BPS Geometry, AdS/CFT, and String Theory

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A Dissertation
Presented to the Faculty
of Princeton University
in Candidacy for the Degree
of Doctor of Philosophy

Recommended for Acceptance
by the Department of
Physics

September, 2006

Abstract

We first study BPS geometries dual to half-BPS states of field theories with 32 supercharges. These field theories are $\mathcal{N} = 4$ SYM on $R \times S^3$, M2 brane theory and M5 brane theory. The dynamics of the half-BPS states in $\mathcal{N} = 4$ SYM is reduced to a matrix quantum mechanics, and they are characterized by droplets on the two dimensional phase space. The geometries dual to these states are specified by boundary conditions on a two-plane, which are identified as the phase space. The phase space gives unified description of half-BPS perturbative and non-perturbative excitations above $AdS_5 \times S^5$. The half-BPS geometries in the M theory case are described by a Toda equation with similar but more complicated boundary conditions on a two-plane.

We then study BPS geometries dual to the vacua of field theories with 16 supercharges. The first class of theory has a bosonic $U(1) \times SO(4) \times SO(4)$ symmetry, they include the mass deformed M2 brane theory, D4 or M5 brane theory on S^3, and intersecting NS5 brane theory. Their vacua are characterized by droplets on a cylinder or torus. The second class of theory has a $SU(2|4)$ symmetry, they include the 0+1d plane-wave matrix model, 2+1d super Yang-Mills theory on $R \times S^2$, $\mathcal{N} = 4$ Super Yang-Mills on $R \times S^3/Z_k$, and NS5 brane theory on $R \times S^5$. The vacua of these theories are described by electrostatic configurations with many disks under external potential. The region between disks describe NS5 brane geometries. Finally we study instanton solutions and superpotentials for the vacua of the plane-wave matrix model and 2+1d super Yang-Mills theory and discuss the emergence of the vacuum geometries.

Acknowledgements

I owe a great amount of thanks to my advisor Juan Maldacena. I thank him for teaching me many things in theoretical physics, introducing me to many concepts, methods or techniques, as well as new problems in string theory. I also thank him for sharing with me his hard work, ideas and projects in the research, and helping me understand the physical interpretations of results, and overcome difficulties during the work. I have benefited from the experience of working with him greatly and in many aspects.

I am very grateful to my collaborator Oleg Lunin who shared with me hard work and also taught me plenty of physics.

The majority of the thesis is based on previously published joint works. The Chapters 2, 3, 4.2, 4.3 were based on published work with Juan and Oleg. The Chapters 4.4, 4.5, 5 were based on published work with Juan. The Chapter 6 is a new and unpublished result.

I thank professors H. Verlinde, I. Klebanov, S. Gubser, A. Polyakov from whom I leaned advanced physics courses, and the director of graduate studies, C. Nappi for many academic advice and support.

I am also thankful for the academic supports from Princeton University and Institute for Advanced Study.

I also thank many other people in string theory I met in various places for helpful discussions or conversations: M. Aganagic, C. Beasley, N. Beisert, I. Bena, R. Bousso, Y. Chuang, J. Friess, P. Gao, E. Gimon, S. Gukov, D. Hofman, Y. Hong, P. Horava, S. Kim, J. Kinney, N. Itzhaki, T. Levi, H. Liu, D. Malyshev, S. Mathur, G. Michalogiorgakis, A. Mints, A. Neitzke, H. Ooguri, V. Pestun, J. Polchinski, L. Rastelli, R. Roiban, P. Shepard,

D. Shih, M. Spradlin, P. Svrcek, I. Swanson, D. Tong, H. Tye, A. Volovich, M. Wijnholt, E. Witten, S. T. Yau, and many others.

I thank R. Austin, S. Park for having me work with them in my experimental project.

I would also like to thank other professors or colleagues I work with when I was a teaching assistant: F. Calaprice, C. Galbiatti, P. Goldbaum, Z. Hasan, Z. Huang, Y. Jau, Y. Liu, W. Man, K. McDonald, J. Olsen, S. Smith, J. Taylor, K. Zhao, and many others.

I want to thank all other friends at Princeton.

Finally, I am especially grateful to various Christian groups at Princeton, to my family, my sister and brother for constant support.

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Chapter 1

Introduction

String theory is a very promising candidate for a theory unifying all the observed interactions and matter in the universe. The theory has a vast number of realistic solutions and their richness is able to subsume both four dimensional general relativity and standard model of particle physics. There are currently various approaches in building low energy phenomenological theories from string theory, and these might be tested in the near future. Some aspects of the cutting-edge experiments in the Large Hadron Collider (LHC) in CERN will presumably address some problems in the Tev Scale physics, for example the electroweak symmetry breaking, superparticles, and large extra dimensions, which are relevant to string theory.

String theory has a very constrained mathematical structure, so the theory is very unique. The theory unifies quantum mechanics and general relativity very naturally. In the 1980s, it was found that there were only five unique superstring theories. A decade later, they were found to be dual to each other via a web of dualities. They all ultimately come from a unique M theory, whose low energy and classical description is the eleven dimensional supergravity. The mathematical formulation of such theory is still not clear and is under much study.

There has been a lot of progress in understanding the nature of string theory. Initially it was thought of as a theory of strings. In the mid 1990s, D-branes were identified as

surfaces where open strings can end, and their low energy dynamics is described by gauge theories. They played an important role in dualities and non-perturbative effects in string theory. Afterwards in 1997, it was proposed [134] that string theory in ten dimensions with fixed boundary conditions should be formulated by boundary quantum gauge theories. An example is that when we put D3-branes in the flat space, they back-react to the geometry, and in a decoupling limit of the D3-branes from the flat space, the D3-branes are equivalently described by a geometry with fluxes, which is the near horizon $AdS_5 \times S^5$ geometry. This is called the AdS/CFT correspondence [134],[86],[181].

This thesis is devoted to some particular aspects of the AdS/CFT correspondence. It studies the emergence of geometries in the bulk string theory from the supersymmetric states (or BPS states) in the boundary gauge theory and the dictionary of their mappings.

In AdS/CFT correspondence, spacetimes in the bulk string theory are emergent from the gauge theory on the boundary. We can formulate the string theory starting from lower dimensional quantum gauge theories. With respect to the dimensionality of the boundary, there are extra spatial dimensions which usually emerge from the eigenvalues of matrices in the gauge theory. States in the bulk gravity are mapped to states in the boundary gauge theory. Specifically, the supersymmetric states in the gauge theory that we will study are described by supersymmetric geometries (or BPS geometries) in the bulk.

The emergence of the bulk geometries from the boundary is in accord with the holographic principle of quantum gravity. In order to quantize gravity, we need to find the correct degrees of freedom of the gravitational system. These turn out to be located at the boundary rather than the bulk. In principle, physics processes that happen in the bulk can all be possibly observed or measured at the boundary. In this way, the numbers of the gravitational degrees of freedom are largely reduced, circumventing the difficult divergence problem in the old approach of quantizing gravity via path integrals of Einstein's action.

The notion of geometry should be modified near the Planck scale in quantum gravity. The classical geometry serves as an approximation to gravity much below the Planck scale. It is analogous to the approximation of the molecules by a fluid. When the classical geometry

breaks down, we can replace it by a better description given by the boundary quantum gauge theory. Conversely, the boundary theory can give rise to classical geometries under some limit. For example, the BPS geometries we will study are described by quantum states in the boundary gauge theory, which are characterized by fermion droplets on a quantum phase space. When the droplets are approximated as an incompressible fluid, we get well-defined smooth classical geometries. When the fluid approximation breaks down, the classical geometry breaks down and is replaced by a full quantum gauge theory description.

In order to study these aspects in AdS/CFT correspondence, we will look at theories that arise naturally in string or M theory in the supersymmetric regime and study their supersymmetric states. Due to the supersymmetry in these theories, we can look at a subsector of the whole states in these theories, which preserve half of the total supersymmetries of the vacua (that is they are half-BPS states). These states are very special, their energies do not receive quantum corrections due to supersymmetric cancellations. As a result their dynamics is rather independent of the regime of the coupling constant in the theory. This is good, since the AdS/CFT correspondence is a strong/weak-coupling duality, which means that the gauge theory side is weakly coupled when the gravity side is strongly curved, and vice versa. Since the states we will study are supersymmetric, their dual description falls into the supergravity regime.

These half-BPS states are in a small subsector of the entire Hilbert space of the boundary quantum gauge theories and can be described by reduced models. For example we will study the $\mathcal{N} = 4$ Supersymmetric Yang-Mills theory (SYM) which is defined on $R \times S^3$ in 3+1 dimensions. The dynamics of a sector of the half-BPS states is reduced on the S^3 to a 0+1 dimensional matrix quantum mechanics. It leads to a quantum free fermion system and is exactly solvable. As a result states are characterized by a two dimensional phase space of the free fermions. On the other hand, in the gravity side, each BPS state in the boundary field theory is described by a BPS geometry in the bulk. These geometries have a large symmetry and contain two S^3s, and asymptote to the $AdS \times S$ geometry near the boundary. After reduction on two spheres, the geometries contain a timelike direction, a radial direction

that is the product of the two radii of the spheres, and another two dimensions. When the radial direction goes to zero, each of the two spheres shrinks at different domains on a two dimensional plane. The geometries are characterized by the two-plane which is identified as the phase space of the free fermions. There are many non-trivial mappings and relations between the two sides of the system.

This system clearly gives an example of the emergent geometry or emergent gravity [32]. The system also gives a unified description of perturbative and non-perturbative excitations above the $AdS \times S$ ground state geometry. Here excitations are not restricted to string or D-brane fluctuations on the $AdS \times S$ background. Also it leads to a very clear picture that the topology-changing processes in the gravity can also be described by the boundary gauge theory. It demonstrates that near the Planck scale in quantum gravity, the classical geometry is replaced by a quantum gauge theory description.

The overview of the thesis is as follows. We first study theories with maximal 32 supercharges in string and M theory in chapters 2 and 3 respectively. These theories include the $\mathcal{N} = 4$ Super Yang-Mills theory on $R \times S^3$, 2+1d M2 brane theory, and 5+1d M5 brane theory. They are associated with or dual to the simplest solutions in type IIB string theory and M theory. Their half-BPS states and half-BPS geometries are characterized by droplets on a two dimensional plane. We then study theories with 16 supercharges in string and M theory in chapters 4, 5 and 6. A class of these theories includes the mass deformed M2 brane theory, D4 brane theory or M5 brane theory on S^3, and intersecting NS5 brane theory, which are studied in chapter 4. Another class of theories are the 0+1d plane-wave matrix model, 2+1d super Yang-Mills theory, $\mathcal{N} = 4$ Super Yang-Mills on $R \times S^3/Z_k$, and NS5 brane theory on $R \times S^5$. They are studied in chapter 5. These theories have even a richer structure than the maximally supersymmetric ones, and have a large number of vacua. Each vacuum of the theory is dual to a BPS geometry with the same symmetry. For the theories in chapter 5, each vacuum in the gravity side is characterized by an electrostatic configuration with many disks under an external potential. We will also study string fluctuations on these vacua. Some of these vacua describe NS5-branes. In chapter 6 we

study the superpotential associated to each vacuum and instantons interpolating between different vacua for the class of theories in chapter 5. These helps us better understand the vacua structure and emergence of the vacuum geometry from the boundary gauge theory. Finally we make a few conclusions in chapter 7.

Chapter 2

Geometry of 1/2 BPS states in $\mathcal{N} = 4$ SYM

2.1 Introduction

This chapter studies 1/2 BPS sector of the $\mathcal{N} = 4$ Supersymmetric Yang-Mills theory (SYM) and its dual gravity description. In the $\mathcal{N} = 4$ SYM, there is a class of local operators which are protected by supersymmetry and annihilated by half of the supercharges. These operators can be written as products of traces of a complex scalar. They are described by a matrix quantum mechanics with only one matrix. This comes from the reduction of $\mathcal{N} = 4$ SYM and BPS considerations. These states can be characterized by free fermion droplets in a two dimensional phase space. In gravity, we find all the 1/2 BPS geometries that correspond to these 1/2 BPS states, with the same symmetry and supersymmetry. In the gravity picture, there is also a two dimensional plane emerged in ten dimensions which is identified as the phase space of the free fermions. In Ch 2.2 we study these supergravity solutions, how they are found, the examples of the solutions, as well as their properties. In Ch 2.3 we analyze these states from $\mathcal{N} = 4$ SYM and derive the free fermion picture. We make a few remarks about this duality in Ch 2.4.

2.2 Gravity description and fermion droplet

2.2.1 The solutions

In this section we find all the geometries dual to the 1/2 BPS chiral primary operators in $\mathcal{N} = 4$ SYM. This class of geometries asymptote to $AdS_5 \times S^5$ while preserving (at least) 1/2 of the supersymmetries of $AdS_5 \times S^5$. The bosonic part of the symmetry is $SO(4) \times SO(4) \times U(1)$. The two $SO(4)$s come from the $SO(4,2)$ and $SO(6)$ respectively. We also have the time translation generator Δ in AdS_5 and an angular momentum generator J on S^5. The $U(1)$ comes from the combination of these two generators $\Delta - J$, and commutes with the preserved supercharges. This is a requirement from saturation of the 1/2 BPS bound $\Delta - J = 0$. These 1/2 BPS geometries are excitations above $AdS_5 \times S^5$ with excitation energy of order J.

We now try to find them from type IIB supergravity equations of motion. The bosonic symmetry implies that the geometry will contain two three-spheres S^3 and $\tilde{S}^3$, and a Killing vector. For the flux, we expect only the five–form field strength F_5 to be excited because the geometries have only D3 brane charges. The geometries therefore have the form

$$ds_{10}^2 = g_{\mu\nu} dx^\mu dx^\nu + e^{H+G} d\Omega_3^2 + e^{H-G} d\tilde{\Omega}_3^2 \tag{2.1}$$

$$F_5 = F_{\mu\nu} dx^\mu \wedge dx^\nu \wedge d\Omega_3 + \tilde{F}_{\mu\nu} dx^\mu \wedge dx^\nu \wedge d\tilde{\Omega}_3 \tag{2.2}$$

where $\mu, \nu = 0, \cdots, 3$.

The self duality condition on the five-form field strength implies that $F_{\mu\nu}$ and $\tilde{F}_{\mu\nu}$ are dual to each other in the four dimensional base space

$$F = e^{3G} *_4 \tilde{F} , \qquad F = dB, \qquad \tilde{F} = d\tilde{B} \tag{2.3}$$

The solutions satisfy the Killing spinor equations

$$\nabla_M \eta + \frac{i}{480} \Gamma^{M_1 M_2 M_3 M_4 M_5} F^{(5)}_{M_1 M_2 M_3 M_4 M_5} \Gamma_M \eta = 0 \tag{2.4}$$

This equation can be analyzed using techniques similar to the ones presented in [76, 74, 87, 78, 77]. We first writes the ten dimensional spinor η as a product of four dimensional

spinors ϵ and spinors $\chi_a,\ \tilde{\chi}_b$ on the spheres [126]. Due to the spherical symmetry the problem reduces to a four dimensional problem involving a four dimensional spinor ϵ. One then constructs a set of differential forms by using spinor bilinears

$$\begin{aligned} &K_\mu = -\bar{\epsilon}\gamma_\mu\epsilon, \qquad L_\mu = \bar{\epsilon}\gamma^5\gamma_\mu\epsilon\ , \qquad \bar{\epsilon} = \epsilon^\dagger\gamma^0 \\ &f_1 = i\bar{\epsilon}\hat{\sigma}_1\epsilon, \quad f_2 = i\bar{\epsilon}\hat{\sigma}_2\epsilon, \qquad Y_{\mu\nu} = \bar{\epsilon}\gamma_{\mu\nu}\hat{\sigma}_1\epsilon \end{aligned} \tag{2.5}$$

Using the reduced Killing spinor equations in four dimensions from (2.4), one can show that

$$\nabla_\mu f_1 = -e^{-\frac{3}{2}(H-G)}\tilde{F}_{\mu\nu}K^\nu = \frac{e^{-\frac{3}{2}(H+G)}}{2}\epsilon_{\mu\nu\lambda\rho}F^{\lambda\rho}K^\nu \tag{2.6}$$

$$\nabla_\mu f_2 = -e^{-\frac{3}{2}(H+G)}F_{\mu\nu}K^\nu \tag{2.7}$$

$$\begin{aligned} \nabla_\nu K_\mu &= -e^{-\frac{3}{2}(H+G)}\left[F_{\mu\nu}f_2 - \frac{1}{2}\epsilon_{\mu\nu\lambda\rho}F^{\lambda\rho}f_1\right] \\ &= -e^{-\frac{3}{2}(H+G)}F_{\mu\nu}f_2 - e^{-\frac{3}{2}(H-G)}\tilde{F}_{\mu\nu}f_1 \end{aligned} \tag{2.8}$$

$$\nabla_\nu L_\mu = e^{-\frac{3}{2}(H+G)}\left[-\frac{1}{2}g_{\mu\nu}F_{\lambda\rho}Y^{\lambda\rho} - F_\mu{}^\rho Y_{\rho\nu} - F_\nu{}^\rho Y_{\rho\mu}\right] \tag{2.9}$$

Another interesting set of spinor bilinears involves taking the the spinor and its transpose

$$\omega_\mu = \epsilon^t\Gamma^2\gamma_\mu\epsilon\ , \quad d\omega = 0 \tag{2.10}$$

which is a closed form. By Fierz rearrangement identities we can show[1]

$$K\cdot L = 0\ , \qquad L^2 = -K^2 = f_1^2 + f_2^2 \tag{2.11}$$

It can be shown that K^μ is a Killing vector and $L_\mu dx^\mu$ is a locally exact form. We thereby can choose a coordinate y so that $dy = -L_\mu dx^\mu$. We also choose one of the coordinates along the Killing vector to be the variable t. The remaining are two coordinates which will be denoted by $x_i,\ (i = 1, 2)$. By reducing all the equations and constraints we listed above gradually (for more details see [126]), one can relate the various functions appearing in the metric and flux to a single function $z(x_i, y)$, which obeys a simple linear differential

[1] We found a useful summary of these identities in [154].

equation. The y coordinate is very special since it is the product of the radii of the two S^3s. Finally the solutions are written as

$$
\begin{aligned}
ds^2 &= -h^{-2}(dt+V_i dx^i)^2 + h^2(dy^2+dx^i dx^i) + ye^G d\Omega_3^2 + ye^{-G}d\tilde{\Omega}_3^2 && (2.12)\\
h^{-2} &= 2y\cosh G, && (2.13)\\
y\partial_y V_i &= \epsilon_{ij}\partial_j z, \qquad y(\partial_i V_j - \partial_j V_i) = \epsilon_{ij}\partial_y z && (2.14)\\
z &= \frac{1}{2}\tanh G && (2.15)\\
F &= dB_t \wedge (dt+V) + B_t dV + d\hat{B}\ , \\
\tilde{F} &= d\tilde{B}_t \wedge (dt+V) + \tilde{B}_t dV + d\hat{\tilde{B}} && (2.16)\\
B_t &= -\frac{1}{4}y^2 e^{2G}, \qquad\qquad \tilde{B}_t = -\frac{1}{4}y^2 e^{-2G} && (2.17)\\
d\hat{B} &= -\frac{1}{4}y^3 *_3 d(\frac{z+\frac{1}{2}}{y^2})\ , \qquad d\hat{\tilde{B}} = -\frac{1}{4}y^3 *_3 d(\frac{z-\frac{1}{2}}{y^2}) && (2.18)
\end{aligned}
$$

where $i = 1, 2$ and $*_3$ is the flat space epsilon symbol in the three dimensions parameterized by x_i and y. The important function is z, which obeys the linear differential equation

$$\partial_i\partial_i z + y\partial_y(\frac{\partial_y z}{y}) = 0 \tag{2.19}$$

Now we consider regularity of the solutions. Since the product of the radii of the two 3-spheres is y, at $y = 0$ either the sphere shrinks. We find that y has to combine with the sphere to form R^4 locally, when either the sphere shrinks. For example, if the $\tilde{S}^3$ shrinks, the solution is non-singular if $z = \frac{1}{2}$ at $y = 0$. Then we see that z will have an expansion $z \sim \frac{1}{2} - e^{-2G} = \frac{1}{2} - y^2 f(x) + \cdots$, where $f(x)$ will be positive with our boundary conditions. From this we find that $e^{-G} \sim yc(x)$. So we see that the metric in the y direction and the second 3-sphere directions becomes

$$\frac{1}{2y\cosh G}dy^2 + ye^{-G}d\tilde{\Omega}_3^2 \sim c(x)(dy^2 + y^2 d\tilde{\Omega}_3^2) \tag{2.20}$$

which forms locally form R^4. In addition we see that h remains finite and the radius of the first sphere also remains finite. One can also show that V remains finite by using the explicit expression we write below. The situation for another sphere S^3 to shrink is exactly similar, and $z = -\frac{1}{2}$ at $y = 0$. We will also show below that the solution is non-singular

at the boundary of the two regions, because it approaches plane-wave geometry there. To summarize, the regularity condition for the solutions is

$$z = \pm\frac{1}{2}, \qquad \tilde{S}^3 \text{ or } S^3 \text{ shrinks to zero at } y = 0 \tag{2.21}$$

The solutions are specified by dividing the x_1, x_2 plane at $y = 0$ into two regions where $z = \pm\frac{1}{2}$.

In fact the transformation $z \to -z$ and an exchange of the two three–spheres is a symmetry of the equations. This corresponds to a particle hole transformation in the fermion system. This will not be a symmetry of the solutions if the fermion configuration itself is not particle-hole symmetric, or the asymptotic boundary conditions are not particle-hole symmetric (as in the $AdS_5 \times S^5$ case). These two signs corresponds to the fermions and the holes, and the x_1, x_2 plane corresponds to the phase space.

After defining $\Phi = z/y^2$ the equation (2.19) becomes the Laplace equation in six dimensions for Φ with spherical symmetry in four of the dimensions, y is then the radial variable in these four dimensions. The boundary values of z on the $y = 0$ plane are charge sources for this equation in six dimensions. It is then straightforward to write the general solution once we specify the boundary values. We find

$$z(x_1, x_2, y) = \frac{y^2}{\pi}\int_{\mathcal{D}} \frac{z(x_1', x_2', 0)dx_1'dx_2'}{[(\mathbf{x}-\mathbf{x}')^2 + y^2]^2} = -\frac{1}{2\pi}\int_{\partial\mathcal{D}} dl\; n_i' \frac{x_i - x_i'}{[(\mathbf{x}-\mathbf{x}')^2 + y^2]} + \sigma \tag{2.22}$$

$$V_i(x_1, x_2, y) = \frac{\epsilon_{ij}}{\pi}\int_{\mathcal{D}} \frac{z(x_1', x_2', 0)(x_j - x_j')dx_1'dx_2'}{[(\mathbf{x}-\mathbf{x}')^2 + y^2]^2} = \frac{\epsilon_{ij}}{2\pi}\oint_{\partial\mathcal{D}} \frac{dx_j'}{(\mathbf{x}-\mathbf{x}')^2 + y^2} \tag{2.23}$$

where in the second expressions for z, V_i we have used that $z(x_1', x_2', 0)$ is locally constant and we have integrated by parts to convert integrals over droplets $\mathcal{D}$ into the integrals over the boundary of the droplets $\partial\mathcal{D}$. In these expressions n_i is the unit normal vector to the droplet pointing towards the $z = \frac{1}{2}$ regions, σ is a contribution from infinity which arises in the case that z is constant outside a circle of very large radius (asymptotically $AdS_5 \times S^5$ geometries). $\sigma = \pm\frac{1}{2}$ when we have $z = \pm\frac{1}{2}$ asymptotically. The contour integral in (2.23) is oriented in such a way that the $z = -\frac{1}{2}$ region is to the left. We see from the second expression for V in (2.23) that V is finite as $y \to 0$ in the interior of a droplet. We also see

from (2.23) that V is a globally well defined vector field.[2] This is important since we want the time direction parameterized by t to be well defined, so we don't have NUT charge.

2.2.2 Examples

Before going to more details about the properties of the solutions, we first present some explicit examples. These include the IIB plane-wave, $AdS_5 \times S^5$, and 1/2 BPS extremal one-charge limit of AdS_5 black holes.

Let us now consider a simple solution which is associated to the half filled plane. We have the boundary conditions

$$z(x_1', x_2', 0) = \frac{1}{2}\text{sign } x_2' \tag{2.24}$$

From this data we can compute the entire function $z(x_2, y)$ using (2.22), (2.23)

$$z(x_2, y) = \frac{1}{2}\frac{x_2}{\sqrt{x_2^2 + y^2}}, \qquad V_1 = \frac{1}{2}\frac{1}{\sqrt{x_2^2 + y^2}}\ , \quad V_2 = 0 \tag{2.25}$$

Inserting this into the general ansatz (2.12) and performing the change of coordinates

$$y = r_1 r_2, \quad x_2 = \frac{1}{2}(r_1^2 - r_2^2) \tag{2.26}$$

we obtain the usual form of the metric for the plane wave [39]

$$ds^2 = -2dtdx_1 - (r_1^2 + r_2^2)dt^2 + d\vec{r}_1^2 + d\vec{r}_2^2 \tag{2.27}$$

We see that the final solution is smooth, despite the fact that on the $y = 0$ plane V diverges at the boundary between two regions ($x_2 = 0$ in this case). In fact, this computation shows that, in general, the boundary between two regions is smooth,. because locally the boundary region looks like the plane wave and therefore we will get a non-singular metric, see figure 2.1.

Let us now recover the familiar $AdS_5 \times S^5$ geometry. In this case it is convenient to introduce a function $\tilde{z} = z - \frac{1}{2}$. The Laplace equation for $\tilde{z}/y^2$ has sources on a disk of

[2] In the cases that we consider, where at most the x_1 coordinate is compact, there are no compact two cycles in the x_1, x_2, y space. So we do not have any compact two cycles on which we could find a non-zero integral of dV.

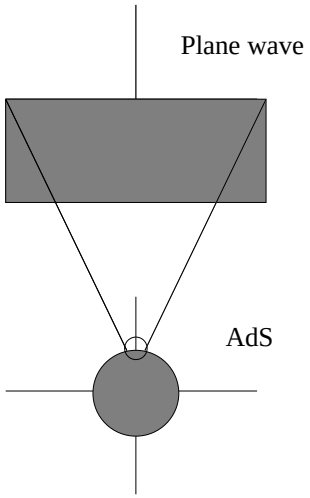

Figure 2.1: Plane wave configurations correspond to filling the lower half plane. This can be understood from the fact that the plane wave solution is a limit of the $AdS \times S$ solution.

radius r_0. We choose polar coordinates r, ϕ in the x_1, x_2 plane. We obtain

$$\begin{aligned} \tilde{z}(r,y) &= -\frac{y^2}{\pi}\int_{\text{Disk}} \frac{r'dr'd\phi}{[r^2+r'^2-2rr'\cos\phi+y^2]^2} \\ \tilde{z}(r,y;r_0) &\equiv \frac{r^2-r_0^2+y^2}{2\sqrt{(r^2+r_0^2+y^2)^2-4r^2r_0^2}} - \frac{1}{2} \qquad (2.28) \\ V_\phi &= -r\sin\phi V_1 + r\cos\phi V_2 = -\frac{1}{2\pi}\int_{\partial\mathcal{D}} \frac{rr'\cos\phi' d\phi'}{r^2+r'^2+y^2-2rr'\cos\phi'} \\ V_\phi(r,y;r_0) &\equiv -\frac{1}{2}\left(\frac{r^2+y^2+r_0^2}{\sqrt{(r^2+r_0^2+y^2)^2-4r^2r_0^2}} - 1\right) \qquad (2.29) \end{aligned}$$

Inserting this into the general ansatz and performing the change of coordinates

$$y = r_0 \sinh\rho \sin\theta, \quad r = r_0\cosh\rho\cos\theta, \quad \tilde{\phi} = \phi - t \qquad (2.30)$$

we see that we get the standard $AdS_5 \times S^5$ metric

$$ds^2 = r_0[-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\Omega_3^2 + d\theta^2 + \cos^2\theta d\tilde{\phi}^2 + \sin^2\theta d\tilde{\Omega}_3^2] \qquad (2.31)$$

So we see that $r_0 = R^2_{AdS} = R^2_{S^5}$, and the area of the disk is proportional to N .

Now that we have constructed the solution for a circular droplet, we can construct in a trivial way the solutions that are superpositions of circles, see figure 2.2(a). Among these the ones corresponding to concentric circles have an extra Killing vector. These lead to

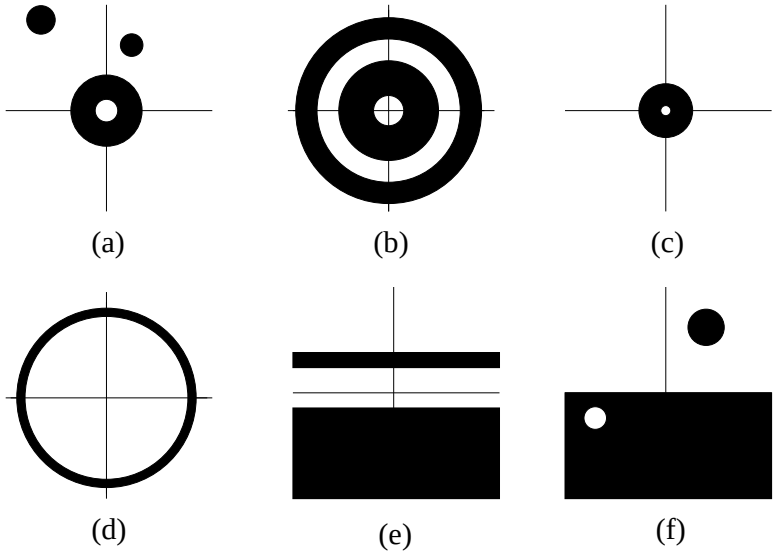

Figure 2.2: We see various configurations whose solutions can be easily constructed as superpositions of the $AdS_5 \times S^5$ solution and the plane wave solution. In (a) we see an example of the type of configurations that can be obtained by superimposing the circular solution (2.28). In (b) we see generic configurations that lead to solutions which have two Killing vectors and lead to static configurations in AdS. In (c) we see the solution corresponding to a superposition of D3 branes wrapping the $\tilde{S}^3$ in S^5. In (d) we see the configuration resulting from many such branes, which can be thought of as a superposition of branes on the S^3 of AdS_5 uniformly distributed along the angular coordinate $\tilde{\phi}$ of S^5. In (e) we see a configuration that can be viewed as an excitation of a plane wave with constant energy density. In (f) we see a plane wave excitation with finite energy.

time independent configurations in AdS. All other solutions will depend on $\phi = t + \tilde{\phi}$ where t is the time in AdS and $\tilde{\phi}$ is an angle on the asymptotic S^5, see (2.31). The solutions corresponding to concentric circles are therefore superpositions of (2.28) and (2.29)

$$\tilde{z} = \sum_i (-1)^{i+1} \tilde{z}(r, y; r_0^{(i)}), \qquad V_\phi = \sum_i (-1)^{i+1} V_\phi(r, y; r_0^{(i)}) \tag{2.32}$$

Here $r_0^{(1)}$ is the radius of the outermost circle, $r_0^{(2)}$ the next one, etc (see figure 2.2(b)). Let us discuss the solution corresponding to a single black ring 2.2(c). When the white hole in the center is very small, this can be viewed as branes wrapping a maximal $\tilde{S}^3$ in S^5. When the area of this hole, N_h, is smaller than the original area, N, of the droplet ($N_h \ll N$), the solution will locally look like an $AdS_5 \times S^5$ solution near the hole. When we increase the number of branes wrapped on $\tilde{S}^3$ in S^5 the area of the holes becomes very large and in

the limit we get a rather thin ring, which could be viewed as a superposition of D3 branes wrapping an S^3 in AdS_5 [3] , see figure 2.2(d).

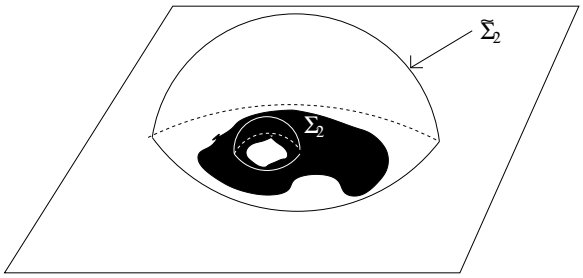

Figure 2.3: We can construct a five-manifold by adding the sphere $\tilde{S}^3$ fibered over the surface $\tilde{\Sigma}_2$. This is a smooth manifold since at the boundary of $\tilde{\Sigma}_2$ on the $y = 0$ plane the sphere $\tilde{S}^3$ is shrinking to zero. The flux of F_5 is proportional to the area of the black region inside $\tilde{\Sigma}_2$. Another five manifold can be constructed by taking Σ_2 and adding the other three–sphere S^3. The flux is proportional to the area of the white region contained inside Σ_2.

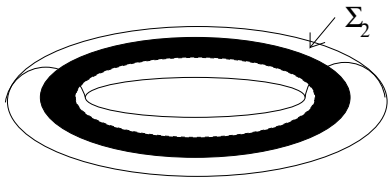

Figure 2.4: We see here an example of a two dimensional surface, Σ_2, that is surrounding a ring. If we add the three–sphere $\tilde{S}^3$ fibered over Σ_2 we get a five manifold with the topology of $S^4 \times S^1$.

None of the solutions described here has a horizon and they are all regular solutions. A singular solution was considered in [151]. That solution was obtained as the extremal limit of a charged black hole in gauged supergravity [25, 26]. Since it is a BPS solution it obeys our equations. We find that the boundary conditions on the $y = 0$ plane are such that we have a disk, similar to the one we have in AdS but the fermion density is not -1 but $-1/(1 + q)$ where q is the charge parameter of the singular solution. Of course, the solution is singular because it violates our boundary condition, but it could be viewed as an

[3]Note that this seems to disagree with a proposal in [21] for realizing a larger radius AdS space inside a smaller radius AdS space.

approximation to the situation where we dilute the fermions, or we consider a uniform gas of holes in the disk, which agrees with the picture in [151]. Due to the superposability of the function $\tilde{z} = z - \frac{1}{2}$, since we know the boundary value for $\tilde{z}$, the solution can be written simply

$$\tilde{z}(r, y; r_0) = \frac{1}{1+q}\left[\frac{r^2 - r_0^2 + y^2}{2\sqrt{(r^2 + r_0^2 + y^2)^2 - 4r^2 r_0^2}} - \frac{1}{2}\right] \tag{2.33}$$

$$\tilde{z}(0, y; r_0) = -\frac{1}{1+q} \text{ or } 0, \qquad \text{when } \; r < r_0 \text{ or } r > r_0 \tag{2.34}$$

This was also discussed in [84],[44].

There is also another related solution corresponding to the $SO(4)$ invariant Coulomb branch of the $\mathcal{N} = 4$ SYM [122]. This solution arises when we have a dilute distribution of droplets, and when we keep the areas of the droplets fixed and scale their separations to infinity. Under this limit, the solution becomes multi-center D3 brane solution [126].

2.2.3 Charges and topology

In this section, we analyze the flux quantization associated to the area of the droplets, we derive the charges Δ and J of the solutions, and discuss their topologies.

From the $AdS_5 \times S^5$ solution in (2.31), we see that $r_0 = R^2_{AdS} = R^2_{S^5}$. In fact, under an overall scaling of the coordinates $(x_i, y) \to \lambda(x_i, y)$ the metric scales by a factor λ. This is what we expect since the total area of the droplets is equal to the number of branes, a fact which we will demonstrate later. By comparing the value of the AdS radius we obtained in (2.31) and the standard answer, $R^4_{AdS} = 4\pi l_p^4 N$, we can write the precise quantization condition on the area of the droplets in the 12 plane as[4]

$$(\text{Area}) = 4\pi^2 l_p^4 N \ , \qquad \text{or} \quad \hbar = 2\pi l_p^4 \tag{2.35}$$

where N is an integer, and we have defined an effective $\hbar$ in the x_1, x_2 plane, where we think of the x_1, x_2 plane as phase space.

[4]We define $l_p = g^{\frac{1}{4}}\sqrt{\alpha'}$.

Let us analyze the topology of the solutions. This analysis is somewhat similar to that used in toric geometry. As long as $y \neq 0$ we have two S^3s. Let us denote these two spheres as S_3 and $\tilde{S}_3$. At the $y = 0$ plane the first sphere shrinks in a non-singular fashion if $z = -\frac{1}{2}$ while the second sphere, $\tilde{S}^3$, shrinks if $z = \frac{1}{2}$. Both spheres shrink at the boundary of the two regions. In fact there is a shrinking S^7 at these points, since the geometry is locally the same as that of a pp-wave. For example, in the $AdS_5 \times S^5$ solution the second sphere, $\tilde{S}_3$, shrinks at $y = 0$ outside the circle, this is the three-sphere contained in S^5. On the other hand the three-sphere contained in AdS_5 shrinks at $y = 0$ inside the circular droplet. Consider a surface $\tilde{\Sigma}_2$ on the $(y, \vec{x})$ space that ends at $y = 0$ on a closed, non-intersecting curve lying in a region with $z = \frac{1}{2}$ see figure 2.3 . We can construct a smooth five dimensional manifold by fibering the second three sphere, $\tilde{S}^3$, on $\tilde{\Sigma}_2$. This is a smooth manifold which is topologically an S^5.

We can now measure the flux of the five-form field strength F_5 on this five-sphere. Looking at the expressions for the field strength (2.2) in terms of the four dimensional gauge field (2.18), (2.16) we find that the spatial components are given by $\tilde{F}|_{spatial} = d(\tilde{B}_t V) + d\hat{\tilde{B}}$. Since $B_t V$ is a globally well defined vector field the flux is given by

$$\tilde{N} = -\frac{1}{2\pi^2 l_p^4} \int d\hat{\tilde{B}} = \frac{1}{8\pi^2 l_p^4} \int_{\tilde{\Sigma}_2} y^3 *_3 d\left(\frac{z - \frac{1}{2}}{y^2}\right) = \frac{(\text{Area})_{z=-\frac{1}{2}}}{4\pi^2 l_p^4} \tag{2.36}$$

where $\tilde{\Sigma}_2$ is the two surface in the three dimensional space spanned by y, x_1, x_2. This expression gives the total charge inside this region for the Laplace equation, which in turn is equal to the total area with $z = -\frac{1}{2}$ contained within the contour on which $\tilde{\Sigma}_2$ ends at $y = 0$, see figure 2.3. Note that (2.36) leads to the quantization of area, (2.35). In the $AdS_5 \times S^5$ case there is only one non-trivial five–sphere and this integral gives the total flux. This flux is quantized in the quantum theory.

We can consider an alternative five-cycle by considering a surface that ends on the $y = 0$ plane on a region with $z = -\frac{1}{2}$ (see figure 2.3). The flux over this five-manifold is given by

$$N = \frac{1}{2\pi^2 l_p^4} \int d\hat{B} = -\frac{1}{8\pi^2 l_p^4} \int_{\Sigma_2} y^3 *_3 d\left(\frac{z + \frac{1}{2}}{y^2}\right) = \frac{(\text{Area})_{z=\frac{1}{2}}}{4\pi^2 l_p^4} \tag{2.37}$$

and it measures the total area of the other type, with $z = \frac{1}{2}$, contained in this region. If these fluxes are non-zero, then these spheres are not contractible. So if we have a large number of droplets, we have a complicated topology for the solution. In addition we can construct other 5-manifolds which are not five–spheres by considering more complicated surfaces. For example we get the five-manifold with topology $S^4 \times S^1$ from the surface depicted in figure 2.4.

These geometries realize very clearly the geometric transitions [120, 175] that arise when one adds branes to a system. Adding a droplet of fermions to an empty region corresponds to adding branes that are wrapped on the sphere that originally did not shrink at $y = 0$. Once we have a new droplet this sphere will now shrink in the interior of the droplet. In the process we have also created a new non-contractible cycle of topology S^5 that consists of the three-sphere that was shrinking before and the two dimensional manifolds (or "cups") depicted in figure 2.3.

An interesting property of the solutions is their energy or their angular momentum J. These are equal to each other due to the BPS condition $\Delta = J$. As explained in [31], this energy is the energy of the fermions in a harmonic oscillator potential minus the zero-point energy of the ground state of N fermions[5] . From the gravity solution it is easier to read off the angular momentum. This involves computing the leading terms in the $g_{\phi+t,t}$ components of the metric, and we get

$$\begin{aligned} \Delta &= J = \frac{1}{16\pi^3 l_p^8}\left[\int_{\mathcal{D}} d^2x(x_1^2+x_2^2) - \frac{1}{2\pi}\left(\int_{\mathcal{D}} d^2x\right)^2\right] \\ &= \int_{\mathcal{D}} \frac{d^2x}{2\pi\hbar}\frac{\frac{1}{2}(x_1^2+x_2^2)}{\hbar} - \frac{1}{2}\left(\int_{\mathcal{D}}\frac{d^2x}{2\pi\hbar}\right)^2 \end{aligned} \tag{2.38}$$

where $\mathcal{D}$ is the domain where $z = -\frac{1}{2}$, which is the domain where the fermions are. Using the definition of $\hbar$ in (2.35) we see that this is the quantum energy of the fermions minus the energy of the ground state.

[5]Equivalently we can express it as the angular momentum of the quantum Hall problem.

2.3 Gauge theory description and matrix quantum mechanics

In this section we consider the same set of 1/2 BPS states in the gauge theory side, that is from the $U(N)$ $\mathcal{N}=4$ SYM on $S^3 \times R$. The theory contains one gauge field A_μ, six scalars, and four Weyl fermions in the adjoint, they are all $N \times N$ Hermitian matrices. The vacuum preserves a $PSU(2,2|4)$ superconformal symmetry, which is the same as that of the $AdS_5 \times S^5$ geometry corresponding to a circular droplet in section 2.2.2. Other 1/2 BPS states are excitations above the vacuum. These are chiral primary operators that are the highest weight states in the $PSU(2,2|4)$ representations. These local operators are built out of products of traces of a single complex scalar $Z = \frac{1}{\sqrt{2}}(X_4 + iX_5)$, where X_4 and X_5 are two adjoint scalars. These operators take the multi-trace form

$$\prod_{a=1}^{k} \text{tr } Z^{n_a} \tag{2.39}$$

where all the exponents n_a are bounded by N, because trace of Z with higher exponent can be written in terms of products of traces with exponents no larger than N by the Cayley-Hamilton identities. The ordering of the traces in the operator does not matter, because different traces commute. The conformal weight and R-charge are both equal to the total number of exponents, therefore the states satisfy

$$\Delta - J = 0 \tag{2.40}$$

which is the same for the gravity solutions discussed before.

