# Second-Order Families of Minimal Lagrangians in $\mathbb{CP}^3$

by

Michael Bell

Department of Mathematics
Duke University

Date: ________________________
Approved:

Robert Bryant, Supervisor

Lenhard Ng

Hubert Bray

Leslie Saper

Dissertation submitted in partial fulfillment of the requirements for the degree of
Doctor of Philosophy in the Department of Mathematics
in the Graduate School of Duke University
2019

# Abstract

Second-Order Families of Minimal Lagrangians in $\mathbb{CP}^3$

by

Michael Bell

Department of Mathematics
Duke University

Date: ______________________
Approved:

______________________
Robert Bryant, Supervisor

______________________
Lenhard Ng

______________________
Hubert Bray

______________________
Leslie Saper

An abstract of a dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Mathematics in the Graduate School of Duke University
2019

# Abstract

In this thesis we analyze families of minimal Lagrangian submanifolds of complex projective 3-space $\mathbb{CP}^3$ whose fundamental cubic forms are invariant under a non-trivial subgroup of SO(3) in its natural action on the the second fundamental form, regarded as a cubic. There is a classification of stabilizer types of such fundamental cubics, which shows there are precisely five families of such cubic forms: Those with stabilizers containing $\mathrm{SO}(2), A_4, S_3, \mathbb{Z}_2$, and $\mathbb{Z}_3$. We use the method of moving frames, along with exterior differential systems techniques to prove existence of minimal Lagrangian submanifolds with each stabilizer type. We also attempt to integrate the resulting structure equations to give explicit examples of each.

# Contents


**Abstract** **iv**

**Acknowledgements** **vii**

**1 Introduction** **1**

**2 Preliminaries** **4**

2.1 Lie Groups and the Maurer-Cartan Form . . . . . . . . . . . . . . . . . 4

2.1.1 Maps into Lie Groups . . . . . . . . . . . . . . . . . . . . . . . 6

2.2 Homogeneous Spaces and Moving Frames . . . . . . . . . . . . . . . . . 7

2.2.1 Homogeneous Spaces . . . . . . . . . . . . . . . . . . . . . . . 8

2.2.2 Moving Frames . . . . . . . . . . . . . . . . . . . . . . . . . . 9

2.3 Exterior Differential Systems . . . . . . . . . . . . . . . . . . . . . . . 11

**3 Structure Equations and Problem Setup** **17**

3.1 The Structure Equations . . . . . . . . . . . . . . . . . . . . . . . . . 17

3.1.1 $\mathbb{CP}^3$ as a Homogeneous Space . . . . . . . . . . . . . . . . . . 17

3.1.2 Moving Frames for $\mathbb{CP}^3$ . . . . . . . . . . . . . . . . . . . . . . 19

3.2 Frame Adaptations and the Second Fundamental Form . . . . . . . . . 22

3.2.1 Summary . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 25

3.2.2 A Bonnet-type Result . . . . . . . . . . . . . . . . . . . . . . . 26

**4 Second-Order Families** **28**

4.1 Classification of SO(3)-Stabilizer Types . . . . . . . . . . . . . . . . . 28

4.2 SO(2)-stabilizer . . . 29

4.2.1 Integrating the Structure Equations . . . 32

4.3 $A_4$-stabilizer . . . 34

4.3.1 Integrating the Structure Equations . . . 35

4.4 $S_3$-Stabilizer . . . 37

4.4.1 Integrating the Structure Equations . . . 40

4.5 $\mathbb{Z}_2$-stabilizer . . . 49

4.6 $\mathbb{Z}_3$-stabilizer . . . 56

**5 Conclusion** **59**

**Bibliography** **60**

**Biography** **62**

# Acknowledgements

I'd like to thank my family for always supporting me, and my advisor, Robert Bryant, for his patience and guidance. I'd also like to thank my friends and peers and Duke: Gavin Ball, Jesse Madnick, Ryan Gunderson, Chester Lian, Greg Herschlag, Robert Ravier, Matthew Beckett, and William Duncan.

# 1

# Introduction

In this thesis, we study minimal Lagrangian submanifolds of complex projective 3-space whose second fundamental form satisfies a set of pointwise geometric conditions. Let $(M, \omega)$ be a symplectic manifold - a $2n$-dimensional smooth manifold equipped with a symplectic form $\omega$. A $n$-dimensional submanifold $L^n \subset M$ is called *Lagrangian* if $\omega$ vanishes identically when pulled back to $L$. The study of Lagrangian submanifolds has a rich history in geometry and topology, and they arise naturally in the context of classical mechanics and mathematical physics. If $M$ happens to be a Kähler manifold—a symplectic manifold together with a compatible complex structure $J$ and Riemannian metric $g$—then the Lagrangian condition is equivalent to $J(T_xL) = T_x^\perp L$, for each $x \in L$, where $T_x^\perp L \subset T_xM$ is the normal space to $L$ at $x$. If $L$ is both Lagrangian with respect to $\omega$ and minimal with respect to $g$ then $L$ is called *minimal Lagrangian*, and such submanifolds are the main objects of interest in this work.

Minimal Lagrangians have been studied extensively, and a variety of explicit examples are known. Bryant [Bry87] studied minimal Lagrangians in a Kähler-Einstein manifold $(M^{2n}, g)$ and showed that in the analytic category every sub-Lagrangian

$(n-1)$-dimensional manifold $N \subset M$ lies in an unique $S^1$-family of minimal Lagrangian submanifolds and that every minimal Lagrangian that contains $N$ is a member of this family in some neighborhood of $N$. A wealth of new examples were found after Harvey and Lawson's seminal paper [HL82] in which they introduced the notion of *calibrated submanifolds*—certain submanifolds of a Riemannian manifold determined by a closed differential form called a calibration. The calibrated condition is only first-order, yet all calibrated submanifolds are automatically minimal, which is a second-order condition.

One class of examples of calibrated geometries are the *special Lagrangian* submanifolds, which are distinguished minimal Lagrangian submanifolds of Calabi-Yau manifolds. In their original paper, Harvey and Lawson give examples of special Lagrangian submanifolds in $\mathbb{C}^n$ that are invariant under various symmetry groups, as well as those arising as normal bundles of other submanifolds in $\mathbb{R}^n$. In [Law89] Lawlor found examples foliated by quadric surfaces (ellipsoids), and these examples were later extended by Harvey [Har90] and Joyce. Joyce's work was part of a series of papers describing new constructions using symmetry methods [Joy02], evolution equations [Joy01a], and integrable systems techniques [Joy01b].

Of particular interest to us is the work by Robert Bryant [Bry00] in which he classifies the special Lagrangian 3-folds in $\mathbb{C}^3$ whose second fundamental form has nontrivial SO(3)-stabilizer. More explicitly, he shows that for a special Lagrangian in $\mathbb{C}^3$, one can interpret the second fundamental form as a traceless, symmetric cubic form called the *fundamental cubic.* He determines which subgroups of SO(3) can appear as the pointwise stabilizer of such a cubic form and then uses Cartan-Kähler analysis to classify the special Lagrangian submanifolds whose fundamental cubic is stabilized by each such subgroup. In her thesis, Ionel [Ion03] extends this idea to special Lagrangians in $\mathbb{C}^4$ whose fundamental cubic has nontrivial SO(4)-stabilizer.

Since $\mathbb{CP}^3$ is not Ricci-flat, it is not a Calabi-Yau manifold, and the notion of special Lagrangian submanifolds no longer makes sense. However, it is a Kähler-Einstein manifold, and many interesting examples of minimal Lagrangian submanifolds of such are already known. Furthermore, it is easy to show that when $L \subset \mathbb{CP}^3$ is minimal Lagrangian, it gives rise to a symmetric, traceless cubic form just as in the special Lagrangian case studied by Bryant. Thus, it is natural to perform the analogous analysis and investigate those minimal Lagrangians in $\mathbb{CP}^3$ that have nontrivial SO(3)-stabilizer.

This dissertation is organized in the following manner: In Chapter 2, we give a brief overview of the necessary techniques and concepts used in our analysis, including Lie groups, homogeneous spaces, the method of the moving frame, and Cartan's theorem for augmented coframes. In Chapter 3, we discuss moving frames for minimal Lagrangian submanifolds of $\mathbb{CP}^3$ and derive the structure equations for such frames. Finally, we review the classification of the possible SO(3)-stabilizers of symmetric, traceless cubic forms and give normal forms for each. In Chapter 4, we carry out the analysis for each of these stabilizer types. Finally, we conclude in Chapter 5 with a discussion of unresolved problems and possible future work.

# 2

# Preliminaries

In this chapter, we set notation and give an overview of the necessary background required for the rest of the thesis. We begin with a brief review of Lie groups and Maurer-Cartan forms, followed by discussions on homogeneous spaces, moving frames, and theorems on augmented coframings. We conclude with an overview of Lagrangian geometry.

## 2.1 Lie Groups and the Maurer-Cartan Form

Let $G$ be a Lie group and $\mathfrak{g} \cong T_eG$ its Lie algebra, where $e$ is the identity element of $G$. Each $g \in G$, defines a diffeomorphism, $L_g : G \to G$ by

$$L_g(h) = gh$$

called the *left multiplication map*. The following construction is of central importance in the theory of moving frames.

**Definition 2.1.** Let $G$ be a Lie Group with Lie algebra $\mathfrak{g}$. The *Maurer-Cartan form* of $G$ is the $\mathfrak{g}$-valued 1-form $\omega : TG \to \mathfrak{g}$ defined at a point $g \in G$ by

$$\omega(X_g) = (L_{g^{-1}})_*(X_g) \tag{2.1}$$

where $X_g \in T_gG$.

The Maurer-Cartan form is *left-invariant*, meaning $L_g^*(\omega) = \omega$ for all $g \in G$. To see this, let $g, h \in G$, and $X_g \in T_gG$. Then, $L_{h*}(X_g) \in T_{hg}G$ so that

$$\begin{aligned} L_h^*(\omega)(X_g) &= \omega(L_{h*}(X_g))) \\ &= L_{(hg)^{-1}*} \circ L_{h*}(X_g) \\ &= L_{g^{-1}*}(X_g) \\ &= \omega(X_g) \end{aligned}$$

proving left-invariance. In fact, the Maurer-Cartan form can be equivalently defined as the unique left-invariant $\mathfrak{g}$-valued 1-form on $G$ that is the identity on $T_eG$.

The Maurer-Cartan form satisfies the well-known *Maurer-Cartan equation* [Gri74]

$$d\omega = -\tfrac{1}{2}[\omega, \omega]. \tag{2.2}$$

Both the Maurer-Cartan form and the Maurer-Cartan equation take simpler forms when the Lie group $G$ is actually a matrix group. Let $\mathbf{g} : G \hookrightarrow M_n(\mathbb{R})$ be the inclusion map that realizes the abstract Lie group $G$ as a subgroup of the $n \times n$ matrices. We can think of $\mathbf{g}$ as essentially the identity map, and $\mathbf{g}(h) = [\mathbf{g}(h)^i_j]$ as giving the matrix coordinates of the abstract group element $h \in G$. In this case, the Maurer-Cartan form is matrix-valued and can be expressed as

$$\omega = \mathbf{g}^{-1} d\mathbf{g}. \tag{2.3}$$

This notation is slightly confusing at first glance, but at a point $h \in G$, we have

$$\omega_h = \mathbf{g}(h)^{-1} d(\mathbf{g}(h))$$

since $\mathbf{g}$ merely identifies abstract group elements with their matrix representations, $d(\mathbf{g}(h))$ acts also as the identity, sending abstract tangent vectors $X \in T_hG$ with their representations as elements of $d(\mathbf{g}(h))(X) \in T_{\mathbf{g}(h)}M_n(\mathbb{R})$. So the product in (2.3) is a matrix multiplication, and left multiplication by $\mathbf{g}(h)^{-1}$ is precisely $L_{\mathbf{g}(h)^{-1}*}$ in the case of matrix groups, so the definitions (2.1) and (2.3) coincide.

Since $d(A^{-1}) = -A^{-1}dAA^{-1}$ when $A$ is matrix-valued, we see that

$$\begin{aligned} d\omega &= d(\mathbf{g}^{-1}d\mathbf{g}) \\ &= d(\mathbf{g}^{-1}) \wedge d\mathbf{g} \\ &= (-\mathbf{g}^{-1}d\mathbf{g}\mathbf{g}^{-1}) \wedge d\mathbf{g} \\ &= -(\mathbf{g}^{-1}d\mathbf{g}) \wedge \mathbf{g}^{-1}d\mathbf{g}, \end{aligned}$$

so that

$$d\omega = -\omega \wedge \omega. \tag{2.4}$$

### *2.1.1 Maps into Lie Groups*

Many problems in differential geometry, particularly in the method of moving frames, reduce to the problem of finding maps from a smooth manifold $M$ to a Lie group $G$. The following theorems of Cartan show how the Maurer-Cartan form classifies such maps.

The first concerns the problem of determining when two such maps are *congruent*:

**Theorem 2.2** (Cartan)**.** Let $f_1, f_2 : M \to G$ be two smooth maps of a connected manifold $M$ to a Lie group $G$, with Maurer-Cartan form $\omega$. Then $f_1$ and $f_2$ are *congruent* in the sense that there exists a fixed $a \in G$ so that

$$f_1 = L_a \circ f_2$$

if and only if

$$f_1^*\omega = f_2^*\omega.$$

The second theorem is more general and provides sufficient conditions for the existence of such maps, unique up to left translation by a fixed element of $G$:

**Theorem 2.3** (Cartan)**.** Let $M$ be a smooth connected and simply-connected manifold, $G$ a matrix Lie group with Maurer-Cartan form $\omega$. Suppose $\varphi$ is a $\mathfrak{g}$-valued 1-form on $M$ satisfying the Maurer-Cartan equation $d\varphi = -\varphi \wedge \varphi$. Then there exists a map $f : M \to G$ such that $f^*\omega = \varphi$. Moreover, this map is unique up to left translation in $G$ by a fixed element $a \in G$ - any two such maps $f_1, f_2 : M \to G$ satisfy $f_1 = L_a \circ f_2$ for some fixed $a \in G$.

Note that if $M$ is not connected and simply-connected we can still locally apply Theorem 2.3 to connected and simply-connected neighborhoods in $M$ or to $\varphi$ pulled back to the simply-connected cover of $M$. We shall see that a moving frame adapted to the geometry of a given submanifold is a map from the submanifold to a Lie group, so these theorems are essential to the theory. The proof of 2.3 is an elementary application of the Frobenius Theorem [Gri74], and will be omitted here.

## 2.2 Homogeneous Spaces and Moving Frames

The main goal of this thesis is to study special submanifolds of $\mathbb{CP}^3$, a homogeneous space diffeomorphic to the quotient $\mathrm{SU}(4)/\mathrm{U}(3)$. Cartan's method of *moving frames* [Car35] gives a way to analyze and systematically determine differential invariants to submanifolds of homogeneous spaces. Once a Riemannian homogeneous space, say $N$, has been expressed as a coset space $G/H$, the group $G$ of isometries of $N$ may be thought of as a bundle of linear frames of $G/H$, and the Maurer-Cartan equations (2.2-2.4) for $G$ give the structure equations such frames must satisfy. A moving frame along a submanifold $f : M \to G/H$ is a lift of $f$ to a map $F : M \to G$, and the canonical construction of such a map involves the process of *frame adaptation*.

