$ on both sides of (2.66) and rearranging them,

$$\gamma^{(4)} \begin{pmatrix} \hat{\epsilon}_1 \\ \hat{\epsilon}_2 \end{pmatrix} = i \left[\pm \gamma \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \sqrt{1-\gamma^2} (-i \gamma^{(4)} \gamma^{(3)}) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right] \begin{pmatrix} \hat{\epsilon}_1 \\ \hat{\epsilon}_2 \end{pmatrix} . \tag{2.67}$$

This is the first projection condition on the spinors, $\hat{\epsilon}_1$, $\hat{\epsilon}_2$.

Motivated by the studies on the curved domain wall solutions in $\mathcal{N} = 2$ gauged supergravity in five dimensions [40, 43], we impose another projection condition on the spinors. We define an operator, $\Gamma = - i \gamma^{(4)} \gamma^{(3)}$. Noting that $\Gamma^2 = 1$, we assume that it acts on the spinors as

$$\Gamma \begin{pmatrix} \hat{\epsilon}_1 \\ \hat{\epsilon}_2 \end{pmatrix} = \begin{pmatrix} -\cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \hat{\epsilon}_1 \\ \hat{\epsilon}_2 \end{pmatrix} . \tag{2.68}$$

where we have introduced a new field, $\theta = \theta(r)$. By solving the supersymmetry equations and the field equations, we will show that it is consistent to impose the projection condition, (2.68), and the new field, $\theta(r)$, is fully determined. Using (2.68) we rewrite (2.67) as

$$\gamma^{(4)}\,\hat{\epsilon}_i = i\,\Big[\pm\,\gamma\,(\sigma^2)_{ij} \,+\, \sqrt{1-\gamma^2}\,\left(\cos\theta\,(\sigma^1)_{ij} \,+\, \sin\theta\,(\sigma^3)_{ij}\right)\Big]\,\hat{\epsilon}_j\,, \tag{2.69}$$

where σ^a, $a\,=\,1,\,2,\,3$, are the Pauli matrices. For brevity, we will write it as

$$\gamma^{(4)}\,\hat{\epsilon}_i \;=\; S_{ij}\,\hat{\epsilon}_i\,, \tag{2.70}$$

where the components of the matrix, S_{ij}, can be read off from (2.69). Similar projection condition was obtained in $\mathcal{N}$ = 2 gauged supergravity in five dimensions in [40, 43].

Now, with the projection condition, (2.69), we solve the spin-1/2 field variation, (2.61),

$$\delta\,\chi_{abc} \;=\; \sqrt{2}\,\Big[\gamma^\mu\,P_{\mu abcd}\,\epsilon^d \,-\, \frac{1}{2}\,g\,A_{dabc}\,\epsilon^d\Big] \;=\; 0\,. \tag{2.71}$$

When we plug the spinors, (2.58), in (2.71), we obtain

$$(P_{4abcd}\,\eta^d_{(1)})\,\gamma^{(4)}\,\hat{\epsilon}_1 \,+\, (P_{4abcd}\,\eta^d_{(2)})\,\gamma^{(4)}\,\hat{\epsilon}_2 \,-\, \frac{g}{2}\,(A_{dabc}\,\eta^d_{(1)})\,\hat{\epsilon}_1 - \frac{g}{2}\,(A_{dabc}\,\eta^d_{(2)})\,\hat{\epsilon}_2 \;=\; 0\,. \tag{2.72}$$

For any specific choice of abc indices we define

$$P_1 \;=\; P_{4abcd}\,\eta^d_{(1)}\,, \qquad P_2 \;=\; P_{4abcd}\,\eta^d_{(2)}\,, \qquad A_1 \;=\; A_{dabc}\,\eta^d_{(1)}\,, \qquad A_2 \;=\; A_{dabc}\,\eta^d_{(2)}\,. \tag{2.73}$$

It turns out that A_1 and A_2 can have two distinct values,

$$\begin{aligned} A_{d368}\,\eta^d_{(1)} &= -\,\frac{3}{16}\,i\,\cos\psi\,\sinh\chi\,, \\ A_{d368}\,\eta^d_{(2)} &= +\,\frac{3}{16}\,i\,\sin\psi\,\sinh\chi\,, \end{aligned} \tag{2.74}$$

or

$$\begin{aligned} A_{d457}\,\eta^d_{(1)} &= -\,\frac{3}{16}\,i\,\sin\psi\,\sinh\chi\,, \\ A_{d457}\,\eta^d_{(2)} &= -\,\frac{3}{16}\,i\,\cos\psi\,\sinh\chi\,. \end{aligned} \tag{2.75}$$

Similarly, for P_1 and P_2, we have

$$\begin{aligned} P_{4\,368d}\,\eta^d_{(1)} &= +\,\frac{1}{16}\,\Big[i\,\sin\psi\,\chi' - i\,\sinh\chi\,\cos\psi\,\psi' - \cosh\frac{\chi}{2}\,\sin a\,\chi' \\ &\quad + \left(\cosh\frac{\chi}{2}\,\sinh\phi\,\cos a + i\,\sinh\chi\,\cos\psi\,\sinh^2\frac{\phi}{2}\right)a'\Big]\,, \\ P_{4\,368d}\,\eta^d_{(1)} &= -\,\frac{1}{16}\,\Big[i\,\cos\psi\,\chi' - i\,\sinh\chi\,\sin\psi\,\psi' - \cosh\frac{\chi}{2}\,\cos a\,\chi' \\ &\quad + \left(\cosh\frac{\chi}{2}\,\sinh\phi\,\sin a - i\,\sinh\chi\,\sin\psi\,\sinh^2\frac{\phi}{2}\right)a'\Big]\,, \end{aligned} \tag{2.76}$$

or

$$\begin{aligned} P_{4\,457d}\,\eta^d_{(1)} &= +\,\frac{1}{16}\,\Big[i\,\cos\psi\,\chi' - i\,\sinh\chi\,\sin\psi\,\psi' - \cosh\frac{\chi}{2}\,\cos a\,\chi' \\ &\quad + \left(\cosh\frac{\chi}{2}\,\sinh\phi\,\sin a - i\,\sinh\chi\,\sin\psi\,\sinh^2\frac{\phi}{2}\right)a'\Big]\,, \\ P_{4\,457d}\,\eta^d_{(1)} &= +\,\frac{1}{16}\,\Big[i\,\sin\psi\,\chi' - i\,\sinh\chi\,\cos\psi\,\psi' - \cosh\frac{\chi}{2}\,\sin a\,\chi' \\ &\quad + \left(\cosh\frac{\chi}{2}\,\sinh\phi\,\cos a + i\,\sinh\chi\,\cos\psi\,\sinh^2\frac{\phi}{2}\right)a'\Big]\,. \end{aligned} \tag{2.77}$$

Other choice for abc indices gives the same supersymmetry equations in the end. For a particular choice of P_1, P_2, A_1, A_2, (2.72) reduces to

$$\sum_{i=1}^{2}\Big[P_i\,\gamma^{(4)}\,\hat{\epsilon}_i\;-\;\frac{g}{2}\,A_i\,\hat{\epsilon}_i\Big]\;=\;0\,. \tag{2.78}$$

Then plugging the projection condition, (2.70), in (2.78) gives

$$\sum_{i,j=1}^{2}\Big[P_i\,S_{ij}\;-\;\frac{g}{2}\,A_i\,\delta_{ij}\Big]\;\hat{\epsilon}_j\;=\;0\,. \tag{2.79}$$

Since we want to have the maximal supersymmetry, we assume that the spinors, $\hat{\epsilon}_1$, $\hat{\epsilon}_2$, are independent which implies that (2.79) can be solved by having

$$\sum_{i=1}^{2}\Big[P_i\,S_{ij}\;-\;\frac{g}{2}\,A_i\,\delta_{ij}\Big]\;=\;0\,, \tag{2.80}$$

where j = 1, 2. After some calculations, the two complex equations in (2.80) yield four real flow equations,

$$\phi'\;=\;+\;\frac{3}{2}\,g\,\sqrt{1-\gamma^2}\,\cos(a\;-\psi\;+\;\theta)\,\sinh\chi\,, \tag{2.81}$$

$$\chi'\;=\;\mp\;\frac{3}{4}\,g\,\gamma\,\sinh(2\,\chi)\;=\;\pm\,\frac{g}{2}\,\frac{\partial W}{\partial\chi}\,\gamma, \tag{2.82}$$

$$a'\;=\;-\;3\,g\,\sqrt{1-\gamma^2}\,\sin(a\;-\;\psi\;+\;\theta)\,\mathrm{csch}(2\,\phi)\,\sinh\chi\,, \tag{2.83}$$

$$\psi'\;=\;+\;\frac{3}{2}\,g\,\sqrt{1-\gamma^2}\,\sin(a\;-\;\psi\;+\;\theta)\,\tanh\phi\,\sinh\chi\,. \tag{2.84}$$

In appendix C, we show that, with the other choice of P_1, P_2, A_1, A_2, we lead to the same set of flow equations, so that the spin-1/2 field variation is solved without introducing additional projection condition. We also obtained the field equations and presented them in appendix F. Unlike the flow equations, (2.64) and (2.81)-(2.84), which are first order differential equations, the field equations are second order. A lengthy calculation shows that the flow equations are indeed consistent with the field equations, provided that the field, $\theta(r)$, introduced in (2.68) satisfies

$$\theta' \,=\, -\frac{3}{2}\,g\,\sqrt{1-\gamma^2}\,\sin(a\,-\,\psi\,+\,\theta)\,\tanh\phi\,\sinh\chi\,. \tag{2.85}$$

This first order constraint on the field, $\theta(r)$, is a result of the fact that the supersymmetry equations and the field equations cannot be reduced to a first order system. [3]

Also note that the supersymmetry equations imply that in the limit, $l\,\rightarrow\,\infty$, which describes a flat domain wall, we must set ϕ, a, ψ to be constants, *i.e.* the dilaton/axion fields decouple, and vice versa. One can turn on the dilaton/axion fields only in the curved domain wall [40, 43].

We have also checked the integrability of the spin-3/2 field variations for the r- and z-directions, but they do not generate any new constraint on the supersymmetry. The variations for these directions are presented in appendix D. By solving the spin-3/2 field variation for the r-direction,

$$\partial_r\,\hat{\epsilon}_1\,-\,(+\,Q_1\,\hat{\epsilon}_1\,+\,Q_2\,\hat{\epsilon}_2)\,+\,\frac{1}{6}\,g\,W\,\gamma^{(4)}\,\hat{\epsilon}_2\,=0\,, \tag{2.86}$$

$$\partial_r\,\hat{\epsilon}_2\,-\,(-\,\overline{Q}_2\,\hat{\epsilon}_1\,-\,Q_1\,\hat{\epsilon}_2)\,-\,\frac{1}{6}\,g\,W\,\gamma^{(4)}\,\hat{\epsilon}_1\,=0\,, \tag{2.87}$$

[3] Here the role of $\theta(r)$ is effectively to reduce one remaining second order equation to a first order equation.

where

$$Q_1 = -\, i\, \sinh\chi \left[\cos(a\, -\, \psi)\, \phi' \, - \, \frac{1}{2}\, \sin(a\, -\, \psi)\, \sinh(2\, \phi)\, a' \right], \tag{2.88}$$

$$Q_2 = -\, i \left[\sinh\chi \left(\sin(a\, -\, \psi)\, \phi' \, - \, \frac{i}{2}\, \sinh\chi\, \psi' \right) \right. \\ \left. + \, \frac{1}{2} \left(\cos(a\, -\, \psi)\, \sinh(2\, \phi)\, \sinh\chi \, - \, \frac{i}{2} \big(-\, 3\, +\, \cosh(2\, \chi) \big)\, \sinh^2\phi \right) a' \right], \tag{2.89}$$

we obtain the r-dependence of the spinors,

$$\begin{pmatrix} \hat{\epsilon}_1(r) \\ \hat{\epsilon}_2(r) \end{pmatrix} = e^{U/2} \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} e^{\frac{i}{2}\Lambda} & 0 \\ 0 & e^{-\frac{i}{2}\Lambda} \end{pmatrix} \begin{pmatrix} \hat{\epsilon}_1^{(0)} \\ \hat{\epsilon}_2^{(0)} \end{pmatrix}, \tag{2.90}$$

where $\cos\Lambda \;=\; \gamma$. Here $\hat{\epsilon}_i^{(0)}$, i = 1, 2, depend on the AdS_4 part of the coordinates in (2.49), but are independent of the r-coordinate, and satisfy the projection conditions for the flat domain wall,

$$\gamma^{(4)} \begin{pmatrix} \hat{\epsilon}_1^{(0)} \\ \hat{\epsilon}_2^{(0)} \end{pmatrix} = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \hat{\epsilon}_1^{(0)} \\ \hat{\epsilon}_2^{(0)} \end{pmatrix}, \tag{2.91}$$

and

$$(-\, i\, \gamma^{(4)}\, \gamma^{(3)}) \begin{pmatrix} \hat{\epsilon}_1^{(0)} \\ \hat{\epsilon}_2^{(0)} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{\epsilon}_1^{(0)} \\ \hat{\epsilon}_2^{(0)} \end{pmatrix}. \tag{2.92}$$

This explains the fact that all the integrability conditions have been satisifed, $i.e.$ an explicit solution to a system of equations must satisfy all integrabilities automatically.

Before we close this section, let us count the number of supersymmetries the solution has. Each five-dimensional spinor, $\hat{\epsilon}_i$, $i \;=\; 1,\, 2$, has four complex components, so we have sixteen real supercharges in total to begin with. The Majorana-Weyl condition on $\hat{\epsilon}_i$, $i = 1, 2$, halves the number of real supercharges to eight. Then, as we have two projection conditions, (2.67) and (2.68), each halves the number of supersymmetry. Hence, there are two real supersymmetries finally. This is the half of the supersymmetry of the $SU(3)$-invariant flow on the flat domain wall, as the interface of the Janus solution breaks half of the supersymmetry.

2.3.3 The numerical solutions

Now we numerically solve the supersymmetry equations, (2.64) and (2.81)-(2.84). We choose the upper sign for $r > 0$ and the lower sign for $r < 0$ [23].

From the condition, $0 < \gamma < 1$, we have

$$0 < 1 - \frac{9\, e^{-2U}}{l^2\, g^2\, W^2} < 1\,, \tag{2.93}$$

where the right hand side is trivially satisfied. From the left hand side, we have

$$-\frac{1}{3}\, l\, g\, W < e^{-U} < +\frac{1}{3}\, l\, g\, W\,. \tag{2.94}$$

As the superpotential, $W = -\frac{3}{4}\,(1 + \cosh(2\chi))$, satisfies $W > -\frac{3}{4}$, from the left hand side of (2.94), we obtain

$$\frac{1}{4}\, l\, g < e^{-U}\,. \tag{2.95}$$

From the supersymmetry equations, we have

$$U'' = \frac{1}{l^2}\, e^{-2U} - \frac{3}{8}\, g^2\, \sinh^2(2\chi)\,. \tag{2.96}$$

From (2.94), we also have

$$- \frac{3\, e^{-U}}{l\, g} \,<\, W \,<\, \frac{3\, e^{-U}}{l\, g}\,. \tag{2.97}$$

Hence, imposing (2.97) with $\sinh^2(2\chi) \,=\, -\frac{8}{3}\, W \,+\, \frac{16}{9}\, W^2$ in (2.96), we get a condition,

$$-\left(-\frac{5}{l^2}\, e^{-2U} \,+\, \frac{3\, g}{l}\, e^{-U}\right) < U'' < \left(-\frac{5}{l^2}\, e^{-2U} \,+\, \frac{3\, g}{l}\, e^{-U}\right). \tag{2.98}$$

In order to obtain a Janus solution, as it was observed in the previously known Janus solutions, we require the turning point of U to be a minimum, $U'' \,>\, 0$. Then, from the right hand side of (2.98), we obtain

$$e^{-U} \,<\, \frac{3}{5}\, l\, g\,. \tag{2.99}$$

From (2.95) and (2.99), there is a narrow range of initial conditions which gives smooth and nonsingular solutions [23],

$$\frac{1}{4}\, l\, g \,<\, e^{-U} \,<\, \frac{3}{5}\, l\, g\,. \tag{2.100}$$

Outside of this range the solution becomes singular at the domain wall *i.e.* at the origin. A numerical solution in the critical range is plotted in figure 2.1, with the choice of initial conditions, $U(0) \,=\, 0$, $\chi(0) \,=\, 0.01$, $\psi(0) \,=\, 0.1$, $\phi(0) \,=\, 1$, $a(0) \,=\, 0.1$, $\theta(0) \,=\, 0.1$, $l \,=\, 1$, and $g \,=\, 2$. *Note that the five-dimensional dilaton and axion fields, ϕ and a, exhibit the dilaton profile of Janus solutions, i.e. it takes constant values on both sides of the interface, but jumps across the interface.* Indeed we will explicitly identify the solution to be the supersymmetric Janus solution in five dimensions in the next section.

