A Matrix Formulation For Small-x Renormalization-Group Improved Evolution

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After recalling the small-x resummation methods which generalize DGLAP and BFKL approaches to QCD evolution equations, I present a recent $\boldsymbol{k}$-factorized matrix formulation in which quarks and gluons are treated on the same ground and exact NLO and NLx calculations are incorporated. I then produce results for the resummed eigenvalue functions and the splitting function matrix which show an overall gentle matching of resummation effects to fixed order quantities. The shallow dip occurring in previous treatments of P_{gg} is confirmed, and found in P_{gq} also.

1 Generalizing BFKL and DGLAP equations in matrix form

The physical question underlying this talk [1] is, at large, to provide a reliable description of rising *hard* cross sections and structure functions at high energies, and a precise determination of parton splitting functions at small-x, while keeping their well known behaviour at larger-x. More precisely, I will deal with the problem of providing a small-x resummation of parton evolution in matrix form [2], so as to treat by $\boldsymbol{k}$-factorization [3] quarks and gluons on the same ground and in a collinear factorization scheme as close as possible to $\overline{\mathrm{MS}}$.

The issue of a small-x generalization of DGLAP [4] and BFKL [5] evolutions has a long story [6, 7, 8, 9, 10], whose outcome is, at present, a certain consensus on the criteria and the mechanism of the evolution-kernel construction. Here I will summarize their application to the matrix case and the ensuing resummed results for the partonic splitting function matrix.

The BFKL equation [5] predicts rising cross-sections but the leading log prediction overestimates the hard Pomeron exponent, while NLL corrections are large and negative [6], and may make it ill-defined. On the other hand, low order DGLAP evolution is consistent with the rise of HERA structure functions, with marginal problems (hints of a negative gluon density). Therefore, we need to reconcile BFKL and DGLAP approaches: in the last decade, various (doubly) resummed approaches have been devised [7, 8, 9, 10] whose main idea is to incorporate RG constraints in the BFKL kernel, by calculating some effective (resummed) BFKL eigenvalue $\chi_{\mathrm{eff}}(\gamma)$ or the *dual* DGLAP anomalous dimension $\Gamma_{\mathrm{eff}}(\omega)$. So far, only the gluon channel has been treated self-consistently, while the quark channel is added by $\boldsymbol{k}$-factorization of the $q-\bar{q}$ dipole.

The purpose of the matrix approach proposed by M. Ciafaloni, G. Salam, A. Staśto and myself is to generalize DGLAP self-consistent evolution for quarks and gluons in $\boldsymbol{k}$-*factorized matrix* form, so as to be consistent, at small-x, with BFKL gluon evolution. One of the outcomes is to define, by construction, some *unintegrated* partonic densities at any x, even if the general issue of their factorization [11] is not actually treated. The main construction criteria for our matrix kernel are to incorporate exactly NLO DGLAP matrix evolution and the NLx BFKL kernel and to satisfy RG constraints in both ordered and anti-ordered collinear regions, and thus the $\gamma \leftrightarrow 1+\omega-\gamma$ symmetry [7]. An important role is played also by what I will call the *minimal-pole assumption* in the γ- and ω-expansions, as explained below.

Let me recall that the DGLAP evolution equations for the PDFs $f_a(Q^2)$ in the hard scale Q^2 define the anomalous dimension matrix $\Gamma(\omega)$, with the moment index $\omega = \partial/\partial \log 1/x$ conjugated to $\log 1/x$:

$$\frac{\partial}{\partial \log Q^2} f_a = [\Gamma(\omega)]_{ab} f_b \ .$$

On the other hand, the BFKL evolution equation in x for the unintegrated gluon PDF $\mathcal{F}(x, \boldsymbol{k}^2)$ defines the kernel $K(\gamma)$, with $\gamma = \partial/\partial \log \boldsymbol{k}^2$ conjugated to $\log \boldsymbol{k}^2$:

$$\omega \mathcal{F} = \frac{\partial}{\partial \log 1/x} \mathcal{F} = K(\gamma) \mathcal{F} \ .$$