Furthermore, these operators are built out of traces of the zero mode of the field Z on a round S^3, because other higher KK modes correspond to inserting derivatives on the S^3 in the operator and will not be 1/2 BPS. And of course $\bar{Z}$ can not appear in the traces, since they contribute oppositely to the conformal weight and R-charge. We can thereby reduce the problem to a 0+1d quantum mechanics involving only Z. See also [91],[53],[31][6] . At

[6]See also an interesting proposal of the string theory dual of the large N harmonic oscillator of a single matrix in [103].

this stage it is a holomorphic complex matrix model, and it can be equally described by a one Hermitian matrix model [31], after integrating out non-dynamical components. The lagrangian of the matrix quantum mechanics for Z can be written as

$$S = \frac{Vs_3}{g_{ym3}^2} \int dt \ \mathrm{tr} \left(\frac{1}{2} \dot{X}_4^{\ 2} + \frac{1}{2} \dot{X}_5^{\ 2} - \frac{1}{2} X_4^2 - \frac{1}{2} X_5^2 \right) \tag{2.41}$$

where the commutator term is not included since they contribute positively to the BPS bound and are usually stringy non-BPS excitations. We set $A_0 = 0$ and have a Gauss law constraint on $U(N)$ gauge singlet physical states, $\frac{\delta L}{\delta A_0} |\Psi\rangle = 0$.

We then define the usual conjugate momemta $P_4,\ P_5$ for the two scalars. The Hamiltonian ($H = \Delta$) and R-charges J are

$$H = \frac{1}{2} \mathrm{tr} \ (P_4^2 + P_5^2 + X_4^2 + X_5^2), \quad J = \mathrm{tr} \ (P_4 X_5 - P_5 X_4) \tag{2.42}$$

We can also discuss in terms of the complex scalar Z and its conjugate momentum $\Pi = -i \frac{\partial}{\partial Z^\dagger} = \frac{1}{\sqrt{2}} (P_4 + i P_5)$. They satisfy the standard commutation relations $[Z^{mn}, \Pi^{\dagger m' n'}] = i\hbar \delta_{nm'} \delta_{n'm},\ [Z, \Pi] = 0$, so we can define two sets of creation/annihilation operators

$$a^\dagger = \frac{1}{\sqrt{2}} (Z^\dagger - i\Pi^\dagger), \quad b^\dagger = \frac{1}{\sqrt{2}} (Z^\dagger + i\Pi^\dagger) \tag{2.43}$$

The Hamiltonian and R-charge operators are then written as

$$H = \mathrm{tr} \ (a^\dagger a + b^\dagger b), \quad J = \mathrm{tr} \ (a^\dagger a - b^\dagger b) \tag{2.44}$$

So it's clear in order to get 1/2 BPS states with $H = J$, we should keep only the single $a^\dagger$ oscillators but not the other $b^\dagger$ oscillators. We have then in the 1/2 BPS sector a one dimensional harmonic oscillator Hilbert space. It's phase space is two dimensional. The singlet condition tells us we should look at states of products of traces

$$|\Psi\rangle = \prod_{a=1}^{k} \mathrm{tr} \ (a^\dagger)^{n_a} |0\rangle \tag{2.45}$$

Now we want to consider the D-brane description of these states. The wave functions and Hamiltonian are invariant under unitary transformations for Z. We want to diagonalize Z as much as possible, we can put it into the form

$$Z = U(z + T)U^\dagger \tag{2.46}$$

where U is an unitary matrix, $z+T$ is a triangular matrix in which z is the diagonal part and T is the off-diagonal part of the triangular matrix. After integrating out U, the wave function takes the form

$$\Psi \sim \quad \mathcal{N}\ \Delta[z] \prod_{a=1}^{k} (\sum_i {z_i}^{n_a}) e^{-\frac{1}{2}\sum_i z_i \bar{z}_i - \frac{1}{2}\mathrm{tr}(TT^\dagger)} \tag{2.47}$$

up to a normalization factor, and

$$\Delta[z] = \prod_{i<j}(z_i - z_j) \tag{2.48}$$

is a Van-de-monde determinant from integrating out U. The wave function has a totally antisymmetric property under exchange of any z_i, z_j, and thereby it describes fermions. See also similar discussions about the wave function in [173],[183],[32],[53],[31]. The factor from $e^{-\frac{1}{2}\mathrm{tr}(TT^\dagger)}$ can further be integrated out to give an overall constant. So the wave function is in the same form if we were starting from a one Hermitian matrix model [31]. There is an interesting property. The wave function, besides the universal Gaussian factor, is a holomorphic function in z_i, due to the BPS condition.

The system of these 1/2 BPS states are described by wave functions of N free fermions under a harmonic oscillator potential. Because they are identical particles, in the two dimensional phase space they occupy different shapes of droplets. The droplets are incompressible, because the total number N is fixed, this is the same as the condition in 2.2.3 that the total flux quanta are fixed by N. See figure 2.5.

It is possible to use a hydrodynamic approach in the phase space [32]. The droplet is the saddle point approximation of the square of the wave function. In the wave function, the Gaussian factor produces a quadratic potential for eigenvalues, and the other factors give repulsion forces between them. In the large N limit, we can approximate the distribution of eigenvalues in phase space by a density function $\rho(\vec{x})$ for the hydrodynamic fluid in two dimensions, so that $\sum_i z_i \bar{z}_i = \int d^2x \rho(x) \vec{x}^2$ and $N = \int d^2x \rho(x)$. For example, if we consider the ground state droplet, due to the quadratic potential and the Van-de-monde determinant, the density function satisfy the integral equation [32]

$$\vec{x}^2 + c = 2 \int d^2y \rho(y) \log(|\vec{x} - \vec{y}|) \tag{2.49}$$

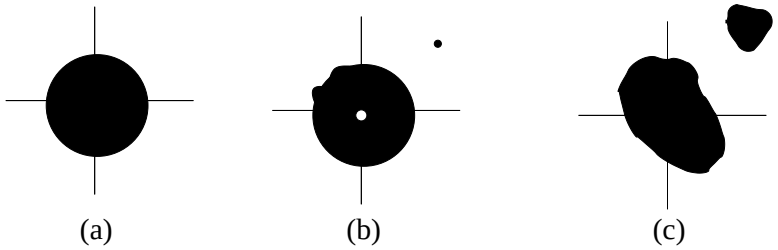

Figure 2.5: Droplets representing chiral primary states. In the field theory description these are droplets in phase space occupied by the fermions. In the gravity picture this is a particular two-plane in ten dimensions which specifies the solution uniquely. In (a) we see the droplet corresponding to the $AdS \times S$ ground state. In (b) we see ripples on the surface corresponding to gravitons in $AdS \times S$. The separated black region is a giant graviton brane which wraps an S^3 in AdS_5 and the hole at the center is a giant graviton brane wrapping an S^3 in S^5. In (c) we see a more general state.

where c is a Lagrange multiplier ensuring the constraint on the total number of fermions. Since in two dimensions $\log(|\vec{x} - \vec{y}|)$ is the Green's function for the Laplace operator, the solution for $\rho(\vec{x})$ is that it is constant within a circular disk, and is zero outside. The other droplets corresponding to coherent states can be similarly described, in which cases the external potential get modified and as a result their shapes are different.

2.4 Discussion

In above two sections, we have analyzed the 1/2 BPS chiral primary states from both the gravity and gauge theory point of view. We find agreement between these two descriptions. In this section we make a few remarks.

In the gauge theory side, these states can be written as products of traces built from one complex scalar Z. The dynamics of these 1/2 BPS states is described by a holomorphic complex matrix quantum mechanics reduced from the $U(N)$ $\mathcal{N} = 4$ SYM on S^3 keeping only the zero mode of the Z. All these 1/2 BPS states can map to the states in this matrix quantum mechanics. This matrix quantum mechanics contains two harmonic oscillators and we only keep one of them due to the BPS condition, so this is finally equivalent to a one Hermitian matrix quantum mechanics [31]. The model can be solved in either closed

string basis where we look at states of the products of traces of creation operators and they are labelled by Young tableaux, or it can be solved in the D-brane or eigenvalue basis, where the wave functions describe free fermions in a harmonic potential. Thus states of the reduced model are described by droplets in a two dimensional phase space. The vacuum of the $U(N)$ $\mathcal{N}=4$ SYM on $S^3 \times R$ is a circular droplet with area N. Other excited states correspond to ripples of the fermion surface, holes or fermions away from the fermions surfaces, and also more dramatic changes of the droplet, etc.

In the gravity side, we solve all the geometries dual to these 1/2 BPS states by considering the same set of symmetry and supersymmetry for these states, and also regularity conditions. The geometry contains two S^3 from the worldvolume and R-symmetry of the $\mathcal{N}=4$ SYM. Besides a direction t, which corresponds to $U(1)_{\Delta-J}$, there is a coordinate y which is the product of the two three-spheres. The remaining are two dimensions x_1, x_2. At $y=0$, each regular solution requires the boundary condition $z=\pm\frac{1}{2}$ on the x_1, x_2 plane where either the two spheres shrinks, and the solutions are determined by specifying these two regions. The $AdS_5 \times S^5$ is a circular droplet. Excited states include supergravity modes, strings, giant gravitons wrapping either spheres, or other completely new geometries.

This system reduces the rather complicated problem to a simple description of the phase space or the x_1, x_2 plane. In the gauge theory side, the quantization of the phase space, corresponds to the quantization of fluxes in the gravity. The flux numbers are the area of the droplets on the x_1, x_2 plane. The energy of the solution J is a second moment of the distribution of the droplets, it matches exactly with the total energy of the fermions corresponding to the same distribution in phase space. The phase space has a symplectic structure and this is also found out in gravity side and leads to a way to quantize the system [139, 80]. Moreover, the phase space description shows particle-hole duality when we switch the two three-spheres and the regions they shrink. The complicated topology of the ten dimensional geometries are also simplified due to this phase space. Finally, we can also compactify this phase space, and this gives rise to other theories related to the original $\mathcal{N}=4$ SYM and will be discussed in chapter 4.

The system gives an unified description of some of the perturbative and non-perturbative excitations above $AdS_5 \times S^5$. The exited states are better described in different ways when we increase the excitation energy J. See figure 2.5. For small excitation energies $J \ll N$, they correspond to supergravity modes propagating in the bulk. When energy increases to $J \sim \sqrt{N}$ in the BMN limit, they have enough partons and become stringy modes. These states are better described by trace formulas, due to the orthogonality. As we increase the excitation energy to the order $J \sim N$, some of the states are better described by D3-branes wrapping either the internal sphere [141] or the AdS sphere [91],[82], and corresponds to adding new droplets in the x_1, x_2 plane. In the gauge theory side the trace formulas breaks down due to non-orthogonality, and states are better described by Shur polynomials or Young tableaux [53] (see also [45]). In the case of D3-branes wrapping the internal sphere, it can be described by determinants or subdeterminants [17]. A single D3-brane wrapping the internal sphere is a state with an additional single column in the Young tableaux, it is a hole excitation. In the gravity side, it corresponds to adding a small bubble in the disk and thereby boosts the total energy. Their energies are bounded by N, the bound is satisfied when the bubble is located at origin, which is the maximal giant graviton. This gives an explanation of the stringy exclusion principle by the finiteness of the fermion sea. On the other hand, a single D3-brane wrapping the AdS sphere is a state with an additional single row in the Young tableaux, and it is a separate fermion excitation. In the gravity side, it corresponds to adding a small droplet away from the disk and their energies are not bounded because it can be sent to infinitely away. We can also have solutions that smoothly interpolate between branes wrapping the sphere and branes wrapping AdS. As we further increase energy to $J \sim N^2$, we expect to have order N of these D3 branes each with order N of energy and their backreaction to the $AdS_5 \times S^5$ cannot be neglected. The best description for them are completely new geometries. In the gauge theory side, they corresponds to large distortions of the disk. It is also interesting to note that some topologically non-trivial excitations with very low energy are better described by low energy gravity modes. This can be seen when we put a small droplet very close to the disk, it is

better described by ripples on the disk or edge states from the point of view of the fermion system.

The system realizes the open-closed string duality. We already see this in the gauge theory description when states can be interpreted in closed string picture and eigenvalue picture. In gravity side, eigenvalues are described by individual small droplets or bubbles, which are N free fermions corresponding to N D3 branes[7] . They can form a large droplet by staying together. When there is a ripple on this large droplet, these fluctuations are better described not by individual fermions, but by their collective excitations (see also [138],[60],[61]). This collective excitation of N D3 branes is a closed string state.

The system exhibit very clearly the geometric transitions [120, 175] that arise when one adds branes to a background. Adding a droplet of fermions to an empty region corresponds to adding branes that are wrapped on the three-sphere that originally does not shrink at the boundary plane. After geometric transition the branes are described by a new droplet on which this three-sphere now shrinks. In the meantime we have also created a new non-contractible cycle of topology S^5. The three-sphere that the branes were wrapping disappears while the additional fluxes through the non-contractible S^5 now turned on.

This system also gives rise to some interesting observations if we change the droplet density in the phase space to be other than one. If we increase the droplet density to exceed one, which violates Pauli exclusion principle, then in the gravity picture, there appear closed time-like curves which violates the causality principle [44]. If we decrease the droplet density to be less than one, like the case of the 1/2 BPS extremal one-charge limit of the black hole in AdS_5, then one can argue that there is an ensemble of smooth geometries with dilute distribution of fermions. The singularity appears when we neglect the details of individual distribution and make a coarse-graining of all the geometries. This is very similar to the view that black hole entropy might arise as sum over different smooth geometries dual to individual microstates [18, 19, 170, 169, 172], see also [30],[34],[140],[133],[132],[20],[5].

[7]The free fermion picture for the D3 branes and closed strings is also reminiscent of the description of the unstable D0 branes and closed strings in the $c = 1$ matrix model [119],[142].

The system also exhibit the emergence of geometry from the matrices and their eigenvalues in the boundary gauge theory [32]. We will discuss more in chapter 5 about this for other similar theories. It demonstrates the holography very clearly and have to some extent a sense of background independence (see also [95],[93]). All the geometries are on the same footing, and only the asymptotic boundary of them is fixed. Moreover, the same plane of phase space can give rise to different asymptotic geometries, if we distribute the droplets in different ways. We will discuss other theories with different asymptotic geometries in chapter 4.

This system is also intimately related to quantum Hall problem. If we redefine the Hamiltonian as $H' = H - J = \Delta - J$, then in terms of this new Hamiltonian, These 1/2 BPS states are the ground states of H' and correspond to the lowest Landau level, and J is given by the angular momentum on the Hall plane. It has been further discussed and extended in [79],[55].

The system gives also an interesting description of the plane wave limit. In terms of the droplets this amounts to zooming in on the edge of a droplet. So the plane wave can be thought of as the ground state of the relativistic fermions, where we fill the lower half plane ($x_2 < 0$), which is an infinite Dirac sea. 1/2 BPS excitations above plane-wave correspond to various particles and/or hole excitations. The lightcone energy of the states is the same as the expression of the energy for a relativistic fermion.

Chapter 3

Geometry of 1/2 BPS states in M2 brane and M5 brane theory

3.1 Introduction

In this chapter we make a similar study of the 1/2 BPS sectors of the M2 brane theory (the 2+1d $\mathcal{N} = 8$ superconformal theory) and the M5 brane theory (the 5+1d (0,2) superconformal theory). The gravity duals of the 1/2 BPS chiral primary operators in these theories are the 1/2 BPS geometries asymptote to $AdS_4 \times S^7$ and $AdS_7 \times S^4$ respectively. We solve all the geometries with the required symmetry and supersymmetry and find that they are described by a continuum Toda equation with the boundary conditions on a plane. This plane is analog to the phase space plane in the IIB case in chapter 2, however the droplet densities are not constant. We make further comment on this in Ch 3.3. While in Ch 3.2, we make detailed study of how the solutions are solved, the examples of them, their charges and topologies, and finally the reduction of the Toda equation to a Laplace equation when there is an extra Killing vector.

3.2 Gravity description, Toda equation and droplet picture

3.2.1 The solutions

In this section we analyze the M-theory solutions corresponding to 1/2 BPS geometries asymptotic to $AdS_{4,7} \times S^{7,4}$. They are dual to 1/2 BPS chiral primary operators in the M2 brane theory and the M5 brane theory. These states preserve 16 supercharges with the supersymmetry group $SU(2|4)$, which is the maximal compact subgroup of the symmetry algebras of the M2 brane or M5 brane theory. The bosonic part of the symmetry is $SO(3) \times SO(6) \times R$. Again, R is the translation generator corresponding to $\Delta - J$. The R generator does not leave the spinor invariant, rather the spinor has non-zero energy under this generator. This R generator should leave the geometries invariant.

We now look for supersymmetric solutions of 11D supergravity which have $SO(6) \times SO(3)$ symmetry

$$ds_{11}^2 = e^{2\lambda}\left(4d\Omega_5^2 + e^{2A}d\tilde{\Omega}_2^2 + ds_4^2\right) \tag{3.1}$$

$$G_{(4)} = G_{\mu_1\mu_2\mu_3\mu_4}dx^{\mu_1} \wedge dx^{\mu_2} \wedge dx^{\mu_3} \wedge dx^{\mu_4} + \partial_{\mu_1}B_{\mu_2}dx^{\mu_1} \wedge dx^{\mu_2} \wedge d^2\tilde{\Omega} \tag{3.2}$$

where $d\Omega_5^2$ and $d\tilde{\Omega}_2^2$ are the metrics on unit radius spheres[1] and $\mu_i = 0, \cdots, 3$.

The equations for the four–form field strength are

$$dG_{(4)} = 0, \qquad d(\overset{\star}{}_{11}G_{(4)}) = 0. \tag{3.3}$$

To find supersymmetric configurations we will solve the equation for Killing spinor

$$\nabla_m\eta + \frac{1}{288}\left[\Gamma_m{}^{npqr} - 8\delta_m^n\Gamma^{pqr}\right]G_{npqr}\eta = 0 \tag{3.4}$$

Following [78] we first perform a reduction on S^5 and on S^2 by decomposing the spinor as

$$\eta = \psi(\theta^a) \otimes e^{\lambda/2}[\chi_+(\theta^\alpha) \otimes \epsilon_+ + \chi_-(\theta^\alpha) \otimes \epsilon_-] \tag{3.5}$$

[1]The factor of 4 in front of the five–sphere metric was inserted for later convenience, and it corresponds to setting the parameter m in appendix F of [126] to $m = \frac{1}{2}$.

where $\psi(\theta^a)$ is a spinor on S^5 and $\chi_+(\theta^\alpha)$, $\chi_-(\theta^\alpha)$ are two component spinors on S^2. The Killing spinor equations are then reduced to a set of equations for the four dimensional spinor ϵ_+, ϵ_-. In order to continue constraining the metric we decompose the Killing spinor in terms of a four dimensional Killing spinor and spinors on S^2 and S^5. So we have an effective problem in four dimensions with a four dimensional one-form field B_μ and two scalars A, λ. A closely related problem was analyzed in [78], where general supersymmetric M-theory solutions with $SO(2,4)\times U(1)$ symmetry were considered. Our solutions preserve more supersymmetries, but after a suitable Wick rotation they are particular examples of the general situation considered in [78] so we can use some of their methods.

Using the equations for the field strength, one can show that

$$G_{\mu_1\mu_2\mu_3\mu_4} = I_1 e^{-3\lambda-2A}\epsilon_{\mu_1\mu_2\mu_3\mu_4} \tag{3.6}$$

with constant I_1. In the solutions related to chiral primaries on $AdS\times S$ or pp-waves the S^2 or the S^5 can shrink, at least in the asymptotic regions. These spheres cannot shrink in a non-singular manner if the flux I_1 were non-vanishing. The reason is that the flux density would diverge at the points where the spheres shrink. So from now on we set $I_1 = 0$.

Then the spinor equations for ϵ_+, ϵ_- are simplified and can be written as two decoupled systems, one for $\epsilon_- + \gamma_5\epsilon_+$ and one for $\epsilon_- - \gamma_5\epsilon_+$. We only need to look at one of them, for example, $\epsilon \equiv \epsilon_+$. We use similar method discussed in chapter 2 and construct bilinears out of four dimensional spinor ϵ:

$$f_1 = \bar\epsilon\epsilon, \quad f_2 = \bar\epsilon\Gamma_5\epsilon, \quad K_\mu = -2\bar\epsilon\gamma_\mu\epsilon, \quad L_\mu = 2m\bar\epsilon\gamma_\mu\Gamma_5\epsilon, \quad Y_{\mu\nu} = \bar\epsilon\gamma_{\mu\nu}\epsilon. \tag{3.7}$$

There are also bilinears involving ϵ^t instead of $\bar\epsilon$, we will consider them later. Taking derivatives of the bilinears, we get

$$\nabla_\mu f_1 = 0, \tag{3.8}$$

$$\nabla_\mu f_2 = L_\mu - 3\partial_\mu\lambda f_2 \tag{3.9}$$

$$\nabla_\nu K_\mu = -2mY_{\mu\nu} + \frac{e^{-3\lambda-2A}}{2}F_{\mu\nu}f_2 \tag{3.10}$$

We will not need the expression for $\nabla_\mu L_\nu$ and $\nabla_\mu Y_{\nu\lambda}$. From equation (3.9) we see that

$$L_\mu dx^\mu = e^{-3\lambda} dy \tag{3.11}$$

Equation (3.10) implies that K^μ is a Killing vector, and we will choose the coordinate t along the vector K^μ. The rest are two coordinates labelled by $x_i, i = 1, 2$.

After reducing other equations gradually (for more details see [126]), one can relates every function in the metric or flux to a single function $D(x_i, y)$, which obeys a non-linear differential equation. The final result is:

$$\begin{aligned} ds_{11}^2 &= -4e^{2\lambda}(1+y^2e^{-6\lambda})(dt+V_i dx^i)^2 + \frac{e^{-4\lambda}}{1+y^2e^{-6\lambda}}[dy^2 + e^D(dx_1^2+dx_2^2)] \\ &\quad +4e^{2\lambda}d\Omega_5^2 + y^2e^{-4\lambda}d\tilde{\Omega}_2^2 \end{aligned} \tag{3.12}$$

$$G_{(4)} = F \wedge d^2\tilde{\Omega} \tag{3.13}$$

$$e^{-6\lambda} = \frac{\partial_y D}{y(1-y\partial_y D)} \tag{3.14}$$

$$V_i = \frac{1}{2}\epsilon_{ij}\partial_j D \qquad \text{or} \qquad dV = \frac{1}{2} *_3 [d(\partial_y D) + (\partial_y D)^2 dy] \tag{3.15}$$

$$F = dB_t \wedge (dt+V) + B_t dV + d\hat{B} \tag{3.16}$$

$$B_t = -4y^3e^{-6\lambda} \tag{3.17}$$

$$\begin{aligned} d\hat{B} &= 2 *_3 [(y\partial_y^2 D + y(\partial_y D)^2 - \partial_y D)dy + y\partial_i\partial_y D dx^i] \\ &= 2\tilde{*}_3[y^2\partial_y(\frac{1}{y}\partial_y e^D)dy + y dx^i \partial_i\partial_y D] \end{aligned} \tag{3.18}$$

where $i, j = 1, 2$, and $*_3$ is the epsilon symbol of the three dimensional metric $dy^2 + e^D dx_i^2$, and $\tilde{*}_3$ is the flat space ϵ symbol. The function D which determines the solution obeys the equation

$$(\partial_1^2 + \partial_2^2)D + \partial_y^2 e^D = 0 \tag{3.19}$$

This is the 3 dimensional continuous version of the Toda equation. Note that (3.19) implies that the expression for $d\hat{B}$ in (3.18) is closed. Notice that the form of the ansatz is preserved under y independent conformal transformations of the 12 plane if we shift D appropriately. Namely

$$x_1 + ix_2 \to f(x_1 + ix_2) \ , \qquad D \to D - \log|\partial f|^2 \tag{3.20}$$

Note that the coordinate y is given in terms of the radii of five–sphere and the two–sphere by $y = R_2 R_5^2/4 = e^{2\lambda} e^{\lambda + A}$. This implies that the two–sphere or five–sphere shrinks to zero size at $y = 0$. Let us first understand what happens when the two–sphere shrinks to zero and the five–sphere remains with constant radius. From the condition that λ remains constant as $y \to 0$ we find that e^D is an x dependent constant at $y = 0$ and in addition we find that $\partial_y D = 0$ at $y = 0$. These conditions ensure that the y coordinate combines with the sphere coordinates in a non-singular fashion. We now can consider the case where the five-sphere shrinks. In this case R_2 is a constant, so that $e^{2\lambda} \sim y$. This happens when $D \sim \log y$ as $y \to 0$. In this case we see that the geometry is non-singular. After redefining the coordinate $y = u^2$, we see that the u and 5-sphere components of the metric become locally the metric of R^6. In summary, we have the following two possible boundary conditions at $y = 0$

$$\partial_y D = 0 \ , \quad D = \text{finite} \ , \qquad S^2 \text{ shrinks} \tag{3.21}$$

$$D \sim \log y \ , \qquad S^5 \text{ shrinks} \tag{3.22}$$

Note that the Toda equation (3.19) has also appeared in the related problem of finding four dimensional hyper-Kahler manifolds with a so called "rotational" Killing vector [41] (see also [15]). In fact the expression for the hyper-Kahler manifold in terms of the solution of the Toda equation can arise as a limit of our expression for the four dimensional part of the metric (3.12).

This equation also arises in the large N limit of the 2d Toda theory based on the group $SU(N)$ in the $N \to \infty$ limit, the $SU(N)$ Dynkin diagram becomes a continuous line parameterized by y [14],[157].

3.2.2 Examples

Let us now discuss some examples. These are the M-theory plane-wave, $AdS_{4,7} \times S^{7,4}$, the elliptic droplets asymptotic to $AdS_{4,7} \times S^{7,4}$, and 1/2 BPS extremal one-charge limits of the $AdS_{4,7}$ black holes.

The simplest example is the pp-wave solution. In this case x_1 is an isometry direction. The necessary change of variables is

$$y = \frac{1}{4} r_5^2 r_2 \tag{3.23}$$

$$x_2 = \frac{r_5^2}{4} - \frac{r_2^2}{2} \tag{3.24}$$

$$e^D = \frac{r_5^2}{4} \tag{3.25}$$

where r_5 and r_2 are the radial coordinates in the first six transverse dimensions and the last three transverse dimensions respectively. In (3.25) we could in principle find D in terms of x_2, y, this involves solving a cubic equation. One can check that D defined in this fashion obeys the Toda equation. It is also easy to see that this expression obeys the appropriate boundary conditions for $x_2 > 0$ and $x_2 < 0$ which represent a half filled plane, and corresponds to the Dirac sea.

Another example is given by the $AdS_7 \times S^4$ solution[2] :

$$e^D = \frac{r^2 L^{-6}}{4 + r^2}, \qquad x = (1 + \frac{r^2}{4}) \cos\theta, \quad 4y = L^{-3} r^2 \sin\theta \tag{3.26}$$

Where θ is a usual angle on S^4 and r is the radial coordinate in AdS_7 and L is the inverse radius of S^4. Notice that in this case the solution asymptotes to $D \sim 0$ at large distances. So we expect that any solution with $AdS_7 \times S^4$ asymptotics can be obtained by solving Toda equation (3.19) with the boundary conditions (3.21), (3.22) and $D \sim 0$ at infinity.

We can similarly describe the solution for $AdS_4 \times S^7$:

$$e^D = 4L^{-6} \sqrt{1 + \frac{r^2}{4}} \sin^2\theta, \qquad x = \left(1 + \frac{r^2}{4}\right)^{1/4} \cos\theta, \qquad 2y = L^{-3} r \sin^2\theta \tag{3.27}$$

Here L is the inverse radius of AdS_4. Notice that in both $AdS_{4,7} \times S^{7,4}$ cases we have circular droplets of the M2 brane type or M5 brane type. Note that these two solutions $AdS_{4,7} \times S^{7,4}$ can be related by analytical continuation and conformal transform of the Toda equation.

Unfortunately the Toda equation is not as easy to solve as the Laplace equation, so it is harder to find new solutions. There are a few other known solutions. These include the

[2]In equations (3.26) and (3.27) we use the polar coordinates in x_1, x_2 plane: $ds_2^2 = dx^2 + x^2 d\psi^2$.

smooth 7d and 4d gauged supergravity solutions and two singular solutions of the 1/2 BPS extremal one-charge limit of $AdS_{7,4}$ black hole.

After some work, both the smooth solution of 7d gauged supergravity and the 1/2 BPS extremal one-charge limit of AdS_7 black hole [129],[54] can be written in the following way

$$
\begin{aligned}
e^D &= m^2 r^2 f/\tilde{F}^2, && (3.28)\\
x_2 + i x_1 &= \left(e^{-\rho}\cos\phi + i e^{\rho}\sin\phi\right)\tilde{F}\cos\theta && (3.29)\\
y &= m^2 r^2 \sin\theta && (3.30)
\end{aligned}
$$

where

$$
\begin{aligned}
\cosh 2\rho &= F', \qquad f = 1 + \frac{F}{2\sqrt{x}}, \qquad x \equiv 4m^4 r^4 && (3.31)\\
\partial_r \tilde{F}(r) &= \frac{2m^2 r \tilde{F}(r)}{f}\cosh 2\rho && (3.32)
\end{aligned}
$$

and every function above is determined by the single function $F(x)$, which satisfy the non-linear differential equation

$$(2\sqrt{x} + F)F'' = 1 - (F')^2 \tag{3.33}$$

where the prime is with respect to $x \geqslant 0$. The boundary condition is $F'|_{x=0} = C \geqslant 1$, where C is a parameter appears in the gravity solution. Every solution is uniquely determined by specifying the value of C.

The regular 7d gauged supergravity solution corresponding to a droplet of elliptical shape of the M5 brane type is given by a nontrivial solution of F with

$$C > 1 \tag{3.34}$$

and $\tilde{F}$ can be simplified to be $\tilde{F} = \sqrt{\frac{c}{\sinh 2\rho}}$, where $c = \sinh 2\rho(r)|_{r=0}$. $C > 1$ corresponds to turning on the off-diagonal charged scalar ρ in the gauged supergravity which sources the solution and make it regular.

On the other hand, the singular 1/2 BPS extremal one-charge limit of AdS_7 black hole is the solution given by $C = 1$ and $F = x + Q$. It's singular because the scalar ρ is not turned on, it can be viewed as a singular distribution of ρ at the singularity.

Similarly, both the smooth solution of 4d gauged supergravity [50] and the 1/2 BPS extremal one-charge limit of AdS_4 black hole (for example in [69]) can be written in the way

$$e^D = 4f\sin^2\theta/\tilde{F}^2 \tag{3.35}$$

$$x_2 + ix_1 = (e^{-\varphi}\cos\phi + ie^{\varphi}\sin\phi)\tilde{F}\,\cos\theta \tag{3.36}$$

$$y = z\sin^2\theta \tag{3.37}$$

where

$$\cosh 2\varphi = F',\ f = 1 + zF,\quad \partial_z\tilde{F} = \frac{z\tilde{F}}{2(1+zF)}\cosh 2\varphi \tag{3.38}$$

and every function above is determined by the single function $F(z)$, which satisfy another non-linear differential equation

$$(z^{-1} + F)F'' = 1 - (F')^2 \tag{3.39}$$

where the prime is with respect to $z \geqslant 0$. The boundary condition is $F'|_{z=0} = \widetilde{C} \geqslant 1$, where $\widetilde{C}$ is a parameter appears in the gravity solution. Again, every solution is uniquely determined by specifying the value of $\widetilde{C}$.

The regular 4d gauged supergravity solution corresponding to an elliptical droplet of the M2 brane type is given by a nontrivial solution of F with

$$\widetilde{C} > 1 \tag{3.40}$$

and $\tilde{F}$ can be simplified to be $\tilde{F} = \sqrt{\frac{c}{\sinh 2\varphi}}$, where $c = \sinh 2\varphi(z)|_{z=0}$. Similarly, $\widetilde{C} > 1$ corresponds to turning on the off-diagonal charged scalar φ which sources the solution and make it regular.

On the other hand, the singular 1/2 BPS extremal one-charge limit of AdS_7 black hole is the given by solution $\widetilde{C} = 1$ and $F = z + 2Q$. It's singular because the scalar φ is not turned on, and it can be viewed as a singular distribution of φ at the singularity.

The examples of the two singular solutions obey our equations but not the regular boundary conditions.

3.2.3 Charges and topology

Let's discuss the topology and charge quantization of these solutions.

We can also separate the 12 plane into droplets where we have one or the other boundary condition above. We can now consider four cycles obtained by fibering the two–sphere over a two-surface Σ_2 on the y, x_1, x_2 space which ends at $y = 0$ in a region where the S^2 shrinks, see figure 2.3. This is a non-singular four-cycle[3] . Since $B_t V$ is a globally well defined vector field, we find that the flux of the four form over this four cycle is given by computing the integral

$$N_5 \sim -\frac{1}{\mathrm{vol}_{S^2}} \int_{\Sigma_4} G_{(4)} = -\int_{\Sigma_2} d\hat{B} = \int_{\mathcal{D}} dx_1 dx_2 2(y^{-1} e^D)|_{y=0} \tag{3.41}$$

where $\mathcal{D}$ is the region in the x_1, x_2 plane with the S^5 shrinking boundary condition, (3.22), which lies inside the surface Σ_2. So the area of this region measures the number of 5-branes in this region.

We can similarly measure the number of two branes by considering the flux of electric field. Namely we consider now a seven cycle which is given by fibering the five–sphere over a two surface Σ_2' which ends on the $y = 0$ in a region where the five–sphere shrinks. Then the electric flux is given by

$$\begin{aligned} N_2 &\sim \frac{1}{\mathrm{vol}_{S^5}} \int_{\Sigma_7} *_{11} G_{(4)} = \int_{\Sigma_2'} [\Phi dV + g_0^{-1} e^{3\lambda - 2A} *_3 dB_t] \\ &= \int_{\mathcal{D}} 2\tilde{*}_3 [y^3 \partial_y^2 (y^{-1} e^D) dy + y^2 \partial_i \partial_y D dx^i] = \int_{\mathcal{D}} dx_1 dx_2 \, 2e^D|_{y=0} \end{aligned} \tag{3.42}$$

where $\tilde{*}_3$ is the flat space ϵ symbol and $\mathcal{D}$ is the region in the 12 plane where the S^2 shrinks which is inside the original Σ_2' surface. This integral counts the number of two branes. If the five–branes were fermions the two-branes are holes. The equation (3.19) implies that the two form we are integrating in (3.42) is closed.

Notice that in both cases the fluxes are given by the area measured with a metric constructed form D. So we first have to solve the Toda equation, (3.19), find D, and only then can we know the number of $M2$ and $M5$ branes associated to the droplets. Note also that any two droplets which differ by a conformal transformation seem to give us the

[3]This four cycle has the topology of a sphere S^4 if Σ_2 is topologically a disk ending at $y = 0$.

same answer. In fact, if we consider circular droplets of different sizes, then a conformal transformation would map them all into a circular droplet of a specific size. The point is that the boundary conditions (3.21), (3.22) do not fix the solution uniquely. Given a solution $D(x_i, y)$, the function $D(x_i, y\lambda) - 2\log\lambda$ is also a solution with the same boundary conditions. We see from (3.41), (3.42) that this change rescales the charges. We expect that this is the only freedom left in determining the solution, but we did not prove this. In other words, we expect that the solution is completely determined by specifying the shapes of the droplets. As opposed to the IIB case, we do not know the correspondence between the precise shape of the droplets in phase space and the shape of the droplets in the $y = 0$ plane[4] . But we expect that their topologies are the same.

3.2.4 Extra Killing vector and reduction to Laplace equation

In the case of solutions with an extra Killing vector we can reduce the problem to a Laplace equation using [178]. Let us consider a translational Killing vector. In the case that the solution is independent of x_1 the equation (3.19) reduces to the two dimensional Toda equation

$$\partial_2^2 D + \partial_y^2 e^D = 0 \tag{3.43}$$

This equation can be transformed to a Laplace equation by the change of variables

$$e^D = \rho^2 \ , \qquad y = \rho\partial_\rho V \ , \qquad x_2 = \partial_\eta V \tag{3.44}$$

Then the equation (3.43) becomes the cylindrically symmetric Laplace equation in three dimensions

$$\frac{1}{\rho}\partial_\rho(\rho\partial_\rho V) + \partial_\eta^2 V = 0 \tag{3.45}$$

The boundary conditions will be discussed in chapter 5.

The pp-wave solution (3.25) can be expressed as

$$V = \rho^2\eta - \frac{2}{3}\eta^3 \tag{3.46}$$

[4]In particular, notice that the densities, given by (3.41) and (3.42), are not constant.

Note that only the region $\eta > 0$ is explicitly seen in the gravity solution. In fact, at $y = 0$, half of the x_2 line is mapped to $\rho = 0$ and the other half to $\eta = 0$. As we consider other solutions of the Laplace equation we expect to find more complicated boundaries, which will be discussed in chapter 5. By solving this equation one can obtain solutions with pp-wave asymptotics that represent particles with nonzero $-p_-$, which are translationally invariant along x^- (this can happen at the level of classical solutions).

One can then compactify x^- and reduce to type IIA. In this way we obtain non-singular geometries that are the gravity duals of the plane-wave matrix model [33]. These solutions were explored in [125] in the Polchinski-Strassler approximation. In chapter 5 we obtain non-singular solutions corresponding to interesting vacua of the plane-wave matrix model. The Young diagrams representing different vacua of the plane-wave matrix model are directly mapped to strips in $y = 0$ plane, just like in the case of M2 brane theory with mass deformation (see figure 5.7). In particular, our solutions make it clear that we have configurations that correspond to D0 branes that grow into NS5 (or M5) branes, as discussed in [137]. Such a solution would come from boundary conditions on the $y = 0$ plane for the Toda equation as displayed in figure 4.2(a). The solution where the D0 branes grow into D2 branes on S^2 is then related to a boundary condition of the form shown in figure 4.2(b). Note that, despite appearances, the topology of these two solutions would be the same. In the type IIA language both solutions would contain a non-contractible S^3 and a non-contractible S^6. These spheres are constructed from the arcs displayed in figure 4.2, together with either S^2 or S^5.

Here we considered plane wave excitations with $p_+ = 0$ and $p_- \neq 0$. Solutions that correspond to a plane wave plus particles with $p_- = 0$ but $p_+ \neq 0$ were discussed in [99], together with their matrix model interpretation.

It is also possible to use this trick, (3.44) when we consider solutions that are rotationally symmetric in the x_1, x_2 plane. The reason is that the plane and the cylinder can be mapped into each other by a conformal transformation, and conformal transformations are a symmetry of the Toda equation (3.19). More explicitly, if we write two dimensional metric

as $dr^2 + r^2 d\phi^2$ and look for solutions which do not depend on ϕ, then three dimensional Toda equation reduces to (3.43) with following replacement: $x_2 \to \log r$, $D \to D + 2\log r$.

3.3 Discussion

We have studied geometries of 1/2 BPS states asymptotic to $AdS_7 \times S^4$ or $AdS_4 \times S^7$ with $SU(2|4)$ symmetry. These are dual to the 1/2 BPS chiral primaries of the M5 brane theory and M2 brane theory. The chiral primaries of the M5 brane theory or (0, 2) SCFT can also be described in terms of Young tableaux with at most N rows [3], as in four dimensional $\mathcal{N} = 4$ SYM. They are in symmetric traceless representations of the $SO(8)$ R-symmetry. Similarly the chiral primaries of the M2 brane theory or the 2+1d $\mathcal{N} = 8$ superconformal field theory are in the symmetric traceless representations of $SO(5)$ R-symmetry with up to N indices. So in terms of labelling of states these are similar to free fermions on a plane.

The situation is qualitatively similar to that of $\mathcal{N} = 4$ SYM. We can consider these conformal theories on $R \times S^5$ or $R \times S^2$, and study the zero mode of a complex scalar out of the 5 or 8 scalars in the M5 brane or M2 brane theory. We expect to have a sort of two dimensional phase space for the underlying dynamics reduced on the spheres. However we do not have the exact dynamics of these theories.

In the gravity side we see a similar x_1, x_2 plane, but the flux density is not constant on this plane. The solutions are governed by a non-linear Toda equation with particular boundary conditions on the x_1, x_2 plane. Nevertheless, the full solutions and their topologies are determined by specifying the shapes of the boundaries between the two regions.

In the weak string coupling limit, their dynamics go over to the NS5 brane theory on $R \times S^5$ or the D2 brane theory on $R \times S^2$ and we can reduce the geometries to the ten dimensional IIA string theory, and these theories and their dualities will be studied in detail in chapter 5.

Chapter 4

Geometry of BPS vacua in field theories with $SO(4) \times SO(4) \times U(1)$ symmetry

4.1 Introduction

In this chapter, we study a class of supersymmetric theories with 16 supercharges, and with a bosonic $U(1) \times SO(4) \times SO(4)$ symmetry. These theories arise when we consider the U daulity of the type IIB solutions in Ch 2. Specifically, we start from the general solutions in type IIB case in Ch 2, and select the configurations that are invariant under x_1, that is, we have an extra Killing vector. We then make a T duality along the x_1, we get solutions in IIA, and we can also uplift the solutions from IIA to M theory. The resulting theories come in three types, corresponding to three different ways of distributing the droplets or more precisely the strips, see figure 4.1. If we fill completely the lower half plane and consider finite strips above it, these are dual to vacua of mass deformed M2 brane theory. If we only have finite strips on the plane, then the vacua correspond to D4 brane theory. Finally, if we distribute the droplets or strips periodically, after two T dualities on both x_1, x_2 and a S

duality, we find that these solutions correspond to vacua of intersecting NS5 brane theory, where two sets of NS5 branes intersect on $R^{1,1}$. In Ch 4.2, we discuss the mass deformed M2 brane theory. In Ch 4.3 we discuss the D4 brane theory. Then we discuss the intersecting NS5 brane theory in Ch 4.4. Finally we study the unusual Poincare supersymmetry algebras in 2+1 or 1+1d worldvolume theorie, such as the mass deformed M2 brane theory.

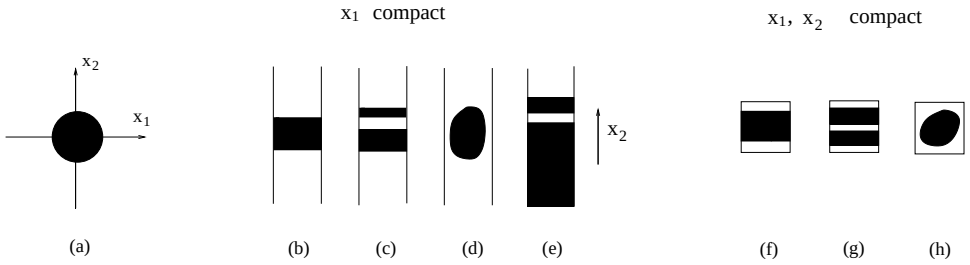

Figure 4.1: In (a) we see a circular droplet in the uncompactified x_1, x_2 plane which corresponds to the vacuum of $\mathcal{N} = 4$ super Yang Mills. In (b,c,d) we show different vacua in the case that we compactify the x_1 coordinate. This "uplifts" $\mathcal{N} = 4$ super Yang Mills to a 4+1 dimensional gauge theory, or more precisely to the $(0, 2)$ six dimensional field theory that lives on $M5$ branes. Figure (d) shows the limit to the $M5$ brane theory when the x_1 dependence recovers. If we compactify also x_2, as in (f,g,h) we get a little string theory whose low energy limit is a Chern Simons theory. If the sizes of x_1 and x_2 are finite, we get the theory on $R \times T^2$ and figures (f,g,h) show different vacua. As we take both sizes to zero, we obtain the theory on $R^{2,1}$. The configuration in (e) corresponds to a vacuum of the theory of M2 branes with a mass deformation.