### 2.2.1 Homogeneous Spaces

**Definition 2.4.** A *homogeneous $G$-space* is a smooth manifold $M$ together with a transitive smooth (left) action by a Lie group $G$. We will sometimes write the action of $g \in G$ on $x \in M$ by $g \cdot x$, or simply $gx$.

**Example 2.5** (Coset Spaces)**.** Let $G$ be a Lie group and $H$ a Lie subgroup of $G$. A *left coset* of $H$ with respect to an element $g \in G$ is the set

$$gH := \{gh \mid h \in H\}.$$

We denote the space of all left cosets of $H$ by $G/H$ and give it the quotient topology induced by the natural projection $\pi : G \to G/H$ that sends $g \in G$ to $gH$. It is a theorem of Cartan that if $H$ happens to be a *closed* subgroup of $G$, then it is a Lie subgroup of $G$. In this case, the quotient manifold theorem [War13] tells us that the coset space $G/H$ is a smooth manifold and has a unique smooth structure for which $\pi : G \to G/H$ is a smooth submersion. In fact, this projection describes $G$ as a principal (right) $H$-bundle over $G/H$:

$$\begin{tikzcd} H \arrow[r] & G \arrow[d, "\pi"] \\ & G/H \end{tikzcd}$$

There is a natural smooth action of $G$ on $G/H$, given by $g_1 \cdot (g_2H) = (g_1g_2)H$, and it is easy to see that this action is transitive, so $G/H$ is indeed a homogeneous space.

Example 2.5 is important because it turns out that every homogeneous space can be realized as a coset space: Let $N$ be a homogeneous $G$-space. For a choice of 'origin', some $x_0 \in N$, the *isotropy subgroup* of $x_0$ is defined to be the set

$$H_{x_0} := \{g \in G \mid g \cdot x_0 = x_0\}.$$

The isotropy subgroup $H_{x_0}$ is a closed Lie subgroup of $G$, and $G/H_{x_0}$ is diffeomorphic to $N$. If a different origin is chosen, say $x_1 \in G$, then their isotropy subgroups are related by conjugation:

$$H_{x_1} = g^{-1} H_{x_0} g$$

where $g \in G$ is any element for which $gx_1 = x_0$. For additional details, see [JMN16].

### *2.2.2 Moving Frames*

As noted above, a $G$-homogeneous space $N$ comes equipped with a natural projection $\pi : G \to N$ and unique smooth structure that makes $\pi$ a surjective submersion. Let $f : M \to N$ be an immersion of an $m$-dimensional manifold to $N$.

**Definition 2.6.** A *moving frame* along $f$ is a lift of the map $f$ to a map $F : M \to G$ such that the following diagram commutes

$$\begin{tikzcd} & G \arrow[d, "\pi"] \\ M \arrow[ur, "F"] \arrow[r, "f"] & N \end{tikzcd}$$

i.e. $\pi \circ F = f$. In other words, a moving frame is a section of the pullback bundle $f^{-1}(G)$ over $M$.

We shall now explain why such a lifting is called a 'moving frame'. It turns out that there is a natural identification of the symmetry group $G$ with the bundle of linear frames of $N$. Let $N$ be an $n$-dimensional homogeneous $G$-space, $\mathbf{o} \in N$ a choice of origin and $H \subset G$ the isotropy subgroup of $\mathbf{o}$. Let $\mathfrak{h} \subset \mathfrak{g}$ is the Lie algebra of $H$, and fix a complementary subspace $\mathfrak{n}$, so that

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}$$

as vector spaces. Since $\pi$ is a submersion, and $\mathfrak{h} = \ker d\pi_e : T_e G \to T_{\mathbf{o}} N$, the map $d\pi_e\big|_{\mathfrak{n}} : \mathfrak{n} \to T_{\mathbf{o}} N$ is an isomorphism.

Choose a reference basis $(E_1, \ldots, E_n)$ of $\mathfrak{n}$, which we think of as left-invariant vector fields on $G$, and so induce vector fields via pushforward,

$$\hat{E}_i(g) := d\pi_g(E_i(g)) \in T_{\pi(g)}N$$

on $N$. At each $g \in G$, the vectors $(\hat{E}_1(g), \ldots, \hat{E}_n(g))$ give a frame of $T_{\pi(g)}N$. For $h_1, h_2 \in H$ we have $h_i \cdot \mathbf{o} = \mathbf{o}$, so that $dh_i\big|_{\mathbf{o}} \in \mathrm{GL}(T_{\mathbf{o}}N)$. Moreover, $d(h_1h_2)\big|_{\mathbf{o}} = dh_1\big|_{\mathbf{o}} \circ dh_2\big|_{\mathbf{o}}$, so we have a representation $H \to \mathrm{GL}(T_{\mathbf{o}}N)$ called the *isotropy representation* of $H$. With a choice of reference basis of $\mathfrak{n}$, we can write $dh\big|_{\mathbf{o}}$ as a matrix, say $A(h)$, with respect to the induced basis $E_i^{\pi}(e)$, and so get the linear isotropy representation $A : H \to \mathrm{GL}(n, \mathbb{R})$. For the constructions below we need to assume that this representation is *almost faithful*, i.e., has a 0-dimensional kernel, so we assume this from now on. If $N$ is actually a Riemannian homogeneous $G$-space, i.e., $G$ is a subgroup of the isometry group of $N$, and it turns out that the linear isotropy representation for Riemannian homogeneous spaces is always faithful. In our case, the linear isotropy representation for $\mathbb{CP}^3 \cong \mathrm{SU}(4)/\,\mathrm{U}(3)$ is only almost faithful: the group of real isometries of $\mathbb{CP}^3$ is actually $\mathrm{SU}(4)/Z$ where $Z$ is a discrete group that is isomorphic to $\mathbb{Z}_4$ and is generated by the matrix $\mathrm{i}I_4 \in \mathrm{SU}(4)$. The method of the moving frame is still valid in this case so we will not need to discuss this detail any further.

Let $\pi : \mathcal{F}_N \to N$ be the *frame bundle* of $N$ - it is the principal right $\mathrm{GL}(n, \mathbb{R})$-bundle whose fiber over a point $x \in N$ consists of all linear frames of $T_xN$. Consider the map $\rho : G \to \mathcal{F}_N$ defined by

$$\rho(g) = (\hat{E}_1(g), \ldots, \hat{E}_n(g)).$$

It is not hard to check that this is a principal bundle map, where the group homomorphism $\phi : H \to GL(n, \mathbb{R})$ is given by $\phi(h) = A(h)$. That is, for $g \in G, h \in H$, we have

$$i(gh) = i(g) \cdot A(h).$$

In summary, choosing a reference basis to the complementary subspace $\mathfrak{n}$ to the Lie algebra $\mathfrak{h}$ of the isotropy group of a point gives the identification of $G$ with $\mathcal{F}_N$. Furthermore, the splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}$ gives a decomposition of the $\mathfrak{g}$-valued Maurer-Cartan form

$$\omega = \omega_{\mathfrak{h}} + \omega_{\mathfrak{n}}$$

where $\omega_{\mathfrak{h}}$ and $\omega_{\mathfrak{n}}$ take values in $\mathfrak{h}$ and $\mathfrak{n}$, respectively. In terms of a reference basis $(E_1, \ldots, E_n)$ of $\mathfrak{n}$, we can further write

$$\omega_{\mathfrak{n}} = E_i \omega^i \tag{2.5}$$

where $\omega^i$ are real-valued, left-invariant 1-forms on $G$. Since $\omega_{\mathfrak{n}}$ annihilates $\mathfrak{h} = \ker d\pi$ the 1-forms $\omega^i$ are all semi-basic for the projection $\pi : G \to N$. In fact, if $F : M \to G$ is any moving frame along a submanifold $f : M \to N = G/H$, then the pullbacks $F^*(\omega^i)$ at any point $x \in M$ span the cotangent space $T^*_x(M)$.

At this stage, there are many possible choices of moving frames which can either be thought of as a lift to $F : M \to G$, or equivalently, a section of the pullback to $M$ of the $H$-bundle $\pi : G \to N = G/H$. A key step in the method of the moving frame is to eliminate this ambiguity through the *frame adaptation* procedure in which one chooses frames that are specially adapted to the geometry of the submanifold in question.

## 2.3 Exterior Differential Systems

During the frame adaptation process, we will be interested the existence of coframings on subbundles of the pullback to $M$ of $\pi : G \to N = G/H$ whose structure equations take a prescribed form. The methods of exterior differential systems (EDS) developed by E. Cartan, specifically a theorem on *augmented coframings*, gives necessary and sufficient conditions for the existence of such coframings. Additionally, it provides information of 'how many' such augmented coframings exist. Exterior differential systems techniques will also be useful in the problem of integrating the structure equations to give explicit descriptions of the minimal Lagrangians in question.

**Definition 2.7.** An *exterior differential system* (EDS) is a pair $(M, \mathcal{I})$ were $M$ is a smooth manifold and $\mathcal{I} \subseteq \Omega^*(M)$ is a differentially closed, graded ideal in the ring of differential forms on $M$. Being differentially closed means that $d\phi \in \mathcal{I}$ whenever $\phi \in \mathcal{I}$. An *integral manifold* of an EDS $(M, \mathcal{I})$ is a submanifold $f : N \to M$ such that $f^*\phi = 0$ for all $\phi \in \mathcal{I}$.

The simplest type of EDS is one for which the differential ideal $\mathcal{I}$ is generated *algebraically* by a set of 1-forms.

**Definition 2.8.** An exterior differential system $(M^{n+r}, \mathcal{I})$ is called *Frobenius of rank* $r$ if at each point in $M$, $\mathcal{I}$ is algebraically generated by exactly $r$ linearly independent 1-forms $\theta_1, \ldots \theta_r \in \Omega^1(M)$. This is equivalent to

$$d\theta_a \equiv 0 \mod \theta_1, \ldots \theta_r$$

for each $1 \leqslant a \leqslant r$. A Frobenius ideal is sometimes referred to as *integrable.*

If $\mathcal{I}$ is a Frobenius ideal, the structure of its integral manifolds is well understood and is given by the well-known

**Theorem 2.9** (Frobenius Theorem)**.** Let $(M^{n+r}, \mathcal{I})$ be a Frobenius EDS of rank $r$. Then at each point $p \in M$, there exists a coordinate system $x^1, \ldots, x^n, y^1, \ldots, y^r$ on a neighborhood $U \subset M$ of $p$ so that $\mathcal{I}\big|_U$ is generated by $\{dy^1, \ldots, dy^r\}$. Moreover, the maximal integral manifolds of $\mathcal{I}$ are $n$-dimensional and are given locally as slices of the form

$$y^1 = c^1, y^2 = c^2, \ldots, y^r = c^r,$$

where the $c^a$ are constants. We say that the integral manifolds of $(M, \mathcal{I})$ depends on $r$ constants.

The maximal integral manifolds of a Frobenius ideal foliate the ambient manifold $M$—every point $p \in M$ lies in a unique maximal connected integral manifold of $\mathcal{I}$, and the space of maximal integral manifolds is called the *leaf space* of $\mathcal{I}$.

Next, we discuss the augmented coframing problem. Cartan's existence theorem relies on the Cartan-Kähler theorem and so is only guaranteed to hold in the real-analytic case, which we assume throughout this work. A surprising variety of problems in differential geometry can be reduced to finding an augmented coframing satisfying a prescribed set of structure equations. For more, see Bryant's notes on exterior differential systems [Bry14].

**Index Convention:** In this section only, we shall use the following index convention: $1 \leqslant i, j, k \leqslant n$, $1 \leqslant \alpha \leqslant s$, and $1 \leqslant \rho, \sigma \leqslant r$.

**Definition 2.10.** An *augmented coframing* on an $n$-dimensional manifold $M$ is a triple $(\omega, a, b)$ consisting of $n$ linearly independent 1-forms $\omega = (\omega^1, \ldots, \omega^n)$ and functions $a = (a^\alpha) : M \to \mathbb{R}^s$ and $b = (b^\rho) : M \to \mathbb{R}^r$. The functions $a = (a^\alpha)$ are known as *primary invariants* of the augmented coframing, and $b = (b^\rho)$ as *derived invariants.*

The augmented coframing problem concerns the following question: Does there exist an augmented coframing $(\omega, a, b)$ on $M^n$ that satisfies the structure equations

$$d\omega^i = -\frac{1}{2} C^i_{jk}(a) \omega^j \wedge \omega^k \tag{2.6}$$

$$da^\alpha = F^\alpha_i(a, b) \omega^i \tag{2.7}$$

where $C^i_{jk} = -C^i_{kj} : \mathbb{R}^s \to \mathbb{R}$ and $F^\alpha_i : \mathbb{R}^{s+r} \to \mathbb{R}$ are given functions. In other words, the form of the structure equations of the 1-forms $(\omega^i)$ and primary invariants $(a^\alpha)$ are prescribed. Since the $b^\rho$ appear only in the derivatives of the primary invariants

$a^\alpha$, and their derivatives are *un*constrained, they are sometimes known as the *free derivatives* of the augmented coframing.

Necessary conditions for the existence of such augmented coframings are given by requiring that $d^2 = 0$ be an identity. Using (2.6), along with the independence of the $\omega^i$, the equations $d^2(\omega^i) = d(C^i_{jk}(a)\omega^j \wedge \omega^k) = 0$ imply

$$F^\alpha_j \frac{\partial C^i_{jk}}{\partial u^\alpha} + F^\alpha_k \frac{\partial C^i_{lj}}{\partial u^\alpha} + F^\alpha_l \frac{\partial C^i_{jk}}{\partial u^\alpha} = (C^i_{mj}C^m_{kl} + C^i_{mk}C^m_{lj} + C^i_{ml}C^m_{jk}) \tag{2.8}$$

while $d^2(a^\alpha) = d(F^\alpha_i(a,b)\omega^i) = 0$ gives

$$0 = \frac{\partial F^\alpha_i}{\partial v^\rho} db^\rho \wedge \omega^i + \frac{1}{2}\left( F^\beta_i \frac{\partial F^\alpha_j}{\partial u^\beta} - F^\beta_j \frac{\partial F^\alpha_i}{\partial u^\beta} - C^l_{ij}F^\alpha_l \right) \omega^i \wedge \omega^j. \tag{2.9}$$

Now, while we do not have expressions for the $db^\rho$, we know we can expand them in terms of the coframe as $\omega^i$ as $db^\rho = G^\rho_i \omega^i$, for some unspecified functions $G^\rho_i(a,b)$ on $\mathbb{R}^{s+r}$. Substituting into (2.9) shows that the $G^\rho_i$ cannot be arbitrary and must satisfy

$$F^\beta_i \frac{\partial F^\alpha_j}{\partial u^\beta} - F^\beta_j \frac{\partial F^\alpha_i}{\partial u^\beta} - C^l_{ij}F^\alpha_l = \frac{\partial F^\alpha_i}{\partial v^\rho} G^\rho_j - \frac{\partial F^\alpha_j}{\partial v^\rho} G^\rho_i. \tag{2.10}$$

If such functions $G^\rho_i$ exist, then setting $d^2(a^\alpha) = 0$ is an identity.