If $\boldsymbol{k}$-factorization is used, DGLAP evolution of the Green's function G corresponds to either the ordered $k \gg k' \gg \cdots \gg k_0$ or the anti-ordered momenta, while BFKL incorporates all possible orderings. At frozen α_s, our RG-improved matrix kernel, generalizing the above evolutions, is expanded in the form $K(\alpha_s, \gamma, \omega) = \alpha_s K_0(\gamma, \omega) + \alpha_s^2 K_1(\gamma, \omega)$ and satisfies the minimal-pole assumption in the γ- and ω-expansion, namely we allow at most simple poles at $\gamma = 0$ (ordered $\boldsymbol{k}$'s) and at $\omega = 0$ (small-x)

$$\begin{aligned} K(\alpha_s, \gamma, \omega) &= (1/\gamma) K^{(0)}(\alpha_s, \omega) + K^{(1)}(\alpha_s, \omega) + \mathcal{O}(\gamma) \\ &= (1/\omega)\ {}_0K(\alpha_s, \gamma) + {}_1K(\alpha_s, \gamma) + \mathcal{O}(\omega) \end{aligned}$$

from which DGLAP anomalous dimension matrix Γ and BFKL kernel χ are derived

$$\Gamma_0 = K_0^{(0)}(\omega) \ ; \qquad \Gamma_1 = K_1^{(0)}(\omega) + K_0^{(1)}(\omega)\Gamma_0(\omega) \ ; \qquad \cdots \tag{1}$$

$$\chi_0 = [{}_0K_0(\gamma)]_{gg} \ ; \qquad \chi_1 = [{}_0K_1(\gamma) + {}_0K_0(\gamma)\ {}_1K_0(\gamma)]_{gg} \ ; \qquad \cdots \tag{2}$$

Such expressions are used to constrain K_0 and K_1 iteratively to yield the known NLO and NLx evolutions, and approximate momentum conservation. Furthermore, the RG constraints in both ordered and anti-ordered collinear regions are met by the $\gamma \leftrightarrow 1 + \omega - \gamma$ symmetry of the kernel which corresponds, in $(\boldsymbol{k}, x)$ space, to the $\boldsymbol{k} \leftrightarrow \boldsymbol{k}'$ and $x \leftrightarrow xk^2/k'^2$ symmetry of the matrix elements and thus relates the ordered and anti-ordered regions mentioned before.

We collect all the previous ideas in the following proposal for the leading-order improved matrix kernel

$$\begin{aligned} K_0 &= \begin{pmatrix} \Gamma^0_{qq}(\omega)\chi^\omega_c(\gamma) & \Gamma^0_{qg}(\omega)\chi^\omega_c(\gamma) + \Delta_{qg}(\gamma, \omega) \\ \Gamma^0_{gq}(\omega)\chi^\omega_c(\gamma) & \left[\Gamma^0_{gg}(\omega) - \frac{1}{\omega}\right]\chi^\omega_c(\gamma) + \frac{1}{\omega}\chi^\omega_0(\gamma) \end{pmatrix} \\ \chi^\omega_c(\gamma) &= \frac{1}{\gamma} + \frac{1}{1 + \omega - \gamma} \\ \chi^\omega_0(\gamma) &= 2\psi(1) - \psi(\gamma) - \psi(1 + \omega - \gamma) \end{aligned}$$

which includes exact LO DGLAP, LLx BFKL, and resums important subleading contributions of both collinear and high-energy type.

Some remarks are in order.

- In the qq, qg and gq entries we find the corresponding LO DGLAP anomalous dimensions $\Gamma_{ab}(\omega)$, multiplied by the collinear kernel $\chi^\omega_c(\gamma)$ which takes into accound collinear splittings in both the collinear and anti-collinear orderings, as can be seen by the presence of the collinear pole $1/\gamma$ and of the anti-collinear one $1/(1 + \omega - \gamma)$.