4.2 DLCQ of IIB plane-wave string theory or mass deformed M2 brane theory

In this section we consider geometries that are dual to the M2 brane theory with a mass deformation [161, 29]. Starting with the usual theory on coincident M2 branes with 32 supercharges, it is possible to introduce a mass deformation that preserves 16 supercharges. This deformation preserves an $SO(4) \times SO(4)$ subgroup of the $SO(8)$ R-symmetry group of the conformal M2 brane theory. One interesting aspect of this theory is that its features are rather similar to those of $\mathcal{N} = 4$ SYM with a mass deformation. Namely, the mass

deformed M2 brane theory also has vacua that are given by dielectric branes [150]. In this case these are M5 branes that are wrapping a 3-sphere in the first four of the eight transverse coordinates or a 3-sphere in the last four of the eight transverse coordinates.

The M2 brane theory with M2 brane number N via IIB/M theory duality becomes DLCQ of IIB string theory in the sector of N units of light-cone momenta [23],[167]. In our case, the mass deformed M2 brane theory via the IIB/M duality is dual to DLCQ of IIB plane-wave string theory. In the latter, the polarized M5 branes wrapping either S^3 are mapped to polarized D3 branes wrapping the same sphere. In order to go from IIB theory to M theory, we first periodically identify along the lightlike Killing direction, $x_1 = x^- \sim x^- + 2\pi R$ in (2.27). The sector with N units of momentum $-p_- = N/R$ is given by the mass deformed M2 brane theory on a torus[1] with N M2 branes, after we T-dualize the x_1 circle and uplift to 11 dimensions.

The vacua of the mass deformed M2 theory is there characterized by the have a cylinder in the x_1, x_2 plane is shown in figure 4.1 (e). In this case we fill the lower half of the cylinder. We get this theory in $R^{2,1}$ after setting the radius of x_1 to zero and taking the strong coupling limit (and doing the obvious U-duality transformations). If the size of x_1 and the string coupling are finite, then we get the theory on $R \times T^2$.

We can obtain these solutions dual to the vacua of mass deformed M2 brane theory by U-dualizing the IIB solutions describing excitations above plane-wave geometry in chapter 1. The explicit form of the solutions is

$$\begin{aligned} ds_{11}^2 &= e^{\frac{4\Phi}{3}}(-dt^2 + dw_1^2 + dw_2^2) \\ &\quad + e^{\frac{-2\Phi}{3}}\left[h^2(dy^2 + dx_2^2) + ye^G d\Omega_3^2 + ye^{-G} d\tilde{\Omega}_3^2\right] \qquad (4.1) \\ e^{2\Phi} &= \frac{1}{h^2 - h^{-2}V_1^2} \qquad (4.2) \\ F_4 &= -d(e^{2\Phi}h^{-2}V_1) \wedge dt \wedge dw_1 \wedge dw_2 \\ &\quad - \frac{1}{4}e^{-2\Phi}[e^{-3G} *_2 d(y^2 e^{2G}) \wedge d\tilde{\Omega}_3 + e^{3G} *_2 d(y^2 e^{-2G}) \wedge d\Omega_3] \qquad (4.3) \end{aligned}$$

where $*_2$ is the flat epsilon symbol in the coordinates y, x_2 and h, G are given by the expres-

[1] There has been another proposal for the DLCQ limit of this theory in [168], which involves a rather different theory.

sions we had above (2.13)–(2.15), (2.19). These functions are determined by considering boundary conditions corresponding to strips that are translation invariant along x_1, see figure 2.2(e) and equations (2.25). Note that since we had translation symmetry along x_1 in the original IIB solution, only the component V_1 is nonzero. The coordinate x_1 does not appear in this M-theory solution because it was U-dualized. So z, V_1 are given by superpositions of solutions of the form (2.25). In other words

$$z(x_2, y) = \sum_i (-1)^{i+1} z^{pp}(x_2 - x_2^i, y), \qquad V_1(x_2, y) = \sum_i (-1)^{i+1} V_1^{pp}(x_2 - x_2^i, y) \tag{4.4}$$

where z^{pp}, V_1^{pp} are the functions in (2.25), and x_2^i is the position of the ith boundary starting from the bottom of the Fermi sea[2] .

The authors of [29] managed to reduce the problem to finding a solution of a harmonic equation, and their solutions and ours are related by change of variables. Our parametrization of the ansatz has the advantage that it is very simple to select out the non-singular solutions.

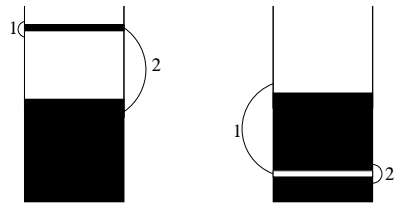

Figure 4.2: We see the configurations corresponding to two of the vacua of the mass deformed M2 brane theory. The vacuum in (a) can be viewed as dielectric M5 branes wrapping the S^3 in the first four coordinates of the eight transverse coordinates. The configuration in (b) corresponds to a vacuum with M5 branes wrapping the S^3 in the second four coordinates. The two geometries have the same topology. Consider arcs in the x_2, y plane that enclose the fermions or the holes and end at $y = 0$. We can construct four spheres by taking one of these arcs and tensoring the S^3 that shrinks to zero at the tip of the arcs. The two different S^3 are denoted in the figure by indices 1 and 2. The flux of F_4 over these four spheres is equal to the number of particles or holes enclosed by the arcs. Note that the horizontal line in this figure does not correspond to a coordinate in the final M-theory geometry.

[2]For odd i the boundary changes from black to white while for even i the boundary changes from white to black. See figure 2.2 (e).

One of the simplest ways to count these vacua is to recall yet another description of this DLCQ theory in terms of a limit of a gauge theory in [149]. According to the description in [149] the vacua are given in terms of chiral primary operators of a particular large N limit of an orbifold theory. It is a simple matter to count those and notice that they are equivalent to partitions of N. This is of course related in a simple manner to the fermion fluid picture for the pp wave. Once we compactify x^- we have fermions on a cylinder, where we fill half the cylinder. The asymptotic conditions automatically imply that we are only interested in states with zero $U(1)$ charge. The $U(1)$ charge is related to the position of the Fermi level. We always choose it such that the total number of particles and holes is zero. The energy of the fermions is the same as the number N of M2 branes. These are relativistic fermions which can be bosonized and the number of states with energy $E = N$ is indeed given by the partitions of N. States which contain highly energetic holes or particles, as shown in figure 4.2, correspond to M5 branes wrapping one or the other S^3. Configurations in between are better thought of as smooth geometries with fluxes.

An interesting fact is that the geometry corresponding to a highly energetic fermion, as in figure 4.2(a), and the geometry corresponding to a highly energetic hole, as in figure 4.2 (b), are topologically the same. The reason is that the geometry contains two distinct S^4s through which we have a non-vanishing flux. Consider for example the configuration in figure 4.2 (a), which can be interpreted as M5 branes wrapping one of the S^3s. One S^4 is the obvious one that is transverse to these branes. The other S^4 arises in an interesting way. Consider the three–sphere that these branes are wrapping. At the center of the space, where one normally imagines the $M2$ branes, this three–sphere is contractible. As we start going radially outwards we encounter the M5 branes, the backreaction of the branes on the geometry will make the S^3 on their worldvolumes contractible. So the end result is that the S^3 contracts to zero on both end points of the interval that goes between the origin and the branes. This produces another S^4. Through this S^4 we have a large flux, which we might choose to view as part of the background flux, the flux that was there before we put in the M2 branes, the flux which is responsible for the mass term on the M2 brane theory.

A configuration with highly energetic holes corresponds to M5 branes wrapping the second S^3. This is topologically the same as the configuration with highly energetic fermions. In other words, the two configurations in figure 4.2 have the same topology. They only differ in the amount of four form flux over the two S^4s.

There is a precise duality under the interchange of the two three–spheres, which maps solutions into each other. Some special solutions will be invariant under the duality. This is particle hole duality in the fermion picture.

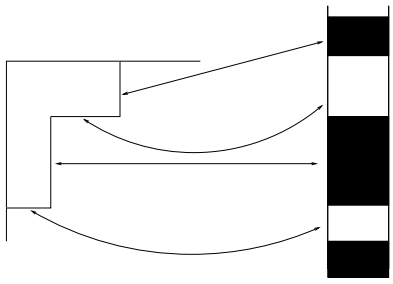

Figure 4.3: Correspondence between the Young diagrams and the states of free fermions. We start from the bottom left of the Young diagram, each time we move right by n boxes we add n holes and each time we move up by n boxes we add n fermions. The energy of the configuration is equal to the total number of boxes of the Young diagram.

These solutions can be related to Young diagrams in a simple way which is pictorially represented in figure 5.7. We start at the bottom of the Young diagram and we move along the boundary. Each time we move up we add as many fermions as boxes, each time we move right we add holes. The Fermi level is set so that the total number of holes is equal to the total number of fermions. Then the energy of the fermion system is equal to the number of boxes, and in our case this is the number of M2 branes. Of course, small curvature solutions are only those where the Young diagram has a small number of corners and a large number of boxes. This is in contrast to the situation encountered in other cases [67, 155] where smooth Young diagrams correspond to smooth macroscopic configurations. In our case, a Young diagram which contains edges separated by few boxes leads to solutions with Planck scale curvature.

In the $\mathcal{N} = 1^*$ theory [177] considered in [159] we expect a similar situation, where geometries will be non-singular but could have large curvatures when some of the fluxes become small.

4.3 D4 brane theory or M5 brane theory on S^3

We now consider D4 brane theory on $R^{1,1} \times S^3$or M5 brane theory on $R^{2,1} \times S^3$. These theories preserve 16 supercharges. These theories can be obtained from the limit of the configurations corresponding to the vacua of mass-deformed M2 brane theory, which describes M5 branes. In the gravity side we can expand around a single strip or several strips, the resulting asymptotic geometry corresponds to a set of $M5$ branes wrapped on $S^3 \times R^{2,1}$.

For example, we can consider a single isolated strip of fermions, as in figure 4.4. If we compactify the x_1 coordinate then the fermion configuration is the same as the one we have in two dimensional QCD on a cylinder. In fact, the dual field theory configuration for a single isolated strip (or single collection of strips) is N M5 branes wrapped on $S^3 \times T^2 \times R$, where N is given by the area of the strip. We can then reduce it to super Yang Mills theory on $S^3 \times S^1 \times R$. The reduction on S^3 leaves us with a gauge theory in two dimensions, which has BPS vacua that are in correspondence with the states of 2d Yang-Mills theory on a circle [144],[143],[65],[66],[147][3] . The role of this 2d Yang-Mills theory characterizing the vacua of D4 brane theory, is very analogous to that of the matrix quantum mechanics for the $N = 4$ SYM discussed in chapter 2. Of course we can decompactify the T^2 or S^1 part of the worldvolumes. When the M5 brane theory is on $R^{1,1} \times S^1 \times S^3$ or $R^{2,1} \times S^3$, the size of x_1 should be taken to zero dual to T duality and the solutions correspond to those in figure 4.1(b,c).

Here we analyze the gravity solutions corresponding to a single strip or several strips as in figure 4.4. In order to characterize the solution we need to give the numbers a_j, b_j which obey $a_j < b_j < a_{j+1} \cdots$. These numbers are the values of x_2 at the boundaries of the black

[3]Recently an interesting connection between 2d Yang-Mills theory and topological strings was proposed in [176].

strips, see figure 4.1(b,c). We have a black strip between a_j and b_j. Then the solution is given by

$$2z = -1 + \sum_j \frac{x-a_j}{\sqrt{(x-a_j)^2+y^2}} - \frac{x-b_j}{\sqrt{(x-b_j)^2+y^2}} \tag{4.5}$$

$$2yV_1 = \sum_j \frac{y}{\sqrt{(x-a_j)^2+y^2}} - \frac{y}{\sqrt{(x-b_j)^2+y^2}} \tag{4.6}$$

$$2z + i2yV_1 = -1 + \sum_j (w_j - z_j) \tag{4.7}$$

$$w_j = \frac{x-a_j+iy}{\sqrt{(x-a_j)^2+y^2}}\,, \qquad z_j = \frac{x-b_j+iy}{\sqrt{(x-b_j)^2+y^2}} \tag{4.8}$$

We see that the complex numbers w_j and z_j lie on the unit circle in the upper half plane.

The ten dimensional solution is

$$ds^2_{IIA} = e^{2\Phi}(-dt^2 + d\tilde{x}_1^2) + \frac{\sqrt{1-4z^2}}{2y}(dy^2 + dx_2^2) + y\sqrt{\frac{1+2z}{1-2z}}d\Omega_3^2 + y\sqrt{\frac{1-2z}{1+2z}}d\tilde{\Omega}_3^2$$

$$e^{-2\Phi} = \frac{1-4z^2-4y^2V_1^2}{2y\sqrt{1-4z^2}} \tag{4.9}$$

$$F_4 = -\frac{e^{-2\Phi}}{4}\left[\frac{(1-2z)^{3/2}}{(1+2z)^{3/2}} *_2 d\left(y^2\frac{1+2z}{1-2z}\right) d\tilde{\Omega}_3 + \frac{(1+2z)^{3/2}}{(1-2z)^{3/2}} *_2 d\left(y^2\frac{1-2z}{1+2z}\right) d\Omega_3\right]$$

$$B_2 = -\frac{4y^2V_1}{1-4z^2-4y^2V_1^2} dt \wedge d\tilde{x}_1 \tag{4.10}$$

where $*_2$ is a flat 2D epsilon symbol. Note that g_{00} is determined in terms of the dilaton. This is related to the fact that the eleven dimensional lift of this solution is lorentz invariant in $2+1$ dimensions.

The M-theory form of the solutions is as in (4.1)-(4.3) with z and V given by (4.5). Here we just give the form of the solution in IIA notation. This solution is a simple U-dualization of (4.1)–(4.3).

We briefly analyze the regularity property of these D4 brane solutions. We will now show that $e^{-2\Phi}$ remains finite and non-zero in the IR region. Of course, in the UV region $\Phi \to \infty$ and we need to go to the eleven dimensional description. Note that away from $y = 0$ the denominator in (4.9) is non-zero. The fact that the numerator is nonzero follows from the representation (4.7) and the fact that w_j, z_j in (4.5) are ordered points on the unit circle on the upper half plane, so the norm $|2z + i2yV_1| < 1$. As we take the $y \to 0$ limit

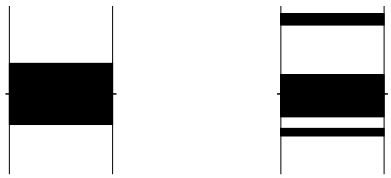

Figure 4.4: We see fermion configurations corresponding to a single isolated strip, or set of strips. These fermions are the same as the ones that appear in 2d QCD on a cylinder, or $SU(N)$ group quantum mechanics. In (a) we display the ground state and in (b) we display an excited state. From the point of view of D4 brane on $S^3 \times S^1 \times R$ these are all supersymmetric ground states.

we see that both the numerator and denominator in (4.9) vanish. We can then expand in powers of y and check that indeed we get a finite, non-zero result, both for $a_j < x_2 < b_j$ and $x_2 = a_j,\ b_j$.

In order to have a look at the asymptotics, we first study the solution given by a single strip. We take a source in a form of a strip

$$\tilde{z}(y=0) = -\theta(x)\theta(1-x) \tag{4.11}$$

Using the general solution above, in the leading order in the large x, y region we find

$$\tilde{z} = -\frac{y^2}{2(x^2+y^2)^{3/2}}, \quad V = -\frac{x}{2(x^2+y^2)^{3/2}}, \quad h^{-2} = \sqrt{2}(x^2+y^2)^{3/4} \tag{4.12}$$

Then introducing polar coordinates in the x, y plane, we find an asymptotic form of the metric

$$ds^2_{IIA} = \sqrt{2}r^{3/2}\left[-dt^2 + dw^2 + d\Omega_3^2\right] + \frac{1}{\sqrt{2}r^{3/2}}\left[dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\tilde{\Omega}_3^2\right] \tag{4.13}$$

which asymptotes at large $r = \sqrt{x^2+y^2}$ to the metric of the D4 brane solution, or M5 brane solution when uplifted to 11 dimensions.

Now we look at the superposition of a number of isolated strips. We find it convenient to study the metric of M5 brane solutions uplifted from the D4 brane solution corresponding to several isolated strips. It is interesting that the first few terms in the large r expansion

are described by 7d gauged supergravity [4] :

$$\begin{aligned} ds_{11}^2 &= \left(\frac{2}{P_0}\right)^{1/3} \Delta^{1/3} \left[\left(r+\frac{P_0}{2}+\frac{P_0^2}{10r}\right)(-dt^2+d\mathbf{w}^2)+rd\Omega_3^2 \right. \\ &\qquad \left. +P_0\frac{dr^2}{2r^2}\left(1-\frac{9P_0}{10r}+\frac{39P_0^2}{100r^2}\right)\right] \\ &+ \left(\frac{P_0}{2}\right)^{2/3} \Delta^{-2/3}(T^{-1})_{IJ}dY^I dY^J \end{aligned} \tag{4.14}$$

Here we parameterized a deformed S^4 by a five dimensional unit vector Y_I ($Y_I Y_I = 1$) such that $Y_5 = \cos\theta$ and $Y_i = \sin\theta\mu_i$, where four dimensional unit vector μ_i parameterizes $\tilde{S}^3$. The matrix T_{IJ} has the form

$$T_{IJ} = \mathrm{diag}(T,T,T,T,T^{-4}), \quad T = 1-\frac{P_0}{10r}+\frac{P_0^4-15P_1^2+20P_0P_2}{100r^2P_0^2}$$

and

$$\begin{aligned} \Delta \equiv Y_I T_{IJ} Y_J &= 1+\frac{(3+5\cos 2\theta)P_0}{20r}+\frac{2\cos^2\theta}{r^2}\left(\frac{P_0^2}{20}-\frac{P_0^4-15P_1^2+20P_0P_2}{50P_0^2}\right) \\ &+ \frac{P_0^4-15P_1^2+20P_0P_2}{100P_0^2r^2}\sin^2\theta \end{aligned} \tag{4.15}$$

The solution is specified by the moments of the distribution:

$$P_n \equiv (n+1)\int_{\mathcal{D}} dx x^n \tag{4.16}$$

By making a shift in coordinate x, we can go to the frame where $P_1 = 0$. In the next order in $1/r$ expansion a generic metric is not described by the ansatz of the gauged supergravity, however if the charges of the solution satisfy the relation $P_3 = -\frac{P_1^3-2P_0P_1P_2}{P_0^2}$, then even in the next order in $1/r$ we excite only fields from the 7d gauged supergravity. So the leading moments of the distribution of the strips are P_0 are P_2. P_0 is the total width of the strips are thereby corresponds to the total number N of the M5 branes (or D4 branes), while P_2 is the second moment of the distribution corresponding to the energy of the solution.

If we start from the M5 brane theory and compactify one of the dimensions, this becomes, at low energies, a 4+1 super Yang-Mills theory on an $S^3 \times R^{1,1}$ where four of the five

[4] For completeness we also give the relation between coordinates x_2, y in (4.1) and r, θ which we use here: $x_2 = r'\cos\theta'$, $y = r'\sin\theta'$, $r' = r + \left(\frac{3P_0}{10}+\frac{P_1\cos\theta}{2P_0}\right) + \frac{3P_0^4-5P_1^2+15P_0P_2-25\cos 2\theta(P_1^2-P_0P_2)}{100rP_0^2}$,
$\theta' = \theta - \frac{P_1\sin\theta}{2rP_0} + \frac{6P_0^2P_1+5\cos\theta(5P_1^2-4P_0P_2)\sin\theta}{40r^2P_0^2}$

transverse scalars have a mass given by the inverse radius of the sphere and the fifth scalar, call it Y, does not have a mass term but it has a coupling of the form $Tr(YF_{01})$ where 01 are the directions in $R^{1,1}$. The number of D4 branes is the total width of the strip (or strips).

Another way to understand the D4 brane theory is that, this 4+1 SYM can also be obtained by compactifying one of the scalars in 3+1 SYM. We consider one of the complex transverse scalars of $\mathcal{N} = 4$ super Yang Mills. When the Yang Mills theory is on $R \times S^3$ the lagrangian contains a term of the form $-\frac{1}{2}(|DZ|^2+|Z|^2)$. We can now write $Z = e^{it}(Y+iX)$. Then the lagrangian becomes

$$\int Tr[-\frac{1}{2}(DX)^2 - \frac{1}{2}(DY)^2 - 2YD_0X] \tag{4.17}$$

We now see that the problem is translational invariant in X. Actually, the problem looks like a particle in a magnetic field.[5] Note that the Hamiltonian associated to this Lagrangian is equal to $H' = H - J$ where H is the original Hamiltonian which is conjugate to translations in the time direction and J is the generator of $SO(6)$ that rotates the field Z. If one compactifies the direction X, using the procedure in [174], we get the five dimensional gauge theory living on D4 branes. This description of the theory is appropriate at weak coupling or long distances on the D4 branes. These theories preserve 16 supercharges. The process of compactifying the coordinate X broke the 32 supersymmetries to 16.

Let us consider the D4 theory on $R \times S^1 \times S^3$. This theory has a large number of supersymmetric vacua. The structure of these vacua is captured by the $1 + 1$ dimensional lagrangian

$$\int Tr[-\frac{1}{4}F^2 - \frac{1}{2}(DY)^2 - YF] \tag{4.18}$$

The space of vacua is the same as the Hilbert space of 2d Yang Mills on a cylinder [126]. All these vacua have zero energy. At first sight we might expect the theory on $R^{1,1} \times S^3$ to have a continuum family of vacua related to possible expectation values for Y. Note, however, that the electric field is given by $E_1 = \frac{\partial L}{\partial \dot{A}_1} \sim F_{01} + 2Y$. For zero energy configurations

[5] Note that the (x_1, x_2) coordinates appearing in the gravity solution correspond to the coordinates (X, Y) in the field theory.

$F_{01} = 0$. So the quantization condition for the electric field will quantize the values of Y. This is good, since, as we explain in the section 4.5 the supersymmetry algebra does not allow massless particles. In fact, the spectrum of states around each of these vacua has a mass gap. We have shown above that the dilaton Φ, as well as the warp factor are bounded in the IR region for any droplet configuration of this type. They never go to zero and the solution is everywhere regular. This is related to the fact that the dual field theory has a mass gap. These vacuum states are characterized by the value of $Tr[E_1^2] \sim Tr[Y^2]$, which in the fermion picture corresponds to the energy of non-relativistic fermions

$$E_{NR} = \int_{Strips} dx \frac{1}{2} x^2 \tag{4.19}$$

In the gravity picture this quantity appears as the leading (angular dependent) deviation from the metric we described above. It is a quantity similar to a dipole moment. This quantity is well defined for the BPS solutions we are considering. It would be nice to know if there is a quantity that is conserved, and it is defined in the full interacting theory, which would reduce to (4.19) on the supersymmetric ground states.

4.4 2+1 Little string theory or intersecting M5 brane theory

Let us discuss the situation when the x_1, x_2 plane is compactified into a two torus, as in figure 4.1(f,g,h). We have a 2 dimensional array of periodic droplets[6] . Let us start first with a description of the gravity solution. An important first step is to find the asymptotic behavior of the solution. The function z goes to a constant at large y. We can find the value of the constant by integrating z over the two torus at fixed y. The result of this integral is independent of y, and we can compute it easily at $y = 0$ where it is given by the difference in areas between the two possible boundary conditions, $z = \pm\frac{1}{2}$. So we find

$$z = \frac{1}{2} \frac{N - K}{N + K} \tag{4.20}$$

[6]In this case, in the large y region, the solution looks similar to the solution we would obtain if we take the full x_1, x_2 plane and we consider a "grey" configuration filled with a fractional density. It shares some similarity but is different from the situation considered in e.g. some of the references in [18, 19, 170, 169, 92, 172, 44, 84, 42, 146, 6] where "grey" regions are finite.

asymptotically, where we used that the areas are quantized due to the flux quantization condition [126], so that N, K are the areas of the fermions and the holes respectively. After doing T-dualities on both circles of the T^2 and an S duality we find that the solution is asymptotic to

$$ds_{10}^2 = -dt^2 + du_1^2 + du_2^2 + N\alpha' d\Omega_3^2 + K\alpha' d\widetilde{\Omega}_3^2 + \frac{NK}{N+K}\alpha' d\rho^2 \tag{4.21}$$

$$e^{\Phi} = g_s\sqrt{NK}\sqrt{N+K}\alpha'^{3/2}e^{-\rho} \tag{4.22}$$

$$H_3 = 2N\alpha' d^3\Omega + 2K\alpha' d^3\widetilde{\Omega} \tag{4.23}$$

We can view this as a little string theory in $1+2$ dimensions. These (asymptotic) solutions are not regular as $\rho \to -\infty$ since the dilaton increases. In that region we should do an S duality and then T-dualities back to the original type IIB description. Then, once we choose a droplet configuration, the solution is regular. This procedure works only if the coordinates u_1, u_2 in (4.21) are compact. Of course, we could also consider the situation when these coordinates are non-compact. In that case we have Poincare symmetry in 2+1 dimensions. In fact, such a solution appears as the near horizon limit of two intersecting fivebranes[7] [112],[75] and was recently studied in [101]. Note that the asymptotic geometry (4.21)-(4.23) is symmetric under

$$K \leftrightarrow N \tag{4.24}$$

which is associated with the symmetry $z \leftrightarrow -z$. So we expect that this is a precise symmetry of the field theory.

It is interesting to start from the D4 brane theory that we discussed above and then compactify one of its transverse directions, the direction Y in (4.18). The lagrangian in (4.18) is not invariant under infinitesimal translations of Y, but it is invariant under discrete translations if the period of Y is chosen appropriately. Following the standard procedure, [174], we obtain a theory in six dimensions which can be viewed as the theory arising on N D5 branes that are wrapping an $R^{1,1} \times S^1 \times S^3$ with K units of RR 3 form flux on S^3. This

[7] One can make a change of variables $e^{2\rho} = \sqrt{N+K}\alpha'^{1/2}r_1 r_2$, $u_2 = \frac{\alpha'^{1/2}}{\sqrt{N+K}}(N\log r_1 - K\log r_2)$, and then r_1 and r_2 become the transverse radial directions of the two sets of fivebranes (intersecting on $R^{1,1}$) respectively in the near horizon geometry, where the number of supersymmetries is doubled.

RR flux induces a level K three dimensional Chern-Simons term. In fact by compactifying Y from (4.18) we get a 2+1 action $\frac{K}{4\pi}\int Tr[-\frac{1}{4}F^2+\omega_{cs}]$ on $R\times T^2$ with Chern Simons term. It turns out that the gauge coupling constant is also set by K. Perhaps a simple way to understand this is that the mass of the gauge bosons, which is due to the Chern Simons term is related by supersymmetry to the mass scale set by the radius of the threesphere, which we can set to one. This implies that $g^2K \sim 1$. This derivation makes sense only when K/N is large and we could be missing finite K/N effects. Notice that in this limit the $\tilde{S}^3$ that is interpreted as the worldvolume of N D5 branes is larger than the other S^3 in (4.21)-(4.23). The gauge theory description is valid in the IR but the proper UV definition of this theory is in terms of the little string theory in (4.21). The theory has a mass gap for propagating excitations but is governed by a $U(N)_K$ Chern Simons theory at low energies. The $U(1)$ factor is free and it should be associated to a "singleton" in the geometric description. On the other hand, it seems necessary to find formulas that are precisely symmetric under $K \leftrightarrow N$. More precisely, in the limit N/K large we get a $U(K)_N$ Chern-Simons theory by viewing the theory as coming from K D5 branes wrapping the other S^3. Interestingly, these two Chern Simons theories are dual to each other [152],[49][8] , which suggests that this is the precise low energy theory for finite N and K. Similar conclusions were reached in [101]. Of course, in our problem we do not have just this low energy theory, we have a full massive theory, with a mass scale set by the string scale. We do not have an independent way to describe it other than giving the asymptotic geometry (4.21)-(4.23), as is the case with little string theories. On the other hand one can show that the symmetry algebra implies that the theory has a mass gap.

We can compute the number of vacua from the gravity side. There we have Landau levels on a torus where we have total flux $N+K$ and we have N fermions and K holes. This gives a total number of vacua

$$D_{grav}(N,K) = \frac{(N+K)!}{K!\ N!} \tag{4.25}$$

[8]Level rank duality, as analyzed in [152, 49], holds up to pieces which comes from free field correlators. This means that we have not checked whether the $U(1)$ factor, as we introduced it here, leads to a completely equivalent theory.

and the filling fraction $\frac{N}{N+K}$. Actually, to be more precise, we derive this Landau level picture as follows. We start from the gravity solutions which are specified by giving the shape of droplets on the torus. We should then quantize this family of gravity solutions. This was done in [139],[80] (see also [138],[60],[61]), who found that the quantization is the same as the quantization for the incompressible fluid we have in the lowest landau level for N fermions in a magnetic field. We now simply compactify the plane considered in [139],[80]. This procedure is guaranteed to give us the correct answer for large N and K. The number of vacua computed from $U(N)_K$ agrees with (4.25) up to factors going like N, K or $N+K$ which we have not computed. These factors are related to the precise contribution of the $U(1)$[9] . In order to compare the field theory answer to the gravity answer one would have to understand properly the role of "singletons", which could give contributions of order N, K, etc. We leave a precise comparison to the future but it should be noted that we have a precise agreement for large N and K where the gravity answers are valid.

We have non-singular gravity solutions if we choose simple configurations for these fermions where they form well defined droplets. The particle hole duality of the Landau problem is the level rank duality in Chern-Simons theory, and is $K \leftrightarrow N$ duality of the full configuration.

4.5 Worldvolume Poincare supersymmetry with nonabelian non-central charges

An unusual property of all the theories we discussed above is that their supersymmetry algebra in 2+1 (or 1+1) dimensions is rather peculiar. In ordinary Poincare supersymmetry the generators appearing in the right hand side of the supersymmetry algebra commute with all other generators. This is actually a theorem for $d \geq 4$ [179]. For this reason they are called *central* charges. On the contrary, the Poincare supersymmetry on the worldvolume theories in our cases contains charges on the right hand side that does not

[9]The number of vacua for $SU(N)_K$ Chern Simons is given by $\frac{(N+K-1)!}{K!(N-1)!}$.

commute with other generators. These charges are non-abelian, and in our case have an internal $SO(4)$ structure related to the R symmetry of the supercharges. In this section we first present an algebra with 8 supercharges and then an algebra with 16 supercharges.

Let us define $(\gamma^\mu)_\alpha^\beta$ as

$$\gamma^0 = i\sigma^2 \ , \quad \gamma^1 = \sigma^1, \qquad \gamma^2 = \sigma^3 \tag{4.26}$$

where σ^i are Pauli matrices. We also define

$$\tilde{\gamma}^\mu_{\alpha\beta} = (\gamma^\mu)^\gamma_\alpha \epsilon_{\gamma\beta} \ , \qquad \tilde{\gamma}^0 = -\delta^{\alpha\beta}, \quad \tilde{\gamma}^1 = -\sigma^3, \qquad \tilde{\gamma}^2 = \sigma^1 \tag{4.27}$$

and we see that $(\tilde{\gamma}^\mu)_{\alpha\beta}$ is symmetric in the indices α, β.

We define supercharges $Q_{\alpha i}$ with i an $SO(4)$ index and α is the 2+1 Lorentz index (spinor of $SO(2,1)$). We can impose the reality condition $Q^\dagger_{\alpha i} = Q_{\alpha i}$.

We start by considering a superalgebra with 8 supercharges given by

$$\begin{aligned}
\{Q_{\alpha i}, Q_{\beta j}\} &= 2\tilde{\gamma}^\mu_{\alpha\beta} p_\mu \delta_{ij} + 2m\epsilon_{\alpha\beta}\epsilon_{ijkl} M_{kl} && (4.28)\\
[p_\mu, Q_{\alpha i}] &= 0, \qquad [p_\mu, p_\nu] = 0, && (4.29)\\
[\Sigma_{\mu\nu}, Q_{\alpha i}] &= \frac{1}{2} (\tilde{\gamma}_{\mu\nu})^\beta_\alpha Q_{\beta i} && (4.30)\\
[M_{jl}, Q_{\alpha i}] &= i(\delta_{ij} Q_{\alpha l} - \delta_{il} Q_{\alpha j}) && (4.31)\\
[M_{ij}, M_{kl}] &= i(\delta_{ik} M_{jl} + \delta_{jl} M_{ik} - \delta_{jk} M_{il} - \delta_{il} M_{jk}) && (4.32)\\
[\Sigma_{\mu\nu}, p_\lambda] &= i(\eta_{\nu\lambda} p_\mu - \eta_{\mu\lambda} p_\nu) && (4.33)\\
[\Sigma_{\mu\nu}, \Sigma_{\lambda\rho}] &= i(\eta_{\nu\lambda}\Sigma_{\mu\rho} + \eta_{\mu\rho}\Sigma_{\nu\lambda} - \eta_{\mu\lambda}\Sigma_{\nu\rho} - \eta_{\nu\rho}\Sigma_{\mu\lambda}) && (4.34)\\
[M_{jl}, p_\mu] &= 0, \qquad [M_{jl}, \Sigma_{\mu\nu}] = 0 && (4.35)
\end{aligned}$$

where m is a constant of dimension of mass, i, j are $SO(4)$ indices and M_{ij} are $SO(4)$ generators. Here we have set $m = 1$ for convenience. This choice is related to the choice of mass scales (e.g. radius of S^3) appearing in the various theories.

In order to check the closure of the superalgebra we need to check the Jacobi identity. The identities involving one bosonic generator will be automatically obeyed since they are just simply stating that objects transform covariantly under the appropriate symmetries. So

the only non-trivial identity that we need to check is the one involving three odd generators. The Jacobi identity is

$$\begin{aligned} & [Q_{\alpha i}, \{Q_{\beta j}, Q_{\gamma l}\}] + [Q_{\beta j}, \{Q_{\gamma l}, Q_{\alpha i}\}] + [Q_{\gamma l}, \{Q_{\alpha i}, Q_{\beta j}\}] \\ = \; & i\epsilon_{\beta\gamma}\epsilon_{jlab}(\delta_{bi}Q_{\alpha a} - \delta_{ai}Q_{\alpha b}) + i\epsilon_{\gamma\alpha}\epsilon_{liab}(\delta_{bj}Q_{\beta a} - \delta_{aj}Q_{\beta b}) + i\epsilon_{\alpha\beta}\epsilon_{ijab}(\delta_{bl}Q_{\gamma a} - \delta_{al}Q_{\gamma b}) \\ = \; & -i\epsilon_{ijla}[\epsilon_{\beta\gamma}Q_{\alpha a} + \epsilon_{\gamma\alpha}Q_{\beta a} + \epsilon_{\alpha\beta}Q_{\gamma a}] - i\epsilon_{ijlb}[\epsilon_{\beta\gamma}Q_{\alpha b} + \epsilon_{\gamma\alpha}Q_{\beta b} + \epsilon_{\alpha\beta}Q_{\gamma b}] \equiv 0 \end{aligned} \tag{4.36}$$

This superalgebra actually appeared in the general classification in [153]. These generators do not commute with the supercharges. So this is a Poincare superalgebra with *non-central* charges [10] where M_{ij} are $SO(4)$ generators. M_{ij} are non-central charges in the superpoincare algebra in 2+1 dimensions. $\Sigma_{\mu\nu}$ is the Lorentz generator in $SO(2,1)$. Notice that the first line is the only non-obvious commutator and is the one stating that we have *non-central* charges.

The $SO(4)$ that appears in (4.28) is the product of an $SU(2)$ that acts on the first S^3 times another $SU(2)$ which acts on the second S^3, where the S^3s we mention here are the three spheres in the geometric description. There are other supercharges which which transforms under another $SO(4)$. The truncation of this algebra to 1+1 dimensions which describes light-cone worldvolume string theories on a class of IIA pp-wave backgrounds is written in (5.63) in chapter 5.

It is interesting to study the particle spectrum for theories based on this superalgebra (4.28). This theory cannot have massless propagating particles. This can be seen as follows. We assume that the massless particle has $p_- = 0$, $p_+ \neq 0$ and $p_2 = 0$. In this case the supersymmetry algebra implies that Q^i_- and M_{ij} annihilate all states in the supermultiplet. On the other hand the Q^i_+ generators arrange themselves into creation and annihilation operators and change the $SO(4)$ quantum numbers in the multiplet. Thus we reached a contradiction. This argument allows Chern Simons interactions since that is a topological theory. So all propagating particles are massive. Let us go to the rest frame of the massive particle, with $p_1 = p_2 = 0$. Then the "little group" (i.e. the truncation of (4.28) to the

[10] This situation, is of course, common in anti-de-Sitter superalgebras. It has also been observed before in some deformations of Euclidean Poincare superalgebras [37].

generators that leave this choice of momenta invariant) is the $\widetilde{SU}(2|2)$ supergroup. The tilde represents the fact that we take the corresponding $U(1)$ to be non-compact. The representation theory of this algebra was studied in [10],[11],[12],[24],[108],[109]. As usual, there are short representations when the BPS bound is obeyed when the mass of the particle is $M = 2m(j_1 + j_2)$, where m is the mass parameter in (4.28).

The superalgebra (4.28)-(4.35) can be reduced to 1+1 dimensions in a trivial fashion, we just set $p_2 = 0$ and remove two of the Lorentz generators. This is the symmetry algebra (5.63) of the sigma model considered in (5.66). The reason this superalgebra arises is the following. Suppose we start with a theory with supergroup $\widetilde{SU}(2|4)$ and we pick a 1/2 BPS state with charge J under generator J in $SO(2) \subset SO(6)$. The supercharges that annihilate this state form the supergroup $\widetilde{SU}(2|2)$. The lightcone string lagrangian (5.66) describes small fluctuations around these BPS states so that the supergroup $\widetilde{SU}(2|2)$ should act on them linearly. Since the worldsheet action is boost invariant along the worldsheet, we find that this supergroup should be extended to (4.28).

Let us give some further examples of theories with this superalgebra. We can construct a 1+1 dimensional SYM with this superalgebra from the plane wave matrix model via matrix theory compactification techniques [174] (also [22]). In fact this 1+1 SYM was constructed in this way by e.g. [56]. Here we will reproduce this result and we will use $SO(9,1)$ gamma matrices and the fermions are $SO(9,1)$ spinors[11] . We will then compactify a scalar of the 1+1 SYM and get a 2+1 super Yang Mills Chern Simons theory satisfying the above superalgebra.

One starts from the plane wave matrix model whose mass terms for the $SO(6)$ scalars takes the form $-\frac{1}{2}(X_a)^2$, where $a = 1, 2, ..., 6$. We have set the mass for the $SO(6)$ scalar to 1. We should write the action so that it is translation invariant in one of the transverse scalars. We can make a field-redefinition for two $SO(6)$ scalars $X_1 + iX_2 = e^{it}(Y + i\phi)$ and

[11] Our convention is different from that of [22] or [33], which use $SO(9)$ gamma matrices.

for fermions $\Psi = e^{\frac{1}{2}\Gamma_{12}t}\theta$. Then the action of plane wave matrix model is

$$\begin{aligned} S &= \frac{1}{g_{YM0}^2}\int dx_0 \mathrm{Tr}\left(-\frac{1}{2}(D_0X_I)^2 - \frac{1}{2}(D_0Y)^2 - \frac{1}{2}(D_0\phi)^2 - \frac{i}{2}\bar{\theta}\Gamma^0 D_0\theta - \frac{1}{2}\bar{\theta}\Gamma_I\left[X_I,\theta\right]\right. \\ &\quad -\frac{1}{2}\bar{\theta}\Gamma_1\left[\phi,\theta\right] - \frac{1}{2}\bar{\theta}\Gamma_2\left[Y,\theta\right] + \frac{1}{2}[\phi,X_I]^2 + \frac{1}{2}[\phi,Y]^2 + \frac{1}{2}[Y,X_I]^2 + \frac{1}{4}[X_I,X_J]^2 - \frac{1}{2}\left(X_a\right)^2 \\ &\quad \left.-\frac{1}{2}2^2\left(X_i\right)^2 + \frac{3}{2}i\,\bar{\theta}\Gamma_{789}\theta + 2i\epsilon^{ijk}\,X_i\,X_j\,X_k - \frac{1}{2}i\,\bar{\theta}\Gamma_0\,\Gamma_{12}\theta - 2Y\,D_0\phi\right) \end{aligned} \tag{4.37}$$

We have 3+4+2 scalars, where the first seven scalars with indices $I = 3, 4, ..., 9$ are split into $a = 3, 4, 5, 6$ and $i = 7, 8, 9$ and the rest two scalars are Y and ϕ.

Then the action becomes translation invariant in the ϕ direction. We now compactify ϕ by replacing ϕ with gauge covariant derivative $\phi \to i\frac{\partial}{\partial x_1} + A_1, -i[\phi, O] \to \partial_1 O - i[A_1, O]$ [174] (also [22]). Plugging this into the original action (4.37) one get the 1+1 dimensional super Yang Mills on $R^{1,1}$ with a mass deformation

$$\begin{aligned} S &= \frac{1}{g_{YM1}^2}\int dx_0 dx_1 \mathrm{Tr}\left(-\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2}(D_\mu X_I)^2 - \frac{1}{2}(D_\mu Y)^2 - \frac{i}{2}\bar{\theta}\Gamma^\mu D_\mu\theta - \frac{1}{2}\bar{\theta}\Gamma_I\left[X_I,\theta\right]\right. \\ &\quad -\frac{1}{2}\bar{\theta}\Gamma_2\left[Y,\theta\right] + \frac{1}{2}[Y,X_I]^2 + \frac{1}{4}[X_I,X_J]^2 - \frac{1}{2}\left(X_a\right)^2 - \frac{1}{2}2^2\left(X_i\right)^2 + \frac{3}{2}i\,\bar{\theta}\Gamma_{789}\theta \\ &\quad \left.+2i\epsilon^{ijk}\,X_i\,X_j\,X_k - \frac{1}{2}i\,\bar{\theta}\Gamma_{012}\theta - Y\epsilon^{\mu\nu}F_{\mu\nu}\right) \end{aligned} \tag{4.38}$$

We have 3+4+1 scalars, with the seven scalars whose indices are $I = 3, 4, ..., 9$, where $a = 3, 4, 5, 6$ and $i = 7, 8, 9$ and another scalar Y, and $\mu = 0, 1$. The theory has super-poincare algebra on $R^{1,1}$ with $SU(2) \times SU(2)$ R symmetry. The first $SU(2)$ rotates the first three scalars $i = 7, 8, 9$ and the second $SU(2)$ is one of the $SU(2)$ factors in the $SO(4)$ rotating the four scalars $a = 3, 4, 5, 6$. In addition, the theory has an $SU(2)$ global symmetry, which is the second $SU(2)$ factor in the $SO(4)$ we have just mentioned. Compactifying along x_1 and taking the compactification size to zero we get back to the plane wave matrix model which has a larger symmetry group. The parameters in the two theories are related by $g_{YM1}^2 = 2\pi R_{x_1} g_{YM0}^2$, where R_{x_1} is the radius of the x_1 circle. The 1+1 SYM constructed from the plane wave matrix model coincides with the DLCQ of the IIA plane wave [171],[97],[98], which was first obtained by [171],[97],[98] from reduction of the supermembrane action under kappa-symmetry fixing condition on 11d maximal plane wave. The action we reproduce here (4.38) is written manifestly Lorentz invariant in 1+1 dimensions.