While the conditions (2.8) and (2.10) are necessary, they are not quite sufficient as it only ensures $d^2 = 0$ holds for the coframe $\omega^i$ and primary invariants $a^\rho$, and new incompatibilities could arise at higher order. To derive conditions that guarantee this is not the case, we need a bit of new terminology.

**Definition 2.11.** Let $(\omega, a, b)$ be an augmented coframing with prescribed structure equations (2.6). Let $u_1, \ldots, u_s$ be a basis of $\mathbb{R}^s$ and $v^1, \ldots, v^n$ be a basis of $(\mathbb{R}^n)^*$. The *tableau of free derivatives* of $(\omega, a, b)$ at a point $(u, v) \in \mathbb{R}^s \times \mathbb{R}^r$ is the linear subspace $A(u,v) \subset \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^s)$ spanned by the $r$ elements

$$\frac{\partial F^\alpha_i}{\partial v^\rho}(u,v)\ u_\alpha \otimes v^i, \qquad 1 \leqslant \rho \leqslant r. \tag{2.11}$$

We can now finally give the theorem.

**Theorem 2.12** (Cartan)**.** Suppose that real analytic functions $C^i_{jk} = -C^i_{kj}$ on $\mathbb{R}^s$ and $F^\alpha_i$ on $\mathbb{R}^{s+r}$ are given satisfying (2.8) and that there exist real analytic functions $G^\rho_i$ on $\mathbb{R}^{s+r}$ that satisfy (2.10). Finally, suppose that the tableaux $A(u,v)$ defined by (2.11) have dimension $r$ and are involutive with Cartan characters $s_i$ $(1 \leqslant i \leqslant n)$ for all $(u,v) \in \mathbb{R}^{s+r}$. Then, for any $(u_0, v_0) \in \mathbb{R}^{s+r}$ there exists an augmented coframing $(\omega, a, b)$ on an open neighborhood $V$ of 0 in $\mathbb{R}^n$ that satisfies (2.6) and has $(a(0), b(0)) = (u_0, v_0)$. Moreover, augmented coframings satisfying (2.6) depend, modulo diffeomorphism, on $s_p$ functions of $p$ variables (in the sense of exterior differential systems), where $s_p$ is the last non-zero Cartan character of $A(u,v)$.

*Proof.* See Theorem 3 in Bryant [Bry14]. $\square$

A special case of this theorem arises in the case that there are no free derivatives appearing in the prescribed structure equations (2.6). That is, we're interested in the existence of an augmented coframing $(\omega, a)$ on an $n$-manifold satisfying the structure equations

$$d\omega^i = -\frac{1}{2} C^i_{jk}(a) \omega^j \wedge \omega^k \tag{2.12}$$

$$da^\alpha = F^\alpha_i(a) \omega^i \tag{2.13}$$

where the functions $C^i_{jk} = -C^i_{kj} : \mathbb{R}^s \to \mathbb{R}$ and $F^\alpha_i : \mathbb{R}^s \to \mathbb{R}$ are given. The necessary conditions

$$F^\alpha_j \frac{\partial C^i_{jk}}{\partial u^\alpha} + F^\alpha_k \frac{\partial C^i_{lj}}{\partial u^\alpha} + F^\alpha_l \frac{\partial C^i_{jk}}{\partial u^\alpha} = (C^i_{mj} C^m_{kl} + C^i_{mk} C^m_{lj} + C^i_{ml} C^m_{jk}) \tag{2.14}$$

and

$$F^\beta_i \frac{\partial F^\alpha_j}{\partial u^\beta} - F^\beta_j \frac{\partial F^\alpha_i}{\partial u^\beta} - C^l_{ij}(a) F^\alpha_l = 0, \tag{2.15}$$

arising from the identity $d^2 = 0$, actually turn out to be sufficient in this case.

**Corollary 2.13** (Cartan)**.** Suppose that real analytic functions $C^i_{jk} = -C^i_{kj}$ on $\mathbb{R}^s$ and $F^\alpha_i$ on $\mathbb{R}^s$ are given satisfying (2.14)-(2.15). Then, for any $u_0 \in \mathbb{R}^s$ there exists an augmented coframing $(\omega, a)$ on an open neighborhood $V$ of 0 in $\mathbb{R}^n$ that satisfies (2.12) and has $a(0) = u_0$. Moreover, any two such augmented coframings satisfying (2.12) and have $a(0) = u_0$ agree in a neighborhood of $0 \in \mathbb{R}^n$, up to a diffeomorphism of $\mathbb{R}^n$ which fixes the origin.

It turns out that the assumption of real analyticity is not required for this corollary to hold. See Theorem 2 in [Bry14] for more.

# 3

# Structure Equations and Problem Setup

In this chapter, we apply the method of the moving frame to minimal Lagrangian submanifolds in $\mathbb{CP}^3$.

## 3.1 The Structure Equations

**Index Convention:** In this section, and for the remainder of this thesis, we shall follow the index convention: $0 \leqslant a, b, c \leqslant 3$, and $1 \leqslant i, j, k \leqslant 3$, together with the Einstein summation convention for these index ranges.

### *3.1.1 $\mathbb{CP}^3$ as a Homogeneous Space*

Complex projective 3-space $\mathbb{CP}^3$ is defined to be the set of one-dimensional linear subspaces of $\mathbb{C}^4$. It can also be defined as the orbit space of the diagonal action of $\mathbb{C}^*$ (the nonzero complex numbers) on the nonzero vectors in $\mathbb{C}^4$:

$$\mathbb{CP}^3 = (\mathbb{C}^4 - \{0\})/\mathbb{C}^*.$$

Let ${}^{\mathrm{t}}(z_0, z_1, z_2, z_3)$ be a nonzero vector in $\mathbb{C}^4$. We will write $[\mathbf{z}]$ to denote the projectivization of $\mathbf{z}$ and shall sometimes write $[\mathbf{z}]$ in *homogeneous coordinates* $[\mathbf{z}] = {}^{\mathrm{t}}[z_0 : z_1 : z_2 : z_3]$.

The matrix group $\mathrm{SU}(4)$ is defined to be the set of $4 \times 4$ complex matrices $A$ satisfying $\det A = 1$ and ${}^{\mathrm{t}}\bar{A}A = A\,{}^{\mathrm{t}}\bar{A} = I$, i.e. the columns of $A$ constitute a special unitary basis of $\mathbb{C}^4$. There is an obvious left action of $\mathrm{SU}(4)$ on $\mathbb{CP}^3$ given by

$$A[\mathbf{x}] = [A\mathbf{x}]$$

for $A \in \mathrm{SU}(4), [\mathbf{x}] \in \mathbb{CP}^3$. It is easily seen that this action is transitive as follows: Let $[\mathbf{x_0}] = {}^{\mathrm{t}}[1 : 0 : 0 : 0]$, and let $[\mathbf{x}] \in \mathbb{CP}^3$ be arbitrary. We can assume, without loss of generality, that the representative $\mathbf{x} \in \mathbb{C}^4$ has been rescaled so that it has unit norm with respect to the standard inner product on $\mathbb{C}^4$. Let $A$ be any matrix in $\mathrm{SU}(4)$ which has $\mathbf{x}$ as its first column, which always exists since any unit vector in $\mathbb{C}^4$ can be completed to a special unitary basis. Then,

$$A[\mathbf{x_0}] = [A\mathbf{x_0}] = [\mathbf{x}]$$

which shows that every point in $\mathbb{CP}^3$ lies in the orbit of $[\mathbf{x_0}]$, so $\mathbb{CP}^3$ is an homogeneous $\mathrm{SU}(4)$-space.

We compute now the isotropy subgroup of the point $[\mathbf{x_0}] = {}^{\mathrm{t}}[1 : 0 : 0 : 0] \in \mathbb{CP}^3$. For $A = [a^i_j] \in \mathrm{SU}(4)$, we have

$$A[\mathbf{x_0}] = {}^{\mathrm{t}}[a^1_1 : a^2_1 : a^3_1 : a^4_1]$$

so $A$ stabilizes $[\mathbf{x_0}]$ if and only if $a^1_1 \neq 0$ and $a^2_1 = a^3_1 = a^4_1 = 0$. Using the fact that $\det A = 1$ and ${}^{\mathrm{t}}\bar{A}A = A\,{}^{\mathrm{t}}\bar{A} = I$, we see that $A$ must be of the form

$$A = \begin{bmatrix} (\det B)^{-1} & 0 & 0 & 0 \\ 0 & & & \\ 0 & & B & \\ 0 & & & \end{bmatrix},$$

where $B$ is a $3 \times 3$ complex matrix satisfying ${}^{\mathrm{t}}\bar{B}B = B\,{}^{\mathrm{t}}\bar{B} = I$, so that $H_{\mathbf{x_0}} \cong \mathrm{U}(3)$. We have therefore shown

$$\mathbb{CP}^3 \cong \mathrm{SU}(4)/\,\mathrm{U}(3). \tag{3.1}$$

Moreover, defining the projection map $\pi : \mathrm{SU}(4) \to \mathbb{CP}^3$ by $\pi([\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]) = [\mathbf{e}_0]$, gives $\mathrm{SU}(4)$ the structure of a right principal $\mathrm{U}(3)$-bundle over $\mathbb{CP}^3$.

### *3.1.2 Moving Frames for* $\mathbb{CP}^3$

As described in Section 2.2.2, in order to interpret $\mathrm{SU}(4)$ as the frame bundle of $\mathbb{CP}^3 \cong \mathrm{SU}(4)/\,\mathrm{U}(3)$, we must first choose a complementary subspace to $\mathfrak{u}(3) \subset \mathfrak{su}(4)$ and then specify a reference basis.

The Lie algebra $\mathfrak{su}(4)$ of $\mathrm{SU}(4)$ consists of all $4 \times 4$ traceless, skew-hermitian matrices, so the Maurer-Cartan form of $\mathrm{SU}(4)$ takes the form

$$\omega = g^{-1}dg = \begin{bmatrix} \omega^0_0 & \omega^0_1 & \omega^0_2 & \omega^0_3 \\ \omega^1_0 & \omega^1_1 & \omega^1_2 & \omega^1_3 \\ \omega^2_0 & \omega^2_1 & \omega^2_2 & \omega^2_3 \\ \omega^3_0 & \omega^3_1 & \omega^3_2 & \omega^3_3 \end{bmatrix} \tag{3.2}$$

where the $\omega^a_b$ are left-invariant complex-valued 1-forms on $\mathrm{SU}(4)$ and satisfy

$$\omega^a_b + \overline{\omega}^b_a = 0, \qquad \sum_{a=0}^{3} \omega^a_a = 0.$$

Rearranging the above equation gives $dg = g\omega$, which gives the following equations:

$$\begin{bmatrix} d\mathbf{e}_0 & d\mathbf{e}_1 & d\mathbf{e}_2 & d\mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{e}_0 & \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix} \begin{bmatrix} \omega^0_0 & \omega^0_1 & \omega^0_2 & \omega^0_3 \\ \omega^1_0 & \omega^1_1 & \omega^1_2 & \omega^1_3 \\ \omega^2_0 & \omega^2_1 & \omega^2_2 & \omega^2_3 \\ \omega^3_0 & \omega^3_1 & \omega^3_2 & \omega^3_3 \end{bmatrix}, \tag{3.3}$$

or

$$d\mathbf{e}_a = \mathbf{e}_b \omega^b_a. \tag{3.4}$$

Furthermore, the Maurer-Cartan equation for matrix groups, $d\omega = -\omega \wedge \omega$, give the additional structure equations

$$d\omega_b^a = -\omega_c^a \wedge \omega_b^c. \tag{3.5}$$

We saw above that the isotropy subgroup of the choice of origin $\mathbf{x_0} = {}^{\mathrm{t}}[1:0:0:0] \in \mathbb{CP}^3$ was the copy of U(3) in SU(4) of the form

$$H_{\mathbf{x_0}} = \left\{ \begin{bmatrix} (\det B)^{-1} & 0 & 0 & 0 \\ 0 & & & \\ 0 & & B & \\ 0 & & & \end{bmatrix} \mid B \in \mathrm{U}(3) \right\} \tag{3.6}$$

so the associated subalgebra $\mathfrak{u}(3) \subset \mathfrak{su}(3)$ is

$$\mathfrak{u}(3) \cong \left\{ \begin{bmatrix} -\operatorname{tr} a & 0 & 0 & 0 \\ 0 & & & \\ 0 & & a & \\ 0 & & & \end{bmatrix} \mid a \in \mathfrak{u}(3) \right\}. \tag{3.7}$$

Let $\mathfrak{n}$ be the complementary subspace to $\mathfrak{u}(3)$ given by

$$\mathfrak{n} \cong \left\{ \begin{bmatrix} 0 & -\bar{z}_1 & -\bar{z}_2 & -\bar{z}_3 \\ z_1 & 0 & 0 & 0 \\ z_2 & 0 & 0 & 0 \\ z_3 & 0 & 0 & 0 \end{bmatrix} \mid Z \in \mathbb{C}^3 \right\} \tag{3.8}$$

which gives the direct sum decomposition

$$\mathfrak{su}(4) = \mathfrak{u}(3) \oplus \mathfrak{n} \tag{3.9}$$

along with the identification of $\mathfrak{n}$ with $\mathbb{C}^3$ by $\begin{bmatrix} 0 & -{}^{\mathrm{t}}Z \\ Z & 0 \end{bmatrix} \leftrightarrow Z$. Taking $(E_1, E_2, E_3)$ as a reference basis (appropriately identified as above), where $E_i$ is the $i$th standard basis vector of $\mathbb{C}^3$, then at a point $g = [\mathbf{e}_a] \in \mathrm{SU}(4)$ the map $d\pi_g(E_1, E_2, E_3)$ gives a frame of $T_{[\mathbf{e}_0]}\mathbb{CP}^3$. The Maurer-Cartan form $\omega$ of $\mathfrak{su}(4)$ splits into components $\omega = \omega_{\mathfrak{u}(3)} + \omega_{\mathfrak{n}}$, and we see from (3.2) that

$$\omega_{\mathfrak{n}} = \omega_0^i \otimes E_i$$

so the $\omega_0^i$ are $\pi$-semibasic. In fact, the forms $\omega_0^1, \omega_0^2, \omega_0^3$ give a unitary basis of the pullbacks of the complex linear (1,0)-forms on $\mathbb{CP}^3$, and it is easy to check that the Hermitian form $h$ defined by

$$h = \omega_0^i \overline{\omega}_0^i \tag{3.10}$$

is invariant under the action of U(3), so it descends to a well-defined tensor on $\mathbb{CP}^3$. In fact, if $h_{FS}$ is the Fubini-Study metric on $\mathbb{CP}^3$, with associated Kähler form $\Omega$, then their pullbacks to SU(4) are

$$\pi^*(h_{FS}) = \omega_0^i \overline{\omega}_0^i \tag{3.11}$$

and

$$\pi^*(\Omega) = \tfrac{\mathrm{i}}{2}\omega_0^i \wedge \overline{\omega}_0^i, \tag{3.12}$$

respectively.