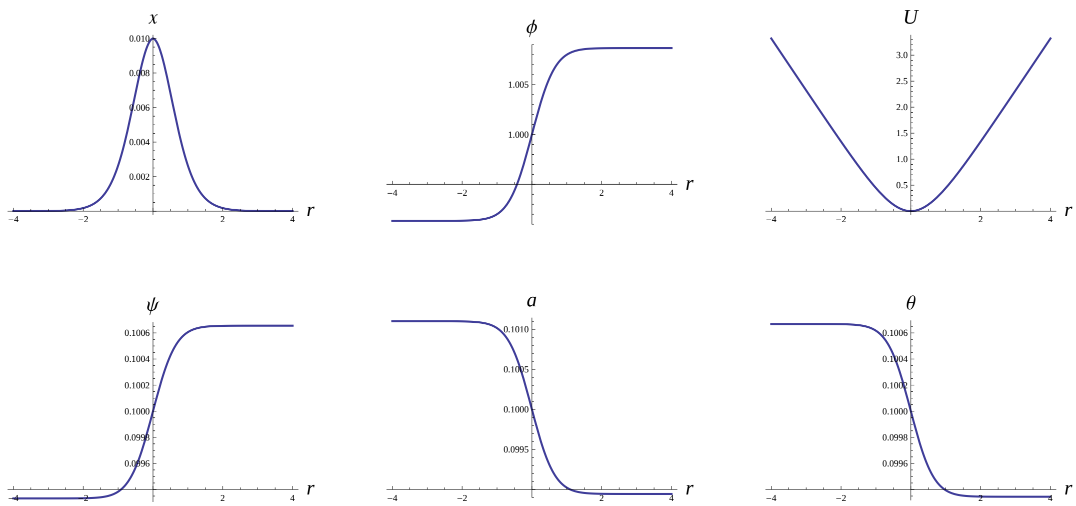

Figure 2.1: *A numerical solution of the supersymmetry equations*

2.4 Super Janus in $\mathcal{N}$ = 2 gauged supergravity in five dimensions [23]

In section 2.2.2, we reviewed a supersymmetric Janus solution, the super Janus, discovered by Clark and Karch in $\mathcal{N}$ = 2 gauged supergravity in five dimensions [23]. In this section we will show that the solution in the $SU(3)$-invariant truncation in the previous section is indeed identical to the super Janus.

Now we prove the equivalence of the super Janus and the solution in the $SU(3)$-invariant truncation. There are four scalar fields living on the scalar manifold, $\frac{SU(2,1)}{SU(2)\times U(1)}$: $\{V,\, \sigma,\, R,\, \alpha\}$ in the super Janus and $\{\chi,\, \psi,\, \phi,\, a\}$ in the $SU(3)$-invariant truncation. We can reparametrize $\{V,\, \sigma,\, R,\, \alpha\}$ in terms of $\{\chi,\, \psi,\, \phi,\, a\}$ by using the inhomogeneous coordinates, ζ_i, i = 1, 2, on the scalar manifold as an intermediate parametrization. We present the details of the reparametrization in appendix E. By employing the reparametrization to the action of the $SU(3)$-invariant truncation, (2.46), we find that it precisely reduces to the action of the super Janus, (2.4). Then, as the supersymmetry equations, (2.12) and (2.13), are for the special case of constant σ and α, they turn out to be the supersymmetry equations of the $SU(3)$-invariant truncation, (2.81)-(2.84), with the constant phases, *i.e.* ψ and a are constant, or more specifically, $a - \psi + \theta = 0$. This proves that the solution of the $SU(3)$-invariant truncation considered in section 2.3 is indeed equivalent to the super Janus.

2.5 Lift of the $SU(3)$-invariant truncation to type IIB supergravity

We uplift the $SU(3)$-invariant truncation in section 2.2 to type IIB supergravity by the consistent truncation ansatz. The consistent truncation ansätze for metric and dilaton/axion fields were presented in [16, 9, 10]. By employing the ansatz, lift of the $SU(3)$-invariant truncation was performed in [10], however, the five-dimensional dilaton/axion fields were suppressed. In this section we will lift the five-dimensional dilaton/axion fields, and as a consequence, we will have nontrivial IIB dilaton/axion fields. We postpone the lift of fluxes to section 2.7.

2.5.1 The metric

The ten-dimensional metric is given by

$$ds^2 \,=\, \Omega^2 \, ds^2_{1,4} \,+\, ds^2_5 \,, \tag{2.101}$$

where $ds^2_{1,4}$ is an arbitrary solution of $\mathcal{N}$ = 8 gauged supergravity in five dimensions. In order to have Janus solution we employ the AdS-domain wall metric, (2.49). The consistent truncation ansatz for the inverse metric of internal space is given by [16, 9, 10]

$$\Delta^{-\frac{2}{3}} g^{pq} = \frac{1}{a^2} K^{IJp} K^{KLq} \widetilde{\mathcal{V}}_{IJab} \widetilde{\mathcal{V}}_{KLcd} \Omega^{ac} \Omega^{bd}, \tag{2.102}$$

where $\widetilde{\mathcal{V}}_{IJab}$ are the inverse coset representatives of the scalar manifold explained in appendix A, K^{IJp} are Killing vectors on round S^5, Ω^{ab} is a $USp(8)$ symplectic form, $\Delta = \det^{1/2}(g_{mp}\hat{g}^{pq})$, and $\hat{g}^{pq}$ is the inverse of the round S^5 metric. The Δ is obtained by taking the determinant on both sides of the ansatz, and $\Omega^2 = \Delta^{-\frac{2}{3}}$ is the warp factor.

To apply the consistent truncation ansatz, we first prepare the proper coordinates in which the $SU(3)$ isometry of internal space is manifest [10]. In Cartesian coordinates, y^I, I = 1, . . . , 6, on $\mathbb{R}^6$, we think of S^5 defined by the surface $\Sigma_I \, (y^I)^2 \; = \; 1$. Let us introduce complex coordinates corresponding to the complex structure, J_{IJ},

$$u^1 \, = \, y^1 \, + \, i \, y^2, \qquad u^2 \, = \, y^5 \, + \, i \, y^6, \qquad u^3 \, = \, y^3 \, + \, i \, y^4 \, . \tag{2.103}$$

We then introduce the complex coordinates where z^i, $i \, = \, 1, \, 2$, are the complex projective coordinates on $\mathbb{CP}_2$, and φ is the $U(1)$ Hopf fiber angle [10],

$$\begin{pmatrix} u^1 \\ u^2 \end{pmatrix} = u^3 \begin{pmatrix} z^1 \\ z^2 \end{pmatrix}, \qquad u^3 \, = \, (1 \, + \, z^1 \, \overline{z}_1 \, + \, z^2 \, \overline{z}_2)^{-1/2} \, e^{-i \, \varphi} \, . \tag{2.104}$$

Convenient real coordinates for the complex coordinates are [10]

$$\begin{pmatrix} z^1 \\ z^2 \end{pmatrix} = - \tan\theta \; g \, (\alpha_1, \, \alpha_2, \, \alpha_3) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{2.105}$$

where g $(\alpha_1,\alpha_2,\alpha_3)$ is an $SU(2)$ invariant matrix in terms of Euler angles, $e.g.$

$$g\,(\alpha_1,\,\alpha_2,\,\alpha_3)\;=\;\begin{pmatrix} e^{-\frac{i}{2}\,(\alpha_1\,+\,\alpha_3)}\cos\left(\frac{\alpha_2}{2}\right) & e^{-\frac{i}{2}\,(\alpha_1\,-\,\alpha_3)}\sin\left(\frac{\alpha_2}{2}\right) \\ -e^{+\frac{i}{2}\,(\alpha_1\,-\,\alpha_3)}\sin\left(\frac{\alpha_2}{2}\right) & e^{+\frac{i}{2}\,(\alpha_1\,+\,\alpha_3)}\cos\left(\frac{\alpha_2}{2}\right) \end{pmatrix}\,. \tag{2.106}$$

With the choice of above coordinates, the lifted metric of internal space reduces to

$$ds_5^2\;=\;\frac{1}{\cosh\chi}\,ds^2_{CP_2}\;+\;\cosh\chi\,(d\varphi\;+\;\frac{1}{2}\,\sin^2\theta\,\sigma_3)^2\,, \tag{2.107}$$

where

$$ds^2_{CP_2}\;=\;d\theta^2\;+\;\frac{1}{4}\,\sin^2\theta\,(\sigma_1^2\;+\;\sigma_2^2\;+\;\cos^2\theta\,\sigma_3^2)\,, \tag{2.108}$$

and σ_i are the left-invariant one-forms of $SU(2)$, (2.18), which satisfy $d\sigma_i\;=\;\frac{1}{2}\,\epsilon_{ijk}\,\sigma_j\wedge\sigma_k$. The warp factor in (2.101) is $\Omega\;=\;\cosh^{1/2}\chi$. As mentioned before, lift of the $SU(3)$-invariant truncation was performed in section 2.9 of [10] without the five-dimensional dilaton/axion fields. Compared to the parametrization of internal space in [10], here we have $\alpha_i\;\to\;-\alpha_i\,,\theta\;\to\;-\theta\,,\varphi\;\to\;-\varphi$. Besides the parametrization, the lifted metric, (2.107), is identical to the one in [10], *i.e.* it is independent of the five-dimensional dilaton/axion fields, ϕ and a.

2.5.2 The dilaton/axion fields

The IIB dilaton/axion fields $(\Phi,\,C_{(0)})$ form a complex scalar, τ, and are related to B by

$$\tau\;=\;C_{(0)}\;+\;i\,e^{-\Phi}\;=\;i\,\frac{1-B}{1+B}\,, \tag{2.109}$$

and f is defined by

$$f\;=\;\frac{1}{\sqrt{1\;-\;|B|^2}}\,. \tag{2.110}$$

The consistent truncation ansatz for the dilaton/axion fields is given by [9]

$$\Delta^{-\frac{4}{3}}\,(SS^T)^{\alpha\beta} \;=\; const\;\times\;\epsilon^{\alpha\gamma}\,\epsilon^{\beta\delta}\,\mathcal{V}_{I\gamma}{}^{ab}\,\mathcal{V}_{J\delta}{}^{cd}\,y^I\,y^J\,\Omega_{ac}\,\Omega_{bd}\,. \tag{2.111}$$

From the ansatz the dilaton/axion field matrix, S, in the $SL(2,\mathbb{R})$ basis reduces to

$$S \;=\; \frac{1}{2\sqrt{1-|B|^2}}\begin{pmatrix} 2+(B+B^*) & i(B-B^*) \\ i(B-B^*) & 2-(B+B^*) \end{pmatrix}, \tag{2.112}$$

where

$$B \;=\; i\,e^{i\,a}\;\tanh\phi\,. \tag{2.113}$$

By changing the basis to $SU(1,1)$, we obtain the dilaton/axion field matrix, V, [9],

$$V \;=\; U^{-1}\,S\,U \;=\; f\begin{pmatrix} 1 & B \\ B^* & 1 \end{pmatrix}, \qquad\qquad U \;=\; \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \tag{2.114}$$

where

$$f \;=\; \cosh\phi\,. \tag{2.115}$$

Then from (2.109) the IIB dilaton and axion fields are

$$\Phi \;=\; \ln\Big(\cosh(2\,\phi)\;-\;\sin(a)\,\sinh(2\,\phi)\Big)\,, \tag{2.116}$$

$$C_{(0)} \;=\; \frac{1}{\sec(a)\,\coth(2\,\phi)\;-\;\tan(a)}\,, \tag{2.117}$$

and we note that they manifestly depend on the five-dimensional dilaton/axion fields, ϕ and a. In fugure 2.2 the IIB dilaton and axion fields are plotted with the identical initial condition as figure 2.1. Note that the dilaton and axion fields exhibit the dilaton profile of Janus solutions. Indeed we will explicitly identify our lifted solution as a special case of the supersymmetric Janus solution in type IIB supergravity in the next section.

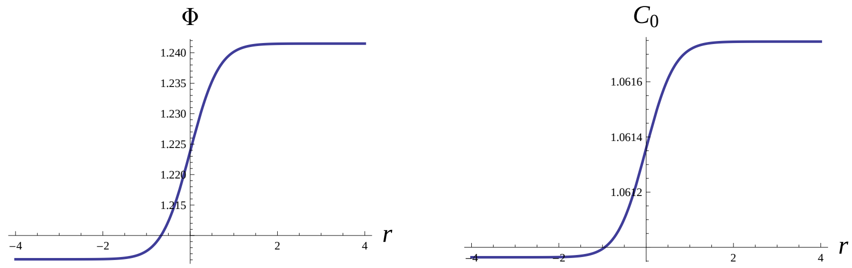

Figure 2.2: *A numerical solution for the dilaton and axion fields*

2.6 Supersymmetric Janus solution in type IIB supergravity [20]

As remarked in the introduction, the supersymmetric Janus solutions in type IIB supergravity were constructed by D'Hoker, Estes and Gutperle in [20, 21, 22] with variety of supersymmetries and isometries. In this section we will show that by choosing metric and dilaton/axion fields to be the lifted ones in section 2.5, the supersymmetric Janus solution with $SU(3)$ isometry in [20] is completely determined, *i.e.* this choice fixes all the IIB fields uniquely including three- and five-form fluxes.

Now we compare the lifted metric, (2.107), and the dilaton/axion fields, (2.113) and (2.115), in section 2.5 with the supersymmetric Janus solution in type IIB supergravity presented in section 2.2.3. By comparing the metric and the dilaton/axion fields, we find that the metric and dilaton/axion field functions in the supersymmetric Janus solution in type IIB supergravity in section 2.2.3 are given by

$$f_1 \; = \; \cosh^{1/2}\chi \,, \tag{2.118}$$

$$f_2 \;=\; \cosh^{-1/2}\chi\,, \tag{2.119}$$

$$f_4 \;=\; e^{U}\,\cosh^{1/2}\chi\,, \tag{2.120}$$

$$B \;=\; i\,e^{i\,a}\,\tanh\phi\,, \qquad\quad f \;=\; \cosh\phi\,. \tag{2.121}$$

We find that by plugging the above set of functions into the field equations, (6.6), (6.13)-(6.16), and the supersymmetry equations, (7.24)-(7.29), in [20], the remaining functions in the solution are completely determined, and we obtain

$$f_5 \;=\; -\frac{\cosh(2\,\chi)-5}{4\cosh^{1/2}\chi}\,, \tag{2.122}$$

$$f_3 \;=\; e^{i\,(a\,-\,\psi)}\,\sinh\phi\,\tanh\chi\,, \tag{2.123}$$

$$g_3 \;=\; -i\,e^{-i\,\psi}\,\cosh\phi\,\tanh\chi\,. \tag{2.124}$$

These functions uniquely fix three- and five-form fluxes in (2.19) and (2.22). Furthermore, we note that this choice of the functions falls into the special case, $a^{DEG} \;=\; 0$, explained in section 2.2.3, and we obtain the hyper-elliptic integral,

$$f_4^2\,\left(\frac{\partial U}{\partial r}\right)^2 \;=\; e^{2\,U} \;+\; \frac{2}{9}\,C_2^2\,e^{-4U} \;+\; \frac{1}{4}\,\left(\frac{2}{9}\,C_2^2\right)^2\,e^{-10\,U} \;-\; 1\,, \tag{2.125}$$

where $\Psi \;=\; e^{U}$ and $\rho \;=\; 1$. This proves that the lifted metric and the dilaton/axion fields from the $SU(3)$-invariant truncation in section 2.5 indeed give a special case of the supersymmetric Janus solution in type IIB supergravity in [20].

2.7 Lift of the $SU(3)$-invariant truncation to type IIB supergravity (continued)

In this section we continue the lift of the $SU(3)$-invariant truncation to type IIB supergravity, and uplift the three- and five-form fluxes which were not considered in section 2.5. The lift formulae for three- and five-form fluxes were proposed in [18], however, we will find that those formulae do not reproduce the correct fluxes for the curved domain walls. We will propose modified lift formulae for three- and five-form fluxes valid for both the flat and the curved domain walls, and check them for some nontrivial cases including the $SU(3)$-invariant truncation.