We pointed out that this theory can be uplifted again making Y periodic. We make the replacement $Y \to i\frac{\partial}{\partial x_2} - A_2, -i[Y,O] \to \partial_2 O + i[A_2,O]$ [174]. The coupling YF_{01}, becomes a Chern-Simons term in 2+1 dimensions. The quantization of the level of the Chern Simons action implies that the compactification radius of Y is quantized. This quantization condition also follows from the fact that the coupling YF_{01} is not invariant under arbitrary shifts of Y, and e^{iS} is periodic only if we shift Y by the right amount. Finally we get the 2+1 dimensional super Yang Mills Chern Simons theory

$$\begin{aligned} S &= \frac{k}{4\pi}\left\{\int \mathrm{Tr}\{-\frac{1}{2}F\wedge *F + A\wedge dA + \frac{2}{3}A\wedge A\wedge A - \frac{i}{12}\bar{\psi}\Gamma_{\mu\nu\lambda}\psi dx^\mu\wedge dx^\nu\wedge dx^\lambda\}\right. \\ &+ \int d^3x \mathrm{Tr}\{-\frac{1}{2}(D_\mu X_I)^2 - \frac{i}{2}\bar{\psi}\Gamma^\mu D_\mu\psi - \frac{1}{2}\bar{\psi}\Gamma^I[X_I,\psi] + \frac{1}{4}[X_I,X_J]^2 \\ &\left. -\frac{1}{2}(X_a)^2 - \frac{1}{2}2^2(X_i)^2 + 2i\epsilon^{ijk}X_iX_jX_k + \frac{i}{4}\epsilon^{ijk}\bar{\psi}\Gamma_{ijk}\psi\}\right\} \end{aligned} \tag{4.39}$$

where we have $3+4$ scalars with indices $I = 3,4,...,9$ split into $a = 3,4,5,6$ and $i,j,k = 7,8,9$, and the worldvolume indices are $\mu,\nu,\lambda = 0,1,2$. The coupling constants is related to the 1+1 SYM by $g^2_{YM2} = 2\pi R_{x_2} g^2_{YM1}$, $\frac{k}{4\pi} = \frac{1}{g^2_{YM2}}$ and $k \in \mathbb{Z}$. So we see that k is the only coupling constant in the theory. When k is large the theory is weakly coupled.

Finally, let us turn our attention to the superalgebra for theories with 16 supercharges. Now we have two $SO(4)$ groups and a second set of supercharges $\widetilde{Q}_{\alpha m}$. We add the anti-commutators

$$\{\widetilde{Q}_{\alpha m}, \widetilde{Q}_{\beta n}\} = 2\tilde{\gamma}^\mu_{\alpha\beta}p_\mu\delta_{mn} + 2m'\epsilon_{\alpha\beta}\epsilon_{mnrs}\widetilde{M}_{rs} \tag{4.40}$$

where $\widetilde{M}_{rs}$ is generator of the second $SO(4)$. The anticommutator of $Q_{\alpha i}$ with $\widetilde{Q}_{\alpha m}$ is zero. The rest of the algebra is rather obvious and is just given by the covariance properties of the indices as in (4.28). In principle we can have $m' \neq m$ in (4.28). In the theories studied here we have $m' = m$. If we want to have BPS states under both Q and $\tilde{Q}$ then we need that m/m' to be a rational number. Note that the little group for a massive particle is $\widetilde{SU}(2|2) \times \widetilde{SU}(2|2)$.

Let us be a little more precise about these two $SO(4)$ groups. The ansatz in [126] above has two three spheres on which two $SO(4)$ group act. Let us call them $SO(4)_i$ with $i = 1,2$. Each of these two groups are $SO(4)_i = SU(2)_{Li} \times SU(2)_{Ri}$. The supercharges

$Q_{\alpha i}$ in (4.28) transform under $SU(2)_{L1} \times SU(2)_{L2}$. The supercharges $\tilde{Q}_{\alpha i}$ transform under $SU(2)_{R1} \times SU(2)_{R2}$. If we quotient any of these theories by a Z_k in $SU(2)_{Ri}$, we get a theory that only has 8 generators as in (4.28).

This algebra with 16 generators is the one that appeared on the worldvolume of theories related to the IIB constructions in this chapter. In the case of the M2 brane theory the two $SO(4)$s are global R-symmetries of the theory. In the case that we consider an M5 on $R^{2,1} \times S^3$ one of the $SO(4)$ groups is a symmetry acting on the worldvolume. When the size of S^3 becomes infinity, the $SO(4)$ that acts on the worldvolume is contracted to ISO(3) and only the translation generators remain in the right hand side of the supersymmetry algebra. Thus, we do not get into trouble with the Haag-Lopuszanski-Sohnius theorem [90] in total spacetime dimension $d \geq 4$.

The dimensional reduction of this algebra to 1+1 dimensions gives the linearly realized symmetries on the lightcone worldsheet of a string moving in the maximally supersymmetric IIB plane wave [39].

Chapter 5

Geometry of BPS vacua in field theories with $SU(2|4)$ symmetry

5.1 Introduction

In this chapter, we study another class of supersymmetric theories with 16 supercharges and a $SU(2|4)$ group. These theories arise naturally when we consider the 11 dimensional solutions in chapter 3, and compactify them to IIA string theory. Again, the configurations we consider have an additional Killing vector in the x_1, x_2 plane, say along the x_1 direction. Then the type of theories come in different ways of distributing droplets, or more precisely strips. Because the M2 droplet and M5 droplet are different, unlike the IIB case in chapter 4, here we have four different theories, rather than three. If we fill the half plane and consider finite strips above it, they are the vacua of the plane-wave matrix model. If we consider finite number of M2 strips, or in ten dimensional point of view, D2 strips, we get the vacua of 2+1d SYM on $R \times S^2$. On the other hand, if we have finite number of M5 strips, or NS5 strips in ten dimensions, then we have the vacua of 5+1d NS5 brane theory on $R \times S^5$. Lastly, if we distribute the strips periodically, after T duality along x_1 direction in the asymptotic region, they asymptote to the geometry dual to $\mathcal{N} = 4$ super Yang Mills on $R \times S^3/Z_k$. These periodic distributions corresponds to the vacua of $\mathcal{N} = 4$

super Yang Mills on $R\times S^3/Z_k$. Except for the NS5 brane theory, all other three theories can be obtained from reduction and truncation of $\mathcal{N}=4$ super Yang Mills on $R\times S^3$, see figure 5.1. In Ch 5.2, we study the general supergravity solutions, their regularity conditions, and general properties. We find that the boundary conditions are in terms of electrostatic configurations of disks under the external potential. This is from the point of view of the Laplace equation. In Ch 5.3 we discuss in the field theory side how to count the number of vacua as well as the excited BPS states by an index for theories with $SU(2|4)$ symmetry. In Ch 5.4, we study the gravity dual of plane-wave matrix model. Especially we focus on the vacua of plane-wave matrix model corresponding to single or mutli-NS5 branes. We find that the stringy excitations around the single NS5 brane vacua consists only 4 bosonic and 4 fermionic oscillators, while compared to multi-NS5 branes these numbers should be doubled. The reason for this is that the dual geometry correspond to single NS5 brane has a throat region that is too massive and low energy strings cannot oscillate on these four directions. In Ch 5.5, 5.6, 5.7, we study the other three theories, that is, the 2+1d SYM on $R\times S^2$, NS5 brane theory on $R\times S^5$, and $\mathcal{N}=4$ super Yang Mills on $R\times S^3/Z_k$ respectively.

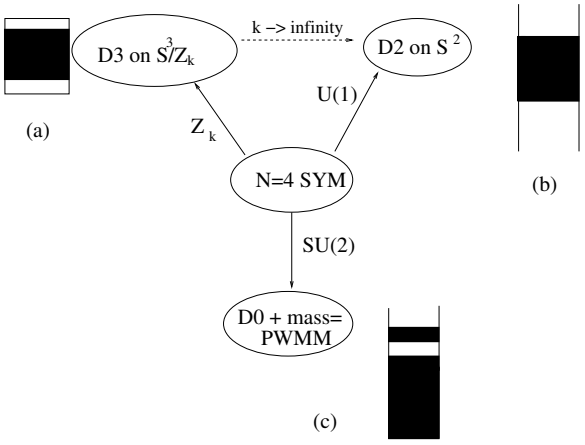

Figure 5.1: Starting from four dimensional $\mathcal{N}=4$ super Yang Mills and truncating by various subgroups of $SU(2)_L$ we get various theories with $\widetilde{SU}(2|4)$ symmetry. We have indicated the diagrams in the x_1, x_2 space that determine their gravity solutions. The x_1, x_2 space is a cylinder, with the vertical lines identified for (b) and (c) and it is a torus for (a).

5.2 Electrostatic description of the vacua

In this section we first analyze the detailed regularity conditions for the gravity duals, we then characterize each vacuum of the gauge theories by a configuration in the gravity, we also study a particular pp-wave limit of the geometries and near BPS spectrum.

All the theories that we have discussed above have the same supersymmetry group $SU(2|4)$. All gravity solutions with this symmetry were classified in [126]. See also chapter 3. The bosonic symmetries, $R \times SO(3) \times SO(6)$, act geometrically. The first generator implies the existence of a Killing vector associated to shifts of a coordinate t. In addition we have an S^2 and an S^5 where the rest of the bosonic generators act. Thus the solution depends only on three variables x_1, x_2, y. The full geometry can be obtained from a solution of the 3 dimensional Toda equation

$$(\partial_{x_1}^2 + \partial_{x_2}^2)D + \partial_y^2 e^D = 0 \tag{5.1}$$

It turns out that $y = R_{S^2} R_{S^5}^2 \geq 0$ where R_{S^i} are the radii of the two spheres. In order to have a non-singular solution we need special boundary conditions for the function D at $y = 0$. In fact, the $x_1,\ x_2$ plane could be divided into regions where the function D obeys two different boundary conditions

$$e^D \sim y \qquad \text{for} \quad y \to 0\ , \qquad S^2 \to 0\ , \qquad \text{M5 region} \tag{5.2}$$

$$\partial_y D = 0 \qquad \text{at} \quad y = 0\ , \qquad S^5 \to 0\ , \qquad \text{M2 region} \tag{5.3}$$

see [126] for further details. The labels M2 and M5 indicate that in these two regions either a two sphere or a five sphere shrinks to zero in a smooth fashion. There are, however, no explicit branes in the geometry. We have a smooth solution with fluxes. However, we can think of these regions as arising from a set of M2 or M5 branes that wrap the contractible sphere. A bounded region of each type in the x_1, x_2 plane implies that we have a cycle in the geometry with a flux related to the corresponding type of brane (see [126] for further details).

The different theories discussed above are related to different choices for the topology of the x_1, x_2 plane. In addition, for each topology the asymptotic distribution of M2 and M5 regions can be different. See figure 5.1. Let us consider some examples. If we choose the x_1, x_2 plane to be a two torus, then we get a solution that is dual to the vacua of the $\mathcal{N} = 4$ super Yang Mills on $R \times S^3/Z_k$, see figure 5.1(a). If the topology is a cylinder, with x_1 compact and the M2 region is localized in the x_2 direction, we have a solution dual to a vacuum of the 2+1 Yang Mills theory on $R \times S^2$, see figure 5.1(b). If we choose a cylinder and we let the M2 region extend all the way to $x_2 \to -\infty$, and the M5 region extend to $x_2 \to +\infty$, and also there are localized M2, M5 strips in between, then we get a solution which is dual to a vacuum of the plane wave matrix model, see figure 5.1(c). Finally, if we consider a cylinder and we have M5 regions that are localized (see figure 5.2(c)) then we get a solution that is dual to an NS5 brane theory on $R \times S^5$, we will came back to this case later.

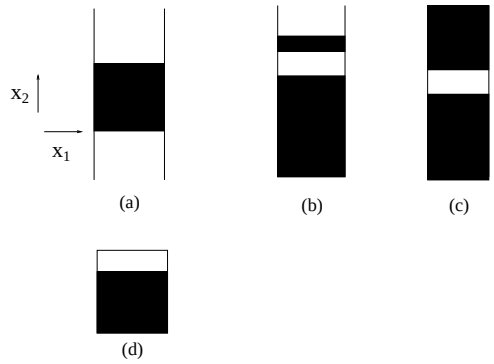

Figure 5.2: Translational invariant configurations in the x_1, x_2 plane which give rise to various gravity solutions. The shaded regions indicate M2 regions and the unshaded ones indicate M5 regions. The two vertical lines are identified. In (a) we see the configuration corresponding to the vacuum of the 2+1 Yang Mills on $R \times S^2$ with unbroken gauge symmetry. In (b) we consider a configuration corresponding to a vacuum of the plane wave matrix model. In (c) we see a vacuum of the NS5 brane theory on $R \times S^5$. Finally, in (d) we have a droplet on a two torus in the x_1, x_2 plane. This corresponds to a vacuum of the $\mathcal{N} = 4$ super Yang Mills on a $R \times S^3/Z_k$.

In principle, we could consider configurations that are not translation invariant, as long as we consider configurations defined on a cylinder or torus as is appropriate. Here we

will concentrate on configurations that are translation invariant along x_1. These will be most appropriate in the regime of parameter space where the 11th direction is small and we can go to a IIA description. So we focus on the region in parameter space where the string coupling is small and the effective 't Hooft coupling is large. If the configuration is translation invariant in the x_1 direction we can transform the non-linear equation (5.1) to a linear equation through the following change of variables [178]

$$y = \rho\partial_\rho V \ , \qquad x_2 = \partial_\eta V \ , \qquad e^D = \rho^2 \tag{5.4}$$

$$\frac{1}{\rho}\partial_\rho(\rho\partial_\rho V) + \partial_\eta^2 V = 0 \tag{5.5}$$

So we get the Laplace equation in three dimensions for an axially symmetric system[1] . The fact that one can obtain solutions in this fashion was observed in [126] and some singular solutions were explored in [13]. Below we will find the precise boundary conditions for V which ensure that we have a smooth solution.

Let us now translate the boundary conditions (5.2) at $y = 0$ into certain boundary conditions for the function V. In the region where $e^D \sim y$ at $y \sim 0$, all that we require is that V is regular at $\rho = 0$, in the three dimensional sense. On the other hand if $y = 0$ but $\rho \neq 0$, then we need to impose that $\partial_y D = 0$. This is proportional to

$$0 = \frac{1}{2}\partial_y D = \rho\frac{\partial\rho}{\partial y} = -\frac{\partial_\eta^2 V}{(\partial_\eta\partial_\rho V)^2 + \left(\partial_\eta^2 V\right)^2} \tag{5.6}$$

We conclude that $\partial_\eta^2 V = 0$. Equation (5.5) then implies that $\partial_\rho^2 V = 0$. Therefore the curve $y = 0,\ \rho \neq 0$, or $\partial_\rho V = 0$, is at constant values of η, since the slope of the curve defined by $\partial_\rho V = 0$ is $\frac{\delta\eta}{\delta\rho} = \frac{-\partial_\rho^2 V}{\partial_\eta\partial_\rho V} = 0$.

If we interpret V as the potential of an electrostatics problem, then $-\partial_\rho V$ is the electric field along the ρ direction. The condition that it vanishes corresponds to the presence of a charged conducting surface. So the problem is reduced to an axially symmetric electrostatic configuration in three dimensions where we have conducting disks that are sitting at positions η_i and have radii ρ_i. See figure 5.3. These disks are in an external electric field

[1] The angular direction of the three dimensional space is not part of the 10 or 11 dimensional spacetime coordinates.

which grows at infinity. If we considered such conducting disks in a general configuration we would find that the electric field would diverge as we approach the boundary of the disks. In our case this cannot happen, otherwise the coordinate x_2 would be ill defined at the rim of the disks. So we need to impose the additional constraint that the electric field is finite at the rim of the disks. This implies that the charge density vanishes at the tip of the disks. This condition relates the charge on the disks Q_i to the radii of the disks ρ_i. So for each disk we can only specify two independent parameters, its position η_i and its total charge Q_i. The precise form of the background electric field depends on the theory we consider (but not on the particular vacuum) and it is fixed by demanding that the change of variable (5.4) is well defined. The relation between the translation invariant droplet configurations in the x_1, x_2 plane and the disks can be seen in figure 5.3.

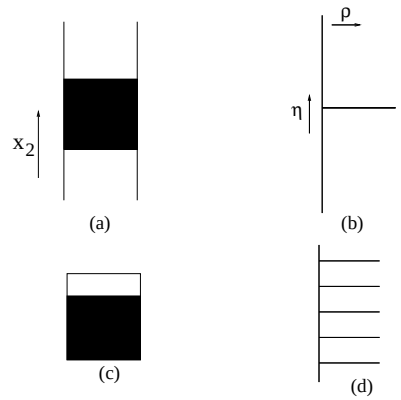

Figure 5.3: Electrostatic problems corresponding to different droplet configurations. The shaded regions (M2 regions) correspond to disks and the unshaded regions map to $\rho = 0$. Note that the x_1 direction in (a), (c) does not correspond to any variable in (b), (c). The rest of the ρ, η plane corresponds to $y > 0$ in the x_2, y variables. In (a),(b) we see the configurations corresponding to a vacuum of 2+1 super Yang Mills on $R \times S^2$. In (c),(d) we see a configuration corresponding to a vacuum of $\mathcal{N} = 4$ super Yang Mills on $R \times S^3/Z_k$. In (d) we have a periodic configuration of disks. The fact that it is periodic corresponds to the fact that we have also compactified the x_2 direction.

Since we are focusing on solutions which are translation invariant along x_1 it is natural to compactify this direction and write the solution in IIA variables. This procedure will make sense as long as we are in a region of the solution where the IIA coupling is small.

The M-theory form of the solutions can be found in [126]. We obtain the string frame solution

$$
\begin{aligned}
ds_{10}^2 &= \left(\frac{\ddot{V}-2\dot{V}}{-V''}\right)^{1/2}\left\{-4\frac{\ddot{V}}{\ddot{V}-2\dot{V}}dt^2+\frac{-2V''}{\dot{V}}(d\rho^2+d\eta^2)+4d\Omega_5^2+2\frac{V''\dot{V}}{\Delta}d\Omega_2^2\right\} \\
e^{4\Phi} &= \frac{4(\ddot{V}-2\dot{V})^3}{-V''\dot{V}^2\Delta^2} \qquad (5.7) \\
C_1 &= -\frac{2\dot{V}'\dot{V}}{\ddot{V}-2\dot{V}}dt \qquad (5.8) \\
F_4 &= dC_3, \qquad C_3 = -4\frac{\dot{V}^2V''}{\Delta}dt\wedge d^2\Omega, \qquad (5.9) \\
H_3 &= dB_2\ , \qquad B_2 = 2\left(\frac{\dot{V}\dot{V}'}{\Delta}+\eta\right)d^2\Omega \qquad (5.10) \\
\Delta &\equiv (\ddot{V}-2\dot{V})V''-(\dot{V}')^2 \qquad (5.11)
\end{aligned}
$$

where the dots indicate derivatives with respect to $\log\rho$ and the primes indicate derivatives with respect to η. $V(\rho,\eta)$ is a solution of the Laplace equation (5.5). For regular solutions, we need to supplement it by boundary condition specified by a general configuration of lines in (ρ,η) plane, like in figure 5.4.

Before we get into the details of particular solutions we would like to discuss some general properties. First note that if we take a random solution of (5.5) we will get singularities. In order to prevent them, we need to be a bit careful. As we explained above we need a solution of an electrostatic problem involving horizontal conducting disks. In addition we need to ensure the positive-definiteness of various metric components, i.e. $\Delta\leq 0$ and $V''\leq 0$, $\ddot{V}-2\dot{V}\geq 0$, $\dot{V}\geq 0$. This is obeyed everywhere if we choose appropriate boundary conditions for the potential at large ρ,η. These boundary conditions imply that there is a background electric field that grows as we go to large ρ,η. For example, if we consider a configuration such as the one in figure 5.3(b), the disk is in the presence of a background potential of the form $V_b\sim\rho^2-2\eta^2$. This background electric field is the same for all vacua, e.g. it is the same in figures 5.3(b) and 5.4(a). For the plane wave matrix model we have an infinite conducting surface at $\eta=0$ and only the region $\eta\geq 0$ is physically significant. In this case the background potential is $V_b\sim\rho^2\eta-\frac{2}{3}\eta^3$. In addition we have finite size disks as seen, for example, in figure 5.4(d) or 5.4(e). In appendix A of [127] we showed that for the

configurations we talk about here (5.7)-(5.11) gives a regular solution. We also show that the dilaton is non-singular and that g_{tt} never becomes zero for the solutions we consider. This ensures that the solutions we have have a mass gap. This follows from the fact that the warp factor never becomes zero so that we cannot decrease the energy of a state by moving it into the region where the warp factor becomes zero. In principle, this argument does not rule out the presence of a small number of massless or tachyonic modes. The latter are, of course, forbidden by supersymmetry. A massless mode would not change the energy of the solution, so it would preserve supersymmetry. On the other hand, once we quantize the charges on the disks we do not have any continuous parameters in our solutions. So we cannot have any massless modes. Of course, this agrees with the field theory expectations since all theories we consider have a mass gap around any of the vacua.

Note that a rescaling of V leaves the ten dimensional metric and B field invariant but rescales the dilaton and the RR fields. This just corresponds to the usual symmetry of the IIA supergravity theory under rescaling of the dilaton and RR fields. There is second symmetry corresponding to rescaling ρ, η and V which corresponds to the usual scaling symmetry of gravity which scales up the metric and the forms according to their scaling dimensions. This allows us to put in two parameters in (5.7)-(5.11) such as an overall charge and the value of the dilaton at its maximum.

More interestingly, we can vary the number of disks, their charges and the distances between each other. See figure 5.4. These parameters are related to different choices of vacua for the different configurations.

All the solutions we are discussing, contain an S^2 and an S^5 and these can shrink to zero at various locations. Using these it is possible to construct three cycles and six cycles respectively by tensoring the S^2 and S^5 with lines in the ρ, η plane. These translate into three cycles and six cycles in the IIA geometry. See figure 5.5. We can then measure the flux of H_3 over the three cycle and call it N_5 and we can measure the flux of $*\tilde{F}_4$ on the six cycle and call it N_2. Using (5.7)-(5.11) or the formulas in [126] we can write them as

$$N_2 = \frac{1}{\pi^3 l_p^6} \int e^D dx_2 \int dx_1 = \frac{2}{\pi^2} \int_0^{\rho_i} \rho^2 \partial_\rho \left(\partial_\eta V|_{\eta_i^+} - \partial_\eta V|_{\eta_i^-} \right) d\rho = \frac{8Q_i}{\pi^2} \tag{5.12}$$

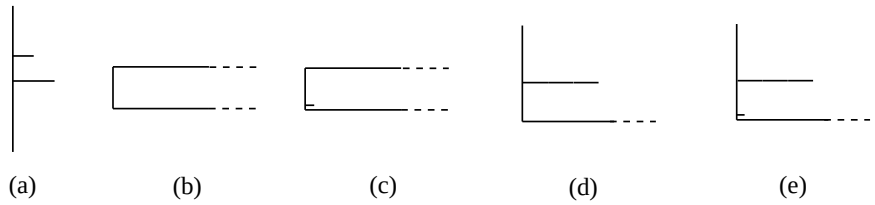

Figure 5.4: In (a) we see a configuration which corresponds to a vacuum of 2+1 super Yang Mills on $R \times S^2$. In (b) we see the simplest vacuum of the theory corresponding to the NS5 brane on $R \times S^5$. In this case we have two infinite conducting disks and only the space between them is physically meaningful. In (c) we have another vacuum of the same theory. If the added disk is very small and close to the the top or bottom disks the solution looks like that of (b) with a few D0 branes added. In (d) we see a configuration corresponding to a vacuum of the plane wave matrix model. In this case the disk at $\eta = 0$ is infinite and the solution contains only the region with $\eta \geq 0$. In (e) we have another vacuum of the plane wave matrix model with more disks.

and

$$N_5 = \frac{1}{2\pi^2 l_p^3} \int y^{-1} e^D dx_2 \int dx_1 = \frac{1}{\pi} \int_{\eta_i + d_i}^{\eta_i} \frac{\rho}{\partial_\rho V} \partial_\eta^2 V|_{\rho=0} d\eta = \frac{2d_i}{\pi} \tag{5.13}$$

In deriving (5.13) we used that near $\rho \to 0$ we can expand $V = f_0(\eta) + \rho^2 f_1(\eta) + \cdots$ and we used the equation for V (5.5) to relate $f_1(\eta)$ to V''. We set $\alpha' = 1$ and $l_p = 1$ for convenience. The quantization conditions (5.12),(5.13) show that N_5 is proportional to the distance between neighboring disks d_i and that N_2 is proportional to the total charge of each disk Q_i. When we solve the electrostatic problem we need to ensure that these parameters are quantized. Strictly speaking the flux given by N_2 is quantized only after we quantize the four form field strength.

The topology of the solutions is related to the topology of the disk configurations. In other words, the number of six cycles and three cycles is related to the number of disks and the number of line segments in between, but is independent of the size of the disks or the distance between the disks.

As we discussed above we will be interested in BPS excitations with angular momentum on S^5. For large, but not too large, angular momentum these are well described by lightlike particles moving in the background (5.7)-(5.11) with angular momentum J along the S^5. In order to minimize their energy, these lightlike geodesics want to sit at a point in the ρ, η

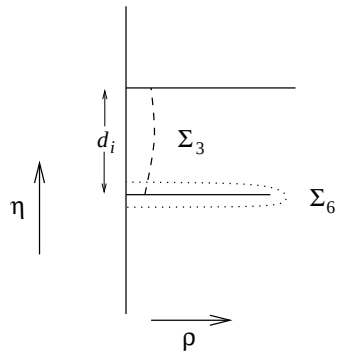

Figure 5.5: We see a configuration associated to a pair of disks. d_i indicates the distance between the two nearby disks. The dashed line in the ρ, η plane, together with the S^2 form a three cycle Σ_3 with the topology of an S^3. The dotted line, together with the S^5 form a six cycle Σ_6 with the topology of an S^6.

space where

$$\frac{|g_{tt}|}{g_{55}} = \frac{\ddot{V}}{\ddot{V} - 2\dot{V}} \geq 1 \tag{5.14}$$

is minimized, where $\sqrt{g_{55}}$ is the radius of the five sphere. It turns out that this is minimized at the tip of the disks, where the inequality in (5.14) is saturated[2] . This corresponds to saturating the BPS condition $E \geq |J|$. In fact, in order to minimize (5.14) we would like to set $\dot{V} = 0$. This occurs at $\rho = 0$ and on the surface of the disks. However, in these cases, also $\ddot{V} = 0$. Expanding the solutions near these regions we find that (5.14) actually diverges at $\rho = 0$, this is because S^5 shrinks at $\rho = 0$. On the disks, (5.14) is bigger than one, except at the tip where it is one. See appendix A for a more detailed discussion.

In order to find the behavior of the solution near these geodesics we expand the solution of the electrostatic problem near the tip of the disks. Near the tip of the disks we have a simple Laplace equation in two dimensions. Namely, we approximate the disk by an infinite half plane. We can then solve the problem by doing conformal transformations. Actually, we can do this whenever we are expanding around a solution at large ρ_0 and we are interested in features arising at distances which are much smaller than ρ_0, but could be larger than the distances between disks, see figure 5.6. So let us first analyze this problem

[2]In the eleven dimensional description the point where (5.14) is minimized lies on the $y = 0$ plane at a local maximum of $e^D|_{y=0}$ in the x_1, x_2 plane.

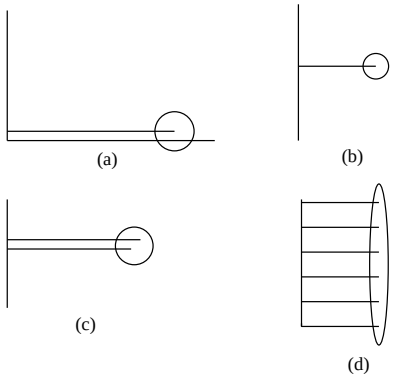

Figure 5.6: In this figure we see the expansion around the region near the tip of the disks. In a generic situation the tip we focus on is isolated, see (b). In other cases, there are other disks nearby that sit close to the tip we are focusing on. In this case we can take a limit where we include the nearby tips. We see such situations in (a),(c) and (d). (d) corresponds to the periodic case. We can focus on a distance that is large compared to the period in η but small compared to the size of the disks.

in general. We can define the complex coordinate

$$z \equiv \xi + i\eta \equiv \rho - \rho_0 + i\eta \tag{5.15}$$

so that we are expanding around the point $(\rho, \eta) = (\rho_0, 0)$. It is actually convenient for our problem to define a complex variable

$$w(z) = 2\partial_z V = (\frac{y}{\rho_0} - ix_2) \tag{5.16}$$

where we also used an approximate form of (5.4). Equation (5.5) implies that w is a holomorphic function of z. We see that w is defined on the right half plane: $Re(w) \geq 0$. Equation (5.5) is simply the statement that the change of variables is holomorphic. Solutions are simply given by finding a conformal transformation that maps the w half-plane into a configuration in z-plane containing various cuts of lines specified by a general configuration, like those in figure 5.6.

For example we could take $z = w^2$. This maps the w half-plane into the z plane with a cut running on the negative real axis. More explicitly, this leads to $V \sim Re(z^{3/2})$. This is the solution near the tip of a disk, see figure 5.6(b).

Once we have found this map we can go back to the general ansatz (5.7)-(5.11) and write the resulting answer. When we do this we note that $\dot{V} \sim \partial_\xi V/\rho_0$ and that $\ddot{V} \sim \partial_\xi^2 V/\rho_0^2$. Since ρ_0 is very large in our limit we keep only the leading order terms in ρ_0. After doing this we find the approximate solution [3]

$$ds_{10}^2 \sim 4\rho_0 \left\{ -(1+\frac{1}{\rho_0} f^{-1} |\partial_w z|^2) dt^2 + d\Omega_5^2 + \frac{f}{\rho_0} \left[dw d\bar{w} + (\frac{w+\bar{w}}{2})^2 d\Omega_2^2 \right] \right\} \quad (5.17)$$

where $f = \frac{\partial_w z + \partial_{\bar{w}} \bar{z}}{2(w+\bar{w})}$.

Let us first consider the specific case where $z = w^2$. This describes the configuration near the tip of the disks. In this case we find that $f = 1$ and the metric in the four dimensional space parametrized by $w, \bar{w}, \Omega_2$ is flat. In addition, we see that (5.14) is indeed saturated at $w = 0$.

Now let us go back to (5.17) and take a general pp-wave limit. We will take $\rho_0 \to \infty$ and scale out the overall factor ρ_0 away from the solution. In other words, we parameterize S^5 as

$$d\Omega_5^2 = d\varphi^2 \cos^2\theta + d\theta^2 + \sin^2\theta d\Omega_3^2 \sim d\varphi^2 (1 - \frac{\vec{r}^{\,2}}{4\rho_0}) + \frac{1}{4\rho_0} d\vec{r}^{\,2} \quad (5.18)$$

where we expanded around $\frac{r}{\sqrt{4\rho_0}} = \theta \sim 0$ and kept $\vec{r}$ finite in the limit. In addition, we set

$$dt = dx^+ , \qquad d\varphi = dx^+ - \frac{1}{4\rho_0} dx^- \quad (5.19)$$

$$-p_+ = E - J , \qquad -p_- = \frac{J}{4\rho_0} \quad (5.20)$$

$$4\rho_0 = R_{S^5}^2 \quad (5.21)$$

where the second line tells us how the generators transform and finally the last line is stating that the parameter ρ_0 is physically the size of the S^5 (we have set $\alpha' = 1$).

[3] The rest of the fields, i.e. the dilaton and fluxes are the the same as in (5.22)-(5.26), with $t = x^+$.

After this pp-wave limit is taken for (5.7)-(5.11), the solution takes the form

$$
\begin{aligned}
ds_{10}^2 &= 2dx^+dx^- - (4f^{-1}|\partial_w z|^2 + \vec{r}^{\,2})(dx^+)^2 + d\ \vec{r}^{\,2} + 4f(dwd\bar{w} + (\frac{w+\bar{w}}{2})^2 d\Omega_2^2) \\
e^{2\Phi} &= 4f \qquad (5.22) \\
B_2 &= i\left[\frac{(w+\bar{w})}{2}(\partial_w z - \partial_{\bar{w}}\bar{z}) - (\ z - \bar{z})\right] d^2\Omega \qquad (5.23) \\
C_1 &= i(w+\bar{w})\frac{(\partial_w z - \partial_{\bar{w}}\ \bar{z})}{(\partial_w z + \partial_{\bar{w}}\ \bar{z})}dx^+ \qquad (5.24) \\
C_3 &= -(w+\bar{w})^3 f dx^+ \wedge d^2\Omega \qquad (5.25) \\
f &\equiv \frac{\partial_w z + \partial_{\bar{w}}\bar{z}}{2(w+\bar{w})} \qquad (5.26)
\end{aligned}
$$

where z is a holomorphic function of w. This is an exact solution of IIA supergravity. When a string is quantized in lightcone gauge on this pp wave it leads to a $(4,4)$ supersymmetric lightcone lagrangian, which will be discussed in section 5.4.2. One can also introduce two parameters by rescaling z and w. Similar classes of IIB pp-wave solutions and their sigma models were analyzed and classified in e.g. [135], [163], [16].

For the single tip solution

$$z = w^2 \qquad (5.27)$$

we get

$$ds_{10}^2 = -2dx^+dx^- - (\vec{r}^{\,2} + 4\vec{u}^{\,2})(dx^+)^2 + d\ \vec{r}^{\,2} + d\vec{u}^{\,2} \qquad (5.28)$$

where $\vec{r}$ and $\vec{u}$ each parameterize R^4. This is a IIA plane wave with $SO(4)\times SO(3)$ isometry and it was considered before in [171],[98],[97].

In conclusion, the expansion of the metric around the trajectories of BPS particles locally looks like a IIA plane wave (5.28) if the tip of the disk is far from other disks. When it is close to other disks we need to use the more general expression (5.22)-(5.26). We will analyze in detail specific cases in the later several sections. In the limit that we boost away the g_{++} component of the metric, the solution (5.22)-(5.26) becomes $R^{5,1}$ times a transverse four dimensional part of the solution which is a superposition of NS5 branes. Notice that f is a solution of the Laplace equation in the four dimensions parametrized by $w, \bar{w}, \Omega_2$. This is related to the fact that we should interpret the space between two closely spaced disks

as being produced by NS5 branes. This will become more clear after we analyze specific solutions in later sections.

The rescaling of J in (5.20) has some physical significance since it will appear when we express the energy of near BPS states in terms of J. In other words, the light cone hamiltonian for a string on the IIA plane wave describes massive particles propagating on the worldsheet. Four of the bosons have mass 1 and the other four have mass 2. The lightcone energy for each particle of momentum n and mass m is

$$(E-J)_n = (-p_+)_n = \sqrt{m^2 + \frac{n^2}{p_-^2}} = \sqrt{m^2 + R^4_{S^5}\frac{n^2}{J^2}}\ , \qquad \alpha' = 1 \tag{5.29}$$

where the masses of the worldsheet fields are $m = 1, 2$ depending on the type of scalar or fermion that we consider on the worldsheet. The subindex n reminds us that this is the contribution from a particle with a given momentum along the string. Since the total momentum along the string should vanish, we need to have more than one particle carrying momentum, each giving rise to a contribution similar to (5.29). Note that the form of the spectrum is completely universal for all solutions, as long as the tip is far enough from other disks. On the the other hand the value of ρ_0 at the tip depends on the details of the solution. It depends not only on the theory we consider but also on the particular vacuum that we are expanding around. In the following sections we will compute the dependence of ρ_0 on the particular parameters of each theory for some specific vacua.

When we can isolate a single disk we can always take pp-wave limit of the solution to the IIA plane wave (5.27), (5.28) near the tip of this single disk. There are many other situations when nearby disks are very close, and we need to include also the region between disks, i.e. the region produced by NS5 branes. In these cases, the geometry parametrized by the second four coordinates $w, \bar{w}, \Omega_2$ is more complicated. We will discuss it in following sections.

As is usual in the gravity/field theory correspondence one has to be careful about the regime of validity of the gravity solutions, and in our case, we should also worry about the following. In the field theory we have many vacua. So we can have tunnelling between the vacua. On the gravity side we have the same issue, we can tunnel between different

solutions of the system. In order to understand this tunnelling problem it is instructive to consider vacua whose solutions are very close to the original solution. Small deformations of a given solution that still preserve all the supersymmetries can be obtained, in the 11d language by considering small "ripples" in the regions connecting M2 and M5 regions. In the IIA description these become D0 branes. For very small excitations these D0 branes sit at $\rho = 0$ at the position of the disks. At these positions it costs zero energy to add the D0 branes. In the electrostatic description we are adding a small disk close to the large disk, as in figure 5.4(c). In order to estimate the tunnelling amplitude we need to understand how we go from a configuration with no D0 branes to a configuration with D0 branes. In a region where we have a finite size three cycle Σ_3 (see figure 5.5) with flux N_5 we can create N_5 D0 branes via a D2 instanton that wraps the Σ_3 (see [136]). We see that such processes will be suppressed if the string coupling in this region is small and the Σ_3 is sufficiently large. These will be discussed in detail in chapter 6.

5.3 Index counting the vacua and BPS states

In this section we turn to the discussion of various field theories with $\widetilde{SU}(2|4)$ symmetry group and the index counting their vacua and excited BPS states.

It is convenient to start with $\mathcal{N} = 4$ super Yang Mills on $R \times S^3$. This theory is dual to $AdS_5 \times S^5$ and its symmetry group is the superconformal group $SU(2,2|4)$. The bosonic subgroup of the superconformal group is $SO(2,4) \times SO(6)$. It is convenient to focus on an $SU(2)_L \subset SO(4) \subset SO(2,4)$. This $SU(2)_L$ is embedded in the $SO(4)$ symmetry group that rotates the S^3 on which the field theory is defined. If we take the full superconformal algebra and we truncate it to the subset that is invariant under $SU(2)_L$ we clearly get a new algebra. This algebra forms the supergroup $\widetilde{SU}(2|4)$, where the tilde here denotes that we take its universal cover. In other words, the bosonic subgroup is $R \times SU(2) \times SU(4)$ [4] . This is the symmetry group of the theories we are going to consider below.[5]

[4] If we replace R by $U(1)$ we get the compact from of $SU(2|4)$.

[5] This symmetry group also appears when we consider 1/2 BPS states in $AdS_{4,7} \times S^{7,4}$ M-theory solutions [126]. A closely related supergroup, $SU(2,2|2)$, is the $\mathcal{N} = 2$ superconformal group in 4 dimensions.

We will get the theories of interest by quotienting $\mathcal{N} = 4$ super Yang Mills by various subgroups of $SU(2)_L$. For example, if we quotient by the whole $SU(2)_L$ group we are left with the plane wave matrix model [115],[121]. We get a reduction to 0+1 dimensions because all Kaluza Klein modes on S^3 carry $SU(2)_L$ quantum numbers except for the lowest ones. The other theories are obtained by quotienting by Z_k and $U(1)_L$ subgroups of $SU(2)_L$. We will discuss these theories in detail below.[6]

Since all theories have a common symmetry group they share some properties. One property that we will discuss in some detail are 1/2 BPS states carrying $SO(6)$ angular momenta. These are states carrying energy E equal to the angular momentum under an $SO(2) \subset SO(6)$ generated by J. The condition $E = J$ is the BPS bound. The fact that these 1/2 BPS states are fully protected follows from the discussion in [58, 114]. Moreover, the arguments in [58, 114] allow us to count precisely these BPS states. Actually, to study BPS states it is convenient to define the index (c.f. [118])

$$I(\beta_i) = Tr\left[(-1)^F e^{-\mu(E-2S-J_1-J_2-J_3)} e^{-\beta_1(E-J_1)} e^{-\beta_2(E-J_2)} e^{-\beta_3(E-J_3)}\right] \tag{5.30}$$

where $S = S_3$ is one of the generators of $SU(2)$, $J_1 = M_{12}$, $J_2 = M_{34}$ and $J_3 = M_{56}$ are $SO(6)$ Cartan generators. Let us explain why (5.30) is an index. Let us consider the supercharge $Q^\dagger = Q^\dagger_{-+++}$, where the indices indicate the charges under (S, J_1, J_2, J_3). This supercharge has $E = 1/2$. This supercharge and its adjoint obey the anticommutation relation

$$\{Q, Q^\dagger\} = \mathcal{U} \equiv E - 2S - (J_1 + J_2 + J_3) \tag{5.31}$$

In addition the combinations $E - J_i$ commute with the supercharges in (5.31). Using the standard arguments (see [180]) any state with nonzero values of $\mathcal{U}$ does not contribute to (5.32). By evaluating (5.30) we will be able to find which BPS representations should remain as we change the coupling. The index (5.30) contains the same information as the indices defined in [58], see [118]. In order to count 1/2 BPS states we can use a simplified

[6] Notice that this truncation procedure is a convenient way to construct the lagrangian, but we cannot get the full quantum spectrum of the plane wave matrix model by restricting to $SU(2)_L$ invariant states of the full $\mathcal{N} = 4$ super Yang Mills theory.

version of (5.30) obtained by taking the limit when $\beta_3 \to \infty$. In this limit the index depends only on $q \equiv e^{-\beta_1-\beta_2}$

$$I_N(q) \equiv \sum_{J=0}^{\infty} D(N,J) q^J = \lim_{\beta_3 \to +\infty} I(\beta_1, \beta_2, \beta_3) \ , \qquad q = e^{-\beta_1-\beta_2} \tag{5.32}$$

where $J = J_3$. This partition function counts the number of 1/2 BPS states $D(N, J)$ in the system. Below we will compute (5.32) for various theories. We will not compute (5.30) here, but it could be computed using the techniques in [4],[160].

In the limit that $1 \ll J \ll N$ we will identify these states as massless geodesics in the geometric description. Notice that, even though we use some of the techniques in [126] to describe the *vacua* of these theories, we do not include backreaction when we consider 1/2 BPS states.[7] We are also going to study the near BPS limit, with J large and $\hat{E} = E - J$ finite. For excitations along the S^5 the one loop perturbative correction is the same as in the $\mathcal{N} = 4$ parent Yang Mills theory. On the gravity side, we will find that, at strong 't Hooft coupling, the result that differs from the naive extrapolation of the weak coupling results. This implies that there exist some interpolating functions in the spectrums. We could similarly study other solutions with large quantum numbers under $SO(6)$, such as the configurations considered in e.g. [73, 148] which have several large quantum numbers. In this theory we could also have BPS and near BPS configurations with large $SO(3)$ spin, which we will not study in this section.

In these theories we have many vacua and, in principle, we can tunnel among the different vacua. In most of the discussion we will assume that we are in a regime in parameter space where we can neglect the effects of tunneling. This tunneling is suppressed in the 't Hooft regime where strings are weakly coupled. Note that despite tunneling the vacua remain degenerate since they all contribute positively to the index (5.32).

[7] The 1/2 BPS states of the theories considered here preserve less supersymmetry than the 1/2 BPS states that were considered in [126]. In other words, the 1/2 BPS states of [126] preserve the same amount of supersymmetry as the *vacua* (which have $J = 0$) of the theories considered in this section. Here we start with theories with 16 supercharges, while [126] started with theories with 32 supercharges.