If we set $\theta_j^i = (\omega_j^i - \delta_j^i \omega_0^0)$, then the structure equations (3.5) yield

$$d\omega_0^i = -\theta_j^i \wedge \omega_0^j \tag{3.13}$$

$$d\theta_j^i = -\theta_k^i \wedge \theta_j^k + \omega_0^i \wedge \overline{\omega}_0^j + \delta_j^i \omega_0^k \wedge \overline{\omega}_0^k \tag{3.14}$$

which shows that the $\theta_j^i$ give connection forms relative to the unitary coframe $\omega_0^i$, and the associated Kähler metric (3.11) has constant holomorphic sectional curvature 4.

Decomposing the $\omega_j^i$ into real and imaginary parts and using the above relations, we can rewrite $\omega$ as follows

$$\omega = g^{-1}dg = \begin{bmatrix} -\mathrm{i}(\beta_{11}+\beta_{22}+\beta_{33}) & -\omega_1 + \mathrm{i}\eta_1 & -\omega_2 + \mathrm{i}\eta_2 & -\omega_3 + \mathrm{i}\eta_3 \\ \omega_1 + \mathrm{i}\eta_1 & \mathrm{i}\beta_{11} & \alpha_{12} + \mathrm{i}\beta_{12} & \alpha_{13} + \mathrm{i}\beta_{13} \\ \omega_2 + \mathrm{i}\eta_2 & -\alpha_{12} + \mathrm{i}\beta_{12} & \mathrm{i}\beta_{22} & \alpha_{23} + \mathrm{i}\beta_{23} \\ \omega_3 + \mathrm{i}\eta_3 & -\alpha_{13} + \mathrm{i}\beta_{13} & -\alpha_{23} + \mathrm{i}\beta_{23} & \mathrm{i}\beta_{33} \end{bmatrix} \tag{3.15}$$

where $\omega_i, \eta_i, \alpha_i, \beta_{ij}$ are all real-valued 1-forms and $\alpha_{ij} = -\alpha_{ji}, \beta_{ij} = \beta_{ji}$. In terms of these components, we have the *first structure equations*:

$$d\omega_i = -\alpha_{ij} \wedge \omega_j + (\delta_{ij} \operatorname{tr} \beta + \beta_{ij}) \wedge \eta_j \tag{3.16}$$

$$d\eta_i = -(\delta_{ij} \operatorname{tr} \beta + \beta_{ij}) \wedge \omega_j - \alpha_{ij} \wedge \eta_j \tag{3.17}$$

and the *second structure equations*

$$d\alpha_{ij} = -\alpha_{ik} \wedge \alpha_{kj} + \beta_{ik} \wedge \beta_{kj} + \omega_i \wedge \omega_j + \eta_i \wedge \eta_j \tag{3.18}$$

$$d\beta_{ij} = -\alpha_{ik} \wedge \beta_{kj} - \beta_{ik} \wedge \alpha_{kj} - \omega_i \wedge \eta_j + \eta_j \wedge \omega_i. \tag{3.19}$$

In terms of these forms, the underlying Kähler form is (omitting pullbacks)

$$\Omega = \omega_i \wedge \eta_i. \tag{3.20}$$

## 3.2 Frame Adaptations and the Second Fundamental Form

Let $(N, J, g, \Omega)$ be a Kähler manifold of real dimension $2n$, where $(J, g, \Omega)$ are the compatible complex, Riemannian, and symplectic structures. An $n$-dimensional submanifold $f : L^n \to N$ is called *Lagrangian* if $f^*(\Omega) = 0$.

Suppose $f : L^3 \to \mathbb{CP}^3$ is a Lagrangian submanifold of $\mathbb{CP}^3$. Our goal is to construct a moving frame along $f$, i.e., a lift $F : L \to \mathrm{SU}(4)$, that incorporates the geometry of the submanifold $L$ and so takes values in a proper subbundle of $\mathrm{SU}(4)$.

**Definition 3.1.** A smooth map $F = [\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] : U \subset L \to \mathrm{SU}(4)$ is called a *moving frame of order zero* along $f$ if

$$\pi \circ F = [\mathbf{e}_0] = f \in L \subset \mathrm{SU}(4).$$

Again, a moving frame along $f$ is nothing but a section of the $\mathrm{U}(3)$-principal bundle $f^{-1}\,\mathrm{SU}(4) \to L$.

Since $L$ is Lagrangian, we have $0 = f^*(\Omega) = (\pi \circ F)^*(\Omega) = F^*(\pi^*(\Omega))$, where $\Omega$ is the Kähler form on $\mathbb{CP}^3$. In terms of the real and imaginary components of the

Maurer-Cartan form (3.20), this is equivalent to

$$0 = F^*(\frac{i}{2}\omega_0^i \wedge \overline{\omega_0^i}) = F^*(\omega^i) \wedge F^*(\eta^i). \tag{3.21}$$

Our first frame adaptation comes from considering only those frames for which $\eta^i = 0$. Such a frame will be called *$L$-adapted.*

**Proposition 3.2.** Given a Lagrangian submanifold $f : L^3 \to \mathbb{CP}^3$, there exists a moving frame $F : L^3 \to \mathrm{SU}(4)$ for which $F^*(\eta^i) = 0$.

*Proof.* Consider the Maurer-Cartan form $\omega$ on SU(4). Fix some a point $g_0 \in \mathrm{SU}(4)$. We have $\omega_{\mathfrak{n}}\big|_{g_0} = (\omega_0^1, \omega_0^2, \omega_0^3) : T_{g_0}\,\mathrm{SU}(4) \to \mathbb{C}^3$. Since $L$ is Lagrangian, the image of $\omega_{\mathfrak{n}}$ is a Lagrangian 3-plane in $\mathbb{C}^3$. Moreover, the U(3)-action on SU(4) induces an action of U(3) on the image of $\omega_{\mathfrak{n}}\big|_{g_0}$. Since the action of U(3) is transitive on the space of Lagrangian 3-planes in $\mathbb{C}^3$, there is some $a \in \mathrm{U}(3)$ for which $\omega_{\mathfrak{n}}\big|_{g_0 a} = (\omega_0^1, \omega_0^2, \omega_0^3) = \mathbb{R}^3$, which is equivalent to $\eta^i = 0$. □

**Proposition 3.3.** Let $F, \tilde{F} : U \subset L \to \mathrm{SU}(4)$ be two $L$-adapted frames. Then $\tilde{F} = Fh$ where $h$ is a smooth map from $U$ to SO(3). In particular, the bundle of $L$-adapted frames is a principal SO(3)-bundle.

*Proof.* Any two frames $F, \tilde{F}$ over $L$ are related by $\tilde{F} = Fh$ for some smooth map $h : U \to \mathrm{U}(3)$

$$h = \begin{bmatrix} (\det a)^{-1} & 0 \\ 0 & a \end{bmatrix}, a \in \mathrm{U}(3).$$

Their respective Maurer-Cartan forms, $\omega, \tilde{\omega}$ are related by

$$\tilde{\omega} = (Fh)^{-1}d(Fh) = h^{-1}F^{-1}dFh + h^{-1}dh = h^{-1}\omega h + h^{-1}dh.$$

Relative to the splitting of the Lie algebra $\mathfrak{su}(4) = \mathfrak{u}(3) \oplus \mathfrak{n}$ given in (3.9), we see that since $h^{-1}dh$ is $\mathfrak{u}(3)$-valued, the $\mathfrak{n}$-valued components of the Maurer-Cartan forms transform as

$$\tilde{\omega}_{\mathfrak{n}} = h^{-1}\omega_{\mathfrak{n}}h.$$

In particular, if we write $\omega_0 = {}^{\mathrm{t}}(\omega_0^1, \omega_0^2, \omega_0^3)$, then the transformation rule becomes

$$\tilde{\omega}_0 = \det(a)^{-1}a^{-1}\omega_0.$$

Suppose now that $F$ is $L$-adapted so that $\overline{\omega_0} = \omega_0$ is real. Then, the condition that $\tilde{F}$ also be $L$-adapted, $\overline{\tilde{\omega}_0} = \tilde{\omega}_0$ becomes $\overline{\det(a)^{-1}}\,\overline{a}^{-1}\overline{\omega}_0 = \det(a)^{-1}a^{-1}\omega_0$, which reduces to

$$\det({}^{\mathrm{t}}aa)\,{}^{\mathrm{t}}aa = I.$$

We see then that, in particular, if $a$ takes values in $\mathrm{SO}(3) \subset \mathrm{U}(3)$ then $L$-adaptation will be preserved. □

We can therefore reduce the $\mathrm{U}(3)$ bundle $f^{-1}\,\mathrm{SU}(4) \to L$ to the $\mathrm{SO}(3)$ bundle of $L$-adapted frames, which we shall denote $\pi : B_L \to L \subset \mathbb{CP}^3$. Since, by definition, $\eta_i = 0$ holds for such frames, the structure equations (3.17) imply

$$0 = d\eta_i = -(\delta_{ij}\operatorname{tr}\beta + \beta_{ij}) \wedge \omega_j, \tag{3.22}$$

and, since the $\omega_i$ remain linearly independent on $L$, Cartan's Lemma implies the existence of functions $h_{ijk} = h_{ikj} = h_{jik}$ on $B_L$ so that

$$(\delta_{ij}\operatorname{tr}\beta + \beta_{ij}) = h_{ijk}\omega_k. \tag{3.23}$$

The second fundamental form of $L$ can then be written

$$\mathrm{I\!I} = h_{ijk}\omega_i\omega_j. \tag{3.24}$$

If we take the trace of the second fundamental form with respect to the induced first fundamental form $I = \omega_i \circ \omega_i$ we see that that $L$ is *minimal* if and only if

$$h_{ikk} = 0$$

for each $1 \leqslant i \leqslant 3$.

We can repackage the information contained in the second fundamental form in the symmetric, traceless, cubic form

$$C = h_{ijk}\ \omega_i \omega_j \omega_k \tag{3.25}$$

which is well-defined on $L$. We shall call $C$ the *fundamental cubic* of the minimal Lagrangian submanifold $L$. When $L$ is minimal, it can be easily checked that the equations (3.23) reduce to

$$\beta_{ij} = h_{ijk}\omega_k$$

where $h_{ijk} = h_{ikj} = h_{jik}$ and $h_{ikk} = 0$ for each $1 \leqslant i \leqslant 3$.

### *3.2.1 Summary*

Here, we collect the relevant constructions and structure equations for $L$-adapted frames. Let $L^3 \subset \mathbb{CP}^3$ be a minimal Lagrangian submanifold. The $L$-adapted frame bundle $\pi : B_L \subset \mathrm{SU}(4) \to L$ is an SO(3)-bundle with $\mathfrak{su}(4)$-valued Maurer-Cartan form

$$\omega = g^{-1}dg = \begin{bmatrix} 0 & -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_1 & \mathrm{i}\beta_{11} & \alpha_{12} + \mathrm{i}\beta_{12} & \alpha_{13} + \mathrm{i}\beta_{13} \\ \omega_2 & -\alpha_{12} + \mathrm{i}\beta_{12} & \mathrm{i}\beta_{22} & \alpha_{23} + \mathrm{i}\beta_{23} \\ \omega_3 & -\alpha_{13} + \mathrm{i}\beta_{13} & -\alpha_{23} + \mathrm{i}\beta_{23} & \mathrm{i}\beta_{33} \end{bmatrix} \tag{3.26}$$

where $\alpha_{ij} = -\alpha_{ji}$, $\beta_{ij} = \beta_{ji}$, $\sum \beta_{ii} = 0$, and moreover, there exists functions $h_{ijk} = h_{ikj} = h_{jik}$, $h_{ikk} = 0$ on $B_L$, so that

$$\beta_{ij} = h_{ijk}\omega_k. \tag{3.27}$$

The Maurer-Cartan equation $d\omega = -\omega \wedge \omega$, plus the symmetries of the $h_{ijk}$ give the first structure equations

$$d\omega_i = -\alpha_{ij} \wedge \omega_j \tag{3.28}$$

and the second structure equations

$$d\alpha_{ij} = -\alpha_{ik} \wedge \alpha_{kj} + \beta_{ik} \wedge \beta_{kj} + \omega_i \wedge \omega_j \tag{3.29}$$

$$d\beta_{ij} = -\alpha_{ik} \wedge \beta_{kj} - \beta_{ik} \wedge \alpha_{kj}. \tag{3.30}$$

### *3.2.2 A Bonnet-type Result*

Let $(L, g, C)$ be a Riemannian 3-manifold with metric $g$ and symmetric cubic form $C$ that is traceless with respect to $g$. A natural question is to ask whether it is possible to isometrically embed $L$ into $\mathbb{CP}^3$ as a minimal Lagrangian submanifold. It turns out that the structure equations (3.29)-(3.30) give necessary and sufficient conditions for such an embedding to exist, unique up to rigid motions. This is similar to Bonnet's theorem, which gives necessary and sufficient conditions for the existence of isometric embeddings of surfaces in $\mathbb{R}^3$ with prescribed first and second fundamental forms.

Let $\omega_1, \omega_2, \omega_3$ be a $g$-orthonormal coframing on some local neighborhood $U \subset L$. By the fundamental theorem of Riemannian geometry there exist unique 1-forms $\alpha_{ij} = -\alpha_{ji}$ for which

$$d\omega_i = -\alpha_{ij} \wedge \omega_j. \tag{3.31}$$

Relative to this this coframing, we can write $C = h_{ijk}\omega_i\omega_j\omega_k$ for functions $h_{ijk}$ on $L$ satisfying $h_{ijk} = h_{ikj} = h_{jik}$, $h_{ikk} = 0$, and define $\beta_{ij} = \beta_{ji} = h_{ijk}\omega_k$.

**Theorem 3.4.** Suppose the forms $\omega_i, \alpha_{ij}, \beta_{ij}$ satisfy (3.29)-(3.30), then there locally exists an isometric immersion of $L$ into $\mathbb{CP}^3$ as a minimal Lagrangian submanifold with $C$ as its fundamental cubic. This immersion is unique up to rigid motion by an element of SU(4).

*Proof.* Let $\omega_i, \alpha_{ij}, \beta_{ij} = h_{ijk}\omega_k$ be defined as above, and suppose (3.28)-(3.30) are satisfied. Construct the $\mathfrak{su}(4)$-valued 1-form $\psi$

$$\psi = \begin{bmatrix} 0 & -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_1 & \mathrm{i}\beta_{11} & \alpha_{12} + \mathrm{i}\beta_{12} & \alpha_{13} + \mathrm{i}\beta_{13} \\ \omega_2 & -\alpha_{12} + \mathrm{i}\beta_{12} & \mathrm{i}\beta_{22} & \alpha_{23} + \mathrm{i}\beta_{23} \\ \omega_3 & -\alpha_{13} + \mathrm{i}\beta_{13} & -\alpha_{23} + \mathrm{i}\beta_{23} & \mathrm{i}\beta_{33} \end{bmatrix}. \tag{3.32}$$

□

The equations (3.28)-(3.30) imply the Maurer-Cartan equation $d\psi = -\psi \wedge \psi$ holds, so by Theorem 2.3 there locally exists a map $F : U \to \mathrm{SU}(4)$, unique up to rigid motion, so that $\psi = F^*(\omega)$, where $\omega$ is the Maurer-Cartan form of SU(4). Then, the composition $f : \pi \circ F : U \to \mathbb{CP}^3$, where $\pi : \mathrm{SU}(4) \to \mathbb{CP}^3$ is the projection, gives an isometric embedding of $U \subset L$ as a minimal Lagrangian submanifold of $\mathbb{CP}^3$.