- In the gg entries we find, besides the gg anomalous dimension, the LO BFKL kernel with ω-shift, and a $1/\omega$ subtraction in order to avoid double counting, because the $\Gamma_{gg}(\omega)$ anomalous dimension contains the collinear part of the high-energy gluon emission.
- K_0 has simple poles in γ (in χ_c^ω and χ_0^ω) and simple poles in ω in the gluon row.
- No ω-poles are present in the quark row, consistently with LO DGLAP and reggeization of the quark at $\omega = -1$. We keep this structure also in K_1.
- At NLO Γ^1_{qq} and Γ^1_{qg} contain $\frac{\alpha_s^2}{\omega}$. Instead of adding such terms in K_1 (see above), we induce them in Γ_1 by adding a proper non-singular $\Delta_{qg}(\gamma,\omega)$ term in the qg entry of K_0, according to Eq. (1).
- K_1 is obtained by adding NLO DGLAP matrix Γ_1 and NLx BFKL kernel χ_1 (in $K_{1,gg}$) with the subtractions due to the γ- and ω- expansions explained before.

Finally, by double-inverse-Mellin transform, we formulate our matrix kernel in $(\boldsymbol{k}, x)$ space. Here, the running coupling is introduced as suggested by the RG and/or the NLx BFKL kernel

$$K(\boldsymbol{k}, \boldsymbol{k}'; x) = \alpha_s(\boldsymbol{k}_>^2) K_0(\boldsymbol{k}, \boldsymbol{k}'; x) + \alpha_s^2(\boldsymbol{k}_>^2) K_1(\boldsymbol{k}, \boldsymbol{k}'; x)$$

where we understand that the scale $\boldsymbol{k}_>^2 \equiv \max(\boldsymbol{k}^2, \boldsymbol{k}'^2)$ is replaced by $(\boldsymbol{k} - \boldsymbol{k}')^2$ in front of the BFKL kernel χ_0^ω.

We remark that reproducing both low order DGLAP and BFKL evolutions provides novel *consistency relations* between the matrix $\boldsymbol{k}$-factorization scheme and the $\overline{\mathrm{MS}}$-scheme. They turn out to be satisfied at NLO/NLx accuracy, while a small violation would appear at NNLO. In fact, the simple-pole assumption in ω-space implies [2] that $[\Gamma_2]_{gq} = (C_F/C_A)[\Gamma_2]_{gg}$ at order α_s^3/ω^2, violated by (n_f/N_c^2)-suppressed terms ($\leq 0.5\%$ for $n_f \leq 6$) in $\overline{\mathrm{MS}}$ [12]. For this reason we do not attempt full inclusion of the NNLO in our scheme.

2 Results for the resummed eigenvalues and the splitting matrix

As first result I show in Fig. 1 the frozen-coupling hard Pomeron exponent, namely the small-x power-growth exponent of cross sections and splitting functions in the linear evolution regime (where saturation effects are not important), having neglected the running of the coupling. This study is instructive because it shows that, for an effective coupling of phenomenological relevance ($\alpha_s \simeq 0.2$), the results we obtain are sensible and stable with respect to the details of the resummation procedure. In particular, the single-channel result coincide with the two-channel result for $n_f = 0$. The inclusion of NLO contributions provides a visible

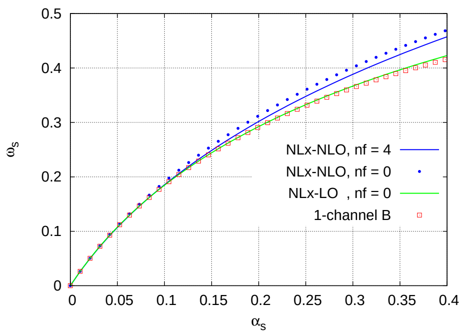

Figure 1: The resummed hard Pomeron exponent ω_s.

enhancement, while the quark contribution slightly lowers the Pomeron estimates.