As an aside, note that the degeneracy of states in $\mathcal{N}=4$ super Yang Mills can be written in various equivalent forms [53, 40]

$$\sum_{N,J=0}^{\infty} p^N q^{N^2/2} q^J D(N,J) = \prod_{n=1}^{\infty}(1+pq^{n-\frac{1}{2}}) \tag{5.33}$$

$$\sum_{J=0}^{\infty} D(N,J) q^J = \frac{1}{\prod_{n=1}^{N}(1-q^n)} \tag{5.34}$$

$$I_{\mathcal{N}=4}(p,q) \equiv \sum_{N,J=0}^{\infty} p^N q^J D(N,J) = \frac{1}{\prod_{n=0}^{\infty}(1-pq^n)} \tag{5.35}$$

In the first form we express it as a system of fermions in a harmonic oscillator potential. In the third form it looks like a system of bosons in a harmonic oscillator potential.

5.4 Plane wave matrix model

5.4.1 General vacua and near-BPS excitations

In this section we discuss the plane wave matrix model [33, 137, 57, 117, 58, 114, 115, 121, 72], both its vacua in the field theory side and their dual gravity descriptions.

This theory arises by truncating the $\mathcal{N}=4$ theory to 0+1 dimensions by keeping all free field theory states that are invariant under $SU(2)_L$ and keeping the same interactions for these states that we had in $\mathcal{N}=4$ super Yang Mills [115]. We keep the zero modes for $SO(6)$ scalars and truncate the gauge field to $A_{\mathcal{N}=4} = X_1\omega_1 + X_2\omega_2 + X_3\omega_3$, where ω_i are three left invariant one-forms on S^3. Thus the X_i are the scalars that transform under $SO(3)$.

This theory has many vacua. These vacua are obtained by setting the scalars X_i equal to $SU(2)$ Lie algebra generators. In fact the vacua are in one to one correspondence with $SU(2)$ representations of dimension N. Suppose that we have $N(n)$ copies of the irreducible representation of dimension n such that

$$N = \sum_n N(n) n \tag{5.36}$$

Each choice of partition of N gives us a different vacuum. So the number of vacua is equal to the partitions of N, $P(N)$.

$\mathcal{N}=4$ super Yang Mills has a unique vacuum. On the other hand, any solution of the plane wave matrix model can be uplifted to a zero energy solution of $\mathcal{N}=4$ super Yang Mills. What do the various plane wave matrix model vacua correspond to in $\mathcal{N}=4$ super Yang Mills? It turns out that these are simply large gauge transformations of the ordinary vacuum. The solutions uplift to $A_{\mathcal{N}=4}=(\omega_1 J_1+\omega_2 J_2+\omega_3 J_3)=-i\,(dg)\,g^{-1}$, were g is an $SU(2)$ group element in the same representation as the J_i. This $SU(2)$ group is parameterizing the S^3. So they are pure gauge transformations from $A_{\mathcal{N}=4}=0$. In summary, in $\mathcal{N}=4$ super Yang Mills these different configurations are related by a gauge transformation. The gauge transformation is not $SU(2)_L$ invariant, even though the actual configurations are $SU(2)_L$ invariant. In the plane wave matrix model they are gauge inequivalent.

As in [137], it is possible to get the 2+1 theory in section 5.5 from a limit in which we take $\tilde{N}$ copies of the representation of dimension n and we take $n\to\infty$. For finite n we get a $U(\tilde{N})$ theory on a fuzzy sphere and in the $n\to\infty$ limit the fuzziness goes away [137].

One can also count the total number of 1/2 BPS states with $SO(6)$ charge J. These are given by the partition function

$$I_{PWMM}(p,q)=\sum_{N,J=0}^{\infty} D_{PWMM}(N,J)p^N q^J=\prod_{m=1}^{\infty} I_{\mathcal{N}=4}(p^m,q)=\frac{1}{\prod_{m=1}^{\infty}\prod_{n=0}^{\infty}(1-p^m q^n)} \tag{5.37}$$

Setting $q=0$ we get that the number of vacua are given by the partitions of N. It is interesting to estimate the large J and N behavior of this index. We obtain

$$D_{PWMM}(N,J)\sim e^{(3.189...)(NJ)^{1/3}} \tag{5.38}$$

where we assumed $J^2/N\gg 1$, $N^2/J\gg 1$. The fact that this is symmetric under $N\leftrightarrow J$ follows from the fact that (5.37) is symmetric under $p\leftrightarrow q$ up to the $n=0$ factor.

Now we turn to some aspects of the gravity solutions corresponding to the plane wave matrix model. In this case we should think of the electrostatic configuration as having an infinite disk at $\eta=0$ and the some finite number of disks of finite size at $\eta_i>0$, see figure 5.4(b). The background electric potential is

$$V_b\sim\rho^2\eta-\frac{2}{3}\eta^3 \tag{5.39}$$

We now consider the boundary conditions that correspond to the solutions dual to the plane wave matrix model and we consider a vacuum corresponding to a single large disk at distance $d \sim N_5$ from the $\eta = 0$ plane. These are the vacua corresponding to N_5 fivebranes. We write the leading solution of the potential in asymptotic region

$$V = \alpha(\rho^2\eta - \frac{2}{3}\eta^3) + \tilde{\Delta} \tag{5.40}$$

Plugging this form to our gravity solution, we find in order to asymptote to D0 brane near horizon geometry $\tilde{\Delta}$ has to take the form

$$\tilde{\Delta} = \frac{P\eta}{(\eta^2 + \rho^2)^{3/2}} \tag{5.41}$$

in the large $\rho,\ \eta$ region. Using the coordinate $r = 4\sqrt{\rho^2 + \eta^2}$ and $t = x_0$ we find that the leading order solution at large r in (5.7)-(5.11) is the standard D0 brane solution [102] at large r, with warp factor

$$Z = \frac{2^8 15P}{r^7\alpha}, \qquad \alpha = \frac{8}{g_s} \tag{5.42}$$

We now need to compute P. We compute the charge and the distance. Since we have images we have $P = 2dQ$. The distance if given in terms of N_5 by (5.13). In order to compute the charge we note that if we have a large disk with a size $\rho_0 \gg N_5$ then the configuration at large distances looks like a single conducting disk at $\eta = 0$ with some extra sources localized near $(\rho, \eta) = (\rho_0, 0)$. We can thus approximate the induced charge on the disk to be the induced charge we would have on the conducting plane at $\eta = 0$ if we had not introduced the disk. This induced charge is given simply by the external potential which is the first term in (5.40). We can thus approximate

$$Q = \frac{1}{4\pi}\int \partial_\eta V_{ext} = \frac{\alpha\rho_0^4}{8}, \qquad d = \frac{\pi}{2}N_5 \tag{5.43}$$

Now we can go back to the expression for Z and write it as

$$Z = \frac{2^5 15\pi\rho_0^4 N_5}{r^7} \tag{5.44}$$

where we are in the regime where the disk is very close to the $\eta = 0$ plane.

We can now compare with the result in [102]

$$Z = \frac{2^7\pi^{9/2}\Gamma(7/2)g_{YM0}^2 N_0}{r^7} = \frac{2^4 15\pi^5 g_{YM0}^2 N_0}{r^7} \tag{5.45}$$

Comparing the two we find

$$\rho_0^4 = \frac{1}{2}\pi^4 g_{YM0}^2 N_2, \qquad \frac{R_{S^5}^2}{\alpha'} = \ 4\rho_0 = 4\left(\frac{\pi^4 g_{YM0}^2 N_2}{2m^3}\right)^{1/4} \tag{5.46}$$

in the strong coupling regime.

So the leading asymptotic form of the solution is

$$V = V_b + P\frac{\eta}{(\rho^2+\eta^2)^{3/2}} \tag{5.47}$$

where we have included the external potential plus the leading dipole moment produced by the disks. The leading contribution is a dipole moment because the conducting disk at $\eta = 0$ gives an image with the opposite charge, so that there is no monopole component of the field at large ρ, η. The total number of D0 branes N_0 is proportional to the dipole moment P. This dipole moment is given by

$$P = 2\sum_i \eta_i Q_i \sim N_0 = \sum_i \left(\sum_{j<i} N_5^j\right) N_2^i \tag{5.48}$$

where the index i runs over the various disks. Notice that the difference between neighboring disks $d_i = \eta_{i+1} - \eta_i$ is proportional to the fivebrane charge. So the distances d_i are quantized. This formula, (5.48), should be compared to (5.36) by identifying $n \sim d_i$ and $N(n) = N_2^i$. The exact proportionality can be found in above discussions.

In [125] this problem was analyzed using technique developed by Polchinski and Strassler in [159], which consists in starting with configurations of D0 branes smeared on two spheres. In our language, this is a limit when we replace the disks by point charges sitting at $\rho = 0$. This approximation is correct as long as the distance between the disks is much bigger than the sizes of the disks and we look at the solution far away from the disks[8] .In this appendix,

[8]We can make this relation more precise as follows. Suppose that the potential in the asymptotic region behaves as $V = \rho^2\eta - \frac{2}{3}\eta^3 + \Delta$, where will treat Δ as a perturbation. Then from the IIA ansatz (5.7)-(5.11) we can write the solution as in [125] and find the warp factor Z in [125]. This gives $Z = \frac{-1}{2\rho^2\eta}(\frac{2}{\rho}\partial_\rho\Delta + \partial_\eta^2\Delta)$

we discuss the charge N_2 and N_5 and asymptotic matching of the solutions dual to vacua of the plane wave matrix model. We then discuss the interpolating function f in the leading gravity approximation..

From the field theory point of view it looks like the simplest vacuum is the one with all $X = 0$. This case corresponds to having N_0 copies of the trivial (dimension one) representation of $SU(2)$. In the gravity description this corresponds to having a single disk at a distance of one unit from the conducting surface at $\eta = 0$, see figure 5.6(a). Unfortunately, since this vacuum corresponds to a single fivebrane, the gravity approximation will not be good near the fivebrane. We will focus on this situation in the next section. However, we can consider vacua corresponding to many copies of dimension N_5 representations of $SU(2)$. These involve N_5 fivebranes and we will be able to give interesting solutions, at least in the region relevant for the description of near BPS states. It should be possible to extrapolate these solutions to smaller values of N_5 using conformal field theory. Let us now study the case that we have only a single disk at a small distance from the infinite disk at $\eta = 0$. So we consider a situation with $N_5 \ll N_0$. Based on the discussion in [137] we expect that the N_0 D0 branes blow up into N_5 NS5 branes. Of course, our solution will be smooth, but we will see that there is a sense in which we have N_5 fivebranes. The appearance of fivebranes is probably connected with the picture in [32] for 1/2 BPS states in terms of eigenvalues that lie on a five sphere. We will discuss more about this in chapter 6.

The explicit solutions for the plane-wave matrix model is very difficult to solve due to that the solutions are generally given by integral equation or coupled integral equations if there are more than one pair of the disks. Nevertheless, we can take a large disk limit and then it is sovable in terms of conformal transform. We will discuss the solutions in the large disk limit, which corresponds to the pp-wave limit.

---

and $B_2 = \frac{1}{\rho}\partial_\rho\left(\frac{4\eta^2}{\rho}\partial_\rho\Delta + 2\eta\partial_\eta\Delta - 2\Delta\right)d^2\Omega$. We get $\Delta = \frac{P\eta}{(\rho^2+\eta^2)^{3/2}}$ by comparing the leading order approximation $Z = \frac{R^7}{r^7}$ and the fluxes $H_3 = \alpha r^{-7}(T_3 - \frac{7}{3}V_3)$, $G_6 = g_s^{-1}\alpha r^{-7}(\frac{1}{3} *_9 T_3 - \frac{7}{3} *_9 V_3)$, which are dual to the mass terms (see [125]), where $r = (\rho^2 + \eta^2)^{1/2}$, and ρ and η are radial variables in $SO(6)$ and $SO(3)$ directions. If we replace Z by the expression corresponding to multi-center D0 branes uniformly smeared on several S^2s, then we get $\Delta = \sum_i Q_i\left[\frac{1}{[(\eta-\eta_0^{(i)})^2+\rho^2]^{1/2}} - \frac{1}{[(\eta+\eta_0^{(i)})^2+\rho^2]^{1/2}}\right]$, this is precisely the limit when the disks above the $\eta = 0$ plane are treated as point charges.

If we have n disks above the infinite disk, then the general solution is

$$\begin{aligned} \partial_w z &= \frac{(w-ia_1)(w-ia_2)\cdots(w-ia_n)}{(w-ic_1)(w-ic_2)\cdots(w-ic_n)}(-ia_{n+1}) \\ &\quad \text{with} \quad a_1 < c_1 < a_2 < c_2 < \cdots < c_{n-1} < a_n < c_n < a_{n+1} \end{aligned} \tag{5.49}$$

where $w = ia_i$ are the location of n tips and $w = ic_i$ are the locations of n sets of fivebranes.

Now we study the simplest vacua of a pair of disks. We can get the solution from above formula for one disk, and do a rescaling. The solution is

$$\partial_w z = i\frac{(w-ia)}{w} \tag{5.50}$$

In this case, the function f in (5.22)-(5.26) becomes

$$f = \frac{a}{2|w|^2} \tag{5.51}$$

and the contribution to g_{++} is

$$4f^{-1}|\partial_w z|^2 = \frac{8}{a}|w-ia|^2 \tag{5.52}$$

So we see that we get the near horizon region of fivebranes. The contribution (5.52) to the g_{++} metric component gives rise to a potential on the lightcone gauge string worldsheet. This potential localizes the string at some particular position along the throat. Writing $w = iae^{\phi+i\theta}$, the 10 dimensional solution is [9]

$$\begin{aligned} ds_{10}^2 &= -2dx^+dx^- + d\vec{r}^2 - \vec{r}^2 dx^{+2} - 4N_5(e^{2\phi} + 1 - 2e^{\phi}\cos\theta)dx^{+2} \\ &\quad + N_5(d\phi^2 + d\theta^2 + \sin^2\theta d\Omega_2^2) \end{aligned} \tag{5.53}$$

$$e^{\Phi} = g_s e^{-\phi} \tag{5.54}$$

$$C_1 = -\frac{1}{g_s}2\sqrt{N_5}(e^{2\phi} - e^{\phi}\cos\theta)dx^+ \tag{5.55}$$

$$C_3 = \frac{1}{g_s}N_5^{3/2}2e^{\phi}\sin^3\theta dx^+ \wedge d^2\Omega \tag{5.56}$$

$$H_3 = 2N_5\sin^2\theta d\theta \wedge d^2\Omega \tag{5.57}$$

where g_s is the value of the dilaton at the tip. By performing a boost $x^{\pm} \to \lambda^{\pm 1}x^{\pm}$ with $\lambda \to 0$ we set to zero all non-trivial terms involving dx^+ and we recover the usual fivebrane

[9] We set $\alpha' = 1$ here.

near horizon geometry [47], [48]. By taking a limit of small ϕ and θ we find the IIA plane-wave in (5.28).

An important parameter is the size of the S^5 in string theory units at the tip of the disks. This can be approximated as [10]

$$\frac{R^2_{S^5}}{\alpha'} = 4\pi \left(\frac{g^2_{YM0} N_2}{2m^3}\right)^{1/4}, \qquad N_2 = \frac{N_0}{N_5}, \qquad m = 1 \tag{5.58}$$

where m is the mass of the $SO(6)$ scalars and is set to 1. N_0 is the number of D0 branes or the rank of the gauge group in the plane wave matrix model. Our gravity approximation is good when we are in the regime of interest, $N_5 \ll N_0$, and the size of S^5 in string unit is large. From this result we can compute the spectrum of near BPS excitations with large angular momentum J. For fluctuations in the directions parametrized by $\vec{r}$ in (5.53) the spectrum is

$$(E-J)_n = \sqrt{1 + (4\pi)^2 \left(\frac{g^2_{YM0} N_0}{2m^3 N_5}\right)^{1/2} \frac{n^2}{J^2}} = 1 + 4\pi^2 \left(\frac{2g^2_{YM0} N_0}{m^3 N_5}\right)^{1/2} \frac{n^2}{J^2} + \cdots \tag{5.59}$$

Under general principles, in the t' Hooft limit, with N_5 fixed, we expect the spectrum to be of the form

$$(E-J)_n = 1 + f\left(\frac{g^2_{YM0} N_0}{m^3 N_5}, N_5\right) \frac{n^2}{J^2} + \cdots \tag{5.60}$$

in the large J limit.

The $N_5 = 1$ case has been analyzed perturbatively up to four loops in [72]. In our conventions[11] their result reads

$$\begin{aligned} f_{pert}\left(\frac{g^2_{YM0} N_0}{m^3}, N_5 = 1\right) &= \frac{2\pi^2 g^2_{YM0} N_0}{m^3} \left[1 - \frac{7}{8}\frac{g^2_{YM0} N_0}{m^3} + \frac{71}{32}\left(\frac{g^2_{YM0} N_0}{m^3}\right)^2 \right. \\ &\quad \left. - \frac{7767}{1024}\left(\frac{g^2_{YM0} N_0}{m^3}\right)^3 + \ldots\right] \end{aligned} \tag{5.61}$$

Of course we expect that the function f interpolates smoothly between the weak coupling result (5.61) and the strong coupling result (5.59).

[10] Our normalization of the action is $S = \frac{1}{g^2_{YM0}} \int \mathrm{Tr}\{\frac{1}{2}(D_0 Y_i)^2 - \frac{1}{2}m^2 Y_i^2 + \frac{1}{4}[Y_i, Y_j]^2 + \cdots\}$ where Y_i are the $SO(6)$ scalars. The dimensionless parameter is g^2_{YM0}/m^3.

[11] The relations between their variables and ours are $4\Lambda = \left(\frac{2}{M}\right)^3 N = \frac{g^2_{YM0} N_0}{m^3}$, $8\pi^2 \Lambda_r = f$.

Our gravity solutions are not valid for $N_5 = 1$, especially in the region relevant for this computation. On the other hand, we see that the quantity that determines f is the radius of the fivesphere. We can think of this solution as follows. Let us first use an approximation similar to that used by Polchinski and Strassler [159], [125]. In this case we approximate the solution by smearing D0 branes on a fivesphere, which we interpret as a fivebrane which carries D0 brane change. We then determine the size of the fivebrane by coupling it to the external fields that are responsible for inducing the mass on the D0 worldvolume. This gives the radius of the fivebrane. In fact, this was computed in [137] where the formula similar to (5.58) was found (the precise numerical factors were not computed in [137]). So it is natural to believe that (5.59) will still be the correct answer for $N_5 = 1$. In other words, the coupling constant "renormalization" that was found in [72] is interpreted here as a physical quantity giving us the size of the fivebrane in the gravity description at strong coupling. This is the situation for the first four coordinates. The fact that a single fivebrane has no near horizon region also suggests that something drastic happens to the second four directions that are transverse to the single fivebrane. We observed this feature for $N_5 = 1$ also from gauge theory side and will explain it in the next section.

Finally, let us discuss the issue of tunneling between different vacua. In general we can tunnel between the different vacua of the matrix model. But the tunneling can be suppressed in some regimes. For example, let us consider the case we discussed above where we consider the vacuum corresponding to a single large disk at a distance N_5 from the $\eta = 0$ plane, see figure 5.6(a). From the gravity point of view we can take one unit of charge from the large disk and put some other disks. Charge is not conserved in the process, but N_0 should be conserved. Reducing the charge of the large disk by one unit we are left with N_5 D0 branes to distribute in the geometry. So, for example, we can put another disk at a distance of one unit from the $\eta = 0$ plane with N_5 units of charge. In the geometry this transition is mediated by a D-brane instanton. The geometry between the original disk and the $\eta = 0$ plane can be approximated by the solution in section 5.6. That solution contains a non-contractible Σ_3, see figure 5.5. If we wrap an Euclidean D2 brane on this Σ_3 we find

that, since there is flux N_5 through it, we need N_5 D0 branes ending on it [136]. Thus, this instanton describes the creation of N_5 D0 branes. Its action is proportional to the action of the Euclidean D2 brane. This process describes the tunneling between the vacua in figures 5.4(d) and 5.4(e). If the volume of the Σ_3 is sufficiently large and the string coupling is sufficiently small this process will be suppressed. In order for this to be the case we need to arrange the field theory parameters appropriately. Notice that there is no instanton that produces a smaller number of D0 branes. This also agrees with the field theory. If we start with the vacuum with many copies of the N_5 dimensional representation of $SU(2)$, then we can take one of these representations and partition those N_5 D0 branes into lower dimensional representations. This is basically the process described by the above instanton. In other words, the fact that the D-brane instanton produces N_5 D0 branes matches with what we get in the field theory. More details on the instantons will be discussed in the chapter 6.

5.4.2 Single and multi-NS5 brane vacua and excitations

In previous sections we have mainly analyzed the near BPS states associated to string oscillations in the the first four dimensions, which are described by free massive fields on the worldsheet. In this section we mainly focus on the second four dimensions which are associated to fivebrane geometries. Since the spectrum depends on the vacuum we expand around, we will focus on the large J near BPS excitations around some particular vacua of the plane wave matrix model. We will consider first the $N_5 = 1$ vacuum and then the $N_5 > 1$ vacua, both from the gauge theory and gravity points of view. We also make some remarks about the simplest vacuum of the 2+1 super Yang Mills on $R \times S^2$.

Let us start by discussing the trivial vacuum of the matrix model, where we expand around the classical solution where all $X = 0$. This is the vacuum we denote by $N_5 = 1$ and which should correspond to a single fivebrane. When we expand around this vacuum we have 9 bosonic and 8 fermionic excitations which form a single representation of $\widetilde{SU}(2|4)$, corresponding to the Young supertableau in figure 5.7(a). Our notation for $\widetilde{SU}(2|4)$ repre-

sentations follows the one in [10],[11],[12],[24],[58]. We are interested in forming single trace excitations which should correspond to single string states in the geometry. For example, we can consider the state created by the field Z of the form $Tr[Z^J]$ [12] , where $Z = Y^5 + iY^6$.[13] This state is BPS and it belongs to the doubly atypical (or very short) representation whose Young supertableau is shown in figure 5.7(b). As in [33] we can consider near BPS states by writing states of roughly the form $\sum_l Tr[Y^i Z^l Y^j Z^{J-l}] e^{i2\pi \frac{ln}{J}}$ where $i, j = 1, \cdots 4$. We can view each insertion of the field Y^i as an "impurity" that propagates along the chain formed by the Z oscillators. These impurities are characterized by the momentum $p = n/J$ and a dispersion relation $\epsilon(p)$, where ϵ is the contribution of this impurity to $\hat{E} \equiv E - J$. Here we are thinking about a situation where we have an infinitely long chain where boundary effects can be neglected. These fields have $\epsilon(p = 0) = 1$, we can think of this as the "mass" of the particles. This is an exact result and can be understood as a consequence of the Goldstone theorem. Namely, when we pick the field Z and we construct the ground state of the string with powers of Z we are breaking $SO(6)$ to $SO(4)$. The excitations Y^i, $i = 1, \cdots, 4$ correspond to the action of the broken generators. This is a fact that does not even require supersymmetry. In other words, we are simply rotating the state $tr[Z^J]$. It is also useful to consider the supersymmetry that is preserved by this chain. Out of the supergroup $\widetilde{SU}(2|4)$ our choice of Z leaves an $SU(2)_G \times \widetilde{SU}(2|2)$ subgroup[14] that acts on the excitations that propagate along the string. The group $SU(2)_G$ together with one of the $SU(2)$ subgroups in $\widetilde{SU}(2|2)$ forms the $SO(4)$ in $SO(6)$ that rotates the first four dimensions. The second $SU(2)$ subgroup of $\widetilde{SU}(2|2)$ is the $SU(2)$ factor in $\widetilde{SU}(2|4)$ and rotates the three scalars X^i. We can use $\widetilde{SU}(2|2)$ to classify these excitations. The non-compact $U(1)$ in $\widetilde{SU}(2|2)$ corresponds to the generator $\hat{E} = E - J$ and gives us the mass of the particle. The fields Y^i belong to the fundamental representation of $\widetilde{SU}(2|2)$ whose Young supertableau is in figure 5.7(c). In addition they transform in the spin one half representation of $SU(2)_G$. We can think of these excitations as "quasiparticles" that propagate along the string. The

[12] We denote the field Z and its creation operator by the same letter.

[13] In this section Y^i, $i = 1, \cdots, 6$ are the scalars that transform under $SO(6)$ and X^i are the ones transforming under $SO(3)$.

[14] The $_G$ subindex indicates that it is global symmetry that commutes with supersymmetry.

properties of these quasiparticles were studied in great detail in [72] where the dispersion relation and particular components of the S-matrix were computed to four loops. These quasiparticles contain four bosons and four fermions.

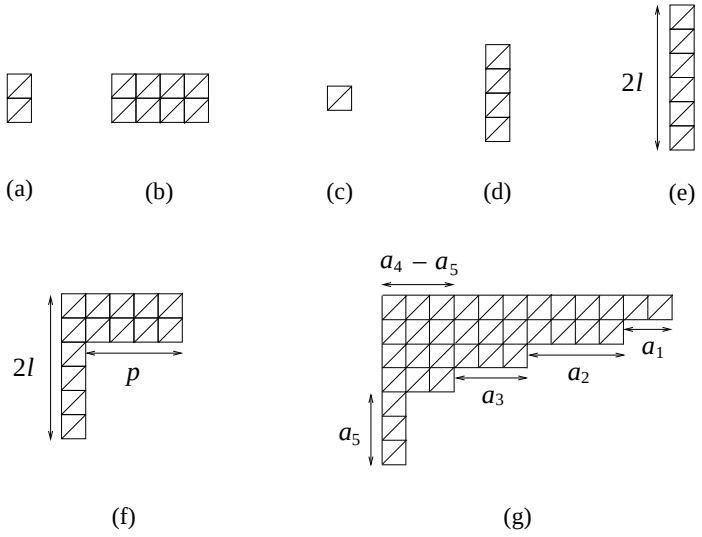

Figure 5.7: Young supertableaux corresponding to various representations of $SU(2|4)$ or $SU(2|2)$ discussed in the text. In (e) and (f), $2(l-2) = a_5, p = a_2$ for $SU(4|2)$ Dynkin labels. Figure (g) shows the correspondence between supertableau and Dynkin labels for a general physically allowed representation $(a_1, a_2, a_3|a_4|a_5)$ of $SU(4|2)$, see also [10],[11],[12],[24],[58].

So far we have been discussing mainly the fields Y^i and the fermions which have $E-J=1$. What about the other fields in the theory? There are four other elementary fields which have $\Delta - J = 2$. These are the three scalars X^i of $SO(3)$ and the field $\bar{Z}$ plus four fermions. Naively, we might think that these would lead to mass two impurities that propagate along the string. This is not the case. Actually, what happens is that they mix with the fields that we have already described and do not lead to new quasiparticles [27]. For example an insertion of the field $\bar{Z}$, such as $tr[\bar{Z}Z^{J+1}]$ mixes with the states $tr[Y^iZ^lY^iZ^{J-l}]$. The result of this mixing is such that the resulting spectrum can be fully understood in terms of two quasiparticles of mass one that propagate along the string. Something similar happens with the insertion of the $SO(3)$ scalar X^i, which mixes with the insertion of two fermions of individual mass one. In fact, the one loop Hamiltonian in this sector is a truncation of the one loop Hamiltonian of $\mathcal{N}=4$ SYM in [28]. So the results we are mentioning here follow

in a direct way from the explicit diagonalization undertaken in [27]. The final conclusion of this discussion, is that in perturbation theory we have a chain which contains impurities with mass one, that transform in the fundamental of $\widetilde{SU}(2|2)$ and fundamental of $SU(2)_G$. We have four bosons and four fermions, which can be viewed as the Goldstone modes of the symmetries broken by the BPS operator $tr[Z^J]$. This spectrum is compatible with the index (5.30) evaluated on single trace states.

Let us now discuss what happens at large 't Hooft coupling. The radius of the fivebrane is given by (5.58) (with $N_5 = 1$). In addition, we have seen that the near BPS states are described by the pp-wave geometry (5.53)-(5.57) which corresponds to the near horizon region of N_5 fivebranes. The first four transverse dimensions correspond to the motion of the string in the direction of the fivebranes and the spectrum contains particles that transform in the fundamental of $\widetilde{SU}(2|2)$ and the fundamental of $SU(2)_G$ as we had in the weak coupling analysis. The dispersion relation is given by the usual relativistic formula (5.59).

On the other hand, when we consider the fate of the last four transverse dimensions we run into trouble with the geometric description. We see that the solution (5.53)-(5.57) does not make sense for $N_5 = 1$ since a single fivebrane is not supposed to have a near horizon region [47]. The reason is that the near horizon region involves a bosonic WZW model with level $k = N_5 - 2$ and this theory is unitary only if $N_5 - 2 \geq 0$. In our context, we also have RR fields that try to push the string into the near horizon region. Since for $N_5 = 1$ we do not have such a region, the simplest assumption is that the second four dimensions are somehow not present in out pp-wave limit. This would agree with what we saw in the weak coupling analysis above, where we did not have any quasiparticles propagating along the string corresponding to the second four dimensions. Of course, a string quantized in lightcone gauge is not Lorentz invariant in six dimensions. But perhaps this is not a problem in this case, since the presence of RR fields breaks Lorentz invariance. Nevertheless, one would like to understand the background in a more precise way in the covariant formalism, so that one can ensure that we have a good string theory solution.

In order to find a better defined string theory we need to consider $N_5 > 1$. So, let us consider what happens when we expand around the vacuum of the plane wave matrix model corresponding to $N_5 > 1$. This is the vacuum where the matrices X_i are the generators of the dimension N_5 representation of $SU(2)$. We would like to understand the similarities and differences between these vacua and the $N_5 = 1$ vacuum. When we expand around these vacua we find that we have N_5 $\widetilde{SU}(2|4)$ supermultiplets, the ones whose Young supertableaux are given in figure 5.7(e) with $l = 1, \cdots, N_5$ [58]. We can view them as the Kaluza Klein modes on a fuzzy S^2. The subsector of this theory where we consider only excitations of the first Kaluza Klein mode is the same as the one we had in the $N_5 = 1$ sector. In fact, the one loop Hamiltonian for these excitations is exactly the same as the one we had for the $N_5 = 1$ case. This can be seen as follows. Since these modes are proportional to the identity matrix in the $N_5 \times N_5$ space that gives rise to the fuzzy sphere we see that their interactions are the same as the ones we had around the $N_5 = 1$ vacuum. The only difference could arise when we consider diagrams that come from one loop propagator corrections. But the value of these propagator corrections is determined by the condition that the energy of the state $tr[Z^J]$ is not shifted, since it is a BPS state. One difference, relative to the expansion around the $N_5 = 1$ vacuum is that the one loop Hamiltonian is proportional to $g^2_{YM0}N_2/N_5$ as opposed to $g^2_{YM0}N_0$ (where $N_0 = N_2N_5$). More precisely, we find that the function f in (5.60) has the form

$$f(\frac{g^2_{YM0}N_0}{m^3N_5}, N_5) = 2\pi^2\frac{g^2_{YM0}N_0}{m^3N_5^2} + \cdots \tag{5.62}$$

for small 't Hooft coupling. We obtain this result as follows. First we notice that $\hat{E} = 1$ excitations are given by diagonal matrices in the $N_5 \times N_5$ blocks that produce the fuzzy sphere. These matrices are $N_2 \times N_2$ matrices. In other words, the relevant fields can be expressed as $Y^i = \mathbf{1}_{N_5\times N_5} \otimes \tilde{Y}^i$ where $\tilde{Y}^i$ are $N_2 \times N_2$ matrices, with $N_2 \equiv N_0/N_5$. Then the action truncated to the $\tilde{Y}^i$ fields looks like the $N_5 = 1$ action except that we get an extra factor of N_5 from the trace over the diagonal matrix $\mathbf{1}_{N_5\times N_5}$. This effectively changes the coupling constant $g^2_{YM0} \to \tilde{g}^2_{YM0} = g^2_{YM0}/N_5$. Since the $\tilde{Y}^i$ fields are $N_2 \times N_2$ matrices we see that corrections in this subsector will be proportional to $\tilde{g}^2_{YM0}N_2$. Notice that (5.62) it

involves a different combination of N_0 and N_5 than the one that appears at strong coupling (5.59). So the interpolating function in (5.60) should have a non-trivial N_5 dependence. In summary, at one loop, the excitations built out of impurities in the first Kaluza Klein harmonic on the fuzzy S^2 give rise to four bosonic and fermionic quasiparticles of mass $\hat{E} = 1$ as we had in the $N_5 = 1$ case.

Let us now focus on the second Kaluza Klein mode, given by the supermultiplet of $\widetilde{SU}(2|4)$ in figure 5.7(d). This multiplet contains four bosonic and four fermionic states of mass $\hat{E} = E - J = 2$. These eight states transform in the $\widetilde{SU}(2|2)$ representation of figure 5.7(a). Let us describe how the bosonic states arise. We expand $Z = \tilde{Z} + \mathcal{J}^i Z_i + \cdots$ in fuzzy sphere Kaluza Klein harmonics using the $N_5 \times N_5$ matrices $\mathcal{J}^i$ which give a representation of $SU(2)$ (see [59]). Three of the states correspond to the impurities Z_i and they are in the $(1,0)$ representations of $SU(2) \times SU(2) \subset \widetilde{SU}(2|2)$ and they are singlets of $SU(2)_G$. The fourth state, denoted by Φ, arises when we expand $X^i = \mathcal{J}^i(\mathbf{1} + \Phi) + \cdots$. This has $E = 2$ and spin zero under all $SU(2)$s. It gives rise to an excitation with $\hat{E} = E - J = 2$ and spin zero. In addition to these four bosonic states we have their fermionic partners. When we consider BPS states with $E - 2S - \sum_i J_i = 0$, the only bosonic state that contributes is Z_+, which has $S = 1$. Thus the state $tr[Z_+ \tilde{Z}^J]$ is BPS. In order to ensure that its energy is not corrected we need to check that it cannot combine with other BPS states. The analysis in [58] tells us which representations this could combine with. By looking explicitly at the ones arising when we construct single trace states we can see that these other representations are not present. This is a result that is exact in the planar limit. In appendix G of [127] we use the index defined in (5.30) to prove the above statement.

What we learned is that for $N_5 > 1$, as opposed to the case with $N_5 = 1$, we have a new quasiparticle of mass two propagating along the string. In fact, the same argument would go through for the case of 2+1 SYM on $R \times S^2$ in section 5.5, expanded around the trivial vacuum, and the new supermultiplet correspond to the three derivatives D_i, $i = 0, 1, 2$ and the seventh scalar Φ, and their fermionic partners. They have mass 2 and correspond to the second four coordinates of the IIA plane wave. In all these cases we have

extra quasiparticles propagating along the string. This agrees with the fact that in string theory we have eight transverse directions for the string. The first four dimensions behave as we discussed above and its presence is ensured by the $SO(6)$ symmetry. The details of the second four dimensions depend on the vacuum we expand around. So let us concentrate more on these second four dimensions.

We will now discuss the two dimensional field theory that describes the second set of four transverse dimensions for a string in light cone gauge moving in the pp-wave geometry (5.53)-(5.57). The target space for this two dimensional theory is $R \times S^3$ with an H_3 flux on the S^3 equal to N_5 and a linear dilaton in the R direction. These are the dimensions parametrized by ϕ, θ, Ω_2 in (5.53). In addition we have a potential which localizes the string at some point along the throat and at a point in S^3. This potential arises from the g_{++} component of the metric in (5.53). Ignoring the potential for a moment we see that we have a the conformal field theory describing the throat of N_5 fivebranes [47]. The potential breaks the $SO(4)$ rotation symmetry of the throat region to $SO(3)$. The resulting sigma model has (4,4) supersymmetry on the worldsheet. When the potential is non-zero the supersymmetry in the 1+1 dimensional worldsheet theory is of a peculiar kind [153]. In ordinary global (4,4) supersymmetry the supercharges transform under an $SU(2) \times SU(2)$ R-symmetry but those symmetries do not appear in the right hand side of the supersymmetry algebra[15] . Let us denote the supercharges by $Q^i_\pm$, where $i = 1, \cdots, 4$ are $SO(4) = SU(2) \times SU(2)$ indices, and $\pm$ indicates two dimensional chirality. The anti-commutators of these supercharges have the form

$$\{Q^i_+, Q^j_+\} = \delta^{ij}(E+P) \ , \qquad \{Q^i_-, Q^j_-\} = \delta^{ij}(E-P) \ , \qquad \{Q^i_+, Q^j_-\} = m\epsilon^{ijkl}J^{kl} \quad (5.63)$$

where J^{kl} are the $SO(4)$ generators and m is a dimensionful parameter which we can set to one. This parameter is related to the scale entering in the potential and determines the mass of BPS particles which carry $SO(4)$ quantum numbers. When the potential is set to zero we set $m = 0$ and we get the ordinary commutation relations we expect for the usual

[15]Notice that here we are talking about the global $(4,4)$ supersymmetry. These are the modes G^i_0 of the superconformal algebra generated by G^i_n. Some of the $SU(2)$ currents do appear in the anticommutators of some of the G^i_n, $n \neq 0$.

(4,4) supersymmetry algebra. Let us denote the algebra (5.63) by $(4,4)_m$. Notice that this is a Poincare superalgebra which contains non-abelian charges in the right hand side. This is possible in total spacetime dimension $d \leq 3$ [153] but not in $d > 3$ [179]. This algebra is a dimensional reduction of a Poincare superalgebra in 2+1 dimensions that we have discussed in detail in 4.5.

Note that the potential implies that the light cone energy is minimized (and it is zero) when the string sits at $\phi = \theta = 0$. There is just a finite energy gap of the order of $N_5|p_-|$ preventing it from going into the region $\phi \to -\infty$ where the pp-wave approximations leading to (5.53)-(5.57) break down[16] . Potentials in models preserving (4,4) supersymmetry were studied in [105],[104],[7] for models based on hyperkahler manifolds. Here we are interested in models with non-zero H flux. In fact, for the general solution (5.22)-(5.26) we can write down the string theory in lightcone gauge

$$
\begin{aligned}
S &= S_1 + S_2 && (5.64)\\
S_1 &= \int dt \int_0^{2\pi\alpha'|p_-|} d\sigma d^2\theta \frac{1}{2}\left[D_+ R^i D_- R^i + R^i R^i\right] && (5.65)\\
S_2 &= \int dt \int_0^{2\pi\alpha'|p_-|} d\sigma d^2\theta \left\{ \frac{1}{2} f(W,\bar{W})(D_+ W D_- \bar{W} + D_+ \bar{W} D_- W) + z(W) + \bar{z}(\bar{W}) + \right. \\
&\quad \left. + [f(W,\bar{W})(W+\bar{W})^2 g_{ij}(\Theta) + B_{ij}(\Theta, W, \bar{W})] D_+ \Theta^i D_- \Theta^j \right\} && (5.66)
\end{aligned}
$$

where S_1 describes the first four coordinates and consists of four free massive superfields. S_2 is the action describing the second four coordinates. We have written the action in $\mathcal{N} = 1$ superspace, by picking one special supercharge. Note that this particular supercharge, say $Q^1_\pm$, obeys the usual super Poincare algebra, therefore we can use the usual superspace formalism. The B field and the function f are simply the ones in (5.22)-(5.26). The theory has $(4,4)_m$ supersymmetry. We have not shown this explicitly from the lagrangian (5.66) but we know this from the supergravity analysis. Compared to the usual WZW action for a system of fivebranes, the only new term is the potential term. Note that RR fields in (5.22)-(5.26) are such that four of the fermions are free, which are the ones included in S_1, and the remaining four are interacting and appear in S_2 in (5.66).

[16] In the region $\phi \to -\infty$ we need to use the fivebrane solution in section 5.6.

Let us first study the theory (5.66) for large N_5. In that case, we can expand the fields around the minimum of the potential. If we keep only quadratic fluctuations we have four free bosons and fermions. In order to characterize these particles we go to their rest frame. Setting $P = 0$ we find that (5.63) reduces to the $\widetilde{SU}(2|2)$ algebra. These particles transform in the representation with two boxes as in figure 5.7(a) (but now viewed as a representation of $\widetilde{SU}(2|2)$). In terms of $SU(2) \times SU(2)$ quantum numbers we have $(1,0)+(1/2,1/2)+(0,0)$ where particles with half integer spin are fermions. This is a short representation, with energy $\hat{E} = 2$. In fact, if we consider a closed string and a superposition of two such particles with zero momentum we can form states that transform in the representations given in figure 5.7(e), which are also protected. As we make N_5 smaller these protected representations have to continue having the same energy. Of course, this argument only works perturbatively in $1/N_5$ since N_5 is not a continuous parameter and we can have jumps in the number of protected states as we change N_5. In order to figure out more precisely which representations are protected it is convenient to introduce an index defined by

$$\mathcal{I}(\gamma) = Tr[(-1)^F 2S_3 e^{-\hat{\mu}(\hat{E}-S_3-\tilde{S}_3)} e^{-\gamma \hat{E}}] \tag{5.67}$$

where S_3 and $\tilde{S}_3$ are generators in each of the two $SU(2)$ groups. We use the letter $\mathcal{I}$ to distinguish (5.67) from (5.30). One can argue that only short representations contribute and that the final answer is independent of $\hat{\mu}$, see appendix G of [127]. We can compute this for large N_5 using the free worldsheet theory and we obtain

$$\mathcal{I}(\gamma)|_{N_5=\infty} = \sum_{n=1}^{\infty} e^{-2n\gamma} \tag{5.68}$$

Since N_5 is not a continuous parameter we see that as we make N_5 smaller (5.68) could change but only by terms that are non-perturbative in the $1/N_5$ expansion. Thus for N_5 fixed and large we expect that the corrections would affect only terms of the form $e^{-(\text{const})N_5\gamma}$.