# 4

# Second-Order Families

## 4.1 Classification of SO(3)-Stabilizer Types

In the previous chapter, we showed that, given a Lagrangian submanifold $f : L \to \mathbb{CP}^3$, we can construct the SO(3)-bundle of $L$-adapted frames $\pi : B_L \subset f^{-1}\,\mathrm{SU}(4) \to L$ on which $\eta_i = 0$ and $\beta_{ij} = h_{ijk}\omega_k$ for traceless, symmetric functions $h_{ijk}$ defined on $B_L$. Further, the second fundamental form of $L$ gives rise to the traceless, symmetric fundamental cubic $C = h_{ijk}\omega_i\omega_j\omega_k$. Note that the forms $\omega_i$ and coefficients $h_{ijk}$ both depend on the particular frame $g \in B_L$, on which SO(3) acts transitively on the fibers. Since the coefficients are symmetric and traceless, we can view $C$ as taking values in $\mathcal{H}_3(\mathbb{R}^3)$, the space of degree 3 homogeneous, harmonic polynomials in 3 variables.

The space $\mathcal{H}_3(\mathbb{R}^3)$ is an irreducible SO(3)-module, where the action is given by

$$(A \cdot f)(\mathbf{x}) = f(\mathbf{x}A)$$

where $A \in \mathrm{SO}(3)$, $f \in \mathcal{H}_3(\mathbb{R}^3)$, and $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$. We wish to study those minimal Lagrangians whose fundamental cubics have nontrivial SO(3)-stabilizers. Bryant [Bry00] gives the following classification of such cubics:

**Proposition 4.1.** The SO(3)-stabilizer of $h \in \mathcal{H}_3(\mathbb{R}^3)$ is nontrivial if and only if $h$ lies in the SO(3)-orbit of exactly one of the following polynomials

1. $0 \in \mathcal{H}_3(\mathbb{R}^3)$, whose stabilizer is SO(3).

2. $rx(2x^2 - 3y^2 - 3z^2)$ for some $r > 0$, whose stabilizer is SO(2).

3. $6sxyz$, for some $s > 0$, whose stabilizer is $A_4 \subset \mathrm{SO}(3)$.

4. $s(y^3 - 3yz^2)$ for some $s > 0$, whose stabilizer is $S_3 \subset \mathrm{SO}(3)$.

5. $rx(2x^2 - 3y^2 - 3z^2) + 6sxyz$, for some $r, s > 0$ such that $r \neq s$, whose stabilizer is $\mathbb{Z}_2 \subset \mathrm{SO}(3)$.

6. $rx(2x^2 - 3y^2 - 3z^2) + s(y^3 - 3yz^2)$, for some $r, s > 0$ satisfying $s \neq r\sqrt{2}$, whose stabilizer is $\mathbb{Z}_3 \subset \mathrm{SO}(3)$.

We now apply the method of moving frames along with exterior differential systems techniques, to analyze the local existence and generality of minimal Lagrangians in $\mathbb{CP}^3$ whose fundamental cubic at each point has one of the non-trivial symmetries listed above. Once local existence is established, we analyze the structure equations further in an attempt to gain more insight into the nature of these examples. In our analysis of the structure equations, most computations were done using the computer algebra software MAPLE in a manner analogous to the techniques used in [Bry00].

## 4.2 SO(2)-stabilizer

**Theorem 4.2.** Minimal Lagrangian submanifolds $L \subset \mathbb{CP}^3$ whose cubic form has an $SO(2)$ symmetry at each point exist locally, and depend on two constants.

*Proof.* Let $L \subset \mathbb{CP}^3$ be a minimal Lagrangian submanifold whose fundamental cubic has an SO(2)-symmetry at every point. By Proposition 4.1, there is a positive real-analytic funciton $r : L \to \mathbb{R}^+$ such that the equation

$$C = r\omega_1(2\omega_1^2 - 3\omega_2^2 - 3\omega_3^2) \tag{4.1}$$

defines a SO(2)-subbundle $B_1 \subset B_L$ of the $L$-adapted frame bundle $B_L \subset \mathrm{SU}(4) \to L$. On the subbundle $B_1$, we have

$$\begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{bmatrix} = \begin{bmatrix} 2r\omega_1 & -r\omega_2 & -r\omega_3 \\ -r\omega_2 & -r\omega_1 & 0 \\ -r\omega_3 & 0 & -r\omega_1 \end{bmatrix}. \tag{4.2}$$

Since $B_1$ is a SO(2)-bundle, the connection forms $\alpha_{12}$ and $\alpha_{31}$ are semibasic, so they can be written

$$\begin{aligned} \alpha_{12} &= t_{31}\omega_1 + t_{32}\omega_2 + t_{33}\omega_3 \\ \alpha_{31} &= t_{21}\omega_1 + t_{22}\omega_2 + t_{23}\omega_3 \end{aligned} \tag{4.3}$$

for functions $t_{ij}$ on $B_1$. Moreover, there exists functions $r_i$ on $B_1$ so that

$$dr = r_1\omega_1 + r_2\omega_2 + r_3\omega_3. \tag{4.4}$$

Substituting (4.2), (4.3), and (4.4) into the structure equations

$$d\beta_{ij} = -\alpha_{ik} \wedge \beta_{kj} - \beta_{ik} \wedge \alpha_{kj} \tag{4.5}$$

and using the identities $d\omega_i = -\alpha_{ij} \wedge \omega_j$ leads to polynomial equations in these variables, which can be solved, yielding

$$\begin{aligned} \alpha_{31} &= t\omega_3 \\ \alpha_{12} &= -t\omega_2 \\ dr &= -4rt\omega_1. \end{aligned} \tag{4.6}$$

The relations $d\alpha_{ij} = -\alpha_{ik} \wedge \alpha_{kj} + \beta_{ik} \wedge \beta_{kj} + \omega_i \wedge \omega_j$ imply

$$dt = (3r^2 - t^2 - 1)\omega_1 \tag{4.7}$$

In summary, given a minimal Lagrangian $L \subset \mathbb{CP}^3$ with fundamental cubic (4.1), the subbundle $B_1 \to L$ is an SO(2) bundle with augmented coframing $\omega_1, \omega_2, \omega_3, \alpha_{23} = -\alpha_{32}$ with primary invariants $r, t$ satisfying the structure equations

$$d\omega_1 = 0 \tag{4.8}$$

$$d\omega_2 = -\alpha_{23} \wedge \omega_3 + t\omega_1 \wedge \omega_2 \tag{4.9}$$

$$d\omega_3 = -\alpha_{32} \wedge \omega_2 - t\omega_3 \wedge \omega_1 \tag{4.10}$$

$$d\alpha_{23} = (r^2 + t^2 + 1)\omega_2 \wedge \omega_3 \tag{4.11}$$

$$dr = -4rt\omega_1 \tag{4.12}$$

$$dt = (3r^2 - t^2 - 1)\omega_1. \tag{4.13}$$

Since $d^2 = 0$ holds identically for all of these quantities, and there are no free derivatives, the existence of such coframings is guaranteed by Corollary 2.13. Thus, for any two constants $\bar{r}, \bar{t}$, there exists an open neighborhood $U$ of $0 \in \mathbb{R}^4$ on which there exists four linearly independent one forms $\omega_1, \omega_2, \omega_3, \alpha_{23}$ and functions $r, t$ that satisfy the structure equations (4.8) - (4.13) as well as

$$r(0) = \bar{r}, \qquad t(0) = \bar{t}.$$

These functions and forms are real-analytic and unique in a neighborhood of 0 up to real-analytic diffeomorphisms fixing the origin. We see that such minimal Lagrangian germs depend on two constants in the sense of exterior differential systems.

Conversely, given such $(\omega_i, r, t)$ on a 3-manifold $L$, one can define $\alpha_{ij} = -\alpha_{ji}$ by the first two equations in (4.6), $\beta_{ij} = \beta_{ji}$ by (4.2), and construct the $\mathfrak{su}(4)$-valued 1-form

$$\psi = \begin{bmatrix} 0 & -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_1 & \mathrm{i}\beta_{11} & \alpha_{12} + \mathrm{i}\beta_{12} & \alpha_{13} + \mathrm{i}\beta_{13} \\ \omega_2 & -\alpha_{12} + \mathrm{i}\beta_{12} & \mathrm{i}\beta_{22} & \alpha_{23} + \mathrm{i}\beta_{23} \\ \omega_3 & -\alpha_{13} + \mathrm{i}\beta_{13} & -\alpha_{23} + \mathrm{i}\beta_{23} & \mathrm{i}\beta_{33} \end{bmatrix}. \tag{4.14}$$

The Maurer-Cartan equation $d\psi = -\psi \wedge \psi$ holds, and so by Theorem 2.3, there exists (at least locally) an immersion, unique up to an element of SU(4), of $L$ into $\mathbb{CP}^3$ as a minimal Lagrangian with fundamental cubic (4.1). □

### *4.2.1 Integrating the Structure Equations*

We should note that this case has been analyzed by other authors in [DL05] where they study minimal Lagrangians in complex $n$-dimensional space forms, whose cubic form has an SO($n-1$) symmetry. Their existence results agree with ours, and they show that these examples are, in fact, complete.

We were not able to fully integrate this case, but can make the following observations about the solutions.

We begin by observing that the equations (4.12)-(4.13) imply that if $(r,t) = (\frac{1}{\sqrt{3}}, 0)$ at any point then the structure equations reduce to

$$\begin{aligned} d\omega_1 &= 0 \\ d\omega_2 &= -\alpha_{23} \wedge \omega_3 \\ d\omega_3 &= -\alpha_{32} \wedge \omega_2 \\ d\alpha_{23} &= \tfrac{4}{3}\omega_2 \wedge \omega_3 \\ dr &= 0 \\ dt &= 0. \end{aligned} \tag{4.15}$$

So $(r,t) = (\frac{1}{\sqrt{3}}, 0)$ on the entire solution, and since the coefficients appearing in (4.15) are all constants these examples are homogeneous. If we assume $(r,t) \neq (\frac{1}{\sqrt{3}}, 0)$, then the structure equations imply that

$$\omega_1 = \frac{dr}{-4rt} = \frac{dt}{3r^2 - t^2 - 1} \tag{4.16}$$

so that $(3r^2 - t^2 - 1)dr + 4rtdt = 0$, which can be integrated to obtain

$$\frac{r^2 + t^2 + 1}{\sqrt{r}} = K$$

for some positive constant $K$. If we let $r = s^2$, the equation above becomes

$$s^4 + t^2 + 1 = Ks. \tag{4.17}$$

Thus, the image of the map $(s, t) : B_3 \to \mathbb{R}^2$ on any solution for which $(r, t) \neq (\frac{1}{\sqrt{3}}, 0)$ is an algebraic curve.

Furthermore, the equation $d\omega_1 = 0$ implies that the EDS given by $\mathcal{I} = \{\omega_1\}$ is integrable, so gives rise to a foliation of $L$ whose leaves are 2-dimensional. On the leaves of this foliation, the structure equations for $(\omega_2, \omega_3, \alpha_{23})$ become

$$d\omega_2 \equiv -\alpha_{23} \wedge \omega_3 \mod \omega_1 \tag{4.18}$$

$$d\omega_3 \equiv -\alpha_{32} \wedge \omega_2 \mod \omega_1 \tag{4.19}$$

$$d\alpha_{23} \equiv (r^2 + t^2 + 1)\omega_2 \wedge \omega_3 \mod \omega_1. \tag{4.20}$$

Setting $\xi_i = \sqrt{s}\omega_i$ for $i = 2, 3$, and using the fact that $r^2 + t^2 + 1 = Ks$, these can be rewritten as

$$d\xi_2 \equiv -\alpha_{23} \wedge \xi_3 \tag{4.21}$$

$$d\xi_3 \equiv -\alpha_{32} \wedge \xi_2 \tag{4.22}$$

$$d\alpha_{23} \equiv K\xi_2 \wedge \xi_3 \tag{4.23}$$

which are the structure equations for a surface of constant Gauss curvature $K$ with metric $g_K = \xi_2^2 + \xi_3^2$ and connection form $\alpha_{23}$. Thus, the metric $g = \omega_1^2 + \omega_2^2 + \omega_3^2$ on $L$ can be written as a warped product on $\mathbb{R} \times S^2$

$$g = \omega_1^2 + \tfrac{1}{s}(\xi_2^2 + \xi_3^2). \tag{4.24}$$

## 4.3 $A_4$-stabilizer

**Theorem 4.3.** Up to rigid motion, there is exactly one Minimal Lagrangian submanifolds $L \subset \mathbb{CP}^3$ whose fundamental cubic has an $A_4$ symmetry.

Note that in the corresponding case for special Lagrangian 3-folds in $\mathbb{C}^3$ whose cubics have $A_4$ symmetry, Bryant [Bry00] shows that no such examples can exist.

*Proof.* Let $L \subset \mathbb{CP}^3$ be a minimal Lagrangian submanifold whose fundamental cubic has an $A_4$-symmetry at every point. We assume, without loss of generality, that $C$ is nowhere vanishing on $L$. By Proposition 4.1, there is a positive real-analytic function $r : L \to \mathbb{R}^+$ such that the equation

$$C = 6r\omega_1\omega_2\omega_3 \tag{4.25}$$

defines an $A_4$-subbundle $B_2 \subset B_L$ of the $L$-adapted frame bundle $B_L \subset \mathrm{SU}(4) \to L$. On the subbundle $B_2$, we have

$$\begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{bmatrix} = \begin{bmatrix} 0 & r\omega_3 & r\omega_2 \\ r\omega_3 & 0 & r\omega_1 \\ r\omega_2 & r\omega_1 & 0 \end{bmatrix}. \tag{4.26}$$

Since $B_2$ is an $A_4$-bundle, the relations $\alpha_{12} \equiv \alpha_{31} \equiv \alpha_{23} \equiv 0 \mod \{\omega_1, \omega_2, \omega_3\}$ hold, so can be written

$$\begin{aligned} \alpha_{12} &= t_{31}\omega_1 + t_{32}\omega_2 + t_{33}\omega_3 \\ \alpha_{31} &= t_{21}\omega_1 + t_{22}\omega_2 + t_{23}\omega_3 \\ \alpha_{23} &= t_{11}\omega_1 + t_{12}\omega_2 + t_{13}\omega_3 \end{aligned} \tag{4.27}$$

for some functions $t_{ij}$ on $B_2$. Furthermore, there exists functions $r_i$ on $B_2$ so that

$$dr = r_1\omega_1 + r_2\omega_2 + r_3\omega_3 \tag{4.28}$$

Substituting (4.26), (4.27), and (4.28) into the structure equations

$$d\beta_{ij} = -\alpha_{ik} \wedge \beta_{kj} - \beta_{ik} \wedge \alpha_{kj} \tag{4.29}$$

and using the identities $d\omega_i = -\alpha_{ij} \wedge \omega_j$ leads to polynomial equations in the $t_{ij}, r_i$, which since we are assuming $r > 0$, can be solved to get

$$\alpha_{ij} = 0 \qquad\qquad dr = 0, \tag{4.30}$$

in particular, $r$ must be constant. Substituting (4.26) and (4.30) into the structure equations

$$d\alpha_{ij} = -\alpha_{ik} \wedge \alpha_{kj} + \beta_{ik} \wedge \beta_{kj} + \omega_i \wedge \omega_j \tag{4.31}$$

leads to the equation

$$r^2 - 1 = 0. \tag{4.32}$$

Since $r > 0$ by assumption, we get exactly one solution for which $r = 1$.

