CMB ANISOTROPIES CAUSED BY GRAVITATIONAL WAVES: DEPENDENCE ON COSMOLOGICAL PARAMETERS AND PRIMORDIAL MAGNETIC FIELD INFLUENCE

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CMB anisotropies in the cosmic microwave background radiation due to gravity waves are investigated. An initial spectrum of gravity waves may have been induced during an epoch of inflation. We study the propagation of gravitational waves in different cosmological models and the dependence of the anisotropies induced in the CMB on the cosmological parameters. We then assume the presence of primordial chaotic magnetic field which sources tensor perturbations. Taking into account the anisotropies in the CMB generated by this process, we constrain the allowed amplitude of the magnetic field as function of the spectral index

1 Introduction

The theoretical and observational determination of anisotropies in the cosmic microwave background radiation (CMB) has recently attracted a lot of attention. One has justified hopes to measure the CMB anisotropies to a precision of a few percent or better within the next ten years. Furthermore, if initial fluctuations are induced during a primordial inflationary period and no external sources induce perturbations at later times, CMB anisotropies can be calculated by linear cosmological perturbation theory to very good accuracy. Since the detailed results depend not only on the primordial spectrum but also on the parameters of the cosmological model considered, the anisotropy spectrum may provide a mean to determine these parameters to an accuracy of a few percent.

As it is well known, tensor perturbations can by generated during inflation. They play an important role in large scale structure formation and contribute in $\Delta T/T$ anisotropies[1] . Hence, the question how gravitational wave contributions depend on model parameters is very important.

During inflation also primordial magnetic fields may be generated. Such fields are necessary to explain the $\sim \mu$Gauss fields observed in galaxies. A number of mechanisms have been proposed for the origin of these seed fields[2,3,4]. Given a small seed field at late times, two different mechanisms attempt to explain its amplification into magnetic fields of order 10^{-6} Gauss in our galaxy: the adiabatic compression of magnetic flux lines can amplify a seed field of order 10^{-9} Gauss to the present observable values; the far more efficient (and controversial) galactic dynamo mechanism might amplify seed fields as small as 10^{-20}Gauss. Clearly, to make some progress in identifying which one of these mechanisms is responsible for galactic magnetic fields, one would like to find a constraint of the seed field before it has been processed by local, galactic dynamics. In particular, if one were able to obtain an upper bound which is lower than 10^{-9} Gauss, than this would be strong evidence that adiabatic compression is insufficient to amplify the primordial magnetic fields to their present values.

Primordial fields of 10^{-9} may have left their traces in the CMB, but we shall never be able to constrain tiny seed fields on the order of 10^{-13}Gauss using CMB anisotropies. A number of methods have been proposed in the past few years for measuring a cosmological magnetic field using the CMB: the effect on the Doppler peaks[5], Faraday rotation on small[6] and large[7] scales, vortical imprints[8] and [9], can all lead to anisotropies of order 10^{-9} to 10^{-8} Gauss.

The influence of homogeneous magnetic field was discussed in different works[5,6,8,9]. There is no fundamental reason to discard the possibility of a homogeneous magnetic field, all physical mechanisms proposed to date lead to the presence of stochastic magnetic fields with no homogeneous term; in this work we consider stochastic magnetic fields.

2 Tensor anisotropies in the CMB: A parameter study

We discuss the model dependence of anisotropies due to gravitational waves for models with a total density parameter $\Omega = 1$ which are thus spatially flat. However, we vary the contributions of cold dark matter (CDM), hot dark matter (HDM) and a cosmological constant, which are given in terms of the parameters Ω_C , Ω_H and Ω_Λ. We also vary the number of degrees of freedom for massless neutrinos and hot particles.

2.1 The models

The basics of linear perturbations of Friedmann universes can be found in [11]. We determine the evolution of tensor perturbations in spatially flat Universes which contain a fraction of cold dark matter (CDM), hot dark matter (HDM) and a cosmological constant Λ, such that: $\Omega_0 = \Omega_H + \Omega_C + \Omega_\Lambda = 1$, where Ω_0 denotes the density parameter today, i.e. at t_0. We neglect the contribution of photons, massless neutrinos and baryons (which may be included in CDM) to Ω_0.