Now, let us compare this with the expectations from the gauge theory side. In order to find protected representations on the gauge theory side it is convenient to use the index (5.30). Since we are focusing on single trace states we can compute (5.30) just for single

trace states. For the case that we expand around the vacuum corresponding to N_2 $SU(2)$ representations of dimension N_5 we get

$$I_{s.t.\ N_5} = I_{s.t\ N_5=1} + \frac{e^{-2N_5(\beta_1+\beta_2+\beta_3)}}{(1-e^{-2N_5(\beta_1+\beta_2+\beta_3)})} - \frac{e^{-2(\beta_1+\beta_2+\beta_3)}}{(1-e^{-2(\beta_1+\beta_2+\beta_3)})} \tag{5.69}$$

$$I_{s.t\ N_5=1} = \frac{e^{-\beta_2-\beta_1}}{1-e^{-\beta_2-\beta_1}} + \frac{e^{-\beta_3-\beta_1}}{1-e^{-\beta_3-\beta_1}} + \frac{e^{-\beta_3-\beta_2}}{1-e^{-\beta_3-\beta_2}} \tag{5.70}$$

Details of this computation are in appendix G of [127]. Let us summarize here some of the results. In appendix appendix G of [127] we show that for the $N_5 = 1$ case we simply get the contributions expected from summing over the representations in figure 5.7(b). These contributions have the form expected from the BPS states on the string theory side coming from the first four transverse dimensions, the dimensions along the fivebrane. So we expect that the extra contribution in (5.69) should correspond to the contribution of the second set of four dimensions. In other words, it should be related to the BPS states in the two dimensional field theory ((5.66) with (5.50)) describing the second four transverse dimensions. In order to extract that contribution it is necessary to match the extra contribution we observe in (5.69) to the contributions we expect from protected representations. In other words, we can compute the index I for various protected representations and we can then match them (5.69). In appendix G of [127] we computed this index for atypical (short) representations and we show that (5.69) can be reproduced by summing over representations of the form shown in figure 5.7(f). In terms of the notation introduced in [58] (see figure 5.7(g)), which uses the Dynkin labels, we expect representations with $(a_1, a_2, a_3|a_4|a_5) = (0, p, 0|a_5 + 1|a_5)$ with $p \geq 0$ and $a_5 = 2(n-1)$, $n = 1, \cdots$ but $n \neq 0\ mod(N_5)$. All values of p and n that are allowed appear once. Representations with various values of p contribute with states that can be viewed as arising from the product of representations of the form in figure 5.7(b) and 5.7(e). The ones in figure 5.7(b) were identified with the first four transverse dimensions. So we interpret the sum over p as producing strings of various lengths given by the total powers of Z, plus the BPS states which are associated to the first four (free) dimensions on the string. So we conclude that the BPS states that should be identified with the second four dimensions should be associated

to the sum over n. Thus we expect from gauge theory side that the field theory on the string associated to the second four dimensions should have an index given by

$$\mathcal{I}_{expected} = \sum_{n=1}^{\infty} e^{-2n\gamma} - \sum_{n=1}^{\infty} e^{-2nN_5\gamma} \tag{5.71}$$

$$\mathcal{I}_{expected}|_{N_5=\infty} = \mathcal{I}(\gamma)|_{N_5=\infty} \tag{5.72}$$

We included the details of derivation in appendix G of [127]. So we see that this differs from (5.68) by a non-perturbative terms in $1/N_5$ of the form $e^{-2N_5\gamma}$. We view (5.71) as the gauge theory prediction for BPS states on the string theory side. Here we have checked that this matches the string theory in a $1/N_5$ expansion, but it would be nice to obtain the second term in (5.71) (which could be viewed as a non-perturbative correction to (5.68)) from an analysis of the two dimensional field theory based on the WZW model plus linear dilaton theory with a potential. These theories have a large group of symmetries and the theories with no potential are solvable. It would be nice if there is integrability in (5.66).

5.5 2+1 SYM on $R \times S^2$

We then come to the 2+1d SYM on $R \times S^2$. This field theory can be constructed as follows. We start with $\mathcal{N} = 4$ super Yang Mills on $R \times S^3$ and we truncate the free field theory spectrum to states that are invariant under $U(1)_L \subset SU(2)_L$, where $SU(2)_L$ is one of the $SU(2)$ factors in the $SO(4)$ rotation group of the S^3. This results in a theory that lives in one less dimension. It is a theory living on $R \times S^2$. This theory was already considered in [137] by considering the fuzzy sphere vacuum of the plane wave matrix model and then taking a large N limit that removed the fuzzyness and produced the theory on the ordinary

sphere. Here we reproduce it as a $U(1)_L$ truncation from $\mathcal{N}=4$ super Yang Mills[17]

$$\begin{aligned} S &= \frac{1}{g_{YM2}^2}\int dt \frac{d^2\Omega}{\mu^2}\ \mathrm{tr}\left(-\frac{1}{4}F^{mn}F_{mn}-\frac{1}{2}(D_mX^a)^2-\frac{1}{2}(D_m\Phi)^2+\frac{i}{2}\bar{\Psi}\Gamma^m D_m\Psi\right. \\ &\quad +\frac{1}{2}\bar{\Psi}\Gamma^a[X^a,\Psi]+\frac{1}{2}\bar{\Psi}\Gamma^\Phi[\Phi,\Psi]+\frac{1}{4}[X_a,X_b]^2+\frac{1}{2}[\Phi,X^a]^2-\frac{\mu^2}{8}X_a^2 \\ &\quad \left.-\frac{\mu^2}{2}\Phi^2-\frac{3i\mu}{8}\bar{\Psi}\Gamma^{012\Phi}\Psi-\mu\Phi dt\wedge F\right) \end{aligned} \tag{5.73}$$

where $m=0,1,2$, $a=4,\cdots,9$ and $(\Gamma^m,\Gamma^\Phi,\Gamma^a)$ are ten dimensional gamma matrices. We see that out of the seven transverse scalars of the maximally supersymmetric Yang Mills theory we select one of them, Φ, which we treat differently than the others. This breaks the $SO(7)$ symmetry to $SO(6)$ while still preserving sixteen supercharges. The radius of S^2 has size μ^{-1} and we have used the two dimensional metric with this radius to raise and lower the indices in (5.73). For our purposes it is convenient to set $\mu=2$, since this is the value we obtain by doing the $U(1)_L$ truncation of $\mathcal{N}=4$ super Yang-Mills on an S^3 of radius one.

The vacua are obtained by considering zero energy states. We write the field strength along the directions of the sphere as $F=fd^2\Omega$. We then see that Φ and f combine into a perfect square in the lagrangian

$$-\frac{1}{2}(f+\mu\Phi)^2 \tag{5.74}$$

For zero energy vacua this should be set to zero. Since the values of f are quantized, so are the values of the Φ field at these vacua. We can first diagonalize Φ and then we can see that its entries are integer valued. So a vacuum is characterized by giving the value of N integers $n_1,\cdots n_N$. The number of vacua is infinite, so we will not write an index. Nevertheless we will see that the gravity solutions reflect the existence of these vacua.

The dimensionless parameters characterizing this theory are N and the value of the 't Hooft coupling at the scale of the two sphere $g_{eff}^2N\equiv\frac{2\pi g_{YM2}^2N}{\mu}$, where μ^{-1} is the size of the sphere. The size of the sphere is a dimensionful parameter which just sets the overall

[17] We write the metric of $R\times S^3$ as $ds^2=-dt^2+\frac{1}{4}[d\theta^2+\sin^2\theta d\phi^2+(d\psi+\cos\theta d\phi)^2]$, where $\theta\in[0,\pi],\phi\in[0,2\pi],\psi\in[0,4\pi]$. We neglect the ψ dependence of all fields and we write the gauge field in calN=4 SYM as $A_{\mathcal{N}=4}=A+\Phi(d\psi+\cos\theta d\phi)$, where A is the $2+1$ dimensional gauge field.

energy scale. We set $\mu = 2$, so that the energy of BPS states with angular momentum J in $SO(6)$ is equal to $E = J$.

Notice that the large k limit of the theory in section 5.7 gives us the theory analyzed here. The values of N are the same and

$$g_{eff}^2 N = \frac{2\pi g_{YM2}^2 N}{\mu} = g_{YM3}^2 N k \tag{5.75}$$

where g_{YM3} is the Yang Mills coupling in the original $\mathcal{N} = 4$ theory in section 5.7. So we see that the limit involves taking $k \to \infty,\ g_{YM3}^2 \to 0$ while keeping g_{YM2}^2 fixed.

If one takes the strong coupling limit of this theory, by taking $g_{YM2} \to \infty$, we expect to get the theory living on M2 branes on $R \times S^2$. This theory has 32 supersymmetries and is the familiar theory associated with $AdS_4 \times S^7$. In this limit we find that the theory has full $SO(8)$ symmetry. When we perform this limit we find that the energy E of the theory in this section goes over to $\Delta - \tilde{J}$, where Δ is the ordinary Hamiltonian for the M2 brane theory on $R \times S^2$ and $\tilde{J}$ is the $SO(2)$ generator in $SO(8)$ which commutes with the $SO(6)$ that is explicitly preserved by (5.73). For a single brane, the $N = 1$ case, this can be seen explicitly by dualizing the gauge field strength into an eighth scalar. Then the vacua described around (5.74) are related to the 1/2 BPS states of the M2 brane theory. These should not be confused with the 1/2 BPS sates of the 2+1 dimensional theory (5.73) which would be related to 1/4 BPS states from the M2 point of view.

Now we turn to the gravity side. This solution corresponds to a single disk, as in figure 5.3(b). This disk is in the presence of a background field $V_b \sim \rho^2 - 2\eta^2$. The solution is a bit harder to obtain. We have obtained it by combining our ansatz with the results in [50]. The solution corresponding to single strip of M2 or D2 branes is

$$\begin{aligned} e^D &= 8C(1+r^2)\sin^2\theta \\ x_2 &= \frac{1}{2C}[1 + r\arctan r]\cos\theta, \\ y &= \frac{1}{\sqrt{2C}}[r + (1+r^2)\arctan r]\sin^2\theta \end{aligned} \tag{5.76}$$

and C is a simple rescaling parameter that is associated to the charge of the solution.

The resulting 10 dimensional solution is

$$\begin{aligned}
ds_{10}^2 &= \lambda^{1/3}\left[-8(1+r^2)f dt^2 + 16 f^{-1}\sin^2\theta d\Omega_5^2 + \frac{8rf}{r+(1+r^2)\arctan r}\left(\frac{dr^2}{1+r^2}+d\theta^2\right)\right. \\
&\quad \left. + \frac{2r\left[r+(1+r^2)\arctan r\right]f}{1+r\arctan r} d\Omega_2^2\right] \qquad (5.77)\\
B_2 &= -\lambda^{1/3}\frac{2\sqrt{2}\left[r+(-1+r^2)\arctan r\right]\cos\theta}{1+r\arctan r} d^2\Omega \qquad (5.78)\\
e^{\Phi} &= g_0\lambda^{1/2} 8 r^{\frac{1}{2}}(1+r\arctan r)^{-\frac{1}{2}}[r+(1+r^2)\arctan r]^{-\frac{1}{2}} f^{-\frac{1}{2}} \qquad (5.79)\\
C_1 &= -g_0^{-1}\lambda^{-\frac{1}{3}}\frac{\left[r+(1+r^2)\arctan r\right]\cos\theta}{2r} dt \qquad (5.80)\\
C_3 &= -g_0^{-1}\frac{r[r+(1+r^2)\arctan r]^2 f^2}{\sqrt{2}(1+r\arctan r)} dt\wedge d^2\Omega \qquad (5.81)\\
f &\equiv \sqrt{\frac{2}{r}[r+(\cos^2\theta+r^2)\arctan r]} \qquad (5.82)
\end{aligned}$$

where λ and g_0 are some constants.

This solution is dual to the vacuum of the 2+1 SYM in section 5.5, with $\Phi = 0$ and unbroken $U(N)$ gauge symmetry. The topology of this solution is $R\times B^3\times S^6$, where the boundary of B_3 is the S^2 on which the field theory is defined. Solutions with other configurations of disks have different topology. The solution is also everywhere regular. Expanding for large r we find that (5.77)-(5.82) approaches the D2 brane solution[18] [102] on $R\times S^2$

$$\begin{aligned}
\frac{ds_{10}^2}{\alpha'} &= (\ 6\pi g_{YM2}^2 N)^{1/3}\left[r^{5/2}(-dt^2+\frac{1}{4}d\ \Omega_2^2)+\frac{dr^2}{r^{5/2}}+r^{-1/2}(d\theta^2+\sin^2\theta d\Omega_5^2)\right]\\
e^{\Phi} &= g_{YM2}^2(6\pi g_{YM2}^2 N)^{-1/6} r^{-5/4} \qquad (5.83)\\
C_3 &= -g_{YM2}^{-2} r^5 (6\pi g_{YM2}^2 N)^{-2/3}\frac{1}{4} dt\wedge d^2\Omega
\end{aligned}$$

Comparing with (5.77)-(5.82) we can compute the value of λ and g_0 in terms of Yang Mills quantities. We can then compute the value of the radius of S^5 at $r=0$, $\theta=\pi/2$. This is the point where the BPS geodesics moving along S^5 sits. We find

$$\frac{R_{S^5}^2}{\alpha'} = \left(\frac{6\pi^3 g_{YM2}^2 N}{\mu}\right)^{1/3}, \qquad \mu = 2 \qquad (5.84)$$

[18] Here we have D2 brane on $R\times S^2$, where the radius of the S^2 is $\frac{1}{\mu}$, and we set $\mu = 2$.

The metric expanded around a geodesic with momentum along S^5 is simply the plane wave (5.28). We can now insert (5.84) in the general expression (5.29) to derive the spectrum of near BPS excitations with large J.

Note that the leading correction to $\hat{E} = E - J$ for fluctuations in the transverse directions in the S^5, which are parametrized by $\vec{r}$ in (5.28), has the form

$$(E-J)_n = 1 + \frac{1}{2}\left(\frac{6\pi^3 g_{YM2}^2 N}{\mu}\right)^{2/3} \frac{n^2}{J^2} + ... \tag{5.85}$$

This is the large coupling result from gravity approximation.

Under general principles we expect that the leading order correction in the large J limit in all regimes of the coupling constant should go like

$$(E-J)_n = 1 + f\left(\frac{g_{YM2}^2 N}{\mu}\right) \frac{n^2}{J^2} + \cdots \tag{5.86}$$

At weak coupling we get basically the same answer we had for $\mathcal{N} = 4$, which at one loop order is $f\left(\frac{g_{YM2}^2 N}{\mu}\right) = \frac{\pi g_{YM2}^2 N}{\mu}$. So we see that in this case the function f has to be non-trivial. This is to be contrasted with the behavior in four dimensional $\mathcal{N} = 4$ theory where the function f has the same form at weak and strong coupling [164], see also [83]. Of course it would be very nice to compute this interpolating function from the gauge theory side. We see a similar phenomenon for the plane wave matrix model. This phenomenon is a generic feature of the strong/weak coupling problem, among many others observed in the literature, e.g. the 3/4 problem in the thermal Yang-Mills entropy [85], and the 3-loop disagreement of the near plane wave string spectrum [46],[166], which are results obtained in different regimes of couplings, and are probably explained by the presence of such interpolating functions.

We can have other more general solutions corresponding to multiple disks, as in figure 5.4(a). The different configurations in the disk picture match the different Higgs vacua for scalar Φ as we discussed in section 5.5. One can also consider strings propagating near the tip in a multi-disk solution. In that case, the actual value of the interpolating function f in the strong coupling regime, which is related to the position of the tip of the disk, is not universal, in the sense that it depends on the vacuum we expand around. What is universal,

however, is the fact that the expansion around any of the tips gives us the IIA plane wave (5.28) as long as there are no other nearby disks. The situation when we consider many disks together will be discussed in the next section.

The explicit solutions for multi-disks are very difficult to solve due to integral equation or coupled integral equations. Nevertheless, we can take a large disk limit and then it is solvable in terms of conformal transform.

If we have n nearby disks, then the general solution is

$$\begin{aligned} \partial_w z &= \frac{(w-ia_1)(w-ia_2)\cdots(w-ia_n)}{(w-ic_1)(w-ic_2)\cdots(w-ic_{n-1})} \\ &\quad \text{with} \quad a_1 < c_1 < a_2 < c_2 < \cdots < c_{n-1} < a_n \end{aligned} \tag{5.87}$$

where $w = ia_i$ are the location of n tips and $w = ic_i$ are the locations of $n-1$ sets of fivebranes. The resulting solution (5.22)-(5.26) describes a multi-center configuration of fivebranes on a plane wave. Boosting away the + components of all fields we find that we end up with a multi centered configuration of fivebranes where the $SO(4)$ symmetry is broken to $SO(3)$, in fact all fivebranes are sitting along a line.

If there are two nearby disks, then we can expand the solution near the tips of these disks and also include the fivebrane region between them. This is the $n = 2$ case in above formula. Consider for example a configuration with two nearby disks such as shown in figure 5.6(c). The holomorphic function $z(w)$ in (5.22)-(5.26) is given by

$$\partial_w z = \frac{(w-ia)(w+ib)}{w} \tag{5.88}$$

with a, b real and positive. We see that for $w \approx ia, -ib$ and for $w \to \infty$ we recover the results we expect for single disks (5.27). This transformation maps the w right half plane (with $Re(w) \geq 0$) to the z plane with two cuts. The points $w = ia, -ib$ map to the two tips and $w = 0$ maps to $Re(z) \to -\infty$ between the two disks, which is expected to look like a fivebrane. In fact, we can check that the function f in (5.22)-(5.26) is given by

$$f = 1 + \frac{ab}{|w|^2} \tag{5.89}$$

which means that we have a single center fivebrane solution. The 5-branes are located at $w = 0$ as expected. One can also check that the fivebrane charge is proportional to the distance between two disks as in (5.13)

$$Im(\Delta z) = Im \int_{-ib}^{ia} \partial_w z dw = \pi ab \tag{5.90}$$

In addition we find a contribution to g_{++} of the form

$$4f^{-1}|\partial_w z|^2 = 4\frac{|w - ia|^2|w + ib|^2}{|w|^2 + ab} \tag{5.91}$$

When we consider a string moving on this geometry in light cone gauge we find that (5.91) appears as a potential for the worldsheet fields. Notice that the minima of the potential are precisely at the two tips of the two disks corresponding to $w = ia$ and $w = -ib$ where we can take pp-wave limit.

When $a = b$ we have a symmetric situation where the two disks have precisely the same length (same value of ρ_i). In this case we see that the two minima are on the two sides of the fivebrane at equal distance between them. Notice that the throat region of the fivebrane corresponds to the region between the disks. This throat region is singular in our approximation since the dilaton blows up as $w \to 0$. This is not physically significant since this lies outside the range of our approximation, since $-Re(z)$ diverges. In fact, in the region between the disks we should actually match onto the fivebrane solution (5.93)-(5.96).

If $a \neq b$, say $a > b$ for example, then we have an asymmetric configuration where one disk is larger than the other. The larger disk is the one whose tip is at $w = a$. If $a \gg b$ then we find that the tip corresponding to the smaller disk is in the throat region of the fivebrane while the tip corresponding to the larger disk is in the region far from the fivebrane throat.

5.6 Little string theory on S^5

Now we turn to the field theory that is coming from expanding around the NS brane vacua on of the plane-wave matrix model. This solution is the simplest from the gravity point of view. In this case we consider two infinite disks separated by some distance $d \sim N$,

see figure 5.4(b). We find that the solution corresponds to N IIA NS5 branes wrapping a $R \times S^5$. The solution for V is

$$V = \frac{1}{g_0} I_0(r) \sin\theta \ , \qquad r = \frac{2\rho}{N\alpha'} \ , \quad \theta = \frac{2\eta}{N\alpha'} \tag{5.92}$$

where $I_0(r)$ is a modified Bessel function of the first kind. This leads to the ten dimensional solution[19]

$$\begin{aligned}
ds_{10}^2 &= N\left[-2r\sqrt{\frac{I_0}{I_2}}dt^2 + 2r\sqrt{\frac{I_2}{I_0}}d\Omega_5^2 + \sqrt{\frac{I_2}{I_0}}\frac{I_0}{I_1}(dr^2+d\theta^2) + \sqrt{\frac{I_2}{I_0}}\frac{I_0 I_1 s^2}{I_0 I_2 s^2 + I_1^2 c^2}d\Omega_2^2\right] \\
B_2 &= N\left[\frac{-I_1^2 cs}{I_0 I_2 s^2 + I_1^2 c^2} + \theta\right] d^2\Omega \qquad (5.93) \\
e^{\Phi} &= g_0 N^{3/2} 2^{-1} \left(\frac{I_2}{I_0}\right)^{\frac{3}{4}} \left(\frac{I_0}{I_1}\right)^{\frac{1}{2}} \left(I_0 I_2 s^2 + I_1^2 c^2\right)^{-\frac{1}{2}} \qquad (5.94) \\
C_1 &= -g_0^{-1}\frac{1}{N} 4 \frac{I_1^2 c}{I_2} dt \qquad (5.95) \\
C_3 &= -g_0^{-1}\frac{4 I_0 I_1^2 s^3}{I_0 I_2 s^2 + I_1^2 c^2} dt \wedge d^2\Omega \qquad (5.96)
\end{aligned}$$

where $I_n(r)$ are a series of modified Bessel functions of the first kind.

This solution is also a limit of the a solution analyzed in [126] using 7d gauged supergravity, except that here we solved completely the equations. The gauged supergravity solution in [126] describes an elliptic M5 brane droplet on the x_1, x_2 plane and we can take a limit that the long axis of the ellipse goes to infinity while keeping the short axis finite, this becomes a single M5 strip. This then corresponds to two infinite charged disks in the electrostatic configuration, see figures 5.4(b). See related discussion in chapter 3.

The solution is dual to little string theory (see e.g. [35],[165],[2]) on $R \times S^5$. As we go to the large r region the solution (5.93)-(5.96) asymptotes to

$$\begin{aligned}
ds_{10}^2 &= N\alpha'\left[2r\left(-dt^2 + d\Omega_5^2\right) + dr^2 + (d\theta^2 + \sin^2\theta d\Omega_2^2)\right] \\
e^{\Phi} &= g_s e^{-r} \qquad (5.97) \\
H_3 &= 2N\alpha' \sin^2\theta d\theta \wedge d^2\Omega
\end{aligned}$$

[19] $s = \sin\theta, c = \cos\theta$. We set $\alpha' = 1$. We used the convention in [158] that $\frac{1}{2\pi\alpha'}\int_{\Sigma_3} H_3 = 2\pi N$, to normalize H_3.

So we see that the solution asymptotes to IIA NS5 branes on $R \times S^5$. In addition we have RR fields which are growing exponentially when we go to large r. These fields break the $SO(4)$ transverse rotation symmetry of the fivebranes to $SO(3)$. Since the coupling is also varying exponentially, it turns out that, in the end, the influence of the RR fields on the metric is suppressed only by powers of $1/r$ relative to the terms that we have kept in (5.97) (relative to the H field terms for example).

The solution is everywhere regular. When either S^5 or S^2 shrinks, it combines with r or θ to form locally R^6 or R^4. Note that at $r = 0$ the solution has a characteristic curvature scale given by $\mathcal{R} \sim \frac{1}{\alpha' N}$ and a string coupling of a characteristic size $g_s \sim g_0 N^{3/2}$. The string coupling decreases as we approach the boundary. Thus, if we take g_s small and N large we can trust the solution everywhere. On the other hand if we take g_s large, then we can trust the solution for large r but for small r we need to go to an eleven dimensional description, include x_1 dependence and solve equation (5.1). It is clear from the form of the problem that for very large g_s we will recover $AdS_7 \times S^4$ in the extreme IR if we choose a suitable droplet configuration. More precisely, as increase g_s we will need to go to the eleven dimensional description and include dependence on x_1. Then we can consider a periodic array of circular droplets. As $g_s \to \infty$ each circle becomes the isolated circle that gives rise to $AdS_7 \times S^4$ [126]. There is also a similar gravity picture for the relation between the 2+1 SYM on $R \times S^2$ in section 5.5 and the 3d superconformal M2 brane theory.

In addition we could consider other solutions in the disk picture that correspond to adding more small disks between the infinite disks, as in figure 5.4(c). These correspond to different vacua of this theory.

5.7 $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$

In this section we consider $U(N)$ $\mathcal{N} = 4$ super Yang-Mills theory on $R \times S^3/Z_k$, with $Z_k \subset SU(2)_L$, and $SU(2)_L$ as defined above (see also [94] for a more general discussion).

We can also obtain this theory by starting with the free field content of $\mathcal{N}=4$, projecting out all fields which are not invariant under Z_k and then considering the same interactions for the remaining fields as the ones we had for $\mathcal{N}=4$. Notice that we first project the elementary fields and we then quantize, which is not the same as retaining the invariant states of the original full quantum $\mathcal{N}=4$ theory. This is the standard procedure. The symmetry group of this theory is $\widetilde{SU}(2|4)$.

This theory is parametrized by N, k, and the original Yang Mills coupling g^2_{YM3}. Whereas $\mathcal{N}=4$ SYM on S^3 has a unique vacuum, the theory on S^3/Z_k has many supersymmetric vacua. Let us analyze the vacua at weak coupling. Since all excitations are massive we can neglect all fields except for a Wilson line of the gauge field. More precisely, the vacua are given by the space of flat connections on S^3/Z_k. This space is parametrized by giving the holonomy of the gauge field U along the non-trivial generator of $\pi_1(S^3/Z_k)=Z_k$, up to gauge transformations. We can therefore diagonalize U, with $U^k=1$. So the diagonal elements are kth roots of unity. Inequivalent elements are given by specifying how many roots of each kind we have. So the vacuum is specified by giving the k numbers $n_1, n_2, \cdots n_k$, with $N=\sum_{l=1}^{k} n_l$. Where n_l specifies how many times $e^{i2\pi\frac{l}{k}}$ appears in the diagonal of U. We can also view these different vacua as arising from orbifolding the theory of D-branes on $S^3\times R$ and applying the rules in [68] with different choices for the embedding of the Z_k into the gauge group. The regular representation corresponds to $n_l=N/k$ for all l, and we need to take N to be a multiple of k.

The total number of vacua is then

$$D(N,k)=\frac{(N+k-1)!}{(k-1)!N!} \tag{5.98}$$

It is also interesting to count the total number of 1/2 BPS states with charge J under one of the $SO(6)$ generators. These numbers are encoded conveniently in the partition function

$$I_{S^3/Z_k}(p,q)=\sum_{N=0}^{\infty} p^N I_N(q)=\sum_{N,J=0}^{\infty} D_{S^3/Z_k}(N,J)p^N q^J=[I_{\mathcal{N}=4}(p,q)]^k=\frac{1}{\prod_{n=0}^{\infty}(1-pq^n)^k} \tag{5.99}$$

where $I_{\mathcal{N}=4}(p,q)$ is the index for $\mathcal{N}=4$ super Yang-Mills. In writing (5.99) we used the last representation in (5.35).

We see that even though we counted the vacua (5.98) at weak coupling, the result is still valid at strong coupling since they all contribute to the Witten index. In fact, setting $q=0$ in (5.99) we recover (5.98).

Now we consider some aspects of the gravity solutions describing $\mathcal{N}=4$ super Yang Mills on $R\times S^3/Z_k$. This theory is particularly interesting since it is a very simple orbifold of $\mathcal{N}=4$ SYM, so that one could perhaps analyze in more detail the corresponding spin chains.

Let us start with the simplest solution, which is $AdS_5/Z_k\times S^5$. If the orbifold is an ordinary string orbifold, then there is a Z_k quantum symmetry. On the field theory side, this orbifold corresponds to considering a vacuum where the holonomy matrix U has $n_l=N/k$ (see the notation around (5.98)) and we need to start with an N which is a multiple of k. This is the configuration which corresponds to the regular representation of the orbifold group action in the gauge group, see [94]. This is the simplest orbifold to consider from the string theory point of view. Other choices for the holonomy matrix U, such as $U=1$, lead to an orbifold which is not the standard string theory orbifold. Such an orbifold can be obtained from the string theory one by turning on twisted string modes living at the singularity.

$AdS_5/Z_k\times S^5$ in type IIB can be dualized to an M-theory or IIA configuration which preserves the same supersymmetries as our ansatz. Let us first understand the M-theory description. Let us first single out the circle where Z_k is acting. Then we lift IIB on this circle to M-theory on T^2. This T^2 is parametrized by the coordinates x_1,x_2 of the general M-theory ansatz in [126]. The solution obtained in this fashion is independent of x_1,x_2. The general solution of (5.1) with translation symmetry along x_1,x_2 is[20]

$$e^D = c_1 y + c_2 \tag{5.100}$$

$$c_1 = \frac{g_s k}{2}\,, \qquad c_2 = \frac{\pi g_s N}{4} \tag{5.101}$$

[20] We set $\alpha'=1$.

Equivalently we can view the configuration as an electrostatic configuration where

$$V = -\frac{\pi N}{2k}\log\rho + V_b \ , \qquad V_b = \frac{1}{g_s k}(\rho^2 - 2\eta^2) \tag{5.102}$$

which means that we have a line of charge at the $\rho = 0$ axis in the presence of the external potential V_b.

These solutions are singular at $y = 0$ since we are not obeying (5.2). At $y = 0$ we find that $4\rho_0 = R^2_{S^5} = \sqrt{4\pi g_s N \alpha'^2}$. In the IIB variables this singularity is simply the Z_k orbifold fixed point. We also find that the radius of the two torus is $R_{x_1} = g_s$ and $R_{x_2} = 1/g_s$. This is as we expect when we go from IIB to M theory.

The map between the IIB and IIA solutions is simply a T-duality along the circle where Z_k acts by a shift $\psi \sim \psi + \frac{4\pi}{k}$. If k is sufficiently large it is reasonable to perform this T duality, at least for some region close to the singularity. Once we are in the IIA variables, we can allow the solution to depend on η. In fact, this dependence on η allows us to resolve the singularity and get smooth solutions. The electrostatic problem is now periodic in the η direction. We have a periodic configurations of disks, see figure 5.3(d), in the presence of an external potential of the form V_b in (5.102). Note that the external potential is not periodic in η. This is not a problem since the piece that determines the charge distribution on the disks is indeed periodic in η. Furthermore, the derivatives of V that appear in (5.7)-(5.11) are all periodic in η [21] . In the IIA picture the region between the disks can be viewed as originating from NS fivebranes. These NS fivebranes arise form the A_{k-1} singularity of the IIB solution after doing T-duality [156] (see also [81]). In fact, the period of η is proportional to k, so that we have k fivebranes $N_5 = k$. From this point of view the simplest situation is when all fivebranes are coincident. This corresponds to taking the matrix U proportional to the identity. On the other hand, the standard string theory orbifold corresponds to the case that we have k equally spaced disks separated by a unit distance. In other words, the fivebranes will all be equally spaced. In this case, since we have single fivebranes, we do not expect the geometric description to be accurate. Note

[21] The η dependent piece in (5.102) ensures that as we go over the period of η we go over the period of x_2 which is T-dual to the circle on which the Z_k acted.

that even though we are talking about these fivebranes, the full solution is non-singular. These fivebranes are a good approximation to the solution when we have large disks that are closely spaced, as we will see in detail below. But as we go to $\rho \to 0$ the solution between the disks approaches the NS5 solution (5.93)-(5.96), which is non-singular. The different vacua (5.98) correspond to the different ways of assigning charges n_l (see notation around (5.98)) to the disks that sit at positions labelled by $\eta \sim l$. There are k such special positions on the circle. Only in cases where we have coincident fivebranes can we trust the gravity description. This happens when some of the n_l are zero.

If we take the $k \to \infty$ limit, keeping N finite, then the direction η becomes non-compact and we go back to the configurations considered in the previous section which are associated to the D2 brane theory (2+1 SYM) of section 5.5. This is also what we expected from the field theory description.

We were not able to solve the equations explicitly in this case. On the other hand, there are special limits that are explicitly solvable. These correspond to looking at the large N limit so that the disks are very large and then looking at the solution near the tip of the disks. Let us consider the case where we have a single disk per period of η. We can find the solution by using (5.49) and we get

$$\partial_w z = ik \prod_{n=-\infty}^{\infty} \frac{(w - ian)}{(w - ia(n + \frac{1}{2}))} = k \tanh \frac{\pi w}{a} \tag{5.103}$$

where k is the number of coincident fivebranes. When we insert this into (5.22)-(5.26) we find that the the solution corresponds to a periodic array of k NS fivebranes along spatial direction χ.

$$f = \frac{k \sinh r}{2r(\cosh r + \cos \chi)} = \sum_{n=-\infty}^{\infty} \frac{k}{r^2 + (\chi + \pi + 2\pi n)^2} \tag{5.104}$$

$$r + i\chi \equiv \frac{2\pi}{a} w \ , \qquad \chi \sim \chi + 2\pi \tag{5.105}$$

$$g_{++} = 8k \frac{r}{\sinh r} (\cosh r - \cos \chi) \tag{5.106}$$

The rim of the disks corresponds to $w = ia$ or $r = \chi = 0$ in (5.104). The g_{++} term in the metric (5.22)-(5.26) implies that the lightcone energy is minimized by sitting at these

points. These points lie between the fivebranes, which sit at $r = 0$, $\chi = \pi$. In flat space the T-dual of an A_{k-1} singularity corresponds to the near horizon region of a system of k fivebranes on a circle. Here we are getting a similar result in the presence of RR fields. As $w \to \infty$ the solution (5.22)-(5.26) approaches the one that is the T-dual of the orbifold of a pp-wave with $R^4 \times R^4/Z_k$ transverse dimensions

$$ds_{10}^2 = -2dx^+dx^- - (\vec{r}^{\,2} + \vec{u}^{\,2})(dx^+)^2 + d\,\vec{r}^{\,2} + du^2 + \frac{u^2}{4}d\Omega_2^2 + \frac{k^2}{u^2}d\chi^2 \qquad (5.107)$$

At large u we can T-dual this back to the Z_k quotient of the IIB plane wave.

Let us understand first the theory at the standard string theory orbifold point. This corresponds to the vacuum with $n_l = N/k$, for all $l = 1, \cdots, k$. As we mentioned above, it is useful to view the Yang Mills theory on $R \times S^3/Z_k$ as the orbifold of the theory on the brane according to the rules in [68]. According to those rules we need to pick a representation of Z_k and embed it into $U(N)$. The regular representation then gives rise to the vacuum where all n_l are equal. For this particular choice we can use the inheritance theorem in [38] that, to leading order in the $1/N$ expansion, the spectrum of Z_k invariant states in the orbifold theory is exactly the same as the spectrum of invariant states in $\mathcal{N} = 4$. This ensures that the matching between the string states on the orbifold and those of the Yang Mills theory is the same as the corresponding matching in $\mathcal{N} = 4$. In the IIA description this regular orbifold goes over to a picture where we have k fivebranes uniformly spaced on the circle. In this case we cannot apply our gravity solutions near the fivebranes because we have single fivebranes. Furthermore, we expect that the orbifold picture should be the correct and valid description for string states even close to the orbifold point, as long as the string coupling is small. The spectrum of string states involving the second four dimensions (the orbifolded ones) can be thought of as arising from $E - J = 1$ excitations which get a phase of $e^{\pm i2\pi/k}$ under the generator of Z_k, but we choose a combination of these excitations that is Z_k invariant. This discussion is rather similar to the one in [149], where the $AdS_5 \times S^5/Z_k$ orbifold (see e.g. [110],[123]) was studied.

We can now consider other vacua. These are associated to different representations for the Wilson line. For example, we can choose $n_k = N$ and $n_i = 0$ for $i \neq k$. In this case the

IIA gravity description can be trusted when we approach the origin as long as the 't Hooft coupling is large and k is large enough. Let us describe the physics in the pp-wave limit in more detail for this case. The pp-wave limit that we are considering consists in taking k fixed and somewhat large, so that the gravity description of the k coincident fivebranes is accurate, and then taking J and N to infinity with J^2/N fixed, exactly as in $\mathcal{N} = 4$ super Yang-Mills [33]. In fact, we find that the worldsheet theory in the first four directions is exactly the same as for $\mathcal{N} = 4$ super Yang Mills. In particular, the dispersion relation for lightcone gauge worldsheet excitations is precisely as in $\mathcal{N} = 4$ super Yang Mills [33], with the same numerical coefficient. The theory in the remaining four directions is more interesting. At large distance from the origin the worldsheet field theory is just the orbifold of the standard IIB plane wave [39]. This is what we had for the regular representation vacuum that we discussed above. A string state whose worldsheet if far from the origin, so that its IIB description is good, is a very excited string state. It is reasonable to expect that the spectrum of such states is not very sensitive to the vacuum we choose. This is what we are finding here, since the spectrum in this region is that of the vacuum of the regular representation we discussed above. On the other hand, as we consider string states where the string is closer to the minimum of its worldsheet potential we should use the IIA description in terms of k coincident fivebranes, using the solution in (5.104). In this case the spectrum of excitations on the string worldsheet is rather different than what we had at the standard orbifold point. In this case we have excitations of worldsheet mass $E - J = 2$ which are Z_k invariant. This spectrum matches with what we naively expect from considering impurities propagating on the string for the vacuum we are considering. This vacuum contains only single particle gauge theory excitations with $E - J \geq 2$ for all fields that could be interpreted as excitations that are associated for the second four dimensions. Let us be a bit more explicit. We can identify some of these $E - J = 2$ excitations as the Kaluza Klein modes of Z given by $\epsilon^{\dot\beta\dot\beta'}\partial_{\alpha\dot\beta}\partial_{\alpha'\dot\beta'}Z$. This gives a singlet under $SU(2)_L$, so that the $Z_k \subset SU(2)_L$ acts trivially. So this Kaluza-Klein mode survives the Z_k quotient. The α, α' indices give rise to a spin one mode under $SU(2)_R \subset \widetilde{SU}(2|4)$.

The spin zero mode under both $SU(2)_{L,R}$ vanishes due to the equation of motion. There is a spin zero excitation with $E - J = 2$ which comes from the mode of the four dimensional gauge field along the ψ circle, the circle we are orbifolding. These elementary fields have $E - J = 2$ and are associated to the $E - J = 2$ excitations of the last four dimensions of the IIA plane wave. An analysis similar to the one we discussed for the plane wave matrix model and 2+1 SYM shows that these excitations are exactly BPS and survive in the strong 't Hooft coupling limit.

Other gravity solutions which are asymptotic to AdS_5/Z_k were constructed in [51, 52, 8]. Those solutions have a form similar to that of the Eguchi-Hanson instanton [71] in the four spatial directions. In those solutions fermions are anti-periodic along the ψ direction. In our case, fermions are periodic in the ψ circle. So, the solutions in [51, 52, 8] arise when we consider a slightly different field theory. Namely, when one considers Yang Mills on $R \times S^3/Z_k$ but where the fermions are antiperiodic along the circle on which Z_k acts. (One should also restrict to k even). This theory breaks supersymmetry. The solutions in [51, 52, 8] describe states (probably the lowest energy states) of these other theories. In such cases the orbifold is another state in the same theory, the theory with antiperiodic fermion boundary conditions along ψ. One then expects that localized tachyon condensation, of the form explored in [1], makes the orbifold decay into the solutions described in [51, 52, 8].

Chapter 6

Geometry of BPS vacua and instantons

6.1 Introduction

In this chapter, we study the property of the vacua and their dual BPS geometries for some of the theories studied in chapter 5. We study in detail the vacua and instanton solutions in plane-wave matrix model and 2+1d SYM on $R \times S^2$. We associate each BPS vacuum geometry with a value of a superpotential. The superpotential can be calculated independently from gauge theory and gravity side. The allowed tunnelings between vacua are described by instanton solutions, and the tunneling amplitude is given by the difference of the superpotentials between the two vacua. In Ch 6.2, we get the superpotential from the instanton action in the weak coupling gauge theory side. In Ch 6.3.1, we find embeddings of Euclidean D-branes wrapping some cycles in the gravity dual and compute their action. This gives the instanton action for the case in which the initial and final vacua are very close to each. In Ch 6.3.2 we find that the superpotential for the vacua in gravity side are given by the energy of the electric charge system. The electric charges are from eigenvalues of matrices in gauge theory. We found agreement between the approaches from gauge theory and gravity side, except that the gravity answer is the strong coupling result which subsumes

the answer from classical gauge theory action which is the weak coupling limit. The gravity answer shows quantum repulsion of eigenvalues, which was absent in the weak coupling limit, a feature similar to the Dijkgraaf-Vafa matrix model [62]. In Ch 6.4, we then study the emergence of all the extra spatial dimensions, relative to the boundary gauge theories, in the gravity picture. We also try to explain the emergence of the electrostatic system and its force in the gravity side from gauge theory. Our method is to embed the sector of vacuum geometries into a larger sector of 1/8 BPS geometries associated to three harmonic oscillators.

6.2 Instanton in gauge theory

The $U(N)$ plane wave matrix model has a gauge field A_0, three $SO(3)$ scalars $X_i,\ i = 1, 2, 3$ and six $SO(6)$ scalars $X_a,\ a = 4, 5, ..., 9$ and their fermionic partners. They are all $N \times N$ Hermitian matrices. The action can be found in many places [33],[58],[117],[114],[57]. Here we use the action such that it depends only on a dimensionless coupling constant g^2_{ym0}/m^3, where m is originally the mass of $SO(6)$ scalars. We have rescaled all fields so that the masses of $SO(6)$ scalars, $SO(3)$ scalars and fermions are 1, 2 and 3/2 respectively. The action takes the form

$$S = \frac{1}{g^2_{ym0}/m^3} \int dt\ \mathrm{tr} \left(\frac{1}{2} D_0 X_a D_0 X_a - \frac{1}{2} X_a X_a + \frac{1}{4} [X_a, X_b]^2 ... \right) \tag{6.1}$$

Due to the Myers term and mass term, the classical vacua are fuzzy spheres parametrized by J_i, the N dimensional representation of $SU(2)$ ($[J_i, J_j] = i\epsilon_{ijk} J_k$) , $X_i = 2J_i,\ X_a = 0$. There is also a "trivial" vacuum $X_i = 0,\ X_a = 0$, which were conjectured to be a single NS5 brane wrapping S^5 from gauge theory analysis [137], and the gravity dual of this vacuum were studied in [127] (see also [128],[70]), which are dual to IIA little string theory on $S^5 \times R$, providing further evidence for the identification. From M theory point of view, these are 1/2 BPS M2 and M5 branes preserving half of the supersymmetries of the asympotic plane-wave geometry. This theory and the gravity duals of its vacua were studied in chapter 5.

There could be in principle tunneling between different fuzzy sphere vacua, exchanging some amount of D0 branes. From the 11 dimensional point of view, this process is the transferring of the longitudinal momenta of two fuzzy membranes. In a more concrete analysis by Yee and Yi [182], they found a class of analytical instanton solutions interpolating between a fuzzy sphere vacuum $X_i = 2J_i$ and a trivial vacuum $X_i = 0$. There could be two approaches to the instanton action in the weak coupling gauge theory side. One is to study a bound in the Euclideanized action corresponding to the action of the instatnon (which was done in [182]), and the other is to use superfield formalism and caculate the superpotential for each vacuum. The instanton equation can be read from the superpotential or by supersymmtry transformation conditions.

We do not have an exact way of writing the action of the plane-wave matrix model in superfields at present, but we notice that in the weak coupling theory, the tunneling process merely involves the three $SO(3)$ scalars X_i and their superpartners, but not the $SO(6)$ sector. We can thereby write a "superpotential" for each of the fuzzy sphere vacuum:

$$W = \text{tr } (X_i X_i + i\frac{1}{3}\epsilon_{ijk} X_i X_j X_k) \tag{6.2}$$

It's easy to check that the lagrangian for X_i is correctly produced, and the action is bounded by the superpotential difference between initial and final vacua:

$$\begin{aligned} S_{inst} &= \frac{1}{g_0^2}\int_{-\infty}^{+\infty} d\tau \text{ tr } \left[\frac{1}{2}(D_\tau X_i)^2 + \frac{1}{2}(\partial_{X_i} W)^2\right] &(6.3)\\ &= \frac{1}{g_0^2}\left[\int_{-\infty}^{+\infty} d\tau \text{ tr } \frac{1}{2}(D_\tau X_i + \partial_{X_i} W)^2 - W|_{\tau=-\infty}^{\tau=+\infty}\right] &(6.4)\end{aligned}$$

where we denote $g_{ym0}^2/m^3 = g_0^2$.