### *4.3.1 Integrating the Structure Equations*

Let's summarize what the analysis above showed. For a minimal Lagrangian $f : L \to \mathbb{CP}^3$ whose fundamental cubic $C$ has an $A_4$-stabilizer, we were able to construct a 3-dimensional subbundle $B_2 \subset B_L$ on which

$$C = 6\omega_1\omega_2\omega_3 \tag{4.33}$$

holds with a coframing $(\omega_1, \omega_2, \omega_3)$ satisfying $d\omega_i = 0$. Note that the structure equations $d\omega_i = 0$ imply there locally exists functions $\theta_i$ on $B_2$ so that

$$\omega_i = d\theta_i. \tag{4.34}$$

Our goal is to try to integrate these structure equations to explicitly describe the lift $F : L \to B_2 \subset B_L$ of the map $f : L \to \mathbb{CP}^3$ so that the following diagram commutes.

$$\begin{tikzcd} & B_2 \subset \mathrm{SU}(4) \arrow[d, "\pi"] \\ L \arrow[ur, "F"] \arrow[r, "f"] & \mathbb{CP}^3 \end{tikzcd}$$

Given such a lift, we can write $\mathfrak{su}(4)$-valued Maurer-Cartan form on $B_2$ in terms of the functions $\theta_i$ in (4.34) as

$$F^{-1}dF = \begin{bmatrix} 0 & -d\theta_1 & -d\theta_2 & -d\theta_3 \\ d\theta_1 & 0 & \mathrm{i}d\theta_3 & \mathrm{i}d\theta_2 \\ d\theta_2 & \mathrm{i}d\theta_3 & 0 & \mathrm{i}d\theta_1 \\ d\theta_3 & \mathrm{i}d\theta_2 & \mathrm{i}d\theta_1 & 0 \end{bmatrix}. \tag{4.35}$$

We note that the Maurer-Cartan form takes values in the 3-dimensional abelian Lie subalgebra $\mathfrak{g}_0 \subset \mathfrak{su}(4)$ with basis

$$A_1 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ 1 & 0 & 0 & 0 \\ 0 & \mathrm{i} & 0 & 0 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & \mathrm{i} & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{4.36}$$

so that

$$\omega = A_1 d\theta_1 + A_2 d\theta_2 + A_3 d\theta_3.$$

Our strategy is to write $F$ as a product $F = F_1F_2F_3$ where $F_i$ is a function of $\theta_i$ satisfying $F_i^{-1}dF_i = A_i d\theta_i$, so that

$$F^{-1}dF = F_1^{-1}dF_1 + F_2^{-1}dF_2 + F_3^{-1}dF_3 = \omega.$$

Exponentiating the $A_i$, we find

$$F_1 = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 & 0 \\ \sin(\theta_1) & \cos(\theta_1) & 0 & 0 \\ 0 & 0 & \cos(\theta_1) & \mathrm{i}\sin(\theta_1) \\ 0 & 0 & \mathrm{i}\sin(\theta_1) & \cos(\theta_1) \end{bmatrix} \tag{4.37}$$

$$F_2 = \begin{bmatrix} \cos(\theta_2) & 0 & -\sin(\theta_2) & 0 \\ 0 & \cos(\theta_2) & 0 & \mathrm{i}\sin(\theta_2) \\ \sin(\theta_2) & 0 & \cos(\theta_2) & 0 \\ 0 & \mathrm{i}\sin(\theta_2) & 0 & \cos(\theta_2) \end{bmatrix} \tag{4.38}$$

$$F_3 = \begin{bmatrix} \cos(\theta_3) & 0 & 0 & -\sin(\theta_3) \\ 0 & \cos(\theta_3) & \mathrm{i}\sin(\theta_3) & 0 \\ 0 & \mathrm{i}\sin(\theta_3) & \cos(\theta_3) & 0 \\ \sin(\theta_3) & 0 & 0 & \cos(\theta_3) \end{bmatrix}. \tag{4.39}$$

Their product, $F(\theta_1, \theta_2, \theta_3) = F_1(\theta_1)F_2(\theta_2)F_3(\theta_3)$, takes values in a 3-dimensional abelian subgroup of SU(4), and so is a maximal torus. The full expression for $F$ is cumbersome, and not too enlightening, so we omit it here. The minimal Lagrangian submanifold of $\mathbb{CP}^3$ it induces is given by the composition $f = \pi \circ F$, which recall is the projectivization of the first column of $F$. In homogeneous coordinates, we compute

$$f(\theta_1, \theta_2, \theta_3) = \begin{bmatrix} e^{\mathrm{i}(\theta_1+\theta_2+\theta_3)} + e^{\mathrm{i}(\theta_1-\theta_2-\theta_3)} + e^{\mathrm{i}(-\theta_1+\theta_2-\theta_3)} + e^{\mathrm{i}(-\theta_1-\theta_2+\theta_3)} \\ -\mathrm{i}\big(e^{\mathrm{i}(\theta_1+\theta_2+\theta_3)} + e^{\mathrm{i}(\theta_1-\theta_2-\theta_3)} - e^{\mathrm{i}(-\theta_1+\theta_2-\theta_3)} - e^{\mathrm{i}(-\theta_1-\theta_2+\theta_3)}\big) \\ -\mathrm{i}\big(e^{\mathrm{i}(\theta_1+\theta_2+\theta_3)} - e^{\mathrm{i}(\theta_1-\theta_2-\theta_3)} + e^{\mathrm{i}(-\theta_1+\theta_2-\theta_3)} - e^{\mathrm{i}(-\theta_1-\theta_2+\theta_3)}\big) \\ -\mathrm{i}\big(e^{\mathrm{i}(\theta_1+\theta_2+\theta_3)} - e^{\mathrm{i}(\theta_1-\theta_2-\theta_3)} - e^{\mathrm{i}(-\theta_1+\theta_2-\theta_3)} + e^{\mathrm{i}(-\theta_1-\theta_2+\theta_3)}\big) \end{bmatrix}. \quad (4.40)$$

$\square$

## 4.4 $S_3$-Stabilizer

**Theorem 4.4.** Minimal Lagrangian submanifolds $L \subset \mathbb{CP}^3$ whose fundamental cubic form has an $S_3$ symmetry exist locally and depend on 4 functions of 1 variable.

*Proof.* Suppose $L \subset \mathbb{CP}^3$ is a minimal Lagrangian submanifold whose fundamental cubic has a $S_3$-stabilizer at every point. We assume, without loss of generality, that $C$ is nowhere vanishing on $L$. By Proposition 4.1, there is a positive real-analytic function $s : L \to \mathbb{R}^+$ such that the equation

$$C = s(\omega_2^3 - 3\omega_2\omega_3^2) \quad (4.41)$$

defines a $S_3$-subbundle $B_3 \subset B_L$ of the $L$-adapted frame bundle $B_L \subset \mathrm{SU}(4) \to L$. On this subbundle, we have

$$\begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & s\omega_2 & -s\omega_3 \\ 0 & -s\omega_3 & -s\omega_2 \end{bmatrix}. \quad (4.42)$$

Since $B_3$ is an $S_3$-bundle, the relations $\alpha_{12} \equiv \alpha_{31} \equiv \alpha_{23} \equiv 0 \mod \{\omega_1, \omega_2, \omega_3\}$ hold, so can be written

$$\begin{aligned}\alpha_{12} &= t_{31}\omega_1 + t_{32}\omega_2 + t_{33}\omega_3 \\ \alpha_{31} &= t_{21}\omega_1 + t_{22}\omega_2 + t_{23}\omega_3 \\ \alpha_{23} &= t_{11}\omega_1 + t_{12}\omega_2 + t_{13}\omega_3\end{aligned} \tag{4.43}$$

for some functions $t_{ij}$ on $B_3$. Further, there exists functions $s_1, s_2, s_3$ on $B_3$ so that

$$ds = s_i\omega_i. \tag{4.44}$$

Substituting (4.42), (4.43), and (4.44) into the structure equations

$$d\beta_{ij} = -\alpha_{ik} \wedge \beta_{kj} - \beta_{ik} \wedge \alpha_{kj} \tag{4.45}$$

and using the identities $d\omega_i = -\alpha_{ij} \wedge \omega_j$ leads to polynomial equations in the $t_{ij}, s_i$, which can be solved, leading to relations

$$\alpha_{12} = 3t_1\omega_2 - 3u_1\omega_3 \tag{4.46}$$

$$\alpha_{31} = -3u_1\omega_2 - 3t_1\omega_3 \tag{4.47}$$

$$\alpha_{23} = u_1\omega_1 - t_3\omega_2 + t_2\omega_3 \tag{4.48}$$

$$ds = 3s(t_1\omega_1 + t_2\omega_2 + t_3\omega_3) \tag{4.49}$$

where we have renamed the functions $t_{ij}$ for simplicity of notation. The relations

$$d\alpha_{ij} = -\alpha_{ik} \wedge \alpha_{kj} + \beta_{ik} \wedge \beta_{kj} + \omega_i \wedge \omega_j \tag{4.50}$$

give polynomial relations on the exterior derivatives of the quantities $u_1, t_1, t_2, t_3$, and lead to expressions of the form

$$du_1 = 6t_1u_1\omega_1 + r_2\omega_2 + r_3\omega_3 \tag{4.51}$$

$$dt_1 = \left(\tfrac{1}{3} + 3t_1^2 - 3u_1^2\right)\omega_1 + r_3\omega_2 - r_2\omega_3 \tag{4.52}$$

$$dt_2 = (3t_1t_2 + 2t_3u_1 + r_3)\omega_1 + (v - p_2)\,\omega_2 + (-3t_1u_1 + p_3)\omega_3 \tag{4.53}$$

$$dt_3 = (3t_1t_3 - 2t_2u_1 - r_2)\omega_1 + (3t_1u_1 + p_3)\omega_2 + (v + p_2)\,\omega_3 \tag{4.54}$$

where

$$v = -s^2 + \tfrac{9}{2}t_1^2 + \tfrac{1}{2}t_2^2 + \tfrac{1}{2}t_3^2 + \tfrac{15}{2}u_1^2 + \tfrac{1}{2} \tag{4.55}$$

and $p_2, p_3, r_2, r_3$ are functions on $B_3$. In the language of Section 2.3, $B_3 \to L$ is an $S_3$ bundle on which $\omega_1, \omega_2, \omega_3$ constitute an augmented coframing satisfying the structure equations

$$d\omega_1 = -u_1\omega_2 \wedge \omega_3 \tag{4.56}$$

$$d\omega_2 = -3t_1\omega_1 \wedge \omega_2 - 2u_1\omega_3 \wedge \omega_1 + t_3\omega_2 \wedge \omega_3 \tag{4.57}$$

$$d\omega_3 = -2u_1\omega_1 \wedge \omega_2 + 3t_1\omega_3 \wedge \omega_1 - t_2\omega_2 \wedge \omega_3 \tag{4.58}$$

with primary invariants $(s, u_1, t_1, t_2, t_3)$ satisfying the structure equations (4.49) and (4.51)-(4.54) where $(p_2, p_3, r_2, r_3)$ are the free derivatives. It is easy to check that the relations on the covariant derivatives of the free derivatives arising from ensuring $d^2 = 0$ is an identity are solvable (so that (2.10) is satisfied). We only need to check the tableau of free derivatives for involutivity.

At a point $(u, v) \in \mathbb{R}^5 \times \mathbb{R}^4$, the tableau of free derivatives for $(s, u_1, t_1, t_2, t_3)$ with respect to $(p_2, p_3, r_2, r_3)$ is the vector subspace $A(u, v) \subset \mathrm{Hom}(\mathbb{R}^3, \mathbb{R}^5)$ given by

$$A(u,v) = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & x_3 & x_4 \\ 0 & x_4 & -x_3 \\ x_4 & -x_1 & x_2 \\ -x_3 & x_2 & x_1 \end{bmatrix} \mid x_1, x_2, x_3, x_4 \in \mathbb{R} \right\}. \tag{4.59}$$

The tableau is 4-dimensional and does not depend depend on the point $(u, v) \in \mathbb{R}^5 \times \mathbb{R}^4$. Moreover, it is involutive with Cartan characters $(s_0, s_1, s_2, s_3) = (5, 4, 0, 0)$. Thus the hypotheses of Theorem 2.12 are satisfied, which shows such augmented coframings exist and depend on 4 functions of 1 variable in the sense of exterior differential systems.

□

### *4.4.1 Integrating the Structure Equations*

We begin by noting that the equations (4.57) and (4.58) imply that the differential system $\{\omega_2 = 0, \omega_3 = 0\}$ is integrable, and so locally there is a submersion $B_3 \to X$, where $X$ is the leaf space of the distribution $\{\omega_2 = 0, \omega_3 = 0\}$ and whose fibers are the 1-dimensional leaves. The complex-valued 1-form $\varphi = \omega_2 + \mathrm{i}\omega_3$ is well-defined on $X$, up to a multiple. In fact, the structure equations

$$\begin{pmatrix} d(\omega_2) \\ d(\omega_3) \end{pmatrix} = \begin{pmatrix} -3t_1\omega_1 & 2u_1\omega_1 + t_3\omega_2 - 2t_2\omega_3 \\ -2u_1\omega_1 - 2t_3\omega_2 + 2t_2\omega_3 & -3t_1\omega_1 \end{pmatrix} \wedge \begin{pmatrix} \omega_2 \\ \omega_3 \end{pmatrix}, \quad (4.60)$$

show that $\varphi$ satisfies the equation

$$d\varphi = [-3t_1\omega_1 - \mathrm{i}(2u_1\omega_1 - t_3\omega_2 + t_2\omega_3)] \wedge \varphi. \quad (4.61)$$

Hence, $\varphi$ gives a well-defined conformal structure on the leaf space $X$.

Consider now the function $z = 3(u_1 - \mathrm{i}t_1)$ on $B_3$. Note that the structure equations (4.51) - (4.52) imply $z$ satisfies the equation

$$dz = -\mathrm{i}(1 - z^2)\omega_1 - 3(r_2 - \mathrm{i}r_3)(\omega_2 + \mathrm{i}\omega_3) \quad (4.62)$$

so that in particular

$$dz \equiv -\mathrm{i}(1 - z^2)\omega_1 \mod (\omega_2 + \mathrm{i}\omega_3). \quad (4.63)$$

Equation (4.63) tells us that that if $z = \pm 1$ at any point on a leaf of $\{\omega_2 = 0, \omega_3 = 0\}$, then it must equal that value on the entire leaf. While we have not been able to integrate the structure equations generally for this case, we are able to show that in the special case $z = \pm 1$ the associated minimal Lagrangians correspond to *superminimal* surfaces in the 1-dimensional quaternionic projective space $\mathbb{HP}^1 \cong S^4$ as analyzed by Bryant in [Bry82]. We consider only the case $z = +1$ below, as the analysis of the $z = -1$ case is analogous.

*Special Case:* $z = +1$

**Index Convention**: In this section, we use the following index ranges: $0 \leqslant a, b, c \leqslant 4$, $1 \leqslant i, j, k \leqslant 3$, $1 \leqslant \alpha, \beta, \gamma \leqslant 2$.