2.7.1 The three-form flux

The three-form flux is defined by, *e.g.* [50, 51],

$$\begin{aligned} G_{(3)} &= dC_{(2)} - \tau\, dB_{(2)} \\ &= dC_{(2)} - (C_{(0)} + i\, e^{-\Phi})\, dB_{(2)} \\ &= (dC_{(2)} - C_{(0)}\, dB_{(2)}) - i\, e^{-\Phi}\, dB_{(2)} \\ &= F_{(3)} - i\, e^{-\Phi}\, H_{(3)}\,, \end{aligned} \tag{2.126}$$

where $C_{(2)}$ and $B_{(2)}$ are RR and NSNS two-form gauge potentials respectively, and we also define

$$F_{(3)} = dC_{(2)} - C_{(0)}\, dB_{(2)}\,, \tag{2.127}$$

$$H_{(3)} = dB_{(2)}\,. \tag{2.128}$$

By examining flow solutions in the flat domain walls, a lift formula for the two-form gauge potential was proposed in [18],

$$B^{\alpha}{}_{pq} \,=\, k\, L^2\, \mathcal{M}^{\alpha\beta}\, (y^K\, \mathcal{V}_{K\alpha}{}^{ab}) \left(\mathcal{V}_{IJab}\, \frac{\partial y^I}{\partial \xi^p}\, \frac{\partial y^J}{\partial \xi^q} \right) \,, \tag{2.129}$$

where y^I are the Cartesian coordinates for an $\mathbb{R}^6$ embedding of S^5, ξ^p are the intrinsic coordinates on the S^5, $\mathcal{M} \,=\, S\, S^T$, and S is given in (2.112). However, if we apply the formula to the $SU(3)$-invariant truncation with dilaton and axion fields, it does not produce the correct two-form gauge potential found in (2.22) with (2.123) and (2.124). What we obtain from (2.129) is a complicted expression and even cannot be expressed in a simple manner by combination of Ω_2 and $\overline{\Omega}_2$, as the correct two-form gauge potential is, hence, we do not present the result here. By empirical observation we propose a modified lift formula for two-form gauge potential,

$$B_{\alpha\, pq} \,=\, -\frac{i}{\sqrt{2}}\, \Delta^{-\frac{4}{3}}\, (y^K\, \mathcal{V}_{K\alpha}{}^{ab}) \left(\mathcal{V}_{IJab}\, \frac{\partial y^I}{\partial \xi^p}\, \frac{\partial y^J}{\partial \xi^q} \right) \,, \tag{2.130}$$

where Δ is the warp factor, and $B_1 \,=\, B_{(2)}$, $B_2 \,=\, C_{(2)}$. We have verified that this lift formula indeed produces the correct two-form gauge potential in section 2.5. There is also another combination of two-form gauge potentials,

$$A_{(2)} \,=\, C_{(2)} \,-\, \tau\, B_{(2)} \,=\, \frac{e^{i\psi}\, \tanh\chi}{\cosh\phi \,+\, i\, e^{ia} \sinh\phi}\, \overline{\Omega}_2 \,, \tag{2.131}$$

where

$$\Omega_2 \,=\, \frac{1}{12}\, e^{-3\, i\, \varphi}\, \sin\theta \left(2\, i\, d\theta \,\wedge\, (\sigma_1 \,+\, i\, \sigma_2) \,+\, \frac{1}{2}\, \sin(2\, \theta)\, (\sigma_1 \,+\, i\, \sigma_2) \,\wedge\, \sigma_3 \right) \,, \tag{2.132}$$

is the holomorphic (2,0)-form of the internal space [10].

2.7.2 The five-form flux

The lift formula for five-form flux was also proposed in [18], however, we will find that it does not reproduce the correct five-form flux for the $SU(3)$-invariant truncation with dilaton/axion fields in section 2.5. In this subsection we propose a modified lift formula for five-form flux from empirical observations.

We consider the metric,

$$ds_{1,4}^2 \,=\, e^{2\,U(\mu)}\, ds_4^2 \,+\, d\mu^2\,, \tag{2.133}$$

where $ds_{1,4}^2$ is any solution of $\mathcal{N} = 8$ gauged supergravity in five-dimensions, and vol_5 denotes the unit volume form of $ds_{1,4}^2$, and vol_4 of ds_4^2. We define the geometric W-tensors,

$$\widetilde{W}_{ab} \,=\, -\, \epsilon^{\alpha\beta}\, y^I\, y^J\, \Omega^{cd}\, \mathcal{V}_{I\alpha ac}\, \mathcal{V}_{J\beta bd}\,, \tag{2.134}$$

$$\widetilde{W}_{abcd} \,=\, +\, \epsilon^{\alpha\beta}\, y^I\, y^J\, \mathcal{V}_{I\alpha ab}\, \mathcal{V}_{J\beta cd}\,, \tag{2.135}$$

and the geometric scalar potential,

$$\widetilde{\mathcal{P}} \,=\, -\frac{g^2}{32}\left(2\, W_{ab}\, \widetilde{W}^{ab} \,-\, W_{abcd}\, \widetilde{W}^{abcd}\right). \tag{2.136}$$

The geometric superpotential, $\widetilde{W}$, is one of the eigenvalues of $\widetilde{W}_{ab}$.

Before presenting the modified lift formula, let us review the lift formula proposed in [18],

$$F_{(5)} \,=\, \mathcal{F} \,+\, *\mathcal{F}\,, \tag{2.137}$$

where

$$\mathcal{F} \,=\, d\,(\widetilde{W}\, vol_4)\,. \tag{2.138}$$

Applying this formula to the $SU(3)$-invariant truncation with dilaton and axion fields, with $\widetilde{W} = W$, $\varphi_i = \chi$ and $vol_4 = e^{4U}\,\epsilon_{\mu\nu\rho\sigma}$ in this case, we obtain

$$\begin{aligned}
\mathcal{F} &= d(W vol_4) \\
&= \frac{\partial W}{\partial \varphi_i}\frac{\partial \varphi_i}{\partial r}\,dr \wedge vol_4 + 4W\frac{\partial U}{\partial r}\,dr \wedge vol_4 \\
&= \left(\frac{\partial W}{\partial \varphi_i}\frac{\partial \varphi_i}{\partial r} + 4W\frac{\partial U}{\partial r}\right) vol_5 \\
&= \left(\frac{\partial W}{\partial \varphi_i}\left(\frac{g}{2}\frac{\partial W}{\partial \varphi_i}\gamma\right) + 4W\left(-\frac{g}{3}W\gamma\right)\right) vol_5 \\
&= 4\left(\frac{g}{8}\left|\frac{\partial W}{\partial \varphi_i}\right|^2 - \frac{g}{3}W^2\right)\gamma\, vol_5 \\
&= 4\,\mathcal{P}\,\gamma\, vol_5\,.
\end{aligned} \tag{2.139}$$

where $\mathcal{P}$ is the scalar potential and γ is from the supersymmetry equations invoked when taking the derivative of the geometric superpotential. However, it is not the correct five-form flux, (2.19) with (2.122), as the correct one does not have the factor of γ.[4]

Now we propose the modified lift formula for five-form flux,

$$\mathcal{F} = \frac{32}{g^2}\,\widetilde{\mathcal{P}}\,vol_5 + \frac{\partial \widetilde{W}}{\partial \xi^p}\,d\xi^p \wedge vol_4, \tag{2.140}$$

where ξ^p are the intrinsic coordinates of internal space.

[4] In fact the five-form flux in (2.19) with (2.122), is also the correct five-form flux in [10], which is the lift of the $SU(3)$-invariant truncation in the flat domain wall *i.e.* without dilaton and axion fields. Hence, for the flat domain wall, $\gamma = 1$, and the formula produces the correct five-form flux in [10].

By employing the lift formula to the $SU(3)$-invariant truncation with dilaton and axion fields, we obtain that $\widetilde{\mathcal{P}} = \mathcal{P}$, $\widetilde{W} = W$, so $\frac{\partial \widetilde{W}}{\partial \xi^p} = 0$. Hence, the five-form flux is

$$F_{(5)} = \cosh^2\chi \left(\cosh(2\chi) - 5 \right) vol_5 - \frac{\cosh(2\chi) - 5}{2\cosh^2\chi} J_2 \wedge J_2 \wedge (\eta + A) , \quad (2.141)$$

where J_2 is the Kähler form, and η is the one-form dual to the Reeb Killing vector to be explained more in section 2.8. This is indeed the five-form flux found in section 2.6. We believe that the modified lift formula, (2.140), generates the correct five-form fluxes for all the flat domain wall cases that the lift formula in [18] was tested. So far we have verified that it does produce the correct five-form flux for the $SU(2)\times U(1)$-invariant truncation in section 2.3 of [10] which is more nontrivial case with $\widetilde{\mathcal{P}} \neq \mathcal{P}$, $\widetilde{W} \neq W$ and $\frac{\partial \widetilde{W}}{\partial \xi^p} \neq 0$.

However, the lift formula only gives the terms of five-form flux which do not involve the gauge field, A_μ, in five dimensions. For the complete five-form flux, we will just present the flux obtained by using the results in [50, 51],

$$\begin{aligned} F_{(5)} &= \cosh^2\chi \big(\cosh(2\chi) - 5 \big) vol_5 - \frac{1}{2} * \mathcal{K} \wedge (\eta + A) - *(dA) \wedge J_2 \\ &\quad - \frac{\cosh(2\chi) - 5}{2\cosh^2\chi} J_2 \wedge J_2 \wedge (\eta + A) - \frac{1}{4\cosh^4\chi} \mathcal{K} \wedge J_2 \wedge J_2 \\ &\quad - dA \wedge J_2 \wedge (\eta + A) , \end{aligned} \quad (2.142)$$

where

$$\mathcal{K} = -\sinh^2(2\chi) \left(\partial_\mu \psi + \sinh^2\phi\, \partial_\mu a + g A_\mu \right) dx^\mu . \quad (2.143)$$

In this section we proposed the lift formulae for three- and five-form fluxes, (2.130) and (2.140). However, we should stress that we have not derived them from a consistent truncation of type IIB supergravity, but have constructed them based on empirical observations. It is possible that some modification would be needed in the general case, as they are modifications of the formulae in [18].

2.8 Type IIB supergravity on Sasaki-Einstein manifolds [50, 51]

Recently there has been notable development in consistent truncation of type IIB supergravity on Sasaki-Einstein manifolds [50, 51, 52, 53]. In this section, we will show that the $SU(3)$-invariant truncation of $\mathcal{N}$ = 8 gauged supergravity in five dimensions and its lift to type IIB supergravity in section 2.3, 2.5 and 2.7 provide a particular example of consistent truncation in [50, 51].

Locally the Sasaki-Einstein metric can be written as [50, 51]

$$ds^2\,(SE_5) \,=\, ds^2\,(KE_4) \,+\, \eta \otimes \eta\,, \tag{2.144}$$

where $ds^2\,(KE_4)$ is a local Kähler-Einstein metric with positive curvature and η is a globally defined one-form dual to the Reeb Killing vector. There are also a globally defined Kähler two-form J_2 and a (2, 0)-form complex structure Ω_2, and they satisfy

$$d\eta \;= 2\,J_2\,, \tag{2.145}$$

$$d\Omega_2 \;= 3\,i\,\eta \,\wedge\, \Omega_2\,. \tag{2.146}$$

The type IIB metric is then given by [50, 51]

$$ds^2 \,=\, e^{\frac{2}{3}\,(4U\,+V)}\,ds^2_{(E)} \,+\, e^{2U}\,ds^2\,(KE_4) \,+\, e^{2V}\,(\eta \,+\, A) \otimes (\eta \,+\, A)\,, \tag{2.147}$$

where $ds^2_{(E)}$ is an arbitrary metric on an external five-dimensional spacetime, U and V are scalar functions [5] and A is a one-form defined on the external five-dimensional spacetime.

In [50, 51] it was shown that the consistent truncation of type IIB supergravity on Sasaki-Einstein manifolds leads to $\mathcal{N}$ = 4 gauged supergravity coupled to two vector mulptiplets in five dimensions. In section 5.3 and 5.4 of [50] and section 3.4.8 of [51], a particular truncation is presented, and for instance, the five-dimensional action for the particular truncation is [6]

$$\begin{aligned}\mathcal{L}_{kin} &= -\frac{1}{2}\partial_\mu\sigma\,\partial^\mu\sigma - \frac{1}{8}\sinh^2(2\sigma)\,(\partial_\mu\theta - \frac{1}{2}e^{\Phi}\,\partial_\mu C_{(0)} - 3\,A_\mu)^2 \\ &\quad - \frac{1}{8}\cosh^2\sigma\,(\partial_\mu\Phi\,\partial^\mu\Phi + e^{2\,\Phi}\,\partial_\mu C_{(0)}\,\partial^\mu C_{(0)})\,, \qquad (2.148)\\ \mathcal{P} &= +\frac{3}{32}g^2\,\left(\cosh^2(2\sigma) - 4\,\cosh(2\sigma) - 5\right)\,, \qquad (2.149)\end{aligned}$$

where σ and θ are five-dimensional scalar fields, [7] and Φ and $C_{(0)}$ are dilaton and axion fields of type IIB supergravity respectively. It seems that the axion field is charged under the gauge field in the kinetic term, however, it is only an artifact of this parametrization. The kinetic term in terms of the projective coordinates on the coset manifold, (E.8), shows that the $SL(2,\mathbb{R})$ invariant complex scalar field is not charged under the gauge field. This truncation without the dilaton/axion fields was used to construct a holographic superconductor in [27, 28].

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[5] Here U and V have nothing to do with the warp factor, U, in (2.49) and the scalar field, V, in (2.10).

[6] In the truncation in section 3.4.8 of [51], the dilaton/axion fields were not considered.

[7] Here σ and θ have nothing to do with the scalar field, σ, in (2.10) and the phase, θ in (2.69).

We found that the following reparametrization of the particular truncation precisely reproduces the five-dimensional action, (2.46), and the lifted IIB fields of the $SU(3)$-invariant truncation in section 2.5 and 2.7, [8]

$$\begin{aligned}\sigma &= \chi\,,\\ \theta &= \text{Tan}^{-1}\left(\frac{\cos\psi\,-\,\sin(a\,-\,\psi)\,\tanh\phi}{\sin\psi\,-\,\cos(a\,-\,\psi)\,\tanh\phi}\right)\,,\\ \Phi &= \ln\Big(\cosh(2\,\phi)\,-\,\sin(a)\,\sinh(2\,\phi)\Big)\,,\\ C_{(0)} &= \frac{1}{\sec(a)\,\coth(2\,\phi)\,-\,\tan(a)}\,.\end{aligned}\tag{2.150}$$

This proves that the $SU(3)$-invariant truncation of $\mathcal{N}=8$ gauged supergravity and its lift indeed provides a particular example of type IIB supergravity on Sasaki-Einstein manifolds in [50, 51].

2.9 The $SU(2)\times U(1)$-invariant flows with dilaton and axion fields

In this chapter, we studied the $SU(3)$-invariant truncation with dilaton and axion fields, and showed that this truncation and its uplift have the $SU(3)$-invariant supersymmetric Janus solution as their solutions.

On the other hand, as mentioned in section 2.2.2, according to the classification of Janus solutions in type IIB supergravity [47], there are four kinds of solutions with $SO(6)$, $SU(3)$, $SU(2)\times U(1)$ and $SO(3)\times SO(3)$ isometries, and each of them has zero, four, eight, and sixteen real supersymmetries, respectively. The one with $SU(3)$-isometry is what we have constructed in this chapter. However, the supersymmetric

[8] We refer to appendix E for this reparametrization

Janus solution with isometry of $SU(2)\times U(1)$ has not been constructed explicitly so far. Interestingly, there are the $SU(2)\times U(1)$-invariant flows with $\mathcal{N}$ = 1 [10] and $\mathcal{N}$ = 2 [9] supersymmetry in (1.3) and (1.4), respectively. Hence, it is natural to try to include dilaton and axion fields to the $SU(2)\times U(1)$-invariant flow to construct the Janus solution.