There are two, frozen α_s, resummed eigenvalue functions: $\omega = \chi_\pm(\alpha_s, \gamma)$, corresponding to the leading and subleading anomalous dimensions $\gamma_\pm(\omega, \alpha_s)$, as depicted in Fig. 2.

The leading eigenvalue function shows fixed points at $\gamma = 0, 2$ and $\omega = 1$, corresponding to momentum conservation in both collinear and anti-collinear limits. Due to the matrix structure for $n_f \neq 0$, a new subleading eigenvalue χ_- appears. The n_f-dependence of $\chi_+(\alpha_s, \gamma)$ is modest, and the NLx-LO scheme recovers the known gluon-channel result (in agreement with [9]) at $n_f = 0$. Finally, a level crossing of χ_- and χ_+, present in the $n_f = 0$ limit, disappears at $n_f = 4$.

The results for the splitting function matrix $P_{ab}(x)$, including running coupling effects, are shown in Fig. 3 for $\alpha_s = 0.2$, and compared to NLO entries. The NLO$^+$ scheme includes, besides NLO, also NNLO terms of order α_s^3/ω^2, while scheme B refers to previous results [8] for the gluon channel only. We have numerically checked that the infrared cutoff independence insures (matrix) collinear factorization. We note that, at intermediate $x \simeq 10^{-3}$, the resummed P_{gg} and P_{gq} entries show a *shallow dip*, similarly to the one-channel case. Furthermore, the small-x rise of the novel P_{qg} and P_{qq} entries is delayed down to $x \simeq 10^{-4}$. Finally, the scale uncertainty band (for a rescaling parameter $0.25 < x_\mu^2 < 4$) is larger for the (small) P_{qa} entries, as perhaps expected from the fact that, in this case, the constant small-x behaviour starts at NLO.

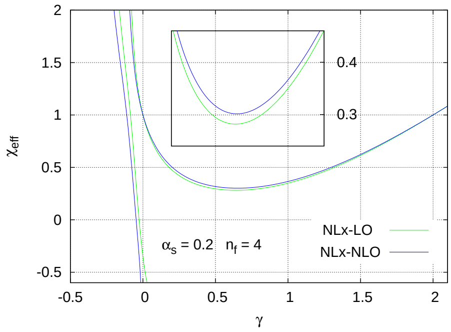

Figure 2: The resummed eigenvalue functions $\chi_\pm$.

To sum up, we have proposed a small-x evolution scheme in matrix form in which quarks and gluons are treated on the same ground and the splitting functions are already (closely) in the $\overline{\text{MS}}$ scheme. We fix the NLO/NLx matrix factorization scheme by further requiring *ordering-anti-ordering symmetry* and *minimal poles*. We find that the Hard Pomeron and the leading eigenvalue function are stable, with modest n_f-dependence, while a new subleading eigenvalue is obtained. The resummed splitting functions P_{ga} show a shallow dip, and the small-x increase of P_{qa} is delayed to $x \simeq 10^{-4}$. Overall, we find a *gentle matching* of low order with resummation. In order to complete this program, we still need coefficients with comparable accuracy; but we could take first the LO impact factors with exact kinematics [13] for preliminary studies.

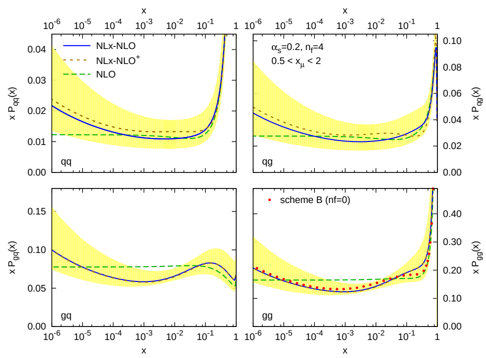

Figure 3: The resummed splitting functions P_{ab}.

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