**Theorem 4.5.** Minimal Lagrangians $L \subset \mathbb{CP}^3$ whose fundamental cubic has an $S_3$ symmetry and satisfy $z = \pm 1$ exist locally and depend on 2 functions of 1 variable. These examples give rise to *superminimal* surfaces in $\mathbb{HP}^1 \simeq S^4$.

*Proof.* Setting $u_1 = \pm\frac{1}{3}, t_1 = 0$, we see that the structure equations

$$0 = du_1 = 6t_1u_1\omega_1 + r_2\omega_2 + r_3\omega_3$$

$$0 = dt_1 = \left(\tfrac{1}{3} + 3t_1^2 - 3u_1^2\right)\omega_1 + r_3\omega_2 - r_2\omega_3$$

imply $r_2 = r_3 = 0$. Hence, in this case we are reduced to an augmented coframing $\omega_1, \omega_2, \omega_3$ with primary invariants $(s, t_2, t_3)$ and free derivatives $(p_2, p_3)$ satisfying the structure equations

$$\begin{aligned}
d\omega_1 &= \mp\tfrac{1}{3}\omega_2 \wedge \omega_3 \\
d\omega_2 &= \mp\tfrac{2}{3}u_1\omega_3 \wedge \omega_1 + t_3\omega_2 \wedge \omega_3 \\
d\omega_3 &= \mp\tfrac{2}{3}u_1\omega_1 \wedge \omega_2 - t_2\omega_2 \wedge \omega_3 \\
ds &= 3s(t_2\omega_2 + t_3\omega_3) \\
dt_2 &= \pm\tfrac{2}{3}t_3\omega_1 + (v - p_2)\,\omega_2 + p_3\omega_3 \\
dt_3 &= \mp\tfrac{2}{3}t_2\omega_1 + p_3\omega_2 + (v + p_2)\,\omega_3
\end{aligned}$$

where

$$v = -s^2 + \tfrac{1}{2}t_2^2 + \tfrac{1}{2}t_3^2 + \tfrac{4}{3}. \tag{4.64}$$

The associated tableau of free derivatives can easily be shown to be involutive with Cartan characters $(s_0, s_1, s_2, s_3) = (3, 2, 0, 0)$. $\square$

The examples in this case all give rise to *superminimal* surfaces in $S^4$, which is isometric to $\mathbb{HP}^1$, projective quaternionic 1-space. In [Bry82], Bryant shows a correspondence between superminimal surfaces in $\mathbb{HP}^1$ and holomorphic curves in $\mathbb{CP}^3$ which are horizontal with respect to the *twistor map* $T : \mathbb{CP}^3 \to \mathbb{HP}^1$, which agrees with the function count proved above. To show that our examples do in fact give rise to superminimal surfaces, we begin by reviewing the necessary moving frame constructions for surfaces in $\mathbb{HP}^1$ as developed in [Bry82].

Let $\mathbb{H}$ denote the non-commutative algebra of quaternions, whose elements have a unique representation in the form $q = z_1 + \mathrm{j}z_2$ where $z_1, z_2 \in \mathbb{C}$ and $j \in \mathbb{H}$ satisfies

$$j^2 = -1, \qquad z\mathrm{j} = \mathrm{j}\bar{z} \tag{4.65}$$

for all $z \in \mathbb{C}$. There is a natural inclusion of $\mathbb{C}$ into $\mathbb{H}$, and we regard $\mathbb{H}$ as a complex vector space, where $\mathbb{C}$ acts on the right. Moreover, we have the natural identification $\mathbb{H}$ with $\mathbb{C}^2$ given by $z_1 + \mathrm{j}z_2 \leftrightarrow {}^{\mathrm{t}}(z_1, z_2)$. Conjugation in $\mathbb{H}$, is given by

$$\bar{q} = \bar{z}_1 - \mathrm{j}z_2. \tag{4.66}$$

Let $\mathbb{H}^2$ be the space of pairs ${}^{\mathrm{t}}(q_1, q_2)$, $q_\alpha \in \mathbb{H}$. It has the structure of a *right* $\mathbb{H}$-vector space, where the action is given by

$${}^{\mathrm{t}}(q_1, q_2) \cdot p = {}^{\mathrm{t}}(q_1 p, q_2 p). \tag{4.67}$$

Since $\mathbb{C} \subset \mathbb{H}$, we can view $\mathbb{H}^2 \cong \mathbb{C}^4$ as complex vector spaces, where the identification is given by

$$(z_0 + \mathrm{j}z_1, z_2 + \mathrm{j}z_3) \leftrightarrow (z_0, z_1, z_2, z_3). \tag{4.68}$$

Just as in the complex case, we define the one dimensional *quaternionic projective space* $\mathbb{HP}^1$ as the set of one dimensional subspaces of $\mathbb{H}^2$. We denote the natural projection which takes a nonzero vector $v \in \mathbb{H}^2$ to the quaternionic line spanned by $v$ by

$$v \mapsto [v]_{\mathbf{H}} \in \mathbb{HP}^1.$$

The *complex* line spanned by $v$ also makes sense and, by the identification given in (4.68), can be thought of as an element of $\mathbb{CP}^3$. The associated projection will be denoted $[v]_{\mathbf{C}} \subset \mathbb{CP}^3$, and it is easy to see we have $[v]_{\mathbf{C}} \subset [v]_{\mathbf{H}}$. Thus the map $T : \mathbb{CP}^3 \to \mathbb{HP}^1$ defined by $T([v]_{\mathbf{C}}) = [v]_{\mathbf{H}}$ is well defined, and the fiber $T^{-1}([v]_{\mathbf{H}})$ consists of all complex lines in $[v]_{\mathbf{H}} \simeq \mathbb{C}^2$, so is a copy of $\mathbb{CP}^1$. This describes the *twistor fibration* of Penrose,

$$\begin{tikzcd} \mathbb{CP}^1 \arrow[r] & \mathbb{CP}^3 \arrow[d, "T"] \\ & \mathbb{HP}^1 \end{tikzcd}$$

and $T$ is called the *twistor map.*

The standard $\mathbb{H}$-valued inner product on $\mathbb{H}^2$, $\langle , \rangle : \mathbb{H}^2 \times \mathbb{H}^2 \to \mathbb{H}$ is defined by

$$\langle (q_1, q_2), (p_1, p_2) \rangle = \bar{q}_1 p_1 + \bar{q}_2 p_2 \tag{4.69}$$

and satisfies the identities

$$\langle v, wq \rangle = \langle v, w \rangle q, \quad \overline{\langle v, w \rangle} = \langle w, v \rangle, \quad \langle vq, w \rangle = \bar{q} \langle v, w \rangle. \tag{4.70}$$

The real part $\mathrm{Re}\,\langle , \rangle : \mathbb{H}^2 \times \mathbb{H}^2 \to \mathbb{R}$ gives the standard inner product on $\mathbb{R}^8 \cong \mathbb{H}^2$ and so gives $\mathbb{H}^2$ the structure of real 8-dimensional euclidean space $\mathbb{E}^8$.

The *symplectic group* $\mathrm{Sp}(2)$ is defined to be the isometry group of $(\mathbb{H}^2, \langle , \rangle)$:

$$\mathrm{Sp}(2) = \left\{ A \in \mathrm{GL}(2, \mathbb{H}) \mid {}^{\mathrm{t}}\bar{A} A = I \right\}. \tag{4.71}$$

It is easy to see that $\mathbb{H}^1$ is a homogeneous $\mathrm{Sp}(2)$-space: Choosing the point $[{}^{\mathrm{t}}(1, 0)]_{\mathbf{H}} \in \mathbb{HP}^1$ as an 'origin', the isotropy subgroup $H$ of this point is the group $\mathrm{Sp}(1) \times \mathrm{Sp}(1) \simeq S^3 \times S^3$ (we can identify $S^3$ with the set of unit quaternions). Explicitly, $H \subset \mathrm{Sp}(2)$ consists of matrices of the form $\begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix}$ where $q_\alpha \in \mathbb{H}$ satisfy $q_\alpha \bar{q}_\alpha = 1$ for $\alpha = 1, 2$.

This gives a description of Sp(2) as an $S^3 \times S^3$ bundle over $\mathbb{HP}^1 \cong \mathrm{Sp}(2)/(S^3 \times S^3)$:

$$\begin{tikzcd} S^3 \times S^3 \arrow[r] & \mathrm{Sp}(2) \arrow[d, "\pi_H"] \\ & \mathbb{HP}^1 \end{tikzcd} \tag{4.72}$$

This fibration also implies $\mathbb{HP}^1$ is connected and simply connected.

Elements of Sp(2) can be though of as matrices $A = [\mathbf{f}_1, \mathbf{f}_2]$ whose $\mathbb{H}^2$-valued columns $\mathbf{f}_\alpha$ satisfy

$$\langle \mathbf{f}_\alpha, \mathbf{f}_\beta \rangle = \delta_{\alpha\beta},$$

where $\langle , \rangle$ is the quaternion Hermitian inner product on $\mathbb{H}^2$ defined above. The projection map $\pi_H : \mathrm{Sp}(2) \to \mathbb{HP}^1$ is then given by $\pi_H(\mathbf{f}) = \pi_H([\mathbf{f}_1, \mathbf{f}_2]) = [\mathbf{f}_1]_{\mathbf{H}}$, the quaternionic line spanned by the first column of $A = [\mathbf{f}_1, \mathbf{f}_2] \in \mathrm{Sp}(2)$. We also have a canonical map $\pi_C : \mathrm{Sp}(2) \to \mathbb{CP}^3$ defined by $\pi_C([\mathbf{f}_1, \mathbf{f}_2]) = [\mathbf{f}_1]_{\mathbf{C}} \in \mathbb{CP}^3$.

We now briefly review the theory of moving frames for surfaces in $\mathbb{HP}^1$. The Lie algebra $\mathfrak{sp}(2)$ consists of the $2 \times 2$ quaternion matrices $X$ satisfying ${}^t\overline{X} + X = 0$. If $\phi$ is the $\mathfrak{sp}(2)$-valued Maurer-Cartan form of Sp(2) then it can be decomposed into components

$$\phi = \begin{bmatrix} \phi_1^1 & \phi_2^1 \\ \phi_1^2 & \phi_2^2 \end{bmatrix}$$

where the $\phi_\beta^\alpha$ are $\mathbb{H}$-valued left-invariant 1-forms on Sp(2) and satisfy

$$\phi_\beta^\alpha + \overline{\phi_\alpha^\beta} = 0. \tag{4.73}$$

Following the usual procedure for the method of moving frames, we have the structure equations

$$d\mathbf{f}_\alpha = \mathbf{f}_\beta \phi_\alpha^\beta \tag{4.74}$$

and

$$d\phi_\beta^\alpha = -\phi_\gamma^\alpha \wedge \phi_\beta^\gamma. \tag{4.75}$$

The symmetries (4.73) imply we can decompose the $\phi^\alpha_\beta$ as

$$\begin{bmatrix} \phi^1_1 & \phi^1_2 \\ \phi^2_1 & \phi^2_2 \end{bmatrix} = \begin{bmatrix} i\rho_1 + \mathrm{j}\phi_1 & -\bar{\eta}_1 + j\eta_2 \\ \eta_1 + \mathrm{j}\eta_2 & i\rho_2 + j\phi_2 \end{bmatrix} \tag{4.76}$$

where $\rho_1, \rho_2$ are real-valued 1-forms and $\omega_1, \omega_2, \phi_1$, and $\phi_2$ are complex valued.

As a complementary subspace to the lie algebra of the stabilizer $H \simeq S^3 \times S^3$ in $\mathfrak{sp}(2)$, we take

$$\mathfrak{n} = \left\{ \begin{bmatrix} 0 & -\bar{p} \\ p & 0 \end{bmatrix} \mid p \in \mathbb{H} \right\}.$$

The splitting of $\phi = \phi_{\mathfrak{h}} + \phi_{\mathfrak{n}}$ implies then that the $\phi^2_1 = \eta_1 + \mathrm{j}\eta_2$ are semibasic and the forms $\eta_1, \bar{\eta}_1, \eta_2, \bar{\eta}_2$ are $\pi_H$ semibasic, and pull back to a basis of $T^*\mathbb{HP}^1$ under any moving frame. The structure equations (4.75) for these components take the form

$$\begin{aligned} d\begin{bmatrix} \eta_1 \\ \bar{\eta}_1 \\ \eta_2 \\ \bar{\eta}_2 \end{bmatrix} &= \begin{bmatrix} \mathrm{i}(\rho_1 - \rho_2) & 0 & \bar{\phi}_2 & -\phi_1 \\ 0 & \mathrm{i}(\rho_2 - \rho_1) & -\bar{\phi}_1 & \phi_2 \\ -\phi_2 & \phi_1 & \mathrm{i}(\rho_1 + \rho_2) & 0 \\ \bar{\phi}_1 & -\bar{\phi}_2 & 0 & -\mathrm{i}(\rho_1 + \rho_2) \end{bmatrix} \wedge \begin{bmatrix} \eta_1 \\ \bar{\eta}_1 \\ \eta_2 \\ \bar{\eta}_2 \end{bmatrix} \\ &= -\Psi \wedge \begin{bmatrix} \eta_1 \\ \bar{\eta}_1 \\ \eta_2 \\ \bar{\eta}_2 \end{bmatrix} \end{aligned} \tag{4.77}$$

and furthermore

$$d\Psi + \Psi \wedge \Psi = 2 \begin{bmatrix} \eta_1 \\ \bar{\eta}_1 \\ \eta_2 \\ \bar{\eta}_2 \end{bmatrix} \wedge \begin{bmatrix} \bar{\eta}_1, \eta_1, \bar{\eta}_2, \eta_2 \end{bmatrix} \tag{4.78}$$

Since the symmetric tensor $ds^2 = 4\phi^2_1 \circ \overline{\phi^2_1} = 4(\eta_1 \circ \bar{\eta}_1 + \eta_2 \circ \bar{\eta}_2)$ is invariant under change of frame, it descends to a well-defined metric on $\mathbb{HP}^1$ which the above computations show has constant curvature $+1$. Since $\mathbb{HP}^1$ is connected and simply connected, it is isometric to the unit 4-sphere $S^4$.

Let $X : M^2 \to \mathbb{HP}^1 \simeq S^4$ be an immersion of an oriented surface. Let $\mathcal{F}^0_X \subset \mathrm{Sp}(2)$ be the pullback of the $S^3 \times S^3$ bundle $\pi_H : \mathrm{Sp}(2) \to \mathbb{HP}^1$ along $X$. Local sections

$\sigma : U \subset M \to \mathcal{F}^0_X$ will be called *zeroth order frames* along $X$. The metric $ds^2$ on $\mathbb{HP}^1$ pulls back to give a metric $X^*(ds^2) = X^*(4(\eta_1 \circ \bar{\eta}_1 + \eta_2 \circ \bar{\eta}_2))$ on $M$, as well as a compatible complex structure. We can reduce the bundle $\mathcal{F}^0_X$ to the $S^1 \times S^1$ bundle $\mathcal{F}^1_X \subset \mathrm{Sp}(2) \to M$ of *first-order adapted frames* which are defined by the conditions that for any section $F : U \subset M \to \mathcal{F}^1_X$ we have that $F^*(\eta_2) = 0$ and $F^*(\eta_1)$ pulls back to be a form of type $(1,0)$ on $M$.