Now we consider the $\mathcal{N}$ = 2 supersymmetic flow [9] with dilaton and axion fields. As we have reviewed in section 1.2, this flow in [9] involves two scalar fields, χ and α, dual to a fermion bilinear and a scalar bilinear, respectively, (1.4). This flow flows to $Hades$, however, when uplifted to type IIB supergravity, the singularity is resolved. It describes the Coulomb branch of $\mathcal{N}$ = 4 SYM. We have the coset generators, $\Sigma_{IJK\alpha}$,

$$\Sigma \,=\, \frac{1}{12}\,\Sigma_{IJK\alpha}\, dx^I \,\wedge\, dx^J \,\wedge\, dx^K \,\wedge\, dy^\alpha\,. \tag{2.151}$$

With the complex coordinates, $z_1 \,=\, x^1 \,+\, i\, x^2$, $z_2 \,=\, x^3 \,-\, i\, x^4$, $z_3 \,=\, x^5 \,-\, i\, x^6$, and $z_4 \,=\, y^1 \,+\, i\, y^2$, we have

$$\Sigma \,=\, \sum_{i=1}^{4} \varphi_i \,\left(\Upsilon_i \,+\, \overline{\Upsilon}_i\right)\,, \tag{2.152}$$

where

$$\begin{aligned} \Upsilon_1 \,&=\, -\,dz_1 \,\wedge\, dz_2 \,\wedge\, dz_3 \,\wedge\, dz_4\,, \qquad & \Upsilon_2 \,&=\, -\,d\bar{z}_1 \,\wedge\, d\bar{z}_2 \,\wedge\, dz_3 \,\wedge\, dz_4\,, \\ \Upsilon_3 \,&=\, -\,d\bar{z}_1 \,\wedge\, dz_2 \,\wedge\, d\bar{z}_3 \,\wedge\, dz_4\,, \qquad & \Upsilon_4 \,&=\, -\,dz_1 \,\wedge\, d\bar{z}_2 \,\wedge\, d\bar{z}_3 \,\wedge\, dz_4\,. \end{aligned} \tag{2.153}$$

We also consider two $SL(6,\mathbb{R})$ generators,

$$\Lambda_I{}^J \,=\, diag(\alpha \,+\, \beta\,,\, \alpha \,+\, \beta\,,\, \alpha \,-\, \beta\,,\, \alpha \,-\, \beta\,,\, -2\alpha\,,\, -2\alpha)\,, \tag{2.154}$$

with two noncompact $SL(2,\mathbb{R})$ generators which are dilaton and axion fields, (2.37) and (2.38), as before. We take a subtruncation with two scalar fields, α and χ with dilaton and axion fields, ϕ and a,

$$\alpha \neq 0\,, \qquad \chi = \varphi_1 = \varphi_4 \neq 0\,, \qquad \beta = 0\,, \qquad \varphi_2 = \varphi_3 = 0\,. \tag{2.155}$$

The scalar fields, α and χ, are dual to the field theory operators in (1.4). In this sector the W_{ab} has two eigenvalues, each with degeneracy of four. One of these two eigenvalues is the superpotential,

$$W = -\frac{1}{\rho^2} - \frac{1}{2}\,\rho^4\,\cosh(2\chi)\,, \tag{2.156}$$

where $\rho = e^{\alpha}$. This gives the scalar potential by (2.54). The four eigenvectors for the superpotential, W, are

$$\begin{aligned} \eta^a_{(1)} &= (0,\,0,\,-1,\,0,\,0,\,0,\,0,\,1)\,, \qquad & \eta^a_{(2)} &= (0,\,0,\,0,\,1,\,0,\,0,\,1,\,0)\,, \\ \eta^a_{(3)} &= (1,\,0,\,0,\,0,\,0,\,1,\,0,\,0)\,, \qquad & \eta^a_{(4)} &= (0,\,-1,\,0,\,0,\,1,\,0,\,0,\,0)\,, \end{aligned} \tag{2.157}$$

and they are related to each other by

$$\Omega_{ab}\,\eta^b_{(1)} = -\,\eta^a_{(2)}, \qquad \Omega_{ab}\,\eta^b_{(2)} = +\,\eta^a_{(1)}\,, \tag{2.158}$$

$$\Omega_{ab}\,\eta^b_{(3)} = -\,\eta^a_{(4)}, \qquad \Omega_{ab}\,\eta^b_{(4)} = +\,\eta^a_{(3)}\,, \tag{2.159}$$

where Ω_{ab} is the $USp(8)$ symplectic form given in *e.g.* [8]. As we have two symplectic pairs of spinors, there are two times more supersymmetry than the flow in section 2.3. Hence, it is an $\mathcal{N}$ = 2 supersymmetric RG flow.

2.9.1 The supersymmetry variations (I)

Now we consider the supersymmetry variations. In order to involve the dilaton and axion fields, ϕ and a, as in the previous sections, we employ the AdS-domain wall of (2.49). For simplicity we explicitly assume trivial phase of the dilaton field, $i.e.$ the axion field is trivial, $a' = 0$. Let us consider the integrability conditions of the spin-1/2 field supersymmetry variation,

$$\delta\,\chi_{abc} \;=\; P_{\mu abcd}\,\gamma^{\mu}\,\epsilon^{d} \;-\; \frac{g}{2}\,A_{dabc}\,\epsilon^{d}\,, \tag{2.160}$$

where ϵ^d are the spinors in (2.58). We choose two different components of the spin-1/2 field,

$$\delta\,\chi_{457} \;=\; 0\,, \qquad \delta\,\chi_{368} \;=\; 0\,. \tag{2.161}$$

By iterating these two equations, they reduce to

$$\hat{\epsilon}_i \;-\; m_{ij}\,\gamma^4\,\hat{\epsilon}_j \;=\; 0\,, \tag{2.162}$$

where

$$\begin{aligned} m_{11} &= -\,i\,e^{-4\alpha}\,\mathrm{csch}(2\chi)\,\phi'\,,\\ m_{12} &= 2\,e^{-4\alpha}\,\mathrm{csch}(2\chi)\,\chi'\,,\\ m_{21} &= -\,2\,e^{-4\alpha}\,\mathrm{csch}(2\chi)\,\chi'\,,\\ m_{22} &= i\,e^{-4\alpha}\,\mathrm{csch}(2\chi)\,\phi'\,. \end{aligned} \tag{2.163}$$

If we take other two components of the spin-1/2 field,

$$\delta\,\chi_{457} \;=\; 0\,, \qquad \delta\,\chi_{458} \;=\; 0\,, \tag{2.164}$$

we obtain

$$\hat{\epsilon}_i \,-\, \widehat{m}_{ij}\,\gamma^4\,\hat{\epsilon}_j \;=\; 0\,, \tag{2.165}$$

where

$$\begin{aligned}
\widehat{m}_{11} \;&=\; -\,i\,e^{-4\alpha}\,\mathrm{csch}(2\chi)\,\phi' \\
&\quad +\,\frac{2}{3}\,i\,e^{-10\alpha}\,\mathrm{csch}^2(2\chi)\left[2\left(1-e^{6\alpha}\cosh(2\chi)\right)\chi'+3e^{6\alpha}\sinh(2\chi)\alpha'\right], \\
\widehat{m}_{12} \;&=\; 2\,e^{-4\alpha}\,\mathrm{csch}(2\chi)\,\chi' \,-\,\frac{2}{3}\,e^{-10\alpha}\,\mathrm{csch}^2(2\chi)\,\left(1\,+\,e^{6\alpha}\cosh(2\chi)\right)\phi'\,, \\
\widehat{m}_{21} \;&=\; -\,2\,e^{-4\alpha}\,\mathrm{csch}(2\chi)\,\chi'\,, \\
\widehat{m}_{22} \;&=\; i\,e^{-4\alpha}\,\mathrm{csch}(2\chi)\,\phi'\,.
\end{aligned} \tag{2.166}$$

However, regardless of taking any components, there should be a unique supersymmetry variation. Hence, by comparing m_{12} and $\widehat{m}_{12}$, we conclude that

$$\phi' \;=\; 0\,, \tag{2.167}$$

$i.e.$ the dilaton field should be trivial for the $\mathcal{N}$ = 2 supersymmetric flow. With $\phi' \;=\; 0$ it reduces back to the flow on the flat domain wall in (2.49). Note that we have assumed the axion field, a, to be trivial in this section.

2.9.2 The supersymmetry variations (II)

In the previous subsection, we showed that the $SU(2)\times U(1)$-invariant $\mathcal{N}$ = 2 supersymmetric flow cannot involve nontrivial dilaton field, provided the axion field, a, is trivial. Now, without assuming trivial axion field, we solve the supersymmetry variations, as we have proceeded in section 2.3.2.

First we consider the spin-3/2 field variation. For the t-, x-, y- directions, we obtain

$$U'\,\gamma^{(4)}\,\epsilon_a\;-\;e^{-U}\,\gamma^{(3)}\,\epsilon_a\;-\;\frac{1}{3}\,g\,W_{ab}\,\epsilon^b\;=\;0\,. \tag{2.168}$$

In the same manner as in section 2.3.2, we obtain the same projection condition,

$$\gamma^{(4)}\,\hat{\epsilon}_i=i\,\Big[\pm\gamma\,(\sigma^2)_{ij}\;+\;\sqrt{1-\gamma^2}\,\left(\cos\theta\,(\sigma^1)_{ij}\;+\;\sin\theta\,(\sigma^3)_{ij}\right)\Big]\;\hat{\epsilon}_j\;=\;S_{ij}\,\hat{\epsilon}_j\,, \tag{2.169}$$

where σ^i, $i\;=\;1,\,2,\,3$, are the Pauli matrices.

Now we solve the spin-1/2 field variation in the same manner as in section 2.3.2,

$$\delta\,\chi_{abc}\;=\;\sqrt{2}\left[\gamma^\mu\,P_{\mu abcd}\,\epsilon^d\;-\;\frac{1}{2}\,g\,A_{dabc}\,\epsilon^d\right]\;=\;0\,, \tag{2.170}$$

and obtain

$$\sum_{i,j=1}^{2}\left[P_i\,S_{ij}\;-\;\frac{g}{2}\,A_i\,\delta_{ij}\right]\;\hat{\epsilon}_j\;=\;0\,. \tag{2.171}$$

where S_{ij} is defined in (2.169) and for a specific choice of abc indices,

$$P_1\;=\;P_{4abcd}\,\eta^d_{(1)}\,,\qquad P_2\;=\;P_{4abcd}\,\eta^d_{(2)}\,,\qquad A_1\;=\;A_{dabc}\,\eta^d_{(1)}\,,\qquad A_2\;=\;A_{dabc}\,\eta^d_{(2)}\,. \tag{2.172}$$

Then we try to solve

$$\sum_{i=1}^{2}\left[P_i\,S_{ij}\;-\;\frac{g}{2}\,A_i\,\delta_{ij}\right]\;=\;0\,, \tag{2.173}$$

where j = 1, 2. However, unlike the $SU(3)$-invariant truncation in section 2.3.2, it gives equations involving ϕ', a', χ', α', and they contradict with each other. Hence, they cannot be solved for ϕ', a', χ', α'. On the other hand, if we set $\phi'\;=\;0$, $a'\;=\;0$, the equations reduce to the flow equations on the flat domain wall in [9]. Therefore, by employing the projection conditions on the spinors, (2.169), it seems that the $\mathcal{N}$ = 2 supersymmetric $SU(2)\times U(1)$-invariant flow does not allow nontrivial dilaton and axion fields. However, there can be additional conditions on the spinors, $\hat{\epsilon}_j$, which we have not found yet. It remains as a future work to study if there are any additional projection conditions to solve the supersymmetry variations.

We have also considered the $\mathcal{N}$ = 1 supersymmetric $SU(2)\times U(1)$-invariant flow [10] in (1.4) in the same manner. Like the $\mathcal{N}$ = 2 supersymmetic flow here, by employing the projection conditions, (2.169), $\mathcal{N}$ = 1 supersymmetric flow seems not allow nontrivial dilaton and axion fields.

Chapter 3

Holographic RG flows with gauge fields

3.1 The $SU(3)$-invariant flow with electric potentials

3.1.1 The supersymmetry variations

Due to the top-down models of AdS/CMT, consistent truncation involving gauge fields has become an interesting topic. Motivated by this, the $\mathcal{N} = 1$ supersymmetric $SU(2)\times U(1)$-invariant flow [8, 10] in (1.2) was generalized to involve electric potentials, and a flow interpolating between two global AdS_5 was discovered [37]. In this section, in the same spirit of [37], we study the electrically charged $SU(3)$-invariant flow with and without dilaton and axion fields. [1]

We consider the global AdS background,

$$ds_5^2 = e^{U(r)} \left[f(r)^2 \, dt^2 - \frac{d^2}{4} (\sigma_1^2 + \sigma_2^2 + \sigma_3^2) \right] - \frac{dr^2}{f(r)^2} \, , \tag{3.1}$$

[1]As we have seen in section 2.3.2, there are two superpotentials, (2.52) and (2.53), in the $SU(3)$-invariant truncation. If we set all scalar fields but one to vanish for the superpotential of the $\mathcal{N} = 1$ supersymmetric $SU(2)\times U(1)$-invariant flow in [37], we recover one of the superpotentials of the $SU(3)$-invariant truncation, (2.53). However, only (2.52) gives the correct scalar potential and flow equations. Hence, the electrically charged flow studied in [37] does not guarantee the existence of the electrically charged $SU(3)$-invariant flow, which we are going to consider in this section.

where d is a constant parameter, and σ_j are the $SU(2)$ left-invariant one-forms,

$$\begin{aligned}
\sigma_1 &= \cos\alpha_3\, d\alpha_1 + \sin\alpha_1\, \sin\alpha_3\, d\alpha_2\,, \\
\sigma_2 &= \sin\alpha_3\, d\alpha_1 - \sin\alpha_1\, \cos\alpha_3\, d\alpha_2\,, \\
\sigma_3 &= d\alpha_3 + \cos\alpha_1\, d\alpha_2\,,
\end{aligned} \tag{3.2}$$

which satisfy $d\sigma_i = \frac{1}{2}\,\epsilon_{ijk}\,\sigma_j \wedge \sigma_k$.

The supersymmetry equations are obtained by setting the supersymmetry variations of fermionic fields, *i.e.* the spin-3/2 and spin-1/2 fields, to zero. The bosonic parts of the variations are [14]

$$\delta\,\psi_{\mu a} = D_\mu\,\epsilon_a - \frac{1}{6}\,g\,W_{ab}\,\gamma_\mu\,\epsilon^b - \frac{1}{6}\,H_{\nu\rho ab}\,(\gamma^{\nu\rho}\,\gamma_\mu + 2\,\gamma^\nu\,\delta^\rho_\mu)\,\epsilon^b\,, \tag{3.3}$$

$$\delta\,\chi_{abc} = \sqrt{2}\,\left[\gamma^\mu\,P_{\mu abcd}\,\epsilon^d - \frac{1}{2}\,g\,A_{dabc}\,\epsilon^d - \frac{3}{4}\,\gamma^{\mu\nu}\,H_{\mu\nu[ab}\,\epsilon_{c]|}\right]\,, \tag{3.4}$$

where

$$D_\mu\,\epsilon_a = \partial_\mu\,\epsilon_a + \frac{1}{4}\,\omega_{\mu ij}\,\gamma^{ij}\,\epsilon_a + Q_{\mu a}{}^b\,\epsilon_b\,. \tag{3.5}$$

We define

$$H_{\mu\nu}{}^{ab} = F_{\mu\nu}{}^{ab} + B_{\mu\nu}{}^{ab}\,, \tag{3.6}$$

where

$$F_{\mu\nu}{}^{ab} = F_{\mu\nu IJ}\,\mathcal{V}^{IJab}\,, \qquad B_{\mu\nu}{}^{ab} = B_{\mu\nu}{}^{I\alpha}\,\mathcal{V}_{I\alpha}{}^{ab}\,. \tag{3.7}$$

For the $SU(3)$-invariant truncation, we have

$$F_{\mu\nu IJ} = \partial_\mu\,A_{\nu IJ} - \partial_\nu\,A_{\mu IJ}\,, \qquad B_{\mu\nu}{}^{I\alpha} = 0\,, \tag{3.8}$$

and we only consider the solutions with electric charges,

$$A_{t\,IJ} \;=\; \Phi(r)\, J_{IJ}\,, \tag{3.9}$$

where J_{IJ} is the complex structure, hence, the only non-zero component is

$$H_{rt\,ab} \;=\; \partial_r A_{t\,IJ}\, \mathcal{V}^{IJab} \;=\; \Phi'\, J_{IJ}\, \mathcal{V}^{IJab}\,. \tag{3.10}$$

The eigenvectors, $\eta^a_{(1)}$, $\eta^a_{(2)}$, are related to each other by

$$\Omega_{ab}\, \eta^b_{(1)} \;=\; -\,\eta^a_{(2)}, \qquad \Omega_{ab}\, \eta^b_{(2)} \;=\; +\,\eta^a_{(1)}\,, \tag{3.11}$$

where Ω_{ab} is the $USp(8)$ symplectic form. Then, as in section 2.3.2, the $SU(3)$-invariant five-dimensional spinors are defined by

$$\epsilon^a \;=\; \eta^a_{(1)}\, \hat{\epsilon}_1 \;+\; \eta^a_{(2)}\, \hat{\epsilon}_2\,, \tag{3.12}$$

$$\epsilon_a \;=\; \Omega_{ab}\, \epsilon^b \;=\; -\,\eta^a_{(2)}\, \hat{\epsilon}_1 \;+\; \eta^a_{(1)}\, \hat{\epsilon}_2\,, \tag{3.13}$$

where $\hat{\epsilon}_1$ and $\hat{\epsilon}_2$ are spinors with four complex components. It is convenient to define the quantities, H, Q_t, and the superpotential, W,

$$W_{ab}\, \eta^b_{(1)} \;=\; W\, \eta^a_{(1)}\,, \qquad W_{ab}\, \eta^b_{(2)} \;=\; W\, \eta^a_{(2)}\,, \tag{3.14}$$

$$H_{rt\,ab}\, \eta^b_{(1)} \;=\; -\,H\, \eta^a_{(2)}\,, \qquad H_{rt\,ab}\, \eta^b_{(2)} \;=\; +\,H\, \eta^a_{(1)}\,, \tag{3.15}$$

$$Q_{t\,a}{}^{b}\, \eta^b_{(1)} \;=\; -\,Q_t\, \eta^a_{(2)}\,, \qquad Q_{t\,a}{}^{b}\, \eta^b_{(2)} \;=\; +\,Q_t\, \eta^a_{(1)}\,, \tag{3.16}$$

hence, we have

$$W_{ab}\,\epsilon^b \;=\; W\,\epsilon^a\,, \tag{3.17}$$

$$H_{rt\,ab}\,\epsilon^b \;=\; H_{rt\,ab}\,(\eta^b_{(1)}\,\hat\epsilon_1 \,+\, \eta^b_{(2)}\,\hat\epsilon_2) \;=\; -H\,\eta^a_{(2)}\,\hat\epsilon_1 \,+\, H\,\eta^a_{(1)}\,\hat\epsilon_2 \;=\; H\,\epsilon_a\,, \tag{3.18}$$

$$Q_{t\,a}{}^b\,\epsilon_b \;=\; Q_{t\,a}{}^b\,(-\,\eta^b_{(2)}\,\hat\epsilon_1 \,+\, \eta^b_{(1)}\,\hat\epsilon_2) \;=\; -Q_t\,\eta^a_{(1)}\,\hat\epsilon_1 \,-\, Q_t\,\eta^a_{(2)}\,\hat\epsilon_2 \;=\; -\,Q_t\,\epsilon^a\,. \tag{3.19}$$

Then, we further define the quantities,

$$\Lambda(n) \;=\; e^{-U}\,(2\,H \,+\, \frac{n}{d})\,, \qquad \widetilde\Lambda \;=\; 2\,f^{-1}\,e^{-U}\,(Q_t \,-\, \frac{c}{d})\,. \tag{3.20}$$

The time-dependence of the five-dimensional spinors on the global AdS is given by

$$\partial_t\hat\epsilon_1 \;=\; -\,\frac{c}{d}\,\hat\epsilon_2\,, \qquad \partial_t\hat\epsilon_2 \;=\; +\,\frac{c}{d}\,\hat\epsilon_1\,, \tag{3.21}$$

where c is a constant parameter.