The metric tensor perturbations is given by: h^T_{ij}, which satisfy the conditions

$$h^{Ti}_{i} = 0, \qquad h^{Tj}_{i} k^i = 0 \tag{1}$$

where k is the wave vector which may be set equal to $(0,0,k)$, such that the conditions (1) reduce to

$$h^T_{11} = -h^T_{22} = H, \qquad \text{and} \qquad h^T_{i3} = h^T_{3i} = 0 \ . \tag{2}$$

We describe dynamics of tensor perturbations in a medium containing collision-less particles, whose anisotropic stresses are not damped by collisions. As long as the collision-less component is relativistic, it provides a source for gravitational waves. The evolution equation for tensor perturbations of the metric is given by[11]:

$$\ddot{h}^T_{ij} + 2\frac{\dot{a}}{a}\dot{h}^T_{ij} + k^2 h^T_{ij} = 8\pi G a^2 p \Pi_{ij} \ . \tag{3}$$

Here p is the pressure of the collision-less component and Π denotes the tensor contribution to the anisotropic stresses, which in our case are due to the presence of relativistic, collision-less particles. In our models we have in principle three kinds of collision-less particles: Hot dark matter, massless neutrinos and, after recombination, the photons. Studying the initial conditions, we shall find that for the growing mode anisotropic stresses are extremely small on super horizon scales. However, when the scales relevant for tensor CMB anisotropies ($\lambda \gg t_{dec}$) enter the horizon, $t \gg t_{dec}$ HDM particles are already non relativistic. We may thus neglect their contribution to anisotropic stresses. Hence, we just consider the pressure anisotropy from massless neutrinos and, after recombination, from the photons themselves. For massless particles we solved Liouville equation and derived their anisotropic stress.

We assume standard inflation according to which the initial amplitude of gravitational waves is independent of scale *i.e.*, $\langle |h(t_{in}, k)|^2 \rangle \propto k^{-3}$. It is easy see that on superhorizon scales ($kt \ll 1$), h =const. and the evolution of gravitational waves and as a result $\Delta T/T$ are independent of the model parameters. For scales $kt \approx 1$, the metric perturbations begin to oscillate and eventually ($kt \gg 1$) damp away. The non-zero $\dot{h}$ then induces anisotropic stresses. Very often, these anisotropic stresses have been neglected in the literature. Here we find that their effect is indeed very small. There is typically about 1% additional damping due to the loss of some gravitational wave energy into anisotropic stresses.

The main model dependence is the modification of the damping term ($\dot{a}/a$) in the different backgrounds considered.

The power spectrum, C_l of the CMB anisotropies can given as follows[12]:

$$C_\ell = \frac{2}{\pi} \int dkk^2 |I(\ell, k)|^2 \ell(\ell-1)(\ell+1)(\ell+2) \ , \tag{4}$$

with

$$I(\ell, k) = \int_{t_{dec}}^{t_0} dt \dot{H}(t, k) \frac{j_\ell((k(t_0 - t))}{(k(t_0 - t))^2} \tag{5}$$

where j_ℓ denotes the spherical Bessel function of order ℓ.

Having calculated the metric perturbations h_{ij}^T numerically, we can determine the CMB fluctuation spectrum by means of numerical integration and investigate it's dependence on the model parameters.

2.2 Results

We have investigated 80 models varying the five parameters $(h_0, \Omega_\Lambda, \Omega_H/\Omega_C, \beta_\nu, \beta_H)$, where h_0 is the Hubble parameter H_0 in units of 100km/s/Mpc, β_ν denotes the number of degrees of freedom in massless neutrini and β_H is the corresponding number for hot dark matter particles.

All the models lead to similar gravity wave induced anisotropies which, for reasonable parameter choices, differ by less than about 10%. The changes due to anisotropic stresses are extremely small, on the order of 1% or less. The main difference is caused by a non-zero cosmological constant. A similar effect is obtained if we increase the Hubble parameter. Hot dark matter does not induce significant changes since, at times when the wavelengths leading to substantial CMB anisotropies enter the horizon, hot dark matter is already non-relativistic, resulting in nearly the same expansion law as cold dark matter.

Our results is presented in Fig.1.

The relevant parameters to be considered are thus $\Omega_M = \Omega_C + \Omega_H$, Ω_Λ and h_0.

The variation of the tensor C_ℓ spectrum for different cosmological models with fixed Hubble parameter which are not already excluded by other observations than CMB anisotropies never exceed 10% for $\ell < 60$, while variations of the Hubble parameter can lead to changes in the spectrum of up to 15%. Thus, by reasons of cosmic variance, the statistical relative error in C_ℓ measured from only one point in the universe is always $1/\sqrt{2\ell+1}$. This is a very significant uncertainty, especially for the gravitational wave contribution which peaks around $\ell \sim 20$ and has already dropped by a factor of about 2 at $\ell = 60$ (see Fig. 1).