The instanton equation is the same as setting the square term to zero,

$$D_\tau X_i + \partial_{X_i} W = D_\tau X_i + 2X_i + i\epsilon_{ijk} X_j X_k = 0 \tag{6.5}$$

and a class of analytical solutions were found [182]:

$$X_{block}^i(\tau) = \underset{n_p}{\otimes} X_{n_p \times n_p}^i(\tau) = \underset{n_p}{\otimes} 2J_{n_p \times n_p}^i \frac{1}{1 + e^{2(\tau - \tau_0)}} \tag{6.6}$$

Here X^i_{block} is a $\sum_p n_p \times \sum_p n_p$ block matrix in X^i, and we used $A_0 = 0$ gauge. This solution corresponds to the transition where p fuzzy spheres each with size n_p at $\tau = -\infty$ gradually shrinks to zero and turns to a trivial vacuum $X^i_{block} = 0$ at $\tau = +\infty$. So the instanton action is the difference of the superpotential ΔW between two vacua $X^i(\pm\infty) = 2J_i|_{\pm\infty}$,

$$S_{inst} = -\frac{1}{g_0^2}\Delta W = -\frac{1}{g_0^2}\sum_{i=1,2,3}\frac{4}{3}\left(\text{tr } J_i^2|_{+\infty} - \text{tr } J_i^2|_{-\infty}\right) \tag{6.7}$$

$$= \sum_{n_p}\frac{1}{g_0^2}\frac{n_p(n_p^2-1)}{3} \tag{6.8}$$

where we used the identity that the second Casimir invariant $\sum_i J^2_{i,n\times n} = \frac{(n^2-1)}{4} I_{n\times n}$.

One can define the W for the trivial vacuum $X_i = 0$ as some constant W_0 as a reference point and then the superpotential for an arbitrary fuzzy sphere vacua can be defined as W_0 plus the difference term in (6.7). W_0 will be set to zero for convenience, because we only need to know the differences of W for different vacua.

Suppose we label each vacuum by $N_2^{(i)}$ copies of $N_5^{(i)}$ dimensional irreducible representation. These satisfy $\sum_i N_2^{(i)} N_5^{(i)} = N$. In the large $N_5^{(i)}$ limit, the instanton action will be simplified to

$$S_{inst} = -\frac{1}{g_0^2}\sum_i \frac{1}{3} N_2^{(i)} N_5^{(i)3}|_{-\infty}^{+\infty} \tag{6.9}$$

In the supersymmetric quantum mechanics, the tunneling amplitude of allowed transition between two vacua a and b is given by the difference of two superpotentials

$$P_{a\to b} = P_0 e^{-S_{inst}/\hbar} = P_0 e^{-\frac{1}{g_0^2}(Wa - W_b)} \tag{6.10}$$

where P_0 is a normalization factor. We set $\hbar = 1$ in the second expression and also later discussions. The tunneling amplitude is largely suppressed if g_0^2 is small and the change $\sum_i \frac{1}{3} N_2^{(i)} N_5^{(i)3}|_{-\infty}^{+\infty}$ is large. Such factors will also appear in the instanton corrections to the vevs of some operators [182].

The method for solving general instanton solutions in plane-wave matrix model and its moduli space has been discussed in detail by [182],[9]. Both the instanton equation and

their moduli space reveal similarity with that of domain wall solutions in $\mathcal{N} = 1^*$ SYM [159],[9].

The 2+1 dimensional SYM on $S^2 \times R$ we discuss here is the theory when we reduce the $\mathcal{N} = 4$ SYM on $S^3 \times R$ on the Hopf fiber of the bundle $S^3 \to S^2$ with fiber S^1 [127]. The gauge fields in the $\mathcal{N} = 4$ SYM reduce to the gauge fields in the 2+1 d SYM plus an additional scalar Φ. The $SO(6)$ scalars $X_a,\ a = 4, 5, ..., 9$ reduce to their components invariant on the S^1. There are totally seven scalars. This theory as well as the gravity duals of its vacua were studied in chapter 5. The consistent truncation from $\mathcal{N} = 4$ SYM to this theory was further studied by [100].

The theory can also be obtained from expanding plane-wave matrix model around its fuzzy sphere vacuum, so under the continuum limit (e.g. [106]) of the matrix regularization of the large fuzzy sphere, it becomes a theory on $S^2 \times R$. Suppose we start from the vacuum with N_2 fuzzy spheres, each of size $\overline{N}_5 \gg 1$, we have $N_2\overline{N}_5 = N$, where N is the number of original D0 branes. We expand around this vacuum. Fluctuations of X_i about $2J_i$ $(\overline{N}_5 \times \overline{N}_5)$ are decomposed into gauge fields on the fuzzy sphere and a scalar Φ [137]. The scalar Φ describes the fluctuations of the sizes of the fuzzy spheres around $\overline{N}_5$.

Due to its relation with the plane-wave matrix model, the vacua and their superpotentials analyzed above carry over to the $U(N_2)$ 2+1 d SYM. For the $U(2)$ theory, the vacua are quite simple, which is characterized by one integer n, this is the eigenvalue of $\Phi \sim$diag$(n, -n)$. The tunneling issue was briefly discussed in [137]. It was further studied in detail by [124], who presented the instanton equation and solved a class of analytical solutions for the $U(2)$ theory. We will analyze the general $U(N_2)$ case, for we already know the superpontial for each vacua from the relation to plane-wave matrix model. One can map the instanton solution in plane-wave matrix model to that of the 2+1 SYM theory.

The action of the theory can be written in two ways. One is in terms of Φ and gauge field A. Another is in terms of three scalars Y_i which is a combination of the Φ and gauge fields A

$$Y^i = e^i\Phi + \epsilon^{ijk}e^jA_k, \quad A_i = \epsilon^{ijk}Y_je^k, \quad \Phi = e^iY^i \tag{6.11}$$

where e_i is a unit vector in R^3, $e_i e_i = 1$. The formalism in terms Φ and A is more direct from a gauge theory point of view. The Y_i originates in the plane-wave matrix model, from the fluctuations of the $SO(3)$ scalars around the fuzzy sphere $Y_i = X_i - 2J_i$. The formalism in terms of Y_i is more convenient for seeing its origin from the plane-wave matrix model.

We will write the action as in [127], further, we write it in such a way that it depends only on a dimensionless coupling constant g^2_{ym2}/m, where m was originally the mass of $SO(6)$ scalars. We rescale all fields so that the masses of $SO(6)$ scalars, Φ and fermions are 1, 2 and 3/2 respectively. The action takes the form

$$S = \frac{1}{g^2_{ym2}/m} \int dt \frac{1}{4} d^2\Omega \ \mathrm{tr} \left(\frac{1}{2} D_0 X_a D_0 X_a - \frac{1}{2} X_a X_a + \frac{1}{4} [X_a, X_b]^2 ... \right) \tag{6.12}$$

This is the same in effect as we set the radius of S^2, $1/\mu = 1/2$ in [127].

Because the vacua satisfy an equation $f + 2\Phi = 0$, where f is the S^2 component of the gauge field strength and is quantized into integers times a half, the vacuum is characterized by the eigenvalue of Φ

$$\Phi = \frac{1}{4} \ \mathrm{diag}(n_5^{(1)}, n_5^{(2)}, ..., n_5^{(N_2)}) \tag{6.13}$$

after an unitary transformation [127]. Notice that $n_5^{(i)} = N_5^{(i)} - \overline{N}_5$ ($n_5 \ll \overline{N}_5$) are the fluctuations of the sizes of N_2 fuzzy spheres.

We want to use the superpotential in the plane-wave matrix model to define the one in this theory. Some of the mappings of fields between the two theories are

$$Y_i = X_i - 2J_i, \quad \mathrm{tr}_{\overline{N}_5 \times \overline{N}_5} \to \overline{N}_5 \int d^2\Omega, \quad [J_j,] \to L_j, \quad L_j = -i\epsilon_{ijk} e_k \partial_i \tag{6.14}$$

where e_i is a unit vector in R^3, $e_i e_i = 1$. The superpotential is derived from the one in plane-wave matrix model (6.2) as

$$W = \overline{N}_5 \int d^2\Omega \ \mathrm{tr}_{N_2 \times N_2} (Y_i Y_i + i \frac{1}{3} \epsilon_{ijk} Y_i Y_j Y_k + i \epsilon_{ijk} Y_i L_j Y_k) \ + W_{J_i} \tag{6.15}$$

where the second term is a large constant term

$$W_{J_i} = \sum_{i=1,2,3} \frac{4}{3} \ \mathrm{tr}_{N \times N} J_i^2 \tag{6.16}$$

which corresponds to the superpotential for the vacuum with exact N_2 numbers of fuzzy spheres of size $\overline{N}_5$. The first term is the difference of the superpoential with respect to the vacuum $\Phi = \text{diag}(0, 0, ..., 0)$. The instanton equation is

$$D_\tau Y_i + \partial_{Y_i} W = D_\tau Y_i + 2Y_i + i\epsilon_{ijk} Y_j Y_k + 2i\epsilon_{ijk} Y_i L_j Y_k = 0 \tag{6.17}$$

The solutions to these equations in principle can be lifted from those solutions in the plane-wave matrix model for X_i.

We want to find also the expression in the variables of Φ and gauge field A. As discussed in appendix of [127], that the 2+1d SYM on $R \times S^2$ can be lifted to a 3+1d SYM on $R^2 \times S^2$, which can also be obtained from reduction of a 4+1d SYM on $R^2 \times S^3$ on the Hopf fiber on S^3. Interestingly, [124] noticed that the instanton solution in the Euclideanized 2+1 SYM on $R \times S^2$ is the same as the self-dual Yang-Mills equation on $R^2 \times S^2$ [124]. If we lift the Euclideanized 2+1 SYM on an x_3 direction, we can define the four dimensional gauge field strength $\mathcal{F}$ on $R^2 \times S^2$

$$\mathcal{F} = f d^2\Omega + (-2\Phi) d\tau dx_3, \quad *\mathcal{F} = (-2\Phi) d^2\Omega + f d\tau dx_3 \tag{6.18}$$

Note that the self-dual equation $\mathcal{F} = *\mathcal{F}$ is equivalent to the equation of motion $f + 2\Phi = 0$. The part of the action that does not involve the $SO(6)$ scalars and is inherent from the $SO(3)$ sector in plane-wave matrix model can be written as $\frac{1}{2}\int \mathcal{F} \wedge *\mathcal{F}$ term. So the action is bounded by

$$\begin{aligned} S_E &= \frac{1}{2}\int \mathcal{F} \wedge *\mathcal{F} = \frac{1}{4}\int (\mathcal{F} - *\mathcal{F}) \wedge (*\mathcal{F} - \mathcal{F}) + \frac{1}{2}\int \mathcal{F} \wedge \mathcal{F} & (6.19) \\ &\geqslant \frac{1}{2}\int \mathcal{F} \wedge \mathcal{F} \sim \int d^2\Omega \ \text{tr}\ \Phi^2 & (6.20) \end{aligned}$$

The bound is satisfied only when $\mathcal{F} = *\mathcal{F}$. Thus after matching parameters, we get that the instanton action should be

$$S_{inst} = -\frac{1}{g_0^2}\Delta W = -\frac{1}{g_0^2} 16 N_2 \overline{N}_5 (\text{tr}\ \Phi^2|_{+\infty} - \text{tr}\ \Phi^2|_{-\infty}) = -\frac{1}{g_0^2} N_2 \overline{N}_5 \sum_i n_5^{(i)2}|_{-\infty}^{+\infty} \tag{6.21}$$

This expression is of course consistent with the $U(2)$ case studied by [137],[124].

From the 2+1d SYM point of view, $\overline{N}_5$ is absorbed into the definition of the coupling constant via the relation $\frac{1}{g_{ym2}^2} = \frac{\mu^2 \overline{N}_5}{g_{ym0}^2}$ [137]. So we have the instanton action for the 2+1 SYM

$$S_{inst} = -\frac{1}{g_2^2} 4N_2(\text{tr } \Phi^2|_{+\infty} - \text{tr } \Phi^2|_{-\infty}) = -\frac{1}{4g_2^2} N_2 \sum_i n_5^{(i)2}|_{-\infty}^{+\infty} \tag{6.22}$$

where $g_2^2 = g_{ym2}^2/m$, purely in terms of parameters in the 2+1d SYM.

6.3 Gravity description

6.3.1 Euclidean branes

In this section we turn to the gravity analysis of the superpotential for each vacuum of both theories. In some regimes, the instanton solutions we studied in the previous section can be approximated as Euclidean branes wrapping non-trivial cycles in the dual background [127]. The backgrounds dual to the vacua of these theories have an $SO(3)$ and $SO(6)$ symmetry and thereby contain an S^2 and S^5, and the remaining three coordinates are time, ρ and η, where ρ is a radial coordinate. These backgrounds are regular because the dual theories have mass gaps. The gravity equations of motion reduce to a three dimensional Laplace equation for V, it is in the space of ρ, η if we combine ρ with an S^1 [126]. The regularity condition requires that the location where the S^2 shrinks are disks (or lines if without the S^1) at constant η_i in the ρ, η space, while the location where the S^5 shrinks are the segment of the $\rho = 0$ line between nearby two disks [127]. V is regarded as a electric potential and on the disks there are charges. Due to flux quantization, the charge and η distance are quantized. The charges are distributed in such a way that the disks are equipotential surfaces. The cycle we will embed the Euclidean brane is the 3-cycle at $\rho = 0$ between two disks, where the $\rho = 0$ line combine the S^2 to form a 3-cycle.

Since we use an Euclidean brane in a geometric background, such analysis is only an approximation when the transition is very slight or perturbative in nature. The geometries dual to two different vacua are different, so a non-perturbative analysis is needed, and it will be discussed in next section. The geometries before and after transition are approximated

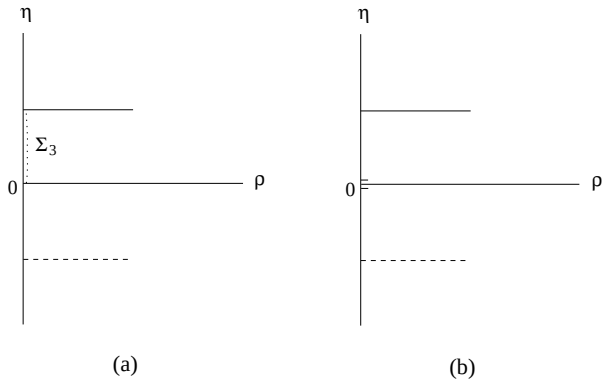

Figure 6.1: The configurations before and after the transition. In (a) there is a large disk with N_2 units of charge above the infinite disk. In (b) the large disk has $N_2 - 1$ units of charge and there is another small disk near origin with N_5 units of charge. There is a Σ_3 cycle that is along $\rho = 0$ and between the disks, formed together with the S^2. Below the infinite disk, there are image disks.

as the same background in this section. The wrapping of the brane should satisfy DBI-WZ action plus global conditions.

We first look at some regimes which obviously can be compared from both sides. We first look at the plane-wave matrix model. We look at in the gauge theory side tunneling from a vacuum with N_2 copies of fuzzy spheres with size N_5, to the vacuum with $N_2 - 1$ copies of fuzzy spheres with size N_5 plus N_5 copies of size 1 (or a single NS5 brane with N_5 D0 branes). We keep N_5 moderately large and N_2 large. In the dual gravity description, each fuzzy sphere is a disk with charge proportional to N_2, and located at a distance $\eta_0 = \frac{\pi}{2}N_5$ above the $\eta = 0$ plane (for more details, see section 2.2 of [127]). In the gravity picture, it corresponds to tunneling from a large disk with charge $Q = \frac{\pi^2}{8}N_2$ above the $\eta = 0$ plane, to a large disk with charge $Q' = \frac{\pi^2}{8}(N_2 - 1)$ plus a very small disk near the origin (see figure 6.1). We consider the small disk and its image as a small dipole with dipole moment $2(Q - Q')\eta_0 = \frac{\pi^3}{8}N_5$ near the origin, and that they do not backreact to the geometry. The total dipole moment, which corresponds to N D0 branes, is conserved.

There is a nontrivial 3-cycle at $\rho = 0$, between the large disk and the $\eta = 0$ plane. The cycle has topology of an S^3, with the S^2 shrinks only at $\eta = 0$ and $\eta = \eta_0$. The Euclidean

D2 brane wraps the S^2 and its Euclidean time τ is embedded along η direction extending from $\eta = 0$ to $\eta = \eta_0$. We should find $\eta(\tau)$ as a function of τ from the DBI-WZ action. Such Euclidean branes preserve half of the total 16 supercharges, since they extend along η and have an additional projection condition for the Killing spinors. We utilize the general background solution in eqn (2.20-2.24) of [127]. It is parametrized by an electric potential V as a function of ρ, η.

The DBI part of the action is

$$S_{DBI}/\tau_2 = -\int d\tau d^2\Omega e^{-\Phi}\sqrt{\det G_\parallel}\sqrt{\det G_\perp + \det(2\pi\alpha' F - B)} \tag{6.23}$$

$$= -\int d\tau 4\pi e^{-\Phi}\sqrt{(-g_{tt}) + g_{\eta\eta}(\partial_\tau\eta)^2}\sqrt{g_{22}^2 + (\pi N_5\alpha' - 2\eta - \frac{2\dot{V}V'}{\Delta})^2} \tag{6.24}$$

where τ_2 is the Euclidean D2 brane tension, $g_{tt}, g_{\eta\eta}, g_{22}$ are various metric components on the t, η, and S^2 directions in the background [127]. $G_\parallel$, $G_\perp$ are the time component and the spherical components of the pull-back metric, and $\det G_\perp = g_{22}^2 \sin^2\theta$, $2\pi\alpha' F - B = (\pi N_5\alpha' - 2\eta - \frac{2\dot{V}V'}{\Delta})\sin\theta d\theta d\phi$ according to flux quantization. We have chosen the gauge that $2\pi\alpha' F - B$ is zero where η is at the top disk. Dot and prime are the derivatives w.r.t. $\log\rho$ and η. The WZ part of the action is

$$S_{WZ}/\tau_2 = -\int [C_3 + (2\pi\alpha' F - B)\wedge C_1] \tag{6.25}$$

$$= -\int d\tau 4\pi \frac{4\dot{V}}{\ddot{V} - 2\dot{V}}[-\dot{V} - \dot{V}'(\frac{1}{2}\pi N_5\alpha' - \eta)] \tag{6.26}$$

Due to the background H_3 flux, the Euclidean D2 brane worldvolume gauge field has a source of N_5 units of magnetic charges, and these are precisely cancelled by the N_5 D0 branes ending on it, which is consistent for wrapping the brane on this cycle. This is similar to the situation of Euclidean D2 branes wrapping on the $SU(2)$ group manifold discussed in [136].

The expression of the action can be simplified if we expand the electric potential near $\rho \approx 0$, as

$$V = \sum_{i=0,even}^{\infty} (-1)^{\frac{i}{2}+1}\frac{K_0^{(i)}}{(i!!)^2}\rho^i = -K_0 + \frac{K}{4}\rho^2 - \frac{K''}{64}\rho^4 + ..., \qquad K \equiv K_0'' \tag{6.27}$$

where K is only a function of η. K has some properties such as $K > 0$, $\frac{-8K}{K''} > 0$, in order to satisfy the regularity condition for the geometry. The action is then

$$\begin{aligned}\frac{S_E}{4\pi\tau_2} &= -\int d\tau \left(\frac{-8K}{K''}\right)^{\frac{1}{2}} \sqrt{[K + K'(\frac{\pi}{2}N_5 - \eta)]^2 - \frac{1}{2}KK''(\frac{\pi}{2}N_5 - \eta)^2}\sqrt{\frac{-8K}{K''} + (\partial_\tau \eta)^2} \\ &\quad + \int d\tau \frac{-8K}{K''}[K + K'(\frac{\pi}{2}N_5 - \eta)] \end{aligned} \tag{6.28}$$

We have set $\alpha' = 1$. Since the action does not explicitly depend on τ, we can have a conserved quantity $H = P\partial_\tau \eta - L = \text{const.} = E$, where $P = \partial_{\partial_\tau \eta} L$. It is obvious that the constant solution $\eta = \frac{\pi}{2}N_5$ is a special case of a class of solutions to the equations, and from this solution we get $E = 0$. Setting $E = 0$, we get the equation of motion

$$\partial_\tau \eta = -\frac{2K(\frac{\pi}{2}N_5 - \eta)}{K + K'(\frac{\pi}{2}N_5 - \eta)} \tag{6.29}$$

The solution for the Euclidean D2 brane is

$$e^{-2\tau} = \frac{K}{(\frac{\pi}{2}N_5 - \eta)} \tag{6.30}$$

There could be an overall constant on the RHS, but it can be absorbed by shifting τ.

We plug in the solution back into the action and integrate out to get the final answer of the Euclidean brane action

$$S_E = -4\pi\tau_2 \int_0^{\frac{\pi}{2}N_5} 2K(\frac{\pi}{2}N_5 - \eta)d\eta \tag{6.31}$$

$$= -\frac{2}{\pi}[V(\eta_0) - V(0) - \eta_0 V'(0)] \tag{6.32}$$

where we used integration by parts several times to arrive at the last expression, and we used $\tau_2 = \frac{1}{4\pi^2}$, $\alpha' = 1$, $\eta_0 = \frac{\pi}{2}N_5$. $V(\eta)$ in the expression is understood as the potential $V(\rho, \eta)$ along the $\rho = 0$ line. The expression (6.32) has the right property of the invariance under the shift of V by a linear term in η, since the gravity solution should be invariant under this shift.

To get more intuition, let's look at the example of two infinite disks, which is the NS5 brane soultion in [127]. We wrap a Euclidean D2 brane on the throat region of the NS5 brane solution. For that solution [127],

$$V = \frac{1}{\widetilde{g}_0} I_0(r)\sin\theta, \qquad r = \frac{2\rho}{N_5}, \qquad \theta = \frac{2\eta}{N_5} \tag{6.33}$$

where $\widetilde{g}_0$ is a constant, so $K = -V''|_{\rho=0} = \frac{4}{N_5^2}\widetilde{g}_0^{-1}\sin\theta$. The solution for the Euclidean brane is then

$$e^{-2\tau} = e^{-2\tau_0}\frac{\sin(\pi-\theta)}{\pi-\theta} \tag{6.34}$$

where τ_0 is a constant. When $\theta = \pi$, $\tau(\pi) = \tau_0$, when $\theta = 0$, $\tau(0) = +\infty$, so this transition corresponds to transferring a small charge (corresponding to one unit of N_2) from top disk to the bottom disk. The Euclidean brane action for this solution is $S_E = \frac{2}{\widetilde{g}_0}$.

Of course, the solution in the form (6.32) works for general disk configurations in the plane-wave matrix model, when there are nearby two disks. The Euclidean D2 branes wrapping on that cycle mediate transferring of charges between two disks while keeping the total dipole moment fixed. It also works for the 2+1d SYM. In that case, the Euclidean D2 brane mediates similar transferring of charges between two nearby disks, while the total dipole moment of system is kept to zero, because the theory is expanded from the plane-wave matrix model and the disks are already in the center of charge frame.

There are issues of the range of validity that need to be addressed. One is that the calculation of the embedding of the Euclidean D-brane needs the gravity backgrounds which are weakly curved, this requires that the two disks where the cycle is in between are quite large, and their separation is also quite large. This needs the parameters $(g_{ym0}^2/m^3)N_2$, N_5 to be relatively large. On the other hand, N_5 had better not be too large, this is because from the Euclidean D2 brane point of view, there are N_5 D0 branes ending on its worldvolume, if N_5 is too large we would consider these N_5 D0 branes as a single NS5 brane, and the brane configuration should be changed. From the disk point of view the small disk near the origin should be much smaller then the top disk, so $N_5 \ll N_2$.

A second issue of the range of validity is that the Euclidean brane calculations only describe the transition between very close vacua or very close backgrounds. The geometric backgrounds before and after the transition are approximated as the same geometry. So in principle we need non-perturbative description which could interpolate between very different backgrounds. This will be discussed more in section 6.3.2.

6.3.2 Electric charge system and superpotential

Now we analyze what the quantity W is in the electrostatic picture. In gauge theory, the instanton action is proportional to the difference in the superpotential of the two vacua as defined by $S_{inst} = -\frac{1}{g_0^2}\Delta W$ in (6.7). In the case of the two vacua that is very similar to each other, we used the approximation of an Euclidean brane action to match the instanton action. Of course, the Euclidean brane action will be the strong-coupling result. For the configuration of two vacua studied in section 6.3.1, involving the exchange of a unit of charge (corresponding to a unit of N_2) between two disks, we arrived at the expression

$$S_E = -\frac{2}{\pi}[V(\eta_0) - V(0) - \eta_0 V'(0)] \tag{6.35}$$

This quantity is proportional to the change of the total energy U in the electrostatic system (including image charges). Note that the relation between D2 brane number and electric charge is $Q = \frac{\pi^2}{8}N_2$ [127], so the charges are in units of $\frac{\pi^2}{8}$, the charge transfer in this process is $\Delta Q = \frac{\pi^2}{8}$. The energy that the unit charge released (including the image charge) is

$$-\Delta U_{\mathrm{charge}} = 2\Delta Q[V(\eta_0) - V(0)] \tag{6.36}$$

On the other hand, the total dipole of the charge system is conserved, the dipole had been lost by the amount $2\Delta Q\eta_0$ by transferring the unit charge from the top disk. After the transferring process, we have the small disk (and also its image disk) near the origin, as in figure 6.1(b). We make an approximation when a pair of small disks near the origin is approximated as a point. We thereby have a small dipole with dipole moment $2\Delta Q\eta_0$ at the origin. The electric field at the origin is $-V'(0)$, which is pointing upward, so the coupling between the dipole and the electric field there gives an negative energy, this is the energy the dipole released:

$$-\Delta U_{\mathrm{dipole}} = (2\Delta Q\eta_0)[-V'(0)] \tag{6.37}$$

So the total energy difference of the system between final and initial configuration is

$$\Delta U_{\mathrm{total}} = -2\Delta Q[V(\eta_0) - V(0) - \eta_0 V'(0)] \tag{6.38}$$

which is just proportional to the Euclidean brane action, in appropriate unit. Of course this definition of the change of the total energy is invariant under the shift of the potential V by a linear term $b\eta$. The energy difference will not be changed due to this shift of V. This is because the charge ΔQ, will release an additional energy $2\Delta Q b\eta_0$, on the other hand, due to that there is an additional electric field downward with magnitude b, the dipole's energy is increased by $2\Delta Q\eta_0 b$, which exactly cancels the additional energy released from the charge. So this expression is consistent with the definition of the change of total energy.

We therefore can identify (in the strong coupling regime) $S_{inst} = S_E = \frac{8}{\pi^3}\Delta U$, and thereby the superpotential for each vacuum can be defined by the energy of the electric configuration corresponding to that vacuum

$$U = -\frac{\pi^3}{8g_0^2}W \tag{6.39}$$

up to an overall constant shift which is not important when comparing difference. The minus sign is due to our conventions. So the superpotential of the system is proportional to the total energy of charges (including image charges)

$$W = -\frac{16g_0^2}{\pi^3}\sum_i Q_i V_i \tag{6.40}$$

and W_0 which corresponds to the single NS5 brane vacuum is set to zero for convenience. i represents different groups of charges at $\eta = \eta_i$ with potential V_i. Now we see that the tunneling amplitude is given by the energy difference between the two configurations of the electric charge system or the eigenvalue system. The amplitude is largely suppressed if the energy difference is large. Note that there are still other parameters like the α' and $\hbar$, which have been set to 1 for convenience. These charges are associated with eigenvalues from the $SO(6)$ scalars and we will discuss this in section 6.4.

It might be useful to notice that the expressions (6.40) can be reduced to integrals of forms in ρ, η space. We define one forms locally $dV = \partial_\eta V d\eta + \partial_\rho V d\rho$, $dq = 2\pi\rho *_2 dV = 2\pi(-\partial_\eta V \rho d\rho + \partial_\rho V \rho d\eta)$, where $*_2$ is flat space epsilon symbol in ρ, η space. The Laplace equation for V is just $d(dq) = 0$.

We choose one-cycle Π_i enclosing each disk, and one-cycle Γ_i between each disk and a reference disk (whose potential is set as zero). We see that

$$Q_i = \int_{\Pi_i} dq, \qquad V_i = \int_{\Gamma_i} dV \tag{6.41}$$

Since Π_i cycle and the S^5 forms a 6-cycle, and Γ_i cycle and S^2 forms a 3-cycle in 9 spatial dimensions, the Π_i and Γ_i cycles do not intersect, $\Pi_i \cap \Gamma_i = 0$. We can thereby form a two-cycle by the cup product of the two cycles, so (6.40) can be written as

$$W = -\frac{16g_0^2}{\pi^3}\int_{\Pi_i\cup\Gamma_i} dq \wedge dV = -\frac{16g_0^2}{\pi^3}\int_{\Pi_i\cup\Gamma_i} |\nabla V|^2 2\pi\rho d\rho \wedge d\eta \tag{6.42}$$

where ∇V is the gradient of V in ρ, η space. So we have defined the superpotential of a boundary gauge theory by purely forms appeared in the bulk of gravity duals. This view is very similar in spirit to the definition of the non-perturbative superpotential in 3+1d $\mathcal{N} = 1$ gauge theories via a flux background after geometric transition [43].

Let's compare the superpotential in the weak coupling case. The charges in the electric system originate from the eigenvalues of the matrices in the $SO(6)$ sector in the gauge theory. We will discuss this more in section 6.4. Classically, these eigenvalues do not interact with each other. They distribute under the external electric potential, which are quadratic ($\sim \rho^2$) in the gravity dual of either theories [127]. So they are sitting on top of each other at fixed position η_i at $\rho = 0$. Quantum mechanically, the eigenvalues repel each other and they are not coincident. They thus form extended disks. The classical and quantum pictures, correspond to strong coupling and weak coupling effect for the superpotential. In the weak coupling analysis, the superpotential does not involve any dependence on the $SO(6)$ sector, so this must get corrections when coupling becomes large.

For the plane-wave matrix model, the background potential is $V_b = \frac{8}{g_s}(\rho^2\eta - \frac{2}{3}\eta^3)$, where $g_s = 4\pi^2 g_{ym0}^2$ [102] and g_s is the string coupling that appears at the asymptotic D0 brane near horizon geometry [125]. Since in the gravity side, m is the unit of energy for $\frac{|g_{tt}|}{g_{55}}$ [127], which is 1 in our case, so we have $m = 1$. In the weak-coupling regime of the gauge theory, since the charges sit at $\rho = 0$ and $\eta = \eta_i$, they feel potential $V_i = -\frac{4}{3\pi^2 g_0^2}\eta_i^3$, so the

superpotential from (6.40) is:

$$W = \frac{64}{3\pi^5}\sum_i Q_i \eta_i^3 = \sum_i \frac{1}{3} N_2^{(i)} N_5^{(i)3} \tag{6.43}$$

where we used $N_2^{(i)} = \frac{8Q_i}{\pi^2}$, $N_5^{(i)} = \frac{2\eta_i}{\pi}$ [127] from flux quantization condition. This is the same as the field theory calculation in weak-coupling regime, c.f. (6.9). The tunneling amplitude is largely suppressed if g_s is small and the change of the energy of the system is large.

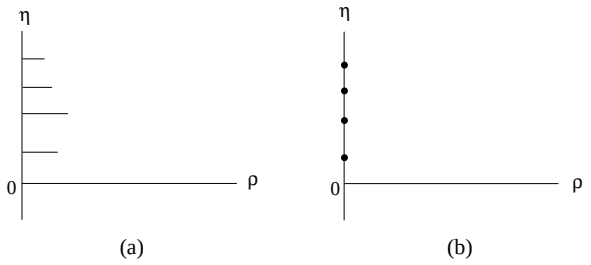

Figure 6.2: Comparison of strong coupling and weak coupling superpotentials. In (a) the charges or eigenvalues are extended into disks due to eigenvalue interactions in strong coupling regime. In (b) they are on top of each other at $\rho = 0$ line and not extended in the $SO(6)$ directions in the weak coupling limit. The superpotential is given by the energy of the system.

The vacua of the 2+1d SYM are inherited from that of plane-wave matrix model. In the disk configuration of the plane-wave matrix model, if we look at the geometry locally near a group of disks that is isolated from all other disks, including their images, then the geometry locally approaches to the solution dual to 2+1d SYM theory. The vacuum of the 2+1 SYM thereby is also characterized by the total energy of the electrostatic system. A special case is that when we start from plane-wave matrix model and look at the configuration where all the disks are very high above $\eta = 0$ plane and they are relatively near each other. Their average position is $\eta_0 = \frac{\pi}{2}\overline{N}_5$. We define new coordinate relative to the average position $\widetilde{\eta} = \eta - \eta_0 = \frac{\pi}{2} n_5$, $(\widetilde{\eta} \ll \eta_0)$ and $n_5 = N_5 - \overline{N}_5$, $(n_5 \ll \overline{N}_5)$. The total number of the D2 brane charges is $N_2 = \sum_{i=disks} N_2^{(i)}$. So in the new coordinate frame, this is the configuration dual to the $U(N_2)$ 2+1d SYM. n_5 is proportional to the location of each disk in the new frame. The case with all $n_5 = 0$ corresponds to a single disk at $\widetilde{\eta} = 0$, and is the

simplest vacuum with unbroken $U(N_2)$. The number $\overline{N}_5$ is used in the matrix regularization for the S^2. The electric potential near this group of disks is

$$V = \frac{2}{\pi^2 g_0^2}(\rho^2\eta - \frac{2}{3}\eta^3) = \frac{2\eta_0}{\pi^2 g_0^2}(\rho^2 - 2\widetilde{\eta}^2) - \frac{4}{3\pi^2 g_0^2}\eta_0^3 + ... \tag{6.44}$$

They basically feel the same external potential $\frac{2\eta_0}{\pi^2 g_0^2}(\rho^2 - 2\widetilde{\eta}^2)$ as in the gravity dual of the 2+1 SYM theory discussed in [127], with a large constant shift due to our expansion. The dot terms are the linear term in $\widetilde{\eta}$ and higher order terms in $\widetilde{\eta}$, which are not important.

In the weak coupling regime, the charges are not extended in the ρ direction. The total energy of the system (excluding the large constant term) gives the superpotential from the relation (6.40)

$$W = -\frac{16 g_0^2}{\pi^3}\sum_i Q_i \ \frac{2\eta_0}{\pi^2 g_0^2}(-2\widetilde{\eta}_i^2) = N_2\overline{N}_5 \sum_i n_5^{(i)2} \tag{6.45}$$

The large constant term from the η_0^3 term in the electric potential, corresponds to $W_{J_i} = \sum_i \frac{4}{3}\ \mathrm{tr}_{N\times N} J_i^2$ term for the fuzzy sphere we are expanding around in plane-wave matrix model, c.f. (6.16).

We thereby find that the weak-coupling superpotential analyzed in 6.2, which does not involve $SO(6)$ scalars, is the situation in the gravity side when the eigenvalue is not extended in the ρ direction, or the $SO(6)$ directions. This is when the eigenvalues do not have interactions. To bring out the interactions, we need quantum effect, and this is shown in the strong coupling regime in the disk picture. These are illustrated in figure 6.2.

Now we briefly discuss some technical aspects of solving the value of the superpotential in the strong-coupling regime, from the disk configurations. We first discuss the situation with a single disk above $\eta = 0$ plane (6.32), and then the situation with some arbitrary number of disks (6.40).

We first discuss the situation of a pair of positive and negative charged disks under the background potential $V_b = \frac{8}{g_s}(\rho^2\eta - \frac{2}{3}\eta^3)$. The size of the disks is ρ_0 and their separation is $2\eta_0$. Apart from the background, the charges have mutual repulsions from other charges on the disk and attractions from the image disk. Suppose the charge density and the potential on the disk is $\sigma(\rho)$ and V_0. We can scale out everything and the problem then is only

determined by the ratio $\kappa = \frac{\eta_0}{\rho_0}$. We make the rescaling $y = \eta/\rho_0,\ x = \rho/\rho_0,\ \ \sigma(\rho) = \frac{8}{g_s}\rho_0^2 g(x),\ V = \frac{8}{g_s}\rho_0^3 v$.

The integral equation in these variables is (which involves dimensionless functions)

$$\{(x^2\kappa - \frac{2}{3}\kappa^3) + \int_0^{2\pi}\int_0^1 [\frac{g(x')x'dx'd\phi}{\sqrt{x^2 - 2xx'\cos\phi + x'^2}} - \frac{g(x')x'dx'd\phi}{\sqrt{x^2 - 2xx'\cos\phi + x'^2 + 4\kappa^2}}]\}_{\forall x\in[0,1]} = \text{const.} \tag{6.46}$$

with the supplementary condition that $g(1) = 0$ due to vanishing of the charge at the edge of the disk. When $g(x)$ is solved, it is also a function of κ.

The integral equation is very difficult to solve due to its complicated kernel. Provided we know the function $g(x)$, since we already know the total charge of the disk, what we need to compute is the potential V on the disk. Since the disk is an equipotential, we can know it from the potential at $\rho = 0$, this consists of a contribution from the background and a contribution from the charges. The value of the superpotential (6.40) is

$$W/(\frac{1}{3}N_2N_5^3) = 1 - \frac{2}{3}\kappa^{-3}2\pi\int_0^1 dx\ g(x)\left[1 - \frac{x}{(x^2+4\kappa^2)^{1/2}}\right] \tag{6.47}$$

It contains two terms. The first term "1" on the RHS gives a contribution to the superpotential that is the same as the weak coupling gauge theory result $W = \frac{1}{3}N_2N_5^3$, it's the result if the charges were all sitting at $\rho = 0$. The second contribution is from the integral term, resulted from the effect of mutual interaction between charges. We do not assume that κ has to be small, so in principle, if we were looking for perturbative corrections to the weak coupling answer, the integral term gives the corrections relative to the first piece, in the large κ regime. The κ depends on N_2 in a very non-linear way. In the small κ limit, it is estimated in [127] that $\rho_0^4 = \frac{1}{2}\pi^4 g_{ym0}^2 N_2,\ \eta_0 = \frac{\pi}{2}N_5$, so $\kappa = \frac{\pi}{2}N_5(\frac{1}{2}\pi^4 g_{ym0}^2 N_2)^{-1/4}$.

The problem of solving the integral equation has been simplified by Abel transforming the kernel in (6.46) as done in [128]. The new integral equation is

$$f(x) - \int_{-1}^1 \frac{1}{\pi}\frac{2\kappa}{4\kappa^2 + (x-t)^2}f(t)dt = 1 - 2\alpha x^2 \tag{6.48}$$

where $f(t)$ is the Abel transform of $g(x)$, $\ f(t) = \frac{2\pi}{\beta}\int_t^1 \frac{rg(r)dr}{(r^2-t^2)^{\frac{1}{2}}}$, $\beta = v_0 + \frac{2}{3}\kappa^3$, and with the additional condition $f(1) = 0,\ f'(1)$ is bounded [128], and α is a constant determined

by these boundary conditions. The new kernel is the one appears in the Love's integral equation [130],[107],[96]. The solution in this form seems very promising. The kernel can even be simplified a little bit more in the small κ limit [107].

Now let's discuss the calculation of the superpotential for the electrostatic configuration with an arbitrary number of disks. Once we specify the charges Q_i of the disks and their heights η_i, what we need to know is the electric potential of each disk V_i, in order to compute the superpotential (6.40). We can thereby use the conformal transform techniques in [127].

In [127], the complex variables were introduced: $w = 2\partial_z V = \partial_\rho V - i\partial_\eta V$ and $z = \rho - \rho_0 + i\eta$. In the large disk limit, the general solution for the plane-wave matrix model is

$$\partial_w z = \prod_{j=1}^{n} \frac{(w - ia_j)}{(w - ic_j)}(-ia_{n+1}) \tag{6.49}$$

where there are n finite disks located at ia_j, and n fivebrane throats located at ic_j, and finally the infinite disk at ia_{n+1}. We can integrate to get the potential of each disk

$$V_i = Re \int w dz = -Re \int_{ia_i}^{ia_{n+1}} \prod_{j=1}^{n} \frac{(w - ia_j)}{(w - ic_j)}(-ia_{n+1}) w dw \tag{6.50}$$

and then plug this in (6.40). The rest is to figure out the relation between a_i, c_i and $N_2^{(i)}, N_5^{(i)}$ [127].

6.4 Emergence of gravity picture and eigenvalue system

In the previous sections, we find that the vacuum tunneling problem is associated to the superpotential of each vacuum, which is in turn related to the emergence of the geometry. The geometries are emergent from matrices in these theories. There are different types of emergence. The emergence of odd dimensional spheres and the even dimensional spheres seem to be different in some cases.

For the plane-wave matrix model, we have to explain the emergence of 9 directions, S^2, S^5, η and ρ. We first discuss the emergence of S^2 in the plane-wave matrix model. This is by the $SU(2)$ commutation relations of J_i. Usually for even dimensional fuzzy spheres, we can use Nambu brackets to define the sphere.

These fuzzy spheres have radial motions, which are characterized by the scalar Φ, most conveniently seen in the 2+1 d SYM. Comparing sections 6.2 and 6.3, we find that the eigenvalues of Φ are proportional to $n_5^{(1)}, n_5^{(2)}, ..., n_5^{(N_2)}$, which maps to $\eta_1, \eta_2, ..., \eta_{N_2}$, which are the heights of each disk in η direction. So the η direction is emerged from the eigenvalues of Φ.

Next we come to the emergence of odd dimensional sphere S^5 and the direction ρ. From intuition, due to the reduction from $\mathcal{N} = 4$ SYM, the $SO(6)$ sector in these theories are quite similar. So the emergence of S^5 should be similar to that of $\mathcal{N} = 4$ SYM [32].

In order to address this problem, we embed these vacuum geometries into the larger sector of 1/8 BPS geometries. These 1/8 BPS geometries are dual to 1/8 BPS states in the gauge theories we analyzed. We look at states satisfying $E - (J_A + J_B + J_C) = 0$, where E is the energy and J_A, J_B, J_C are the three $U(1)$ R-charges associated to the excitations by three complex scalars from the $SO(6)$ scalars. These 1/8 BPS states are in turn embedded in an even larger Hilbert space. Finally our BPS vacuum geometries are the ground states in the 1/8 BPS sector.

Though quite similar, a difference in the theories we analyzed, relative to $\mathcal{N} = 4$ SYM on $S^3 \times R$, is that we have many ground state geometries characterized by a broken gauge symmetry $U(N) \to \prod_i U(N^{(i)})$. The most convenient theory is the 2+1d SYM where a particular vacuum is specified by looking at the eigenvalues of the scalar Φ. If there are $N^{(i)}$ coincident eigenvalues, then there is a $U(N^{(i)})$ symmetry. These eigenvalues form an extended disks in the electric picture, but this could have alternative descriptions if we promote the geometries into large sectors. If the disks are far away from each other, the repulsion of charges on each disk is quite similar to the case of a single disk. In this limit, we study only the case of a single disk, that is, we look at the block of $SO(6)$ scalars corresponding to $N^{(i)}$ and only excite $U(N^{(i)})$ invariant 1/8 BPS states. Then finally we can excite all the blocks and build $\prod_i U(N^{(i)})$ invariant 1/8 BPS states.