Now we consider the spin-3/2 field variation, (3.3). For $\mu \;=\; t,\, p,\, r$, where $p \;=\; x,\, y,\, z$, respectively, the spin-3/2 field variation gives

$$\frac{1}{2}\,(f' \,+\, f\,U')\,\gamma^0\,\gamma^4\,\epsilon_a \,-\, \frac{1}{2}\,\widetilde\Lambda\,\epsilon^a \,-\, \frac{1}{6}\,g\,W\,\gamma^0\,\epsilon^b \,-\, \frac{1}{3}\,\Lambda(0)\,\gamma^4\,\epsilon_a \;=\; 0\,, \tag{3.22}$$

$$\frac{1}{2}\,f\,U'\,\gamma^0\,\gamma^4\,\epsilon_a \,-\, \frac{1}{6}\,g\,W\,\gamma^0\,\epsilon^a \,+\, \frac{1}{6}\,\Lambda(3)\,\gamma^4\,\epsilon_a \;=\; 0\,, \tag{3.23}$$

$$f\,\partial_r\,\epsilon_a \,+\, Q_{r\,a}{}^b\,\epsilon_b \,+\, \frac{1}{6}\,g\,W\,\gamma^4\,\epsilon^b \,+\, \frac{1}{3}\,\Lambda(0)\,\gamma^0\,\epsilon_a \;=\; 0\,. \tag{3.24}$$

Now we consider the spin-1/2 field variation, (3.4). It reduces to

$$\begin{aligned}\gamma^t\,P_{t\,abcd}\,(\eta^d_{(1)}\,\hat\epsilon_1 \,+\, \eta^d_{(2)}\,\hat\epsilon_2) + \gamma^r\,P_{r\,abcd}\,(\eta^d_{(1)}\,\hat\epsilon_1 \,+\, \eta^d_{(2)}\,\hat\epsilon_2) \,+\, \frac{g}{2}\,A_{dabc}\,(\eta^d_{(1)}\,\hat\epsilon_1 \,+\, \eta^d_{(2)}\,\hat\epsilon_2)\\ -\,\frac{3}{2}\,\gamma^{rt}\,H_{rt\,ab}\,(-\,\eta^c_{(2)}\,\hat\epsilon_1 \,+ \eta^c_{(1)}\,\hat\epsilon_2) \;=\; 0\,,\end{aligned} \tag{3.25}$$

With a specific choice of abc indices we define

$$P_{t\,abcd}\eta^d_{(1)} = P_{t1}, \qquad P_{tabcd}\eta^d_{(2)} = P_{t2}, \qquad P_{rabcd}\eta^d_{(1)} = P_{r1}, \qquad P_{rabcd}\eta^d_{(2)} = P_{r2}, \tag{3.26}$$

$$A_{dabc}\eta^d_{(1)} = A_1, \qquad A_{dabc}\eta^d_{(2)} = A_2, \qquad H_{rtab}\eta^c_{(1)} = H_1, \qquad H_{rtab}\eta^c_{(2)} = H_2. \tag{3.27}$$

Then, we have

$$\gamma^t(P_{t1}\hat{\epsilon}_1 + P_{t2}\hat{\epsilon}_2) + \gamma^r(P_{r1}\hat{\epsilon}_1 + P_{r2}\hat{\epsilon}_2) + \frac{g}{2}(A_1\hat{\epsilon}_1 + A_2\hat{\epsilon}_2) - \frac{3}{2}\gamma^{rt}(-H_2\hat{\epsilon}_1 + H_1\hat{\epsilon}_2) = 0, \tag{3.28}$$

and it reduces to

$$\begin{aligned} e^{-U}\, f^{-1}\,\gamma^0\,(P_{t\,1}\,\hat{\epsilon}_1 \,+\, P_{t\,2}\,\hat{\epsilon}_2) \,+\, f\,\gamma^4\,(P_{r\,1}\,\hat{\epsilon}_1 \,+\, P_{r\,2}\,\hat{\epsilon}_2) \,+\, \frac{g}{2}\,(A_1\,\hat{\epsilon}_1 \,+\, A_2\,\hat{\epsilon}_2) \\ -\,\frac{3}{2}\,e^{-U}\,\gamma^4\,\gamma^0\,(-\,H_2\,\hat{\epsilon}_1 \,+\, H_1\,\hat{\epsilon}_2) \,=\, 0\,. \end{aligned} \tag{3.29}$$

3.1.2 The flow equations without dilaton and axion fields

Note that, in the $SU(3)$-invariant truncation, there are four scalar fields, χ, ψ, ϕ, a. The scalar field ψ is the phase of χ, and a is the phase of ϕ. Also ϕ and a are the dilaton and axion fields, respectively. In this section we first solve the supersymmetry variations only with χ. From the spin-1/2 field variation, (3.29), we obtain

$$\begin{aligned} f\,\chi'\,\gamma^4\,\hat{\epsilon}_1 \,-\, \frac{g}{2}\,\frac{\partial W}{\partial \chi}\,\hat{\epsilon}_2 \,+\, \widetilde{X}\,\gamma^0\,\hat{\epsilon}_2 \,=\, 0\,, \\ f\,\chi'\,\gamma^4\,\hat{\epsilon}_2 \,+\, \frac{g}{2}\,\frac{\partial W}{\partial \chi}\,\hat{\epsilon}_1 \,-\, \widetilde{X}\,\gamma^0\,\hat{\epsilon}_1 \,=\, 0\,. \end{aligned} \tag{3.30}$$

Now we collect all the supersymmetry variations,

$$\frac{1}{2}\,(f' + f\,U')\,\gamma^0\,\gamma^4\,\epsilon_a - \frac{1}{2}\,\widetilde{\Lambda}\,\epsilon^a - \frac{1}{6}\,g\,W\,\gamma^0\,\epsilon^b - \frac{1}{3}\,\Lambda(0)\,\gamma^4\,\epsilon_a = 0\,, \tag{3.31}$$

$$\frac{1}{2}\,f\,U'\,\gamma^0\,\gamma^4\,\epsilon_a - \frac{1}{6}\,g\,W\,\gamma^0\,\epsilon^a + \frac{1}{6}\,\Lambda(3)\,\gamma^4\,\epsilon_a = 0\,, \tag{3.32}$$

$$f\,\partial_r\,\epsilon_a + Q_{r\,a}{}^{b}\,\epsilon_b + \frac{1}{6}\,g\,W\,\gamma^4\,\epsilon^b + \frac{1}{3}\,\Lambda(0)\,\gamma^0\,\epsilon_a = 0\,, \tag{3.33}$$

$$f\,\chi'\,\gamma^4\,\epsilon^a - \frac{g}{2}\,\frac{\partial W}{\partial\chi}\,\epsilon_a + \widetilde{X}\,\gamma^0\,\epsilon_a = 0\,, \tag{3.34}$$

where

$$\Lambda(n) = e^{-U}\,\left(3\,\Phi' + \frac{n}{d}\right)\,, \qquad \widetilde{\Lambda} = -\,f^{-1}\,e^{-U}\,\left(3\,g\,\Phi\,\cosh^2\chi + \frac{2c}{d}\right)\,,$$

$$\widetilde{X} = \frac{3\,g}{2}\,e^{-U}\,f^{-1}\,\Phi\,\sinh(2\chi)\,. \tag{3.35}$$

Subtracting (3.32) from (3.31) gives

$$\frac{1}{2}\,f'\,\gamma^0\,\gamma^4\,\epsilon_a - \frac{1}{2}\,\widetilde{\Lambda}\,\epsilon^a - \frac{1}{2}\,\Lambda(1)\,\gamma^4\,\epsilon_a = 0\,. \tag{3.36}$$

We consider the dielectric projection condition on the spinors,

$$\begin{pmatrix} \hat{\epsilon}_1 \\ \hat{\epsilon}_2 \end{pmatrix} + \begin{pmatrix} \cos\xi\,\gamma^0 & -\sin\xi\,\gamma^4 \\ \sin\xi\,\gamma^4 & \cos\xi\,\gamma^0 \end{pmatrix} \begin{pmatrix} \hat{\epsilon}_1 \\ \hat{\epsilon}_2 \end{pmatrix} = 0\,, \tag{3.37}$$

where $\xi = \xi(r)$. With this projection condition we recast the equations, (3.32), (3.34), (3.36), as

$$\cos\xi = \frac{f'}{\Lambda(1)} = -\frac{1}{3}\frac{\Lambda(3)}{f\,U'} = -\frac{2\,\widetilde{X}}{g\,\partial_\chi W} = -\frac{X_\alpha}{f\,\alpha'}\,, \tag{3.38}$$

$$\sin\xi = -\frac{\widetilde{\Lambda}}{\Lambda(1)} = \frac{1}{3}\frac{g\,W}{f\,U'} = -\frac{2\,f\,\chi'}{g\,\partial_\chi W} = -\frac{g}{12}\frac{\partial_\alpha W}{f\,\alpha'}\,. \tag{3.39}$$

From the third equality of (3.38) and the second equality of (3.39),

$$\frac{1}{3}\frac{\Lambda(3) - \Lambda(1)}{f\,U'} = \frac{1}{g}\left(\frac{2\,\widetilde{X}}{\partial_\chi W} + \frac{\widetilde{\Lambda}}{W}\right)\,, \tag{3.40}$$

and from this, we obtain one of the flow equations,

$$U' = -\frac{g}{3\,c}\,W\,. \tag{3.41}$$

Then, from the third equality of (3.39), we obtain

$$\chi' = \frac{g\,c}{2\,f^2}\frac{\partial W}{\partial\chi}\,. \tag{3.42}$$

From the second equality of (3.39), we obtain

$$\Phi' + \frac{1}{d} = \frac{3}{g}\frac{U'}{W}\left(g\,\Phi\,\cosh^2\chi + \frac{2c}{d}\right)\,, \tag{3.43}$$

and using this in the second equality of (3.38),

$$f\,f' = -\frac{2}{3c}\,e^{-2U}\,\Phi\,\left(g\,\Phi\,\cosh^2\chi + \frac{2c}{d}\right)\,. \tag{3.44}$$

One can show that (3.43) and (3.44) are consistent with

$$\Phi \,=\, -\frac{1}{2}\, e^{U} \sqrt{f^2 \,-\, c^2}\,, \tag{3.45}$$

and

$$\frac{d}{dr} \left[e^{3U} \sqrt{f^2 \,-\, c^2} \right] \,=\, \frac{2}{d}\, e^{2U}\,. \tag{3.46}$$

Hence, we find the flow equations for the scalar fields, warp factor, and the electric potential similar to the ones in [37].

3.1.3 The flow equations with dilaton and axion fields

In this section, we consider the supersymmetry variations with all four scalar fields, χ, ψ, ϕ, a. The spin-3/2 field variations, (3.31), (3.32), (3.33), do not get modified by including more scalar fields. For the spin-1/2 field variation, (3.29), with all four scalar fields, we obtain for the real part,

$$\begin{aligned} &\left[f\, \chi' \gamma^4\, \hat{\epsilon}_1 \,-\, \left(\frac{g}{2} \,-\, \frac{1}{3}\, f\, (\psi' \,+\, \sinh^2\phi\, a')\, \gamma^4 \right) \frac{\partial W}{\partial \chi}\, \hat{\epsilon}_2 \,+\, \widetilde{X}\, \gamma^0\, \hat{\epsilon}_2 \right] \cos\psi \\ - &\left[f\, \chi' \gamma^4\, \hat{\epsilon}_2 \,+\, \left(\frac{g}{2} \,-\, \frac{1}{3}\, f\, (\psi' \,+\, \sinh^2\phi\, a')\, \gamma^4 \right) \frac{\partial W}{\partial \chi}\, \hat{\epsilon}_1 \,-\, \widetilde{X}\, \gamma^0\, \hat{\epsilon}_1 \right] \sin\psi \,=\, 0\,, \end{aligned} \tag{3.47}$$

where

$$\widetilde{X} \,=\, \frac{g}{2}\, e^{-U} f^{-1}\, (\Phi_1 \,+\, \Phi_2 \,+\, \Phi_3)\, \sinh(2\chi)\,. \tag{3.48}$$

For the imaginary part, we obtain

$$\phi'\, (\cos a\, \hat{\epsilon}_1 \,+\, \sin a\, \hat{\epsilon}_2) \,+\, \frac{1}{2}\, \sinh(2\phi)\, a'\, (-\sin a\, \hat{\epsilon}_1 \,+\, \cos a\, \hat{\epsilon}_2) \,=\, 0\,, \tag{3.49}$$

which gives

$$\phi' \,=\, 0\,, \tag{3.50}$$

and

$$\phi \,=\, 0 \qquad or \qquad a' \,=\, 0\,. \tag{3.51}$$

Hence, we again conclude that the dilaton field, ϕ, should be trivial.

3.2 The $SU(3)$-invariant flow with magnetic fields

3.2.1 The magnetic brane solutions

Recently, from the AdS/CMT perspective, there were interests in magnetic brane solutions in supergravity [31, 32, 33, 34, 35, 36]. In this section, we review the magnetic brane solutions in [32].

We consider the truncation of type IIB supergravity to a five-dimensional Einstein-Maxwell theory [32]. The truncated Lagrangian is

$$\mathcal{L}_5 \,=\, R \,-\, \frac{1}{4}\, T^{-1}_{ij}\, D_\mu\, T_{jk}\, T^{-1}_{kl}\, D^\mu\, T_{li} \,-\, \frac{1}{8}\, T^{-1}_{ik}\, T^{-1}{}_{jl}\, F^{ij}_{\mu\nu}\, F^{\mu\nu}_{kl} \,-\, V\,, \tag{3.52}$$

where the scalar potential is

$$V \,=\, \frac{g^2}{2}\, \left(2\, T_{kl}\, T_{kl} \,-\, (T_{kk})^2\right)\,, \tag{3.53}$$

and

$$D\mu^i \,=\, d\mu^i \,+\, g\, A^{ij}\, \mu^j\,. \tag{3.54}$$

Here T_{ij} is a symmetric 6×6 unimodular tensor to represent the 20 scalar fields in the $20'$ representation of $SO(6)$. The A^{ij} are the one-form potentials to represent the 15 gauge fields.