3 Tensor anisotropies in the CMB due to a primordial magnetic field

We derive an expression for the angular power spectrum of cosmic microwave background anisotropies due to gravity waves generated by a stochastic magnetic field. We assume a simple

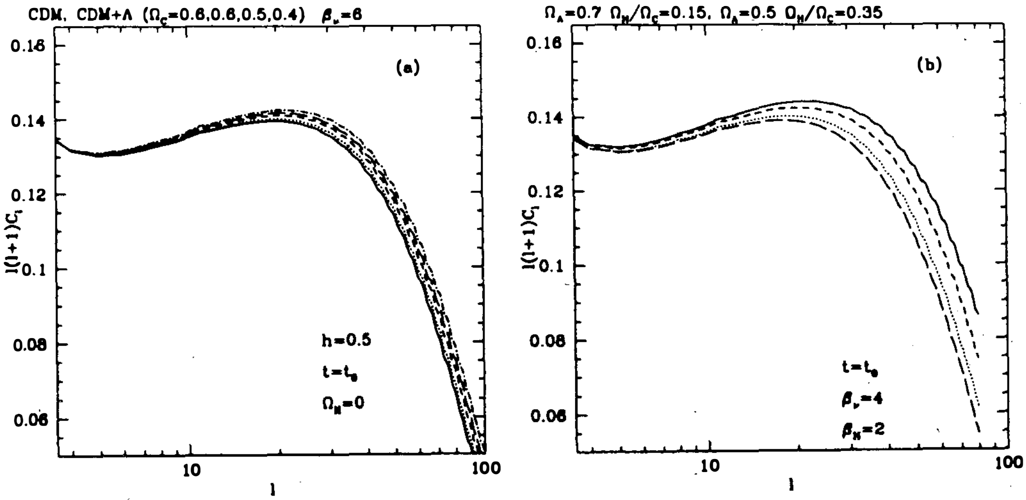

Figure 1: In frame (a), the angular power spectra of CMB anisotropies induced by gravitational waves are shown for models with different values for the cosmological constant. The solid line represents the model $\Omega_\Lambda = 0$ (solid line) and the amplitude increases with increasing Λ. In frame (b), we show the effect of increasing the Hubble parameter. The models chosen are mixed dark matter models with cosmological constant, $\Omega_H/\Omega_{CMD} = 0.15$, $\Omega_\Lambda = 0.7$ with $h_0 = 0.5$ (solid line) and $h_0 = 0.75$ (dotted line); and $\Omega_H/\Omega_{CDM} = 0.35$, $\Omega_\Lambda = 0.5$ with $h_0 = 0.5$ (dashed line) and $h_0 = 0.75$ (long dashes) are shown.

flat model of the Universe, $\Omega_0 = \Omega_C = 1$, neglecting a possible non-zero Ω_Λ, due to the weak dependence on this cosmological parameter (see section 2).

3.1 The model

Although the baryon photon plasma has decoupled, the conductivity of the inter-galactic medium is still effectively infinite. This means that the perturbations due to the Lorentz force due to any velocity perturbations are negligible in comparison to the zeroth order contribution of the Lorentz force itself. In this regime we can effectively decouple the time evolution from the spatial structure such that $\mathbf{B}(\eta, \mathbf{x}) = \mathbf{B}_0(\mathbf{x})/a^2$. We model $\mathbf{B}_0(\mathbf{x})$ as a statistically homogeneous and isotropic random field. Thus, the probability distribution function of $\mathbf{B}_0$ is Gaussian; Such a magnetic field will influence the fluid motion and perturbations of the metric to second order in $\mathbf{B}$. In particular, $\mathbf{B}$ may induce not only scalar, but also vector and tensor perturbations of the same order. We shall investigate the tensor perturbations[13].

The anisotropic stresses induced by a magnetic field can be given by[13]:

$$
\begin{aligned}
|\Pi_B|^2 &= f(k)^2 a^{-4} \quad \text{where} &(6)\\
f(k)^2 &= \int d^3q B^2(q) B^2(|\mathbf{k}-\mathbf{q}|)(1 + 2\gamma^2 + \gamma^2\beta^2) \ , &(7)
\end{aligned}
$$

with $\gamma = \hat{\mathbf{k}} \cdot \hat{\mathbf{q}}$ and $\beta = \hat{\mathbf{k}} \cdot (\widehat{\mathbf{k}-\mathbf{q}})$. We parameterize $B(k)$ in terms of an amplitude and a scale dependence through

$$
B^2(k) = \begin{cases} \frac{(n+3)(2\pi)^5}{4} \frac{B_0^2}{k_c^3} \left(\frac{k}{k_c}\right)^n & \text{for } k < k_c \\ 0 & \text{otherwise} \end{cases} \quad (8)
$$