For our purposes, we can consider first a single disk vacuum in the 2+1d SYM with unbroken $U(N)$ symmetry, and then other cases will be similar. These 1/8 BPS states are

described by the harmonic oscillators of the $SO(6)$ scalars and only s-wave modes of the S^2 are relevant for this class of BPS excitations. We use the methods similar to that of [32],[53],[64]. The action that is relevant for the excitation of $SO(6)$ scalars is the matrix quantum mechanics

$$S = \frac{\pi}{g_{ym2}^2/m} \int dt \ \mathrm{tr} \left(\frac{1}{2} \dot{X}_a \dot{X}_a - \frac{1}{2} X_a X_a + \frac{1}{4} [X_a, X_b]^2 \right) \tag{6.51}$$

where $a, b = 4, 5, ..., 9$ and we have set $\mu = 2$ in the lagrangian. We set $A_0 = 0$ and have a Gauss law constraint on $U(N)$ gauge singlet physical states, $\frac{\delta L}{\delta A_0} |\Psi\rangle = 0$. Furthermore, the commutator terms will not be included in the BPS excitation because they contribute positively to the energy but not the R-charges, they are generically stringy and non-BPS excitations. Then we define the conjugate momenta for X_a, which are P_a. The energy and R-charges are

$$E = \frac{1}{2} \mathrm{tr} \ (P_a P_a + X_a X_a), \quad J_A = \mathrm{tr} \ (P_4 X_5 - P_5 X_4), \tag{6.52}$$

$$J_B = \mathrm{tr} \ (P_6 X_7 - P_7 X_6), \quad J_C = \mathrm{tr} \ (P_8 X_9 - P_9 X_8) \tag{6.53}$$

We can form the usual three complex scalars Z_L, and their conjugate momenta $\Pi_L = -i \frac{\partial}{\partial Z_L^\dagger}$, $(L = A, B, C)$. They satisfy the standard commutation relations $[Z_L^{mn}, \Pi_L^{\dagger m' n'}] = i\hbar \delta_{nm'} \delta_{n'm}$, $[Z_L, \Pi_L] = 0$, so we can define creation/annihilation operators

$$a_L^\dagger = \frac{1}{\sqrt{2}} (Z_L^\dagger - i \Pi_L^\dagger), \quad b_L^\dagger = \frac{1}{\sqrt{2}} (Z_L^\dagger + i \Pi_L^\dagger) \tag{6.54}$$

and their conjugates. The energy and R-charge operators are then written as

$$E = \mathrm{tr} \sum_L (a_L^\dagger a_L + b_L^\dagger b_L), \quad J_L = \mathrm{tr} \ (a_L^\dagger a_L - b_L^\dagger b_L) \tag{6.55}$$

So it's clear in order to get 1/8 BPS states, we should keep only the three $a_L^\dagger$ oscillators but not the three $b_L^\dagger$ oscillators. We have then in the 1/8 BPS subspace of the Hilbert space a three dimensional harmonic oscillator Hilbert space. Its phase space is six dimensional. The singlet condition tells us we should look at states of products of traces

$$|\Psi\rangle = \prod_{i=1}^{k} \mathrm{tr} \ (a_A^\dagger)^{l_i} (a_B^\dagger)^{m_i} (a_C^\dagger)^{n_i} |0\rangle \tag{6.56}$$

with $l_i, m_i, n_i \leqslant N$. Because the commutator terms vanish, by a unitary transformation we can simultaneously put Z_A, Z_B, Z_C into the form of a diagonal matrix plus an off-diagonal triangular matrix. We integrate out the unitary matrix and the off-diagonal triangular matrix. Let the diagonal matrix elements be z_{Ai}, z_{Bi}, z_{Ci}. The wave function corresponding to (6.56) is of the form

$$\Psi \sim \Delta(z_{Ai}, z_{Bi}, z_{Ci}) \prod_{j=1}^{k} (\sum_i {z_{Ai}}^{l_j})(\sum_i {z_{Ai}}^{m_j})(\sum_i {z_{Ai}}^{n_j}) e^{-\frac{1}{2}\sum_i z_{Ai}\overline{z}_{Ai}+z_{Bi}\overline{z}_{Bi}+z_{Ci}\overline{z}_{Ci}} \tag{6.57}$$

up to a normalization factor. $\Delta(z_{Ai}, z_{Bi}, z_{Ci})$ is a Van-de-monde determinant from integrating out unitary matrices. See similar wave functions in [32], and its 1/2 BPS limit [53],[31],[173],[183]. There is an interesting property. The wave function, besides the universal Gaussian factor, is a holomorphic function in z_{Ai}, z_{Bi}, z_{Ci}, due to the BPS condition. This holomorphicity provides some simplifications. Wave functions that depends also on $\overline{z}_{Ai}, \overline{z}_{Bi}, \overline{z}_{Ci}$ are non-BPS. The complex structure is related to the symplectic structure in the phase space C^3. Each z_{Li} coordinates in the phase space are paired by a physical real coordinate and its conjugate momentum, $z_{Li} = x_{Li} + ip_{Li}$ [32].

In the large N limit, the wave function has a thermodynamic interpretation, the wave function squared describes the probability of a particular eigenvalue distribution [32]. The saddle point approximation of the wave function gives a droplet configuration in the phase space. The polynomial part of the wave function provides a repulsion force between eigenvalues. The Gaussian factor gives a spherically symmetric quadratic potential. The ground state should be a spherically symmetric droplet with $SO(6)$ symmetry in C^3, which is bounded by an S^5. Because of the spherical symmetry of the ground state droplet, the S^5 emerges in the gravity description. We see that the eigenvalues are extended in the radial direction of R^6, the ρ coordinate in the gravity description is really mapped from the radial direction of the phase space R^6. In the electric picture, charges are extended along ρ, if we tensor the ρ with the S^5, the charges are really distributing in a six dimensional subspace. This would be more clear if we had all the regular 1/8 BPS geometries of these theories. The S^5 then will be a particular enhanced symmetry for only the ground state geometries.

The thermodynamic interpretation of the wave function tells us that the probability for a particular configuration of eigenvalues is proportional to its Boltzmann factor $e^{-\beta U}$. Here U is the total energy of the eigenvalue system, which should be identified with the total energy of the electrostatic system in section 6.3.2, and β is the inverse temperature or inverse coupling constant. The amplitude for tunneling from one configuration a with energy U_a to another configuration b with energy U_b is proportional to the ratio of their Boltzmann factors, up to a normalization factor, so

$$P_{a\to b} = P_0 e^{-\beta(U_b - U_a)} \tag{6.58}$$

where P_0 is a constant. If we compare (6.58) with (6.10), we find similarity. This gives another support of the identification of the energy of the eigenvalue system with the superpotential, in appropriate units.

What are the droplet description if we start from 1/4 BPS sector or 1/2 BPS sector? They are described by two or one dimensional harmonic oscillators similar to the discussion above. They will have a C^2 or C^1 phase space. The ground state droplet will have $SO(3)$ symmetry in R^4 bounded by an S^3 or $U(1)$ symmetry in R^2 bounded by an S^1. In the electrostatic picture, there is an S^1 isometry from the S^5, that could combine with ρ into a real two dimensional disk. If we had all the 1/2 BPS geometries for these theories, they should have a S^3 isometry out of S^5. The ground states have an enhanced symmetry of S^1 which combines ρ to form a disk. Small ripples on the edge of the disk then describe BPS particles travelling along the S^1 equator of the S^5. We have a further evidence from the 3+1d $U(N)$ $\mathcal{N}=4$ SYM on $R\times S^3/Z_k$ with k/N small. In the electrostatic picture, the vacuum are periodic disks. If we look at the vacuum with one disk in a single period, because k/N is small, they look like charges in a cylinder, and the Laplace equation will be two dimensional in the space of ρ and an S^1. On the disk, the charges feel logarithm repulsions plus an spherical symmetric quadratic potential. The integral equation is exactly the same as the one in eqn (2.16) of [32] for the 1/2 BPS ground state droplet of $\mathcal{N}=4$ SYM on $R\times S^3$. We see their descriptions are consistent and should be exactly the same for $k=1$.

These ideas should be compared to the gravity results. A class of 1/4 BPS and 1/8 BPS geometries in $\mathcal{N} = 4$ SYM, relevant to the matrix model discussed above, have been studied by [63],[131] and by [113]. They have $U(1) \times SO(4) \times SO(2)_R$ [63],[131] and $U(1) \times SO(4)$ symmetries [113] (after analytical continuation of AdS_3) respectively. A Kahler structure were observed in both cases. In the 1/4 BPS case, it was reduced to a five dimensional complex Monge-Ampère equation which is highly non-linear. The boundary condition is roughly to divide regions where the S^3 or S^1 shrinks smoothly. Thereby there could be a droplet configuration in a four dimensional space. However, the regularity condition and the topology of these solutions are very complicated. To understand the differential equation in terms of the emergent forces between eigenvalues is a very challenging problem. Our case of the emergence of the electric picture of eigenvalues from gauge theories may bear some similarity to these more complicated cases.

The emergence of the odd dimensional sphere from phase space discussed above can be generalized to other cases, for example S^7. We consider the emergence of $AdS_4 \times S^7$ from M2 brane theory on $R \times S^2$. Apart from the time and S^2 that are already present, we need to explain the radial coordinate r in AdS_4 and the S^7. We look at the 1/16 BPS states of the M2 brane theory, by looking at a BPS bound with 4 $U(1)$ R-charges out of $SO(8)$. They corresponds to excitation of 4 oscillators associated with the 4 complex scalars. The 1/16 BPS sector has an eight dimensional phase space. The ground state droplet should have an enhanced symmetry of S^7 in the phase space. Similar picture for the 1/2 BPS sector of M2 brane theory has been studied in [126], the emergence of the S^1 (part of S^7) for the ground state in a two-plane has been observed.

Other way of emergence of odd dimensional spheres, similar to the even dimensional ones, have been studied by [162] or [168] using Nambu odd brackets and references therein. Their emergence is similar to the S^2 in our cases.

Chapter 7

Conclusions

In this thesis, we first studied the 1/2 BPS sector of the $AdS_5/\mathcal{N}=4$ SYM duality. We found the regular gravity solutions asymptotic to $AdS_5 \times S^5$ and preserve half of the supersymmetries of $AdS_5 \times S^5$. The gravity solutions are parametrized by incompressible fluid droplets on a two dimensional plane (x_1, x_2 plane). It is the same as the phase space of the quantum mechanics of N free fermions. This quantum mechanics arises from the dual $\mathcal{N}=4$ Super Yang Mills (SYM), when one reduces $\mathcal{N}=4$ SYM on S^3 to a matrix quantum mechanics. The N eigenvalues of the matrix model are fermions. The above description solved explicitly the $AdS_5/\mathcal{N}=4$ SYM correspondence in a reduced half-BPS sector. It demonstrates open-closed string duality, geometric transitions, and holography. It is also related to the coarse-graining of geometries and chronology protection mechanism in AdS/CFT.

We also studied 1/2 BPS excitations above IIB plane-wave, and these are described by relativistic fermions. These gravity solutions via dualities become different vacua of the mass deformed M2 brane theory, which describes M2 branes polarized to M5 branes wrapping two possible 3-spheres.

We found out the explicit superpoincare algebra in 2+1 dimensions for the mass deformed M2 brane theory. This algebra has the feature where on the right hand side of the supercharge anti-commutators, there are non-central charges. These are non-abelian gener-

alizations of the usual abelian central charges. This happens only for spacetime dimension less than 4. In our example the non-central charges are $SO(4)$ generators and the BPS conditions become that the particle mass is bounded by its total spins in two $SU(2)$s. One can reduce this superalgebra to 1+1 dimensions describing the DLCQ of IIB plane-wave string theory. The little group of these Poincare superalgebras gives $SU(2|2)$ supergroup. We have more generators than $SU(2|2)$ since we have poincare generators on the worldvolume. We also found theories of 1+1d SYM, 2+1d Super Yang Mills Chern Simons theory, linear dialton theory, and 2d (4,4) supersymmetric sigma model with non-zero H_3-flux, that satisfy these algebras.

We also studied similar phenomenon for the 1/2 BPS sector of $AdS_7 \times S^4$ or $AdS_4 \times S^7$ in M theory and found the gravity descriptions of these 1/2 BPS states. We hope this will provide some information for the 6d and 3d superconformal theories on M5 and M2 branes. In gravity side we also see a two dimensional phase space but the fermion density is not constant.

We also find the relations of various truncations and reductions of $\mathcal{N} = 4$ SYM which lead to $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$ (lens space), 2+1d SYM on $R \times S^2$ and 0+1d plane wave (BMN) matrix model. Various vacua of the resulting theories are related by gauge transformations when uplifted to $\mathcal{N} = 4$ SYM. These theories all have NS5 brane vacua. We found unified gravity dual descriptions of the vacua of these theories by specifying a plane with many cuts. Many dual gravity solutions have regions of NS5 brane near horizon geometries.

These above theories all have the supergroup $SU(2|4)$. We applied a Witten index in counting the number of 1/2 BPS states of these theories. The index is closely related to the characters of the supergroups and thereby this method is quite general. It can also be generalized to the index counting 1/4, 1/8 BPS states.

We take a general pp-wave limit of the gravity duals of the above theories, near some geodesics, and the worldsheet theories on these backgrounds lead to a class of 2 dimensional (4,4) supersymetric sigma models with non-zero H_3-flux. The non-zero H-flux is associated

with the NS5 branes in the background. The (4,4) supersymmetric sigma models with non-zero H_3-flux are rare, and of course the one with zero H_3-flux are given by hyperKahler geometries.

When we studied near BPS spectra of these backgrounds, we found two new phenomenon. We found that the near BPS spectrum of strings in the background dual to these theories generally involves a nontrivial interpolating function. The interpolating functions have different functional dependences in weak and strong couplings. This also gives a new possibility for understanding the phenomenon of the mismatch in the three-loop level between the near-plane-wave string spectrum and that of the dual dilatation operators.

The second phenomenon we found is the gauge theory interpretation of the single NS5 brane geometry. We found that the string excitations above the single NS5 brane vacua ($N_5 = 1$) of the plane wave matrix model only have four transverse oscillator modes, which means that the four dimensions transverse to the single NS5 brane (the S^3 and the linear dilaton direction) is infinitely massive and the string does not oscillate in these directions. On the other hand for the multi NS5 bane vacua ($N_5 \geq 2$) we found there are eight transverse oscillators, and the extra four comes from the 2nd Klein-Kluza modes on the fuzzy sphere ($N_5 \times N_5$ matrices), and forms a new supermultiplet. In the dual gravity side, we see a similar picture. The gravity solution is the near horizon geometry of N_5 NS5 branes with additional RR fluxes, since the NS5 branes are constructed from D0 branes. It is a deformation of the original NS5 brane near horizon geometry. Under certain rescaling, the RR fluxes can be rescaled away. The worldsheet theory is the deformation of supersymmetric $SU(2)_{N_5}$ WZW model, and under level shift it becomes bosonic $SU(2)_{N_5-2}$ WZW model with fermions and potentials. The geometry has a throat region for $N_5 \geq 2$ and doesn't have a throat region for $N_5 = 1$, due to the level, and this is also what we see from the dual plane wave matrix model side.

Similarly we also found relations between mass deformed M2 brane theory, D4 brane theory or M5 brane theory on S^3, and intersecting NS5 brane theory or the dual linear dilaton theory, in terms of a two-plane (x_1, x_2 plane). We compactified this plane into

a cylinder or a torus in the gravity side and then we see the relation of these theories by compactifying scalars. The vacuum structure of these theories are characterized by solutions of 2d Yang-Mills or 3d Chern-Simons theory. The configuration that corresponds to fermions on the torus x_1, x_2 gives rise to the near horizon geometry of two sets of intersecting NS5 branes intersecting on $R^{1,1}$.

We finally studied how different vacua of the theories with $SU(2|4)$ symmetry described above are interpolated between each other. We defined a superpotential for these vacua and studied the instanton action in gauge theory side. We also found Euclidean D-branes in gravity side describing some of the instantons. We found that the superpotential of each vacuum is given by the energy of the electric charge system in the gravity side and studied the emergence of the vacuum geometries.

References

[1] A. Adams, J. Polchinski, and E. Silverstein. Don't panic! closed string tachyons in ale space-times. *JHEP*, 10:029, 2001.

[2] O. Aharony, M. Berkooz, D. Kutasov, and N. Seiberg. Linear dilatons, ns5-branes and holography. *JHEP*, 10:004, 1998.

[3] O. Aharony, M. Berkooz, and N. Seiberg. Light-cone description of (2,0) superconformal theories in six dimensions. *Adv. Theor. Math. Phys.*, 2:119–153, 1998.

[4] O. Aharony, J. Marsano, S. Minwalla, K. Papadodimas, and M. Van Raamsdonk. The hagedorn / deconfinement phase transition in weakly coupled large n gauge theories. *Adv. Theor. Math. Phys.*, 8:603–696, 2004.

[5] L. F. Alday, J. de Boer, and I. Messamah. What is the dual of a dipole? *Nucl. Phys.*, B746:29–57, 2006.

[6] M. Alishahiha, H. Ebrahim, B. Safarzadeh, and M. M. Sheikh-Jabbari. Semi-classical probe strings on giant gravitons backgrounds. *JHEP*, 11:005, 2005.

[7] L. Alvarez-Gaume and D. Z. Freedman. Potentials for the supersymmetric nonlinear sigma model. *Commun. Math. Phys.*, 91:87, 1983.

[8] D. Astefanesei, R. B. Mann, and C. Stelea. Nuttier bubbles. *JHEP*, 01:043, 2006.

[9] C. Bachas, J. Hoppe, and B. Pioline. Nahm equations, n = 1* domain walls, and d-strings in ads(5) x s(5). *JHEP*, 07:041, 2001.

[10] A. Baha Balantekin and I. Bars. Dimension and character formulas for lie supergroups. *J. Math. Phys.*, 22:1149, 1981.

[11] A. Baha Balantekin and I. Bars. Representations of supergroups. *J. Math. Phys.*, 22:1810, 1981.

[12] A. Baha Balantekin and I. Bars. Branching rules for the supergroup su(n/m) from those of su(n+m). *J. Math. Phys.*, 23:1239, 1982.

[13] D. Bak, S. Siwach, and H.-U. Yee. 1/2 bps geometries of m2 giant gravitons. *Phys. Rev.*, D72:086010, 2005.

[14] I. Bakas. The structure of the w(infinity) algebra. *Commun. Math. Phys.*, 134:487–508, 1990.

[15] I. Bakas and K. Sfetsos. Toda fields of so(3) hyper-kahler metrics and free field realizations. *Int. J. Mod. Phys.*, A12:2585–2612, 1997.

[16] I. Bakas and J. Sonnenschein. On integrable models from pp-wave string backgrounds. *JHEP*, 12:049, 2002.

[17] V. Balasubramanian, M. Berkooz, A. Naqvi, and M. J. Strassler. Giant gravitons in conformal field theory. *JHEP*, 04:034, 2002.

[18] V. Balasubramanian, J. de Boer, V. Jejjala, and J. Simon. The library of babel: On the origin of gravitational thermodynamics. *JHEP*, 12:006, 2005.

[19] V. Balasubramanian, V. Jejjala, and J. Simon. The library of babel. *Int. J. Mod. Phys.*, D14:2181–2186, 2005.

[20] V. Balasubramanian, P. Kraus, and M. Shigemori. Massless black holes and black rings as effective geometries of the d1-d5 system. *Class. Quant. Grav.*, 22:4803–4838, 2005.

[21] V. Balasubramanian and A. Naqvi. Giant gravitons and a correspondence principle. *Phys. Lett.*, B528:111–120, 2002.

[22] T. Banks, W. Fischler, S. H. Shenker, and L. Susskind. M theory as a matrix model: A conjecture. *Phys. Rev.*, D55:5112–5128, 1997.

[23] T. Banks and N. Seiberg. Strings from matrices. *Nucl. Phys.*, B497:41–55, 1997.

[24] I. Bars, B. Morel, and H. Ruegg. Kac-dynkin diagrams and supertableaux. *J. Math. Phys.*, 24:2253, 1983.

[25] K. Behrndt, A. H. Chamseddine, and W. A. Sabra. Bps black holes in n = 2 five dimensional ads supergravity. *Phys. Lett.*, B442:97–101, 1998.

[26] K. Behrndt, M. Cvetic, and W. A. Sabra. Non-extreme black holes of five dimensional n = 2 ads supergravity. *Nucl. Phys.*, B553:317–332, 1999.

[27] N. Beisert. Bmn operators and superconformal symmetry. *Nucl. Phys.*, B659:79–118, 2003.

[28] N. Beisert. The complete one-loop dilatation operator of n = 4 super yang-mills theory. *Nucl. Phys.*, B676:3–42, 2004.

[29] I. Bena and N. P. Warner. A harmonic family of dielectric flow solutions with maximal supersymmetry. *JHEP*, 12:021, 2004.

[30] I. Bena and N. P. Warner. Bubbling supertubes and foaming black holes. 2005.

[31] D. Berenstein. A toy model for the ads/cft correspondence. *JHEP*, 07:018, 2004.

[32] D. Berenstein. Large n bps states and emergent quantum gravity. *JHEP*, 01:125, 2006.

[33] D. Berenstein, J. M. Maldacena, and H. Nastase. Strings in flat space and pp waves from n = 4 super yang mills. *JHEP*, 04:013, 2002.

[34] P. Berglund, E. G. Gimon, and T. S. Levi. Supergravity microstates for bps black holes and black rings. *JHEP*, 06:007, 2006.

[35] M. Berkooz, M. Rozali, and N. Seiberg. Matrix description of m theory on t**4 and t**5. *Phys. Lett.*, B408:105–110, 1997.

[36] N. Berkovits. Covariant quantization of the superstring. *Int. J. Mod. Phys.*, A16:801–811, 2001.

[37] N. Berkovits and N. Seiberg. Superstrings in graviphoton background and n = 1/2 + 3/2 supersymmetry. *JHEP*, 07:010, 2003.

[38] M. Bershadsky, Z. Kakushadze, and C. Vafa. String expansion as large n expansion of gauge theories. *Nucl. Phys.*, B523:59–72, 1998.

[39] M. Blau, J. Figueroa-O'Farrill, C. Hull, and G. Papadopoulos. A new maximally supersymmetric background of iib superstring theory. *JHEP*, 01:047, 2002.

[40] D. Boulatov and V. Kazakovand. Vortex anti-vortex sector of one-dimensional string theory via the upside down matrix oscillator. *Nucl. Phys. Proc. Suppl.*, 25A:38, 1992.

[41] C. P. Boyer and I. Finley, J. D. Killing vectors in selfdual, euclidean einstein spaces. *J. Math. Phys.*, 23:1126, 1982.

[42] A. Buchel. Coarse-graining 1/2 bps geometries of type iib supergravity. 2004.

[43] F. Cachazo, K. A. Intriligator, and C. Vafa. A large n duality via a geometric transition. *Nucl. Phys.*, B603:3–41, 2001.

[44] M. M. Caldarelli, D. Klemm, and P. J. Silva. Chronology protection in anti-de sitter. *Class. Quant. Grav.*, 22:3461–3466, 2005.

[45] M. M. Caldarelli and P. J. Silva. Giant gravitons in ads/cft. i: Matrix model and back reaction. *JHEP*, 08:029, 2004.

[46] J. Callan, Curtis G. et al. Quantizing string theory in ads(5) x s**5: Beyond the pp-wave. *Nucl. Phys.*, B673:3–40, 2003.

[47] J. Callan, Curtis G., J. A. Harvey, and A. Strominger. Supersymmetric string solitons. 1991.

[48] J. Callan, Curtis G., J. A. Harvey, and A. Strominger. World sheet approach to heterotic instantons and solitons. *Nucl. Phys.*, B359:611–634, 1991.

[49] M. Camperi, F. Levstein, and G. Zemba. The large n limit of chern-simons gauge theory. *Phys. Lett.*, B247:549–554, 1990.

[50] Z. W. Chong, H. Lu, and C. N. Pope. Bps geometries and ads bubbles. *Phys. Lett.*, B614:96–103, 2005.

[51] R. Clarkson and R. B. Mann. Eguchi-hanson solitons. 2005.

[52] R. Clarkson and R. B. Mann. Eguchi-hanson solitons in odd dimensions. *Class. Quant. Grav.*, 23:1507–1524, 2006.

[53] S. Corley, A. Jevicki, and S. Ramgoolam. Exact correlators of giant gravitons from dual n = 4 sym theory. *Adv. Theor. Math. Phys.*, 5:809–839, 2002.

[54] M. Cvetic and S. S. Gubser. Phases of r-charged black holes, spinning branes and strongly coupled gauge theories. *JHEP*, 04:024, 1999.

[55] J. Dai, X.-J. Wang, and Y.-S. Wu. Dynamics of giant-gravitons in the llm geometry and the fractional quantum hall effect. *Nucl. Phys.*, B731:285–308, 2005.

[56] S. R. Das, J. Michelson, and A. D. Shapere. Fuzzy spheres in pp-wave matrix string theory. *Phys. Rev.*, D70:026004, 2004.

[57] K. Dasgupta, M. M. Sheikh-Jabbari, and M. Van Raamsdonk. Matrix perturbation theory for m-theory on a pp-wave. *JHEP*, 05:056, 2002.

[58] K. Dasgupta, M. M. Sheikh-Jabbari, and M. Van Raamsdonk. Protected multiplets of m-theory on a plane wave. *JHEP*, 09:021, 2002.

[59] B. de Wit, J. Hoppe, and H. Nicolai. On the quantum mechanics of supermembranes. *Nucl. Phys.*, B305:545, 1988.

[60] A. Dhar. Bosonization of non-relativstic fermions in 2-dimensions and collective field theory. *JHEP*, 07:064, 2005.

[61] A. Dhar, G. Mandal, and N. V. Suryanarayana. Exact operator bosonization of finite number of fermions in one space dimension. *JHEP*, 01:118, 2006.

[62] R. Dijkgraaf and C. Vafa. A perturbative window into non-perturbative physics. 2002.

[63] A. Donos. A description of 1/4 bps configurations in minimal type iib sugra. 2006.

[64] A. Donos, A. Jevicki, and J. P. Rodrigues. Matrix model maps in ads/cft. *Phys. Rev.*, D72:125009, 2005.

[65] M. R. Douglas. Conformal field theory techniques for large n group theory. 1993.

[66] M. R. Douglas. Conformal field theory techniques in large n yang-mills theory. 1993.

[67] M. R. Douglas and V. A. Kazakov. Large n phase transition in continuum qcd in two-dimensions. *Phys. Lett.*, B319:219–230, 1993.

[68] M. R. Douglas and G. W. Moore. D-branes, quivers, and ale instantons. 1996.

[69] M. J. Duff and J. T. Liu. Anti-de sitter black holes in gauged n = 8 supergravity. *Nucl. Phys.*, B554:237–253, 1999.

[70] H. Ebrahim. Semiclassical strings probing ns5 brane wrapped on s**5. *JHEP*, 01:019, 2006.

[71] T. Eguchi and A. J. Hanson. Asymptotically flat selfdual solutions to euclidean gravity. *Phys. Lett.*, B74:249, 1978.

[72] T. Fischbacher, T. Klose, and J. Plefka. Planar plane-wave matrix theory at the four loop order: Integrability without bmn scaling. *JHEP*, 02:039, 2005.

[73] S. Frolov and A. A. Tseytlin. Multi-spin string solutions in ads(5) x s**5. *Nucl. Phys.*, B668:77–110, 2003.

[74] J. P. Gauntlett, J. B. Gutowski, C. M. Hull, S. Pakis, and H. S. Reall. All supersymmetric solutions of minimal supergravity in five dimensions. *Class. Quant. Grav.*, 20:4587–4634, 2003.

[75] J. P. Gauntlett, D. A. Kastor, and J. H. Traschen. Overlapping branes in m-theory. *Nucl. Phys.*, B478:544–560, 1996.

[76] J. P. Gauntlett, D. Martelli, S. Pakis, and D. Waldram. G-structures and wrapped ns5-branes. *Commun. Math. Phys.*, 247:421–445, 2004.

[77] J. P. Gauntlett, D. Martelli, J. Sparks, and D. Waldram. Sasaki-einstein metrics on s(2) x s(3). *Adv. Theor. Math. Phys.*, 8:711–734, 2004.

[78] J. P. Gauntlett, D. Martelli, J. Sparks, and D. Waldram. Supersymmetric ads(5) solutions of m-theory. *Class. Quant. Grav.*, 21:4335–4366, 2004.

[79] A. Ghodsi, A. E. Mosaffa, O. Saremi, and M. M. Sheikh-Jabbari. Lll vs. llm: Half bps sector of n = 4 sym equals to quantum hall system. *Nucl. Phys.*, B729:467–491, 2005.

[80] L. Grant, L. Maoz, J. Marsano, K. Papadodimas, and V. S. Rychkov. Minisuperspace quantization of 'bubbling ads' and free fermion droplets. *JHEP*, 08:025, 2005.

[81] R. Gregory, J. A. Harvey, and G. W. Moore. Unwinding strings and t-duality of kaluza-klein and h- monopoles. *Adv. Theor. Math. Phys.*, 1:283–297, 1997.

[82] M. T. Grisaru, R. C. Myers, and O. Tafjord. Susy and goliath. *JHEP*, 08:040, 2000.

[83] D. J. Gross, A. Mikhailov, and R. Roiban. Operators with large r charge in n = 4 yang-mills theory. *Annals Phys.*, 301:31–52, 2002.

[84] S. S. Gubser and J. J. Heckman. Thermodynamics of r-charged black holes in ads(5) from effective strings. *JHEP*, 11:052, 2004.

[85] S. S. Gubser, I. R. Klebanov, and A. W. Peet. Entropy and temperature of black 3-branes. *Phys. Rev.*, D54:3915–3919, 1996.

[86] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov. Gauge theory correlators from non-critical string theory. *Phys. Lett.*, B428:105–114, 1998.

[87] J. B. Gutowski, D. Martelli, and H. S. Reall. All supersymmetric solutions of minimal supergravity in six dimensions. *Class. Quant. Grav.*, 20:5049–5078, 2003.

[88] J. B. Gutowski and H. S. Reall. General supersymmetric ads(5) black holes. *JHEP*, 04:048, 2004.

[89] J. B. Gutowski and H. S. Reall. Supersymmetric ads(5) black holes. *JHEP*, 02:006, 2004.

[90] R. Haag, J. T. Lopuszanski, and M. Sohnius. All possible generators of supersymmetries of the s matrix. *Nucl. Phys.*, B88:257, 1975.

[91] A. Hashimoto, S. Hirano, and N. Itzhaki. Large branes in ads and their field theory dual. *JHEP*, 08:051, 2000.

[92] P. Horava and P. G. Shepard. Topology changing transitions in bubbling geometries. *JHEP*, 02:063, 2005.

[93] G. T. Horowitz. Spacetime in string theory. *New J. Phys.*, 7:201, 2005.

[94] G. T. Horowitz and T. Jacobson. Note on gauge theories on m/gamma and the ads/cft correspondence. *JHEP*, 01:013, 2002.

[95] G. T. Horowitz and J. Polchinski. Gauge / gravity duality. 2006.

[96] V. Hutson. The circular plate condenser at small separations. *Proc. Cambridge Phil. Soc.*, 59:211–224, 1963.

[97] S.-j. Hyun and H.-j. Shin. N = (4,4) type iia string theory on pp-wave background. *JHEP*, 10:070, 2002.

[98] S.-j. Hyun and H.-j. Shin. Solvable n = (4,4) type iia string theory in plane-wave background and d-branes. *Nucl. Phys.*, B654:114–134, 2003.

[99] N. Iizuka. Supergravity, supermembrane and m(atrix) model on pp- waves. *Phys. Rev.*, D68:126002, 2003.

[100] G. Ishiki, Y. Takayama, and A. Tsuchiya. N = 4 sym on r x s**3 and theories with 16 supercharges. 2006.

[101] N. Itzhaki, D. Kutasov, and N. Seiberg. I-brane dynamics. *JHEP*, 01:119, 2006.

[102] N. Itzhaki, J. M. Maldacena, J. Sonnenschein, and S. Yankielowicz. Supergravity and the large n limit of theories with sixteen supercharges. *Phys. Rev.*, D58:046004, 1998.

[103] N. Itzhaki and J. McGreevy. The large n harmonic oscillator as a string theory. *Phys. Rev.*, D71:025003, 2005.

[104] A. N. Jourjine. Constraints on superpotentials in off-shell extended nonlinear sigma models. *Nucl. Phys.*, B236:181–188, 1984.

[105] A. N. Jourjine. The effective potential in extended supersymmetric nonlinear sigma models. *Ann. Phys.*, 157:489, 1984.

[106] D. Kabat and W. Taylor. Spherical membranes in matrix theory. *Adv. Theor. Math. Phys.*, 2:181–206, 1998.

[107] M. Kac and H. Pollard. The distribution of the maximum of partial sums of independent random variables. *Canad. J. Math.*, 2:375–384, 1950.

[108] V. G. Kac. Lie superalgebras. *Adv. Math.*, 26:8–96, 1977.

[109] V. G. Kac. A sketch of lie superalgebra theory. *Commun. Math. Phys.*, 53:31–64, 1977.

[110] S. Kachru and E. Silverstein. 4d conformal theories and strings on orbifolds. *Phys. Rev. Lett.*, 80:4855–4858, 1998.

[111] V. Kazakov, I. K. Kostov, and D. Kutasov. A matrix model for the two-dimensional black hole. *Nucl. Phys.*, B622:141–188, 2002.

[112] R. R. Khuri. Remark on string solitons. *Phys. Rev.*, D48:2947–2948, 1993.

[113] N. Kim. Ads(3) solutions of iib supergravity from d3-branes. *JHEP*, 01:094, 2006.

[114] N. Kim and J.-H. Park. Superalgebra for m-theory on a pp-wave. *Phys. Rev.*, D66:106007, 2002.

[115] N.-w. Kim, T. Klose, and J. Plefka. Plane-wave matrix theory from n = 4 super yang-mills on r x s**3. *Nucl. Phys.*, B671:359–382, 2003.

[116] N.-w. Kim, A. Pankiewicz, S.-J. Rey, and S. Theisen. Superstring on pp-wave orbifold from large-n quiver gauge theory. *Eur. Phys. J.*, C25:327–332, 2002.

[117] N.-w. Kim and J. Plefka. On the spectrum of pp-wave matrix theory. *Nucl. Phys.*, B643:31–48, 2002.

[118] J. Kinney, J. M. Maldacena, S. Minwalla, and S. Raju. An index for 4 dimensional super conformal theories. 2005.

[119] I. R. Klebanov. String theory in two-dimensions. 1991.

[120] I. R. Klebanov and M. J. Strassler. Supergravity and a confining gauge theory: Duality cascades and chisb-resolution of naked singularities. *JHEP*, 08:052, 2000.

[121] T. Klose and J. Plefka. On the integrability of large n plane-wave matrix theory. *Nucl. Phys.*, B679:127–142, 2004.

[122] P. Kraus, F. Larsen, and S. P. Trivedi. The coulomb branch of gauge theory from rotating branes. *JHEP*, 03:003, 1999.

[123] A. E. Lawrence, N. Nekrasov, and C. Vafa. On conformal field theories in four dimensions. *Nucl. Phys.*, B533:199–209, 1998.

[124] H. K. Lee. Gauge theory and supergravity duality in the pp-wave background. UMI-31-97350.

[125] H. Lin. The supergravity dual of the bmn matrix model. *JHEP*, 12:001, 2004.

[126] H. Lin, O. Lunin, and J. M. Maldacena. Bubbling ads space and 1/2 bps geometries. *JHEP*, 10:025, 2004.

[127] H. Lin and J. M. Maldacena. Fivebranes from gauge theory. 2005.

[128] H. Ling, A. R. Mohazab, H.-H. Shieh, G. van Anders, and M. Van Raamsdonk. Little string theory from a double-scaled matrix model. 2006.

[129] J. T. Liu and R. Minasian. Black holes and membranes in ads(7). *Phys. Lett.*, B457:39–46, 1999.

[130] E. R. Love. The electrostatic field of two equal circular co-axial conducting disks. *The Quarterly Journal of Mechanics and Applied Mathematics*, 2(4):428–451, 1949.

[131] O. Lunin. On gravitational description of wilson lines. *JHEP*, 06:026, 2006.

[132] O. Lunin, J. M. Maldacena, and L. Maoz. Gravity solutions for the d1-d5 system with angular momentum. 2002.

[133] O. Lunin and S. D. Mathur. Ads/cft duality and the black hole information paradox. *Nucl. Phys.*, B623:342–394, 2002.

[134] J. M. Maldacena. The large n limit of superconformal field theories and supergravity. *Adv. Theor. Math. Phys.*, 2:231–252, 1998.

[135] J. M. Maldacena and L. Maoz. Strings on pp-waves and massive two dimensional field theories. *JHEP*, 12:046, 2002.

[136] J. M. Maldacena, G. W. Moore, and N. Seiberg. D-brane instantons and k-theory charges. *JHEP*, 11:062, 2001.

[137] J. M. Maldacena, M. M. Sheikh-Jabbari, and M. Van Raamsdonk. Transverse five-branes in matrix theory. *JHEP*, 01:038, 2003.

[138] G. Mandal. Fermions from half-bps supergravity. *JHEP*, 08:052, 2005.

[139] L. Maoz and V. S. Rychkov. Geometry quantization from supergravity: The case of 'bubbling ads'. *JHEP*, 08:096, 2005.

[140] S. D. Mathur. The quantum structure of black holes. *Class. Quant. Grav.*, 23:R115, 2006.

[141] J. McGreevy, L. Susskind, and N. Toumbas. Invasion of the giant gravitons from anti-de sitter space. *JHEP*, 06:008, 2000.

[142] J. McGreevy and H. L. Verlinde. Strings from tachyons: The c = 1 matrix reloaded. *JHEP*, 12:054, 2003.

[143] A. A. Migdal. Gauge transitions in gauge and spin lattice systems. *Sov. Phys. JETP*, 42:743, 1975.

[144] A. A. Migdal. Recursion equations in gauge field theories. *Sov. Phys. JETP*, 42:413, 1975.

[145] A. Mikhailov. Giant gravitons from holomorphic surfaces. *JHEP*, 11:027, 2000.

[146] G. Milanesi and M. O'Loughlin. Singularities and closed time-like curves in type iib 1/2 bps geometries. *JHEP*, 09:008, 2005.

[147] J. A. Minahan and A. P. Polychronakos. Equivalence of two-dimensional qcd and the c = 1 matrix model. *Phys. Lett.*, B312:155–165, 1993.

[148] J. A. Minahan and K. Zarembo. The bethe-ansatz for n = 4 super yang-mills. *JHEP*, 03:013, 2003.

[149] S. Mukhi, M. Rangamani, and E. P. Verlinde. Strings from quivers, membranes from moose. *JHEP*, 05:023, 2002.

[150] R. C. Myers. Dielectric-branes. *JHEP*, 12:022, 1999.

[151] R. C. Myers and O. Tafjord. Superstars and giant gravitons. *JHEP*, 11:009, 2001.

[152] S. G. Naculich, H. A. Riggs, and H. J. Schnitzer. Group level duality in wzw models and chern-simons theory. *Phys. Lett.*, B246:417–422, 1990.

[153] W. Nahm. Supersymmetries and their representations. *Nucl. Phys.*, B135:149, 1978.

[154] J. F. Nieves and P. B. Pal. Generalized fierz identities. *Am. J. Phys.*, 72:1100–1108, 2004.

[155] A. Okounkov, N. Reshetikhin, and C. Vafa. Quantum calabi-yau and classical crystals. 2003.

[156] H. Ooguri and C. Vafa. Two-dimensional black hole and singularities of cy manifolds. *Nucl. Phys.*, B463:55–72, 1996.

[157] Q.-H. Park. Selfdual gravity as a large n limit of the two-dimensional nonlinear sigma model. *Phys. Lett.*, B238:287, 1990.

[158] J. Polchinski. String theory. vol. 2: Superstring theory and beyond. Cambridge, UK: Univ. Pr. (1998) 531 p.

[159] J. Polchinski and M. J. Strassler. The string dual of a confining four-dimensional gauge theory. 2000.

[160] A. M. Polyakov. Gauge fields and space-time. *Int. J. Mod. Phys.*, A17S1:119–136, 2002.

[161] C. N. Pope and N. P. Warner. A dielectric flow solution with maximal supersymmetry. *JHEP*, 04:011, 2004.

[162] S. Ramgoolam. On spherical harmonics for fuzzy spheres in diverse dimensions. *Nucl. Phys.*, B610:461–488, 2001.

[163] J. G. Russo and A. A. Tseytlin. A class of exact pp-wave string models with interacting light-cone gauge actions. *JHEP*, 09:035, 2002.

[164] A. Santambrogio and D. Zanon. Exact anomalous dimensions of n = 4 yang-mills operators with large r charge. *Phys. Lett.*, B545:425–429, 2002.

[165] N. Seiberg. New theories in six dimensions and matrix description of m- theory on t**5 and t**5/z(2). *Phys. Lett.*, B408:98–104, 1997.

[166] D. Serban and M. Staudacher. Planar n = 4 gauge theory and the inozemtsev long range spin chain. *JHEP*, 06:001, 2004.

[167] S. Sethi and L. Susskind. Rotational invariance in the m(atrix) formulation of type iib theory. *Phys. Lett.*, B400:265–268, 1997.

[168] M. M. Sheikh-Jabbari. Tiny graviton matrix theory: Dlcq of iib plane-wave string theory, a conjecture. *JHEP*, 09:017, 2004.

[169] P. G. Shepard. Black hole statistics from holography. *JHEP*, 10:072, 2005.

[170] P. J. Silva. Rational foundation of gr in terms of statistical mechanic in the ads/cft framework. *JHEP*, 11:012, 2005.

[171] K. Sugiyama and K. Yoshida. Type iia string and matrix string on pp-wave. *Nucl. Phys.*, B644:128–150, 2002.

[172] N. V. Suryanarayana. Half-bps giants, free fermions and microstates of superstars. *JHEP*, 01:082, 2006.

[173] Y. Takayama and A. Tsuchiya. Complex matrix model and fermion phase space for bubbling ads geometries. *JHEP*, 10:004, 2005.

[174] I. Taylor, Washington. D-brane field theory on compact spaces. *Phys. Lett.*, B394:283–287, 1997.

[175] C. Vafa. Superstrings and topological strings at large n. *J. Math. Phys.*, 42:2798–2817, 2001.

[176] C. Vafa. Two dimensional yang-mills, black holes and topological strings. 2004.

[177] C. Vafa and E. Witten. A strong coupling test of s duality. *Nucl. Phys.*, B431:3–77, 1994.

[178] R. S. Ward. Einstein-weyl spaces and su(infinity) toda fields. *Class. Quant. Grav.*, 7:L95–L98, 1990.

[179] S. Weinberg. The quantum theory of fields. vol. 3: Supersymmetry. Cambridge, UK: Univ. Pr. (2000) 419 p.

[180] E. Witten. Constraints on supersymmetry breaking. *Nucl. Phys.*, B202:253, 1982.

[181] E. Witten. Anti-de sitter space and holography. *Adv. Theor. Math. Phys.*, 2:253–291, 1998.

[182] J.-T. Yee and P. Yi. Instantons of m(atrix) theory in pp-wave background. *JHEP*, 02:040, 2003.

[183] T. Yoneya. Extended fermion representation of multi-charge 1/2-bps operators in ads/cft: Towards field theory of d-branes. *JHEP*, 12:028, 2005.