The first ansatz interpolating between AdS_5 and $AdS_3 \times T^2$ is

$$ds_5^2 = -U(r)\,dt^2 + \frac{dr^2}{U(r)} + e^{2V(r)}\left((dx^1)^2 + (dx^2)^2\right) + e^{2W(r)}\,dy^2\,, \tag{3.55}$$

with

$$T_{ij} = \begin{pmatrix} T_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & T_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & T_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & T_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & T_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & T_3 \end{pmatrix}, \tag{3.56}$$

$$F_{(2)}^{ij} = \begin{pmatrix} 0 & -\lambda_1 & 0 & 0 & 0 & 0 \\ \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\lambda_3 \\ 0 & 0 & 0 & 0 & \lambda_3 & 0 \end{pmatrix} \mathcal{F}_{(2)}, \tag{3.57}$$

and

$$\mathcal{F}_{(2)} = \mathcal{B}\,dx^1 \wedge dx^2\,. \tag{3.58}$$

By solving the equations of motion, we obtain

$$ds_5^2 = \frac{r^2}{L^2}\left(-\,dt^2 + (dx^1)^2 + (dx^2)^2 + dy^2\right) + \frac{L^2}{r^2}\,dr^2\,, \tag{3.59}$$

which is the expected ultraviolet solution at $r \to \infty$ where $\mathcal{B} = 0$. At infrared where $\mathcal{B} \neq 0$, we obtain

$$ds_5^2 = -\frac{3\,(r^2 - r_+^2)}{L^2}\,dt^2 + \frac{L^2\,dr^2}{3\,(r^2 - r_+^2)} + \frac{B\,L}{\sqrt{3}}\left((dx^1)^2 + (dx^2)^2\right) + \frac{3\,r^2}{L^2}\,dy^2\,, \tag{3.60}$$

where

$$\begin{aligned} L^{-2} &= \frac{g^2}{3}\left(\frac{1}{T_1} + \frac{1}{T_2} + \frac{1}{T_3}\right)\,, \\ B^2 &= \frac{1}{4}\left(\frac{\lambda_1^2}{T_1^2} + \frac{\lambda_2^2}{T_2^2} + \frac{\lambda_3^2}{T_3^2}\right)\mathcal{B}^2\,. \end{aligned} \tag{3.61}$$

This is the product of a BTZ black hole and a torus. At zero temperature limit, we obtain $AdS_3 \times T^2$.

The second ansatz interpolating between AdS_5 and $AdS_2 \times T^3$ is

$$ds_5^2 = -U(r)\,dt^2 + \frac{dr^2}{U(r)} + e^{2V(r)}\left((dx^1)^2 + (dx^2)^2 + (dx^3)^2\right)\,, \tag{3.62}$$

with T_{ij} identical to the previous one and

$$F^{ij}_{(2)} = \begin{pmatrix} 0 & -\lambda_1\,\mathcal{F}^1_{(2)} & 0 & 0 & 0 & 0 \\ \lambda_1\,\mathcal{F}^1_{(2)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda_2\,\mathcal{F}^2_{(2)} & 0 & 0 \\ 0 & 0 & \lambda_2\,\mathcal{F}^2_{(2)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\lambda_3\,\mathcal{F}^3_{(2)} \\ 0 & 0 & 0 & 0 & \lambda_3\,\mathcal{F}^3_{(2)} & 0 \end{pmatrix}\,, \tag{3.63}$$

with

$$\mathcal{F}^1_{(2)} \,=\, \mathcal{B}_1\, dx^2 \,\wedge\, dx^3\,,$$
$$\mathcal{F}^2_{(2)} \,=\, \mathcal{B}_2\, dx^3 \,\wedge\, dx^1\,,$$
$$\mathcal{F}^3_{(2)} \,=\, \mathcal{B}_3\, dx^1 \,\wedge\, dx^2\,. \tag{3.64}$$

By solving the equations of motion, we obtain

$$ds_5^2 \,=\, g^2\, r^2 \left(-\, dt^2 \,+\, (dx^1)^2 \,+\, (dx^2)^2 \,+\, (dx^3)^2\right) \,+\, \frac{dr^2}{g^2\, r^2}\,, \tag{3.65}$$

which is the expected ultraviolet solution at $r \,\rightarrow\, \infty$ where $\mathcal{B} \,=\, 0$. At infrared where $\mathcal{B} \,\neq\, 0$, we obtain

$$ds_5^2 \,=\, -\,8\, g^2\, (r^2 \,-\, r_+^2)\, dt^2 + \frac{dr^2}{8\, g^2\, (r^2 \,-\, r_+^2)} + \frac{B}{g\,\sqrt{2}} \left((dx^1)^2 \,+\, (dx^2)^2 \,+\, (dx^3)^2\right)\,, \tag{3.66}$$

where $T_1 \,=\, T_2 \,=\, T_3 \,=\, 1$ and $L^{-1} \,=\, g$. At zero temperature limit, we obtain $AdS_2 \,\times\, T^3$.

3.2.2 Configurations of magnetic fields

Motivated by the magnetic brane solutions in the previous section, we study the holographic RG flows in the presence of magnetic fields. To be specific we study the $SU(3)$-invariant truncation with magnetic fields. Instead of the global AdS, we consider the Poincaré AdS background,

$$ds_5^2 \,=\, e^{U(r)}\, \left[f(r)^2\, dt^2 \,-\, dx^2 \,-\, dy^2 \,-\, dz^2\right] \,-\, \frac{dr^2}{f(r)^2}\,. \tag{3.67}$$

In the $SU(3)$-invariant truncation there is only one gauge field, and we have two choices,

$$A_x \,=\, A_1(r)\,, \qquad A_y \,=\, A_2(r)\,, \qquad A_z \,=\, A_3(r)\,, \tag{3.68}$$

or

$$A_x \,=\, A_1(x,y,z)\,, \qquad A_y \,=\, A_2(x,y,z)\,, \qquad A_z \,=\, A_3(x,y,z)\,. \tag{3.69}$$

We will consider the first and second cases in section 3.2.3 and section 3.2.4, respectively.

3.2.3 The $SU(3)$-invariant flow with magnetic fields (case I)

We consider the background in (3.67). We define

$$H_{\mu\nu}{}^{ab} \,=\, F_{\mu\nu}{}^{ab} \,+\, B_{\mu\nu}{}^{ab}\,, \tag{3.70}$$

where

$$F_{\mu\nu}{}^{ab} \,=\, F_{\mu\nu IJ}\,\mathcal{V}^{IJab}\,, \qquad B_{\mu\nu}{}^{ab} \,=\, B_{\mu\nu}{}^{I\alpha}\,\mathcal{V}_{I\alpha}{}^{ab}\,. \tag{3.71}$$

For the $SU(3)$-invariant truncation, we have

$$F_{\mu\nu IJ} \,=\, (\partial_\mu\, A_\nu \,-\, \partial_\nu\, A_\mu)\, J_{IJ}\,, \qquad B_{\mu\nu}{}^{I\alpha} \,=\, 0\,, \tag{3.72}$$

where J_{IJ} is the complex structure. We consider the magnetic components of the gauge field,

$$A_x \,=\, A_1(r)\,, \qquad A_y \,=\, A_2(r)\,, \qquad A_z \,=\, A_3(r)\,. \tag{3.73}$$

Hence, the non-zero components are

$$H_{rxab} = \partial_r A_x \, J_{IJ} \, \mathcal{V}^{IJab} \,, \qquad H_{ryab} = \partial_r A_y \, J_{IJ} \, \mathcal{V}^{IJab} \,, \qquad H_{rzab} = \partial_r A_z \, J_{IJ} \, \mathcal{V}^{IJab} \,. \tag{3.74}$$

Now we solve the supersymmetry variations. From $\delta \psi_{ta} = 0$, from $\delta \psi_{xa} = 0$, $\delta \psi_{ya} = 0$, $\delta \psi_{za} = 0$, and from $\delta \chi_{abc} = 0$, we obtain, respectively,

$$-\frac{1}{6} g \, W \gamma^4 \epsilon^a - \frac{1}{2} (f' + f \, U') \, \epsilon_a + \frac{1}{3} e^{-U} f \, (H_1 \gamma^1 + H_2 \gamma^2 + H_3 \gamma^3) \, \epsilon_a = 0 \,, \tag{3.75}$$

$$\frac{1}{2} f \, U' \gamma^4 \epsilon^a + \frac{1}{6} g \, W \epsilon_a + \frac{1}{3} e^{-U} (Q_1 \gamma^1 + Q_2 \gamma^2 + Q_3 \gamma^3) \, \epsilon_a = 0 \,, \tag{3.76}$$

$$- f \chi' \gamma^4 \epsilon^a + \frac{g}{2} \frac{\partial W}{\partial \chi} \epsilon_a + (X_1 \gamma^1 + X_2 \gamma^2 + X_3 \gamma^3) \, \epsilon_a = 0 \,, \tag{3.77}$$

where

$$H_i = \frac{3}{2} A_i' \,, \qquad Q_i = g \, W A_i \,, \qquad X_i = \frac{g}{2} \frac{\partial W}{\partial \chi} 2 \, e^{-U} A_i \,. \tag{3.78}$$

Now we consider a projection condition on the spinors,

$$\epsilon_a - \sin \xi \, \gamma^4 \epsilon^a + \left(c_1 \gamma^1 + c_2 \gamma^2 + c_3 \gamma^3 \right) \epsilon_a = 0 \,, \tag{3.79}$$

where

$$\sin^2 \xi + c_1^2 + c_2^2 + c_3^2 = 1 \,, \tag{3.80}$$

or

$$c_1^2 + c_2^2 + c_3^2 = \cos^2 \xi \,. \tag{3.81}$$

We recast the equations, (3.75), (3.76), (3.77), as

$$- \sin\xi = \frac{f\,U'}{\frac{1}{3}\,g\,W} = \frac{\frac{1}{3}\,g\,W}{f' + f\,U'} = - \frac{f\,\chi'}{\frac{g}{2}\frac{\partial W}{\partial \chi}}\,, \tag{3.82}$$

$$c_i = \frac{2\,e^{-U}\,Q_i}{g\,W} = - \frac{\frac{2}{3}\,e^{-U}\,f\,H_i}{f' + f\,U'} = \frac{X_i}{\frac{g}{2}\frac{\partial W}{\partial \chi}}\,, \tag{3.83}$$

or after plugging (3.78),

$$- \sin\xi = \frac{f\,U'}{\frac{1}{3}\,g\,W} = \frac{\frac{1}{3}\,g\,W}{f' + f\,U'} = - \frac{f\,\chi'}{\frac{g}{2}\frac{\partial W}{\partial \chi}}\,, \tag{3.84}$$

$$c_i = 2\,e^{-U}\,A_i = - \frac{e^{-U}\,f\,A_i'}{f' + f\,U'} = 2\,e^{-U}\,A_i\,. \tag{3.85}$$

We do not present the procedure, but just the first order system they reduce to,

$$U' = - \frac{1}{3}\,g\,W\,h\,, \tag{3.86}$$

$$\chi' = \frac{g}{2}\,\frac{\partial W}{\partial \chi}\,h\,, \tag{3.87}$$

$$f' = - \frac{1}{3}\,g\,W\left(\frac{1}{fh} - f h\right)\,, \tag{3.88}$$

$$A_i' = - 2\,A_i\,U'\,\frac{1}{f^2 h^2}\,, \tag{3.89}$$

where the new function, $h = h(r)$, can be determined from the condition, (3.80) or (3.81), as

$$4\,e^{-2U}\,(A_1^2 + A_2^2 + A_3^2) = 1 - f^2\,h^2\,. \tag{3.90}$$

From (3.85), we have $A_1 = A_2 = A_3$, hence, (3.90) gives

$$A_i = \frac{1}{2\sqrt{3}}\,e^U\,\sqrt{1 - f^2\,h^2}. \tag{3.91}$$

By differentiating (3.91) and using the flow equations and the conditions known so far, we obtain

$$A_i' \,=\, A_i\, U' \left(-\frac{f}{h}\frac{\partial h}{\partial f}\right)\,, \tag{3.92}$$

and it should be identical to (3.89), hence, we obtain

$$\frac{f}{h}\frac{\partial h}{\partial f} \,=\, \frac{2}{f^2\, h^2}\,. \tag{3.93}$$

By integrating (3.93), we obtain

$$h \,=\, \sqrt{3 \,-\, \frac{2}{f^2}}\,, \tag{3.94}$$

where we have set $h \,=\, 1$ when $f \,=\, 1$. Hence, we obtain the flow equations,

$$U' \,=\, -\frac{1}{3}\, g\, W\, \sqrt{3 \,-\, \frac{2}{f^2}}\,, \tag{3.95}$$

$$\chi' \,=\, \frac{g}{2}\,\frac{\partial W}{\partial \chi}\, \sqrt{3 \,-\, \frac{2}{f^2}}\,, \tag{3.96}$$

$$f' \,=\, -\frac{1}{3}\, g\, W\, \left(\frac{1}{f\sqrt{3 \,-\, \frac{2}{f^2}}} \,-\, f\sqrt{3 \,-\, \frac{2}{f^2}}\right)\,, \tag{3.97}$$

$$A_i \,=\, \frac{1}{2}\, e^U\, \sqrt{1 \,-\, f^2}\,. \tag{3.98}$$

They satisfy the projection condition as they reduce to

$$\sin\xi \,=\, \sqrt{3\, f^2 \,-\, 2}\,, \qquad c_i \,=\, 2\, e^{-U}\, A_i \,=\, \sqrt{1 \,-\, f^2}\,, \tag{3.99}$$

and

$$\sin^2\xi \,+\, c_1^2 \,+\, c_2^2 \,+\, c_3^2 \,=\, (3\, f^2 \,-\, 2) \,+\, 3\,(1 \,-\, f^2) \,=\, 1\,. \tag{3.100}$$

When $f \,=\, 1$, it reduces back to the flat domain wall solution with no magnetic fields.

However, we have also solved the field equations for this truncation, and the flow equations we obtained here were not consistent with the field equations. It remains as a future work to find out why the supersymmetry equations were not sufficient to solve the field equations for this truncation with magnetic fields.

3.2.4 The $SU(3)$-invariant flow with magnetic fields (case II)

We consider the background in (3.67) with the magnetic field,

$$A_x \,=\, A_x(x,y,z)\,, \qquad A_y \,=\, A_y(x,y,z)\,, \qquad A_z \,=\, A_z(x,y,z)\,. \tag{3.101}$$

Hence, the non-zero components of $H_{\mu\nu ab}$ are

$$\begin{aligned} H_{xy\,ab} &= (\partial_x A_y \,-\, \partial_y A_x)\, J_{IJ}\, \mathcal{V}^{IJab}\,, \\ H_{yz\,ab} &= (\partial_y A_z \,-\, \partial_z A_y)\, J_{IJ}\, \mathcal{V}^{IJab}\,, \\ H_{zx\,ab} &= (\partial_z A_x \,-\, \partial_x A_z)\, J_{IJ}\, \mathcal{V}^{IJab}\,. \end{aligned} \tag{3.102}$$

Now we solve the spin-3/2 field supersymmetry variation. From $\delta\,\psi_{ta} \,=\, 0$, we have

$$\frac{1}{2}\,(f' \,+\, f\,U')\,\gamma^4\,\epsilon_a \,-\, \frac{1}{6}\,g\,W\,\epsilon^a \,-\, \frac{1}{2}\,e^{-2U}\,(H_3\,\gamma^1\,\gamma^2 \,+\, H_1\,\gamma^2\,\gamma^3 \,+\, H_2\,\gamma^3\,\gamma^1)\,\epsilon_a \,=\, 0\,, \tag{3.103}$$

and from $\delta\,\psi_{xa} \,=\, 0$, $\delta\,\psi_{ya} \,=\, 0$, $\delta\,\psi_{za} \,=\, 0$, we have

$$\begin{aligned} &\frac{1}{2}\,f\,U'\,\gamma^4\,\epsilon_a \,-\, \frac{1}{3}\,e^{-U}\,(Q_1\,\gamma^1 \,+\, Q_2\,\gamma^2 \,+\, Q_3\,\gamma^3)\,\epsilon^a \,-\, \frac{1}{6}\,g\,W\,\epsilon^a \\ &\qquad\qquad +\, \frac{1}{2}\,e^{-2U}\,(H_3\,\gamma^1\,\gamma^2 \,+\, H_1\,\gamma^2\,\gamma^3 \,+\, H_2\,\gamma^3\,\gamma^1)\,\epsilon_a \,=\, 0\,, \end{aligned} \tag{3.104}$$

where

$$H_1 \,=\, \partial_x\, A_y \,-\, \partial_y\, A_x\,, \qquad H_2 \,=\, \partial_y\, A_z \,-\, \partial_z\, A_y\,, \qquad H_3 \,=\, \partial_z\, A_x \,-\, \partial_x\, A_z\,, \tag{3.105}$$

and

$$Q_1 \,=\, g\, W\, A_x\,, \qquad Q_2 \,=\, g\, W\, A_y\,, \qquad Q_3 \,=\, g\, W\, A_z\,. \tag{3.106}$$

Hence, we have two independent equations from the spin-3/2 field variation, (3.103) and (3.104), and they do not reduce to a unique projection condition. This implies that we have two projection conditions in this case. Solving for the supersymmetry equations remains as a future work.