The normalization is such that $\langle B_0^i(\mathbf{x}) B_0^i(\mathbf{x}) \rangle = B_0^2$. We have included a short wavelength cutoff;

From Eqs. (8) and (7) we can calculate $f(k)$. The integral cannot be computed analytically, but the following result is a good approximation for all wave numbers k:

$$f(k) \simeq \frac{(2\pi)^{11}}{4}\frac{B_0^4}{k_c^3}\begin{cases} \frac{(3+n)^2}{(3+2n)} & \text{for } -3/2 < n \\ \frac{n(n+3)}{(3+2n)}(k/k_c)^{2n+3} & \text{for } -3 < n < -3/2 \end{cases} \tag{9}$$

The magnetic field sources the gravitational waves via its anisotropic stresses (3) and hence anisotropies in the CMB.

Equation (3) with source (7) can be solved using the Wronskian method. For the solution at late time, $\eta > \eta_{eq}$ one finds[13]:

$$\dot{H}(k,t) \simeq 2\pi G\eta_0^2 z_{eq}\ln(z_{in}/z_{eq})kf(k)\frac{j_2(k\eta)}{k\eta}\ . \tag{10}$$

Using an analytical approximation for I [14], we obtain [a]

$$\ell^2 C_\ell \simeq 1.3\times 10^{-3}\left(\frac{B_0}{10^{-9}\text{ Gauss}}\right)^4 \ln^2(z_{in}/z_{eq}) \times$$

$$\begin{cases} \frac{2}{3\pi}\frac{(n+3)^2}{2n+3}(k_c\eta_0)^{-3}\ell^3 & \text{for } -3/2 < n \\ \frac{n(n+3)\Gamma(1-2n)}{(2n+3)\Gamma^2(1-n)2^{(1-2n)}}(k_c\eta_0)^{-(2n+6)}\ell^{6+2n} & \text{for } -3 < n < -3/2\ , \end{cases} \tag{11}$$

(The detailed calculations is presented in [13]).

The cutoff k_c in the power spectrum for **B** actually represents the coherence scale of the magnetic field. A reasonable value for $\lambda_c = 2\pi/k_c$ may be the size of a galaxy, $\lambda_c \sim 0.1h^{-1}$Mpc. The dependence on z_{in} is only logarithmic, we took: $z_{in} = 10^8$.

For a scale invariant spectrum $n \to -3$, (in the exactly scale invariant case, $n \equiv -3$, the integral for B_0 diverges, we describe a spectrum with spectral index slightly different from -3) the dependence on k_c and on ℓ disappear. To obtain a constraint for the magnetic field amplitude, B_0, we compare the predicted value for $\ell^2 C_\ell$ with the one observed by COBE, $C_\ell \ell^2 \simeq 10^{-10}$. As a result we find *e.g.* (for more details see [13]):

$$n = -2.999.\ B_0 \simeq 10^{-11}\text{Gauss} \tag{12}$$

This constraint is obtained if we assume that gravitational waves are the main source of CMB anisotropies. In reality, scalar and vector modes will contribute similarly and we obtain an even stronger limit.

4 Conclusion

The tensor contributions to the C_ℓ power spectrum depends only weakly on cosmological parameters such as the hot dark matter contribution and species of number of collisionless (massive or massless) particles. It will never induce a difference in the C_ℓ spectrum which is larger than cosmic variance. Only an extremely large cosmological constant or a difference in the Hubble parameter can induce changes in the gravitational wave spectrum which are in principle observable but nevertheless small. Thus, the gravitational wave contribution does not contain detailed information about the main cosmological parameters and can not be used to measure them with high accuracy. On the other hand, since this contribution is so model independent, it conserves its information about the initial condition and thus about the amplitude and spectral index which it inherited during, e.g., an inflationary epoch.

[a] To get a numerical estimate for the C_ℓs we use $G^2\eta_0^4 B_0^4 = 144\Omega_B^2$, and $\Omega_B \simeq 2.4\times 10^{-12}h^{-2}\left(\frac{B_0}{10^{-9}\text{Gauss}}\right)^4$

If we will assume the presence of a stochastic primordial magnetic field in the Universe, we such field can source gravitational waves, which will contribute to the CMB anisotropies. From observational data we may constraint the value of B_0, which mainly depends on the spectral index n and the cutoff scale k_c. Assuming the scale invariant spectrum and $\lambda_c = 2\pi/k_c$ equal to the magnetic field coherence scale in the galaxies, the obtained limit of B_0 is 10^{-11} Gauss, which is stronger than all the limits previously obtained for primordial magnetic fields.

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