Chapter 4

Conclusions

We have considered the generalizations of holographic renormalization group flows.

First, we studied the $SU(3)$-invariant truncation of $\mathcal{N}$ = 8 gauged supergravity in five dimensions with dilaton and axion fields and its lift to type IIB supergravity [11]. We showed that the known Janus solutions in five and in ten dimensions, *i.e.* the super Janus in five dimensions [23] and the supersymmetric Janus solution with $SU(3)$ isometry in type IIB supergravity [20], are constructed in a unified way in the framework of $\mathcal{N}$ = 8 gauged supergravity and its lift. Furthermore, we studied the $SU(2)\times U(1)$-invariant $\mathcal{N}$ = 1 and $\mathcal{N}$ = 2 supersymmetric RG flows with dilaton and axion fields.

Second, we studied the $SU(3)$-invariant RG flow with gauge fields. We found the systems of first-order flow equations for the $SU(3)$-invariant flows with electric potentials or magnetic fields. As a future work, it would be interesting to find some charged black hole solutions of these systems. Furthermore, it is natural to revisit the $SU(2)\times U(1)$-invariant $\mathcal{N}$ = 1 supersymmetric RG flow in [37] with magnetic fields, instead of electric potentials, and find some magnetically charged black hole solutions of this truncation.

Appendix A: $\mathcal{N}$ = 8 gauged supergravity in five dimensions

In this appendix we review $\mathcal{N}$ = 8 gauged supergravity in five dimensions with emphasis on the structure of its scalar manifold, $E_{6(6)}/USp(8)$, by following [14]. We will employ the conventions of [14] throughout the paper.

The $SO(6)$ gauged $\mathcal{N}$ = 8 supergravity in five dimensions [12, 13, 14] has local $USp(8)$ symmetry, but global $E_{6(6)}$ symmetry of the ungauged theory is broken. The field content consists of 1 graviton $e_\mu{}^a$, 8 gravitini $\psi_\mu{}^a$, 15 vector fields $A_{\mu IJ}$, 12 two-form tensor fields $B_{\mu\nu}{}^{I\alpha}$, 48 spinor fields χ^{abc}, and 42 scalar fields ϕ^{abcd} where a, b, $\ldots$ are $USp(8)$ indices, I, J, $\ldots$ are $SL(6,\mathbb{R})$, and α, β, $\ldots$ are $SL(2,\mathbb{R})$. Here $SL(6,\mathbb{R})\times SL(2,\mathbb{R})$ is one of the maximal subgroups of $E_{6(6)}$.

The infinitesimal $E_{6(6)}$ transformation in the $SL(6,\mathbb{R})\times SL(2,\mathbb{R})$ basis, (z_{IJ}, $z^{I\alpha}$), in terms of $\Lambda^I{}_J$, $\Lambda^\alpha{}_\beta$, and $\Sigma_{IJK\alpha}$ was already given in (2.30) and (2.31). Exponentiating the transformation in (2.30) and (2.31),

$$z'_{IJ} \;=\; \frac{1}{2}\,U^{MN}{}_{IJ}\,z_{MN} \;+\; \sqrt{\frac{1}{2}}\,U_{P\beta IJ}\,z^{P\beta}\,, \tag{A.1}$$

$$z'^{K\beta} \;=\; U_{P\beta}{}^{K\alpha}\,z^{P\beta} \;+\; \sqrt{\frac{1}{2}}\,U^{IJK\alpha}\,z_{IJ}\,, \tag{A.2}$$

we obtain the coset representatives in the $SL(6,\mathbb{R})\times SL(2,\mathbb{R})$ basis, $U^{IJ}{}_{KL}$, $U^{IJK\alpha}$ and $U_{I\alpha}{}^{J\beta}$. We also have the coset representatives in the $USp(8)$ basis,

$$\mathcal{V}^{IJab} \;=\; \frac{1}{8}\;\left[(\Gamma_{KL})^{ab}\,U^{IJ}{}_{KL} \;+\; 2(\Gamma_{K\beta})^{ab}\,U^{IJK\beta}\right]\,, \tag{A.3}$$

$$\mathcal{V}_{I\alpha}{}^{ab} \;=\; \frac{1}{4}\sqrt{\frac{1}{2}}\;\left[(\Gamma_{KL})^{ab}\,U_{I\alpha}{}^{KL} \;+\; 2(\Gamma_{K\beta})^{ab}\,U_{I\alpha}{}^{K\beta}\right]\,. \tag{A.4}$$

The inverse coset representatives are

$$\widetilde{\mathcal{V}}_{IJab} \;=\; \frac{1}{8}\,[(\Gamma_{KL})_{ab}\,\widetilde{U}_{IJ}{}^{KL} \;+\; 2\,(\Gamma_{K\alpha})_{ab}\,\widetilde{U}_{IJ}{}^{K\alpha}]\,, \tag{A.5}$$

$$\widetilde{\mathcal{V}}^{I\alpha}{}_{ab} \;=\; \frac{1}{4}\sqrt{\frac{1}{2}}\,[(\Gamma_{KL})_{ab}\,\widetilde{U}^{I\alpha KL} \;+\; 2\,(\Gamma_{K\beta})_{ab}\,\widetilde{U}_{I\alpha}{}^{K\beta}]\,. \tag{A.6}$$

Now we consider the action of the theory [14]. The bosonic part of the Lagrangian is

$$e^{-1}\,\mathcal{L} \;=\; -\frac{1}{4}\,R+\mathcal{L}_{kin}+\mathcal{P}-\frac{1}{8}\,H_{\mu\nu ab}\,H^{\mu\nu ab}+\frac{1}{8\,g\,e}\,\epsilon^{\mu\nu\rho\sigma\tau}\,\epsilon_{\alpha\beta}\,B_{\mu\nu}{}^{I\alpha}\,D_\rho B_{\sigma\tau}{}^{I\beta}+\mathcal{L}_{CS}\,, \tag{A.7}$$

where the covariant derivative is defined by

$$D_\mu X_{aI} \;=\; \partial_\mu X_{aI} \;+\; Q_{\mu a}{}^{b}\,X_{bI} \;-\; g\,A_{\mu IJ}\,X_{aJ}\,, \tag{A.8}$$

with the $USp(8)$ connection,

$$Q_{\mu a}{}^{b} \;=\; -\,\frac{1}{3}\,\Big[\widetilde{\mathcal{V}}^{bcIJ}\,\partial_\mu\mathcal{V}_{IJac} \;+\; \widetilde{\mathcal{V}}^{bcI\alpha}\,\partial_\mu\mathcal{V}_{I\alpha ac} + g\,A_{\mu IL}\,\eta^{JL}\,(2\,V_{ae}{}^{IK}\,\widetilde{\mathcal{V}}^{be}{}_{JK} \;-\; \mathcal{V}_{J\alpha ae}\,\widetilde{\mathcal{V}}^{beI\alpha})\Big]\,. \tag{A.9}$$

The kinetic term for scalar fields is defined by

$$\mathcal{L}_{kin} \;=\; \frac{1}{24}\,P_{\mu abcd}\,P^{\mu abcd}\,, \tag{A.10}$$

where

$$P_\mu{}^{abcd} \;=\; \widetilde{\mathcal{V}}^{ab}{}_{IJ}\,D_\mu\mathcal{V}^{IJcd} \;+\; \widetilde{\mathcal{V}}^{abI\alpha}\,D_\mu\mathcal{V}_{I\alpha}{}^{cd}\,. \tag{A.11}$$

The scalar potential is defined by

$$\mathcal{P} \,=\, -\,\frac{1}{32}\,(2W_{ab}\,W^{ab}\,-\,W_{abcd}W^{abcd})\,, \tag{A.12}$$

where

$$W_{abcd} \,=\, \epsilon^{\alpha\beta}\,\eta^{IJ}\,\mathcal{V}_{I\alpha ab}\,\mathcal{V}_{J\beta cd}\,, \tag{A.13}$$

$$W_{ab} \,=\, W^{c}{}_{acb}\,. \tag{A.14}$$

We also define

$$H_{\mu\nu}{}^{ab} \,=\, F_{\mu\nu}{}^{ab} + B_{\mu\nu}{}^{ab}\,, \tag{A.15}$$

where

$$F_{\mu\nu}{}^{ab} \,=\, F_{\mu\nu IJ}\,\mathcal{V}^{IJab}\,, \tag{A.16}$$

$$B_{\mu\nu}{}^{ab} \,=\, B_{\mu\nu}{}^{I\alpha}\,\mathcal{V}_{I\alpha}{}^{ab}\,, \tag{A.17}$$

for the last three terms of Lagrangian.

We adopt the gamma matrix convention of [14], with

$$\{\gamma^i,\,\gamma^j\} \,=\, 2\,\eta^{ij}\,, \tag{A.18}$$

where $\eta^{ij} \,=\, diag\,(+,\,-,\,-,\,-,\,-)$, and γ^0, γ^1, γ^2, γ^3 are pure imaginary as in four-dimensions and $\gamma^4 \,=\, i\,\gamma^5$ is pure real. The matrices γ^0 and γ^5 are antisymmetric and γ^1, γ^2, γ^3 are symmetric. Only in section 2, in order to prevent the confusion with the function, γ, we denote the gamma matrices by $\gamma^{(i)}$ instead of γ^i.

Appendix B: $SU(2,1)$ algebra

The $SU(2,1)$ algebra is given by

$$[L_i,\, L_j] \;=\; i\, f_{ijk}\, L_k \,, \tag{B.1}$$

with the structure constants

$$f_{123} = 1, \qquad f_{147} = f_{165} = f_{246} = f_{257} = f_{345} = f_{376} = \frac{1}{2}, \qquad f_{458} = f_{678} = -\frac{\sqrt{3}}{2}. \tag{B.2}$$

The standard 3-dimensional $SU(2,1)$ generators are obtained by modifying $SU(3)$ Gell-Mann matrices where the Gell-Mann matrices are

$$\lambda_1 \;=\; \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \; \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \; \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{B.3}$$

Multiplying four of the Gell-Mann matrices by i, they close onto an $SU(2,1)$ algebra,

$$\begin{aligned} L_1 \;&=\; \frac{\lambda_1}{2}\,,\; L_2 \;=\; \frac{\lambda_2}{2}\,,\; L_3 \;=\; \frac{\lambda_3}{2}\,,\; L_4 \;=\; i\frac{\lambda_4}{2}\,, \\ L_5 \;&=\; i\frac{\lambda_5}{2}\,,\; L_6 \;=\; i\frac{\lambda_6}{2}\,,\; L_7 \;=\; i\frac{\lambda_7}{2}\,,\; L_8 \;=\; \frac{\lambda_8}{2}\,, \end{aligned} \tag{B.4}$$

where L_1, L_2, L_3 are $SU(2)$ generators, L_4, L_5, L_6, L_7 are $\frac{SU(2,1)}{SU(2)\times U(1)}$ coset generators, and L_8 is a $U(1)$ generator.

The generators in the 27-dimensional representation in section 2.2 corresponding to the 3-dimensional generators are given by

$$L_1 \to \frac{i}{8}(\Sigma^{(3)} - \Sigma^{(4)}),\ L_2 \to \frac{i}{8}(\Sigma^{(3)} + \Sigma^{(4)}),\ L_3 \to \frac{i}{4}(\Lambda^{(5)} - \Lambda^{(8)}),\ L_4 \to \frac{i}{4\sqrt{2}}\Sigma^{(1)},$$
$$L_5 \to \frac{i}{4\sqrt{2}}\Sigma^{(2)},\ L_6 \to \frac{i}{2\sqrt{2}}(\Lambda^{(7)}+\Lambda^{(6)}),\ L_7 \to \frac{i}{2\sqrt{2}}(\Lambda^{(7)}-\Lambda^{(6)}),\ L_8 \to \frac{i}{4\sqrt{3}}(\Lambda^{(5)}+3\Lambda^{(8)}). \tag{B.5}$$

Appendix C: The supersymmetry variations for spin-1/2 fields

In this appendix we reconsider the spin-1/2 field variation, and show that it is solved without introducing additional projection condition. As in (2.73), for a choice of abc indices we define P_1, P_2, A_1, A_2. In the same manner, for another choice of abc indices we define $\widetilde{P}_1$, $\widetilde{P}_2$, $\widetilde{A}_1$, $\widetilde{A}_2$. From the spin-1/2 field variation, as in (**??**), they satisfy

$$P_1\,\gamma^{(4)}\,\hat{\epsilon}_1\,+\,P_2\,\gamma^{(4)}\,\hat{\epsilon}_2\,-\,\frac{g}{2}\,A_1\,\hat{\epsilon}_1\,-\,\frac{g}{2}\,A_2\,\hat{\epsilon}_2\,=0\,, \tag{C.1}$$

$$\widetilde{P}_1\,\gamma^{(4)}\,\hat{\epsilon}_1\,+\,\widetilde{P}_2\,\gamma^{(4)}\,\hat{\epsilon}_2\,-\,\frac{g}{2}\,\widetilde{A}_1\,\hat{\epsilon}_1\,-\,\frac{g}{2}\,\widetilde{A}_2\,\hat{\epsilon}_2\,=0\,. \tag{C.2}$$

By multiplying by A_i, $\widetilde{A}_i$ and subtracting, we obtain

$$(P_1\,\widetilde{A}_2\,-\,\widetilde{P}_1\,A_2)\,\gamma^{(4)}\,\hat{\epsilon}_1\,+\,(P_2\,\widetilde{A}_2\,-\,\widetilde{P}_2\,A_2)\,\gamma^{(4)}\,\hat{\epsilon}_2\,-\,\frac{g}{2}\,(A_1\,\widetilde{A}_2\,-\,\widetilde{A}_1\,A_2)\,\hat{\epsilon}_1\,=0\,, \tag{C.3}$$

$$(P_1\,\widetilde{A}_1\,-\,\widetilde{P}_1\,A_1)\,\gamma^{(4)}\,\hat{\epsilon}_1\,+\,(P_2\,\widetilde{A}_1\,-\,\widetilde{P}_2\,A_1)\,\gamma^{(4)}\,\hat{\epsilon}_2\,-\,\frac{g}{2}\,(A_2\,\widetilde{A}_1\,-\,\widetilde{A}_2\,A_1)\,\hat{\epsilon}_2\,=0\,. \tag{C.4}$$

By rearranging, we obtain

$$\hat{\epsilon}_1\,-\,\frac{2}{g}\,\frac{P_1\,\widetilde{A}_2\,-\,\widetilde{P}_1\,A_2}{A_1\,\widetilde{A}_2\,-\,\widetilde{A}_1\,A_2}\,\gamma^{(4)}\,\hat{\epsilon}_1\,-\,\frac{2}{g}\,\frac{P_2\,\widetilde{A}_2\,-\,\widetilde{P}_2\,A_2}{A_1\,\widetilde{A}_2\,-\,\widetilde{A}_1\,A_2}\,\gamma^{(4)}\,\hat{\epsilon}_2\,=\,0\,, \tag{C.5}$$

$$\hat{\epsilon}_2\,-\,\frac{2}{g}\,\frac{P_1\,\widetilde{A}_1\,-\,\widetilde{P}_1\,A_1}{A_2\,\widetilde{A}_1\,-\,\widetilde{A}_2\,A_1}\,\gamma^{(4)}\,\hat{\epsilon}_1\,-\,\frac{2}{g}\,\frac{P_2\,\widetilde{A}_1\,-\,\widetilde{P}_2\,A_1}{A_2\,\widetilde{A}_1\,-\,\widetilde{A}_2\,A_1}\,\gamma^{(4)}\,\hat{\epsilon}_2\,=\,0\,. \tag{C.6}$$

Hence, the spin-1/2 field variation, (2.61), reduces to

$$\hat{\epsilon}_i \,-\, m_{ij}\,\gamma^{(4)}\,\hat{\epsilon}_j \,=\, 0\,, \tag{C.7}$$

where

$$m_{ij} \,=\, \begin{pmatrix} m_1 \,+\, m_2 \,-\, m_3 & m_4 \,-\, m_5 \\ m_4 \,+\, m_5 & -m_1 \,+\, m_2 \,-\, m_3 \end{pmatrix}, \tag{C.8}$$

and

$$\begin{aligned} m_1 \,&=\, -\,\frac{2}{3\,g}\,i\,\mathrm{csch}\chi\,\Big(\sin(a\,-\,\psi)\,\phi' \,+\, \frac{1}{2}\,\sinh(2\,\phi)\,\cos(a\,-\,\psi)\,a'\Big)\,, \\ m_2 \,&=\, -\,\frac{2}{3\,g}\,\sinh^2\phi\,a'\,, \\ m_3 \,&=\, -\,\frac{2}{3\,g}\,\psi'\,, \\ m_4 \,&=\, +\,\frac{2}{3\,g}\,i\,\mathrm{csch}\chi\,\Big(\cos(a\,-\,\psi)\,\phi' \,+\, \frac{1}{2}\,\sinh(2\,\phi)\,\sin(a\,-\,\psi)\,a'\Big)\,, \\ m_5 \,&=\, +\,\frac{4}{3\,g}\,\mathrm{csch}(2\,\chi)\,\chi'\,. \end{aligned} \tag{C.9}$$

When we plug the supersymmetry equations, (2.81), (2.82), (2.83), (2.84), in (C.7), it reduces to the projection condition, (2.69). Hence, the spin-1/2 field variation, (C.7), does not provide any additional projection condition. In total, we have two projection conditions, (2.67) and (2.68), on the spinors.

Appendix D: The supersymmetry variations for spin-3/2 fields

In this appendix we present the $SU(3)$-invariant truncation of supersymmetry variations for spin-3/2 fields. The variation for t-, x-, y- directions is given in (2.62). For z-direction the variation is given by

$$-2\,e^{-U}\,\gamma^{(3)}\,z\,\partial_z\,\hat{\epsilon}_1\,-U'\,\gamma^{(4)}\,\hat{\epsilon}_1\,+\frac{1}{3}\,g\,W\,\hat{\epsilon}_2\,=\,0\,, \tag{D.1}$$

$$+2\,e^{-U}\,\gamma^{(3)}\,z\,\partial_z\,\hat{\epsilon}_2\,+U'\,\gamma^{(4)}\,\hat{\epsilon}_2\,+\frac{1}{3}\,g\,W\,\hat{\epsilon}_1\,=\,0\,. \tag{D.2}$$

For the variation in the r-direction we need to know the action of $Q_{\mu a}{}^{b}$ tensor on the spinors,

$$Q_{ra}{}^{b}\,\eta_{(1)b}\,=\,+Q_1\,\eta_{(1)a}\,+\,Q_2\,\eta_{(2)a}\,, \tag{D.3}$$

$$Q_{ra}{}^{b}\,\eta_{(2)b}\,=\,-\overline{Q}_2\,\eta_{(1)a}\,-\,Q_1\,\eta_{(2)a}\,, \tag{D.4}$$

where

$$Q_1 = -\,i\,\sinh\chi\left[\cos(a\,-\,\psi)\,\phi'\,-\,\frac{1}{2}\,\sin(a\,-\,\psi)\,\sinh(2\,\phi)\,a'\right], \tag{D.5}$$

$$\begin{aligned} Q_2 = -\,i\,\Big[&\sinh\chi\left(\sin(a\,-\,\psi)\,\phi'\,-\,\frac{i}{2}\,\sinh\chi\,\psi'\right)\\ &+\,\frac{1}{2}\left(\cos(a\,-\,\psi)\,\sinh(2\,\phi)\,\sinh\chi\,-\,\frac{i}{2}\big(-\,3\,+\,\cosh(2\,\chi)\big)\,\sinh^2\phi\right)a'\Big]\,. \end{aligned} \tag{D.6}$$

Then the variation in the r-direction is given by

$$\partial_r \hat{\epsilon}_1 - (+Q_1 \hat{\epsilon}_1 + Q_2 \hat{\epsilon}_2) + \frac{1}{6} g W \gamma^{(4)} \hat{\epsilon}_2 = 0 , \tag{D.7}$$

$$\partial_r \hat{\epsilon}_2 - (-\overline{Q}_2 \hat{\epsilon}_1 - Q_1 \hat{\epsilon}_2) - \frac{1}{6} g W \gamma^{(4)} \hat{\epsilon}_1 = 0 , \tag{D.8}$$

where W is the superpotential in (2.52).

Appendix E: The parametrizations of the scalar manifold

In this paper we have employed several different parametrizations for the four real scalar fields living on the scalar manifold, $\frac{SU(2,1)}{SU(2)\times U(1)}$. In this appendix we summarize the origins of and the relations between different parametrizations.

The coset manifold, $\frac{SU(2,1)}{SU(2)\times U(1)}$, is topologically an open ball in $\mathbb{C}^2$ with the Bergman metric [54],

$$ds^2 \,=\, \frac{d\zeta_1\, d\overline{\zeta}_1 \,+\, d\zeta_2\, d\overline{\zeta}_2}{1 \,-\, \zeta_1\,\overline{\zeta}_1 \,-\, \zeta_2\,\overline{\zeta}_2} \,+\, \frac{(\overline{\zeta}_1\, d\zeta_2 \,+\, \overline{\zeta}_2\, d\zeta_2)(\zeta_1\, d\overline{\zeta}_2 \,+\, \zeta_2\, d\overline{\zeta}_2)}{(1 \,-\, \zeta_1\,\overline{\zeta}_1 \,-\, \zeta_2\,\overline{\zeta}_2)^2}\,, \tag{E.1}$$

which is a Kähler metric with Kähler potential,

$$\mathcal{K} \,=\, -\,\frac{1}{2}\,\ln(1 \,-\, \zeta_1\,\overline{\zeta}_1 \,-\, \zeta_2\,\overline{\zeta}_2)\,. \tag{E.2}$$

The first two parametrizations of the scalar manifold we employed in this paper were the rectangular and angular parametrizations, $\{x_1,\, x_2,\, x_3,\, x_4\}$ in (2.44) and $\{\chi,\, \psi,\, \phi,\, a\}$, respectively, for the the $SU(3)$-invariant truncation in section 2.2. The relation between them is given in (2.45). In terms of the rectangular parametrization, the inhomogeneous coordinates on the scalar manifold are given by

$$\zeta_1 \,=\, \frac{(x_1 \,+\, i x_2)\,\tanh\left(\frac{1}{2}\,\sqrt{x_1^2 \,+\, x_2^2}\right)}{\sqrt{x_1^2 \,+\, x_2^2}}\,\text{sech}\left(\frac{1}{2}\,\sqrt{x_3^2 \,+ x_4^2}\right)\,, \tag{E.3}$$

$$\zeta_2 \,=\, \frac{(x_3 \,+\, i x_4)\,\tanh\left(\frac{1}{2}\,\sqrt{x_3^2 \,+\, x_4^2}\right)}{\sqrt{x_3^2 \,+\, x_4^2}}\,. \tag{E.4}$$

We can reverse the relation to get

$$x_1 = \frac{\zeta_1 + \overline{\zeta}_1}{2\, Z_1}\,, \qquad x_2 = \frac{\zeta_1 - \overline{\zeta}_1}{2\, i\; Z_1}\,, \qquad x_3 = \frac{\zeta_2 + \overline{\zeta}_2}{2\, Z_2}\,, \qquad x_4 = \frac{\zeta_2 - \overline{\zeta}_2}{2\, i\; Z_2}\,, \quad \text{(E.5)}$$

where

$$Z_1 = \frac{\sqrt{\zeta_1\,\overline{\zeta}_1}\,\sqrt{1 + \zeta_2\,\overline{\zeta}_2}}{2\,\tanh^{-1}\sqrt{\zeta_1\,\overline{\zeta}_1}}\,, \qquad Z_2 = \frac{\sqrt{\zeta_2\,\overline{\zeta}_2}}{2\,\tanh^{-1}\sqrt{\zeta_2\,\overline{\zeta}_2}}\,. \quad \text{(E.6)}$$

Before proceeding to the third parametrization, we consider the $SU(3)$-invariant truncation in terms of the complex coordinates, ζ_i, i = 1, 2. When we exponentiate the coset generators in (2.44), if we employ the complex coordinates by (E.5), we can have the action of the $SU(3)$-invariant truncation in terms of the complex coordinates,

$$e^{-1}\,\mathcal{L} = -\frac{1}{4}\,R + \mathcal{L}_{kin} + \mathcal{P} - \frac{3}{4}\,F_{\mu\nu}\,F^{\mu\nu} + \mathcal{L}_{CS}\,. \quad \text{(E.7)}$$

The kinetic term is

$$\mathcal{L}_{kin} = \frac{1}{2}\,g_{i\bar{j}}\,D_\mu\,\zeta_i\,D^\mu\,\overline{\zeta}_{\bar{j}}\,, \quad \text{(E.8)}$$

where the metric is the Bergman metric, (E.1), and the covariant derivative with respect to the gauge field is

$$D_\mu\,\zeta_1 = \partial_\mu\,\zeta_1 + 3\,g\,A_\mu\,\zeta_1\,, \qquad D_\mu\,\zeta_2 = \partial_\mu\,\zeta_2\,. \quad \text{(E.9)}$$

The scalar potential is

$$\mathcal{P} = -\frac{3}{8}\,g^2\,\frac{(1 - |\zeta_2|^2)\,(2 - 3|\zeta_1|^2 - 2|\zeta_2|^2)}{(1 - |\zeta_1|^2 - |\zeta_2|^2)^2}\,. \quad \text{(E.10)}$$

Thirdly, in $\mathcal{N}$ = 2 gauged supergravity in five dimensions, there is another parametrization by the scalar fields, $\{V,\,\sigma,\,R,\,\alpha\}$, which was employed for the super

Janus in section 2.3. In terms of these scalar fields the inhomogeneous coordinates are given by [46, 23]

$$\zeta_1 = \frac{-2\,i\,R\,e^{i\,\alpha}}{1 + R^2 + V + i\,\sigma}\,, \tag{E.11}$$

$$\zeta_2 = \frac{1 - R^2 - V - i\,\sigma}{1 + R^2 + V + i\,\sigma}\,. \tag{E.12}$$

By plugging (E.11), (E.12) into (E.7), we precisely reproduce the action of the super Janus, (2.4). The rest of the truncation, *e.g.* the supersymmetry equations, can also be reparametrized, and they are explained in section 2.3. This reparametrization was used to establish the equivalence of the $SU(3)$-invariant truncation and the super Janus in section 2.3.

Lastly, there is a parametrization by $\{\sigma,\,\theta,\,\Phi,\,C_{(0)}\}$, employed for a particular truncation of type IIB supergravity on Sasaki-Einstein manifolds in section 2.7. The Φ and $C_{(0)}$ are the IIB dilaton and axion fields respectively, and σ and θ are some five-dimensional scalar fields. We briefly mention that by comparing Killing vectors for (2.47) and (2.148), we have found the relation between $\{\sigma,\,\theta,\,\Phi,\,C_{(0)}\}$ and $\{\chi,\,\psi,\,\phi,\,a\}$ in (2.150). Note that the IIB dilaton and axion fields are indeed identical to the ones from the lift in (2.116) and (2.117).

Appendix F: The field equations of the $SU(3)$-invariant truncation

In this appendix, we present the field equations of the $SU(3)$-invariant truncation. Let us consider the action for complex scalar fields and gravity,

$$\mathcal{L} = \sqrt{g}\left(-\frac{1}{4}R + \frac{1}{2}g^{\mu\nu}h_{a\bar{b}}\,\partial_\mu\phi_a\,\partial_\nu\overline{\phi}_{\bar{b}} - \mathcal{P}(\phi_a,\overline{\phi}_{\bar{a}})\right). \tag{F.1}$$

The scalar equations reduce to

$$\frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}\,g^{\mu\nu}\,\partial_\nu\phi^a) + \Gamma^a{}_{bc}\,g^{\mu\nu}\,\partial_\mu\phi^b\,\partial_\nu\phi^c - h^{\bar{b}a}\,\partial_{\bar{b}}\mathcal{P} = 0\,, \tag{F.2}$$

$$\frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}\,g^{\mu\nu}\,\partial_\nu\overline{\phi}^{\bar{a}}) + \Gamma^{\bar{a}}{}_{\bar{b}\bar{c}}\,g^{\mu\nu}\,\partial_\mu\overline{\phi}^{\bar{b}}\,\partial_\nu\overline{\phi}^{\bar{c}} - h^{\bar{a}b}\,\partial_b\mathcal{P} = 0\,, \tag{F.3}$$

where

$$\Gamma^a{}_{bc} = h^{\bar{d}a}\,\partial_c h_{b\bar{d}}\,, \tag{F.4}$$

$$\Gamma^{\bar{a}}{}_{\bar{b}\bar{c}} = h^{\bar{a}d}\,\partial_{\bar{c}} h_{d\bar{b}}\,. \tag{F.5}$$

The Einstein equations are

$$R_{\mu\nu} - \frac{1}{2}R\,g_{\mu\nu} = 2\,T_{\mu\nu}\,, \tag{F.6}$$

where the energy-momentum tensor is

$$T_{\mu\nu} = h_{a\bar{b}}\,\partial_\mu\phi_a\,\partial_\nu\overline{\phi}_{\bar{b}} - g_{\mu\nu}\left(\frac{1}{2}g^{\rho\sigma}h_{a\bar{b}}\partial_\rho\phi_a\partial_\sigma\overline{\phi}_{\bar{b}} - \mathcal{P}(\phi,\overline{\phi})\right). \tag{F.7}$$

For the metric in (2.49), the Einstein equations reduce to

$$3\,(U''\,+\,2\,U'\,U')\,+\,\frac{3}{l^2}\,e^{-2U}\;=\;-\,2\,(\frac{1}{2}\,h_{a\bar{b}}\,\phi'_a\,\overline{\phi}'_{\bar{b}}\,-\,\mathcal{P})\,, \tag{F.8}$$

$$3\,U'\,U'\,+\,\frac{3}{l^2}\,e^{-2U}\;=\;(\frac{1}{2}\,h_{a\bar{b}}\,\phi'_a\,\overline{\phi}'_{\bar{b}}\,+\,\mathcal{P})\,. \tag{F.9}$$

Then, for the $SU(3)$-invariant truncation in (E.7), in terms of the inhomogeneous coordinates, $\{\zeta_1,\,\zeta_2\}$, the field equations reduce to

$$0\;=\;4\,U'\,\zeta_1'\,+\,\zeta_1''\,+\,\frac{2\,\zeta_1'\,(\overline{\zeta}_1\,\zeta_1'\,+\,\overline{\zeta}_2\,\zeta_2')}{1\,-\,\overline{\zeta}_1\,\zeta_1\,-\,\overline{\zeta}_2\,\zeta_2}\,+\,\frac{3\,g^2}{4}\,\frac{1-3\,\overline{\zeta}_1\,\zeta_1\,-\,2\,\overline{\zeta}_2\,\zeta_2}{1-\overline{\zeta}_1\,\zeta_1\,-\,\overline{\zeta}_2\,\zeta_2}\,, \tag{F.10}$$

$$0\;=\;4\,U'\,\zeta_2'\,+\,\zeta_2''\,+\,\frac{2\,\zeta_2'\,(\overline{\zeta}_1\,\zeta_1'\,+\,\overline{\zeta}_2\,\zeta_2')}{1\,-\,\overline{\zeta}_1\,\zeta_1\,-\,\overline{\zeta}_2\,\zeta_2}\,, \tag{F.11}$$

$$\begin{aligned}0\;=\;&3\,(U''\,+\,2\,U'\,U')\,+\,\frac{3}{l^2}\,e^{-2\,U}\\&+2\,\left(\frac{1}{2}\frac{\overline{\zeta'}_1\zeta_1'+\overline{\zeta'}_2\zeta_2'}{1-\overline{\zeta}_1\zeta_1-\overline{\zeta}_2\zeta_2}+\frac{1}{2}\frac{(\overline{\zeta}_1\zeta_1'+\overline{\zeta}_2\zeta_2')(\overline{\zeta'}_1\zeta_1+\overline{\zeta'}_2\zeta_2)}{(1-\overline{\zeta}_1\zeta_1-\,\overline{\zeta}_2\zeta_2)^2}\right.\\&\left.-\frac{3\,g^2}{8}\frac{(1-\overline{\zeta}_2\zeta_2)(2-3\overline{\zeta}_1\zeta_1-2\overline{\zeta}_2\zeta_2)}{(1-\overline{\zeta}_1\zeta_1-\overline{\zeta}_2\zeta_2)^2}\right)\,,\end{aligned} \tag{F.12}$$

$$\begin{aligned}0\;=\;&3\,U'\,U'\,+\,\frac{3}{l^2}\,e^{-2\,U}\\&-\,\left(\frac{1}{2}\frac{\overline{\zeta'}_1\zeta_1'+\overline{\zeta'}_2\zeta_2'}{1-\overline{\zeta}_1\zeta_1-\overline{\zeta}_2\zeta_2}+\frac{1}{2}\frac{(\overline{\zeta}_1\zeta_1'+\overline{\zeta}_2\zeta_2')(\overline{\zeta'}_1\zeta_1+\overline{\zeta'}_2\zeta_2)}{(1-\overline{\zeta}_1\zeta_1-\,\overline{\zeta}_2\zeta_2)^2}\right.\\&\left.-\frac{3\,g^2}{8}\frac{(1-\overline{\zeta}_2\zeta_2)(2-3\overline{\zeta}_1\zeta_1-2\overline{\zeta}_2\zeta_2)}{(1-\overline{\zeta}_1\zeta_1-\overline{\zeta}_2\zeta_2)^2}\right)\,.\end{aligned} \tag{F.13}$$

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