Since $\eta_2 = 0$ holds on $\mathcal{F}^1_X$, the structure equation

$$d\eta_2 = -\phi_2 \wedge \eta_1 + \phi_1 \wedge \bar{\eta}_1 + \mathrm{i}(\rho_1 + \rho_2) \wedge \eta_2 \tag{4.79}$$

implies

$$0 = -\phi_2 \wedge \eta_1 + \phi_1 \wedge \bar{\eta}_1.$$

By Cartan's Lemma, there exist functions $A, B_1, B_2$ on $\mathcal{F}^1_X$ so that

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} A & \overline{B}_1 \\ B_2 & -A \end{pmatrix} \begin{pmatrix} \eta_1 \\ \bar{\eta}_1 \end{pmatrix} \tag{4.80}$$

As Bryant explains in [Bry82], it is easy to check that the immersion $X : M \to \mathbb{HP}^1$ is minimal if and only $A$ vanishes identically. In this case we have

$$\phi_1 = \overline{B}_1 \bar{\eta}_1, \qquad \phi_2 = B_2 \eta_1,$$

and a minimal immersion is called *superminimal with positive spin* if $\phi_1 = 0$ and *superminimal with negative spin* if $\phi_2 = 0$.

Returning now to the special case of $u_1 = \frac{1}{3}, t_1 = 0$, recall that the structure equations

$$0 = du_1 = 6t_1u_1\omega_1 + r_2\omega_2 + r_3\omega_3 \tag{4.81}$$

$$0 = dt_1 = \left(\tfrac{1}{3} + 3t_1^2 - 3u_1^2\right)\omega_1 + r_3\omega_2 - r_2\omega_3 \tag{4.82}$$

imply $r_2 = r_3 = 0$. In this case, the $\mathfrak{su}(4)$-valued Maurer-Cartan form on $B_3$ looks like

$$\omega = \begin{bmatrix} 0 & -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_1 & 0 & -\omega_3 & \omega_2 \\ \omega_2 & \omega_3 & \mathrm{i}s\omega_2 & \frac{1}{3}\omega_1 - t_3\omega_2 + (t_2 - \mathrm{i}s)\omega_3 \\ \omega_3 & -\omega_2 & -\frac{1}{3}\omega_1 + t_3\omega_2 - (t_2 + \mathrm{i}s)\omega_3 & -\mathrm{i}s\omega_2 \end{bmatrix} \tag{4.83}$$

Now, consider the 2-plane $E_2 = \mathbf{e}_0 \wedge \mathbf{e}_1 - \mathbf{e}_2 \wedge \mathbf{e}_3$. The structure equations $d\mathbf{e}_a = \mathbf{e}_b \omega_a^b$ show

$$\begin{aligned} d(E_2) &= (\mathbf{e}_2\omega_2 + \mathbf{e}_3\omega_3) \wedge \mathbf{e}_1 + \mathbf{e}_0 \wedge (\mathbf{e}_2\omega_3 - \mathbf{e}_3\omega_2) \qquad (4.84) \\ &\quad - (-\mathbf{e}_0\omega_2 - \mathbf{e}_1\omega_3 + \mathrm{i}s\mathbf{e}_2\omega_2) \wedge \mathbf{e}_3 - \mathbf{e}_2 \wedge (-\mathbf{e}_0\omega_3 + \mathbf{e}_1\omega_2 - \mathrm{i}s\mathbf{e}_3\omega_3) \\ &= 0. \qquad (4.85) \end{aligned}$$

So the plane $E_2$ is fixed. It is easy to check that the Lie subalgebra of $\mathfrak{su}(4)$ that fixes the plane $E_2$ is the 10-dimensional subalgebra

$$\mathfrak{g}_{E_2} = \left\{ \begin{bmatrix} \mathrm{i}a & x_1 & -\overline{x}_2 & \overline{x}_3 \\ -\overline{x}_1 & \mathrm{i}a & -\overline{x}_3 & x_2 \\ x_2 & x_3 & \mathrm{i}b & -\overline{x}_4 \\ \overline{x}_3 & -\overline{x}_2 & x_4 & -\mathrm{i}b \end{bmatrix} \mid a, b \in \mathbb{R}, x_1, x_2, x_3, x_4 \in \mathbb{C} \right\}.$$

For any such matrix, we can construct $2 \times 2$ quaternion matrix

$$A = \begin{bmatrix} \mathrm{i}a - \mathrm{j}x_1 & -\bar{x}_2 + \mathrm{j}x_3 \\ x_2 + \mathrm{j}x_3 & \mathrm{i}b + \mathrm{j}x_4 \end{bmatrix}$$

which clearly satisfies ${}^{\mathrm{t}}\bar{A} + A = 0$, and so is an element of $\mathfrak{sp}(2)$.

Thus in the special case that $u_1 = \frac{1}{3}, t_1 = 0$, the Maurer-Cartan form $\omega$ is actually $\mathfrak{sp}(2) \subset \mathfrak{su}(4)$-valued and so any moving frame $F = [\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] : U \subset L \to B_3$ takes values in an $\mathrm{Sp}(2) \subset \mathrm{SU}(4)$. On $B_3$, we have the map $\mathbf{e}_0 : B_3 \to \mathbb{C}^4 \simeq \mathbb{H}^2$, along with the two projections $[\mathbf{e}_0]_{\mathbf{C}} \in \mathbb{CP}^3$ and $[\mathbf{e}_0]_{\mathbf{H}} = T \circ [\mathbf{e}_0]_{\mathbf{C}} \in \mathbb{HP}^1 \simeq S^4$.

$$\begin{tikzcd} & B_3 \subset \mathrm{Sp}(2) \arrow[d, "{[\mathrm{e}_0]_C}"] \arrow[dr, "{[\mathrm{e}_0]_H}"] & \\ L \arrow[ur, "F"] \arrow[r, "f"] & \mathbb{CP}^3 \arrow[r, "T"] & \mathbb{HP}^1 \end{tikzcd}$$

The image of $[\mathbf{e}_0]_H : \mathrm{Sp}(2) \to \mathbb{HP}^1$ is a surface, and we claim that it is actually *superminimal*. The $\mathfrak{sp}(2)$-valued Maurer-Cartan form (4.83) has the following form when written as a $2 \times 2$ quaternion matrix:

$$\hat{\omega} = \begin{bmatrix} -\mathrm{j}\omega_1 & -\omega_2 + \mathrm{j}\omega_3 \\ \omega_2 + \mathrm{j}\omega_3 & \mathrm{i}s\omega_2 + \mathrm{j}\tau \end{bmatrix} \tag{4.86}$$

where $\tau = -\frac{1}{3}\omega_1 + t_3\omega_2 - (t_2 + \mathrm{i}s)\omega_3$. Comparing this expression for that of the Maurer-Cartan form for surfaces in $\mathbb{HP}^1$ given in (4.76), we see that the first-order adaptation condition, namely $\eta_2 = 0$ is equivalent to $\phi_1^2 = \overline{\phi_2^1}$ being purely complex. We claim that we can achieve this by conjugation by an element of $S^3 \times S^3$. Consider the $S^3 \times S^3$-valued matrix $Q = \begin{bmatrix} q & 0 \\ 0 & q \end{bmatrix}$ where $q = \frac{1-\mathrm{k}}{\sqrt{2}} = \frac{1-\mathrm{ij}}{\sqrt{2}}$. Since $q$ satisfies $q\mathrm{i}\bar{q} = -\mathrm{j}$ and $q\mathrm{j}\bar{q} = \mathrm{i}$, and since the $\omega_i$ are real, we have

$$Q \begin{bmatrix} -\mathrm{j}\omega_1 & -\omega_2 + \mathrm{j}\omega_3 \\ \omega_2 + \mathrm{j}\omega_3 & \mathrm{i}s\omega_2 + \mathrm{j}\tau \end{bmatrix} \overline{Q} = \begin{bmatrix} -\mathrm{i}\omega_1 & -\omega_1 + \mathrm{i}\omega_3 \\ \omega_2 + \mathrm{i}\omega_3 & \mathrm{i}(-\frac{1}{3}\omega_1 + t_3\omega_2 - t_2\omega_3) - \mathrm{j}s(\omega_2 + \mathrm{i}\omega_3) \end{bmatrix}. \tag{4.87}$$

This is the Maurer-Cartan form of a first-order adapted frame for a surface in $\mathbb{HP}^1 \simeq S^4$. Moreover, a component-wise comparison of this matrix and that found in (4.76) gives

$$\begin{aligned} \rho_1 &= -\omega_1 \\ \rho_2 &= (-\tfrac{1}{3}\omega_1 + t_3\omega_2 - t_2\omega_3) \\ \phi_1 &= 0 \\ \phi_2 &= -s(\omega_2 + \mathrm{i}\omega_3) \\ \eta_1 &= \omega_2 + \mathrm{i}\omega_3 \\ \eta_2 &= 0. \end{aligned}$$

In particular, we have

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -s & 0 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \bar{\eta}_1 \end{pmatrix} \tag{4.88}$$

so that in the notation of (4.80), we have $A = B_1 = 0$ and $B_2 = -s$. In particular, we have shown that the resulting surface $B_2 \subset \mathrm{Sp}(2) \to \mathbb{HP}^1$ is superminimal of positive spin.

We expect to be able to show that the reverse construction is possible, i.e., given a superminimal surface $M \subset \mathbb{HP}^1$, we can recover a minimal Lagrangian $L \subset \mathbb{HP}^1$. This is part of our future work.

## 4.5 $\mathbb{Z}_2$-stabilizer

**Theorem 4.6.** Minimal Lagrangian submanifolds $L \subset \mathbb{CP}^3$ whose fundamental cubic has an $Z_2$ symmetry at every point exist locally, and depend on six constants. These examples are foliated in codimension 1 by $\mathbb{RP}^3$-sections, and these sections are quadric surfaces in $\mathbb{RP}^3$.

*Proof.* Assume $L \subset \mathbb{CP}^3$ is a minimal Lagrangian submanifold whose fundamental cubic has a $\mathbb{Z}_2$-stabilizer at every point. We assume, without loss of generality, that $C$ is nowhere vanishing on $L$. By Proposition 4.1, there are positive real-analytic functions $r, s : L \to \mathbb{R}^+$ with $r \neq s$ such that the equation

$$C = r\omega_1(2\omega_1^2 - 3\omega_2^2 - 3\omega_3^2) + 6s\omega_1\omega_2\omega_3 \tag{4.89}$$

defines a $\mathbb{Z}_2$-subbundle $B_4 \subset B_L$ of the $L$-adapted frame bundle $B_L \subset \mathrm{SU}(4) \to L$. On this subbundle, we have

$$\begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{bmatrix} = \begin{bmatrix} 2r\omega_1 & s\omega_3 - r\omega_2 & s\omega_2 - r\omega_3 \\ s\omega_3 - r\omega_2 & -r\omega_1 & s\omega_1 \\ s\omega_2 - r\omega_3 & s\omega_1 & -r\omega_1 \end{bmatrix}. \tag{4.90}$$

Furthermore, as $B_4$ is an $\mathbb{Z}_2$-bundle, the relations $\alpha_{12} \equiv \alpha_{31} \equiv \alpha_{23} \equiv 0 \mod \{\omega_1, \omega_2, \omega_3\}$

hold on $B_4$, so the $\alpha_{ij}$ can be written

$$\begin{aligned}\alpha_{12} &= t_{31}\omega_1 + t_{32}\omega_2 + t_{33}\omega_3 \\ \alpha_{31} &= t_{21}\omega_1 + t_{22}\omega_2 + t_{23}\omega_3 \\ \alpha_{23} &= t_{11}\omega_1 + t_{12}\omega_2 + t_{13}\omega_3\end{aligned} \tag{4.91}$$

for some functions $t_{ij}$ on $B_4$. Additionally, for $i = 1, 2, 3$ there exist functions $r_i$ and $s_i$ on $B_4$ so that

$$dr = r_i\omega_i, \qquad ds = s_i\omega_i. \tag{4.92}$$

Substituting (4.90), (4.91), and (4.92) into the structure equations

$$d\beta_{ij} = -\alpha_{ik} \wedge \beta_{kj} - \beta_{ik} \wedge \alpha_{kj} \tag{4.93}$$

together with the identities $d\omega_i = -\alpha_{ij} \wedge \omega_j$ give the following relations

$$\begin{aligned}\alpha_{12} &= -st_3\omega_1 + rt_1\omega_2 - st_1\omega_3 \\ \alpha_{31} &= -st_2\omega_1 + st_1\omega_2 - rt_1\omega_3 \\ \alpha_{23} &= \tfrac{1}{2}(st_2 - rt_3)\omega_2 + \tfrac{1}{2}(st_3 - rt_2)\omega_3 \\ d\omega_1 &= st_3\omega_1 \wedge \omega_2 + st_2\omega_3 \wedge \omega_1 \\ d\omega_2 &= -rt_1\omega_1 \wedge \omega_2 - st_1\omega_3 \wedge \omega_1 + \tfrac{1}{2}(rt_3 - st_2)\omega_2 \wedge \omega_3 \\ d\omega_3 &= st_1\omega_1 \wedge \omega_2 + rt_1\omega_3 \wedge \omega_1 + \tfrac{1}{2}(rt_2 - st_3)\omega_2 \wedge \omega_3 \\ dr &= 2t_1(2r^2 + s^2)\omega_1 + s(2rt_3 + st_2)\omega_2 - s(2rt_2 + st_3)\omega_3 \\ ds &= s(6rt_1\omega_1 + (rt_2 + 2st_3)\omega_2 - (rt_3 + 2st_2)\omega_3)\end{aligned} \tag{4.94}$$

where, to avoid denominators and simplify notation, we have introduced the new variables

$$t_1 = -t_{23}/r \tag{4.95}$$

$$t_2 = -t_{21}/s \tag{4.96}$$

$$t_3 = -t_{31}/s. \tag{4.97}$$

There exist functions $u_{ij}$ on $B_4$ so that

$$\begin{aligned} dt_1 &= u_{11}\omega_1 + u_{12}\omega_2 + u_{13}\omega_3 \\ dt_2 &= u_{21}\omega_1 + u_{22}\omega_2 + u_{23}\omega_3 \\ dt_3 &= u_{31}\omega_1 + u_{32}\omega_2 + u_{33}\omega_3. \end{aligned} \tag{4.98}$$

Substituting (4.94) and (4.98) into the identities

$$0 = d^2(\omega_i) = d^2(r) = d^2(s)$$

gives polynomial relations among the $u_{ij}, r, s,$ and $t_i$. These can be solved to give the identities

$$\begin{aligned} dt_1 &= (su_1 - 3r - 3rt_1^2)\omega_1 \\ dt_2 &= -3t_1(rt_2 - st_3)\omega_1 + \left(u_2 - \tfrac{3}{2}rt_2^2\right)\omega_2 + \left(u_3 + \tfrac{3}{2}st_2^2\right)\omega_3 \\ dt_3 &= -3t_1(rt_3 - st_2)\omega_1 - \left(u_3 + \tfrac{3}{2}st_3^2\right)\omega_2 - \left(u_2 - \tfrac{3}{2}rt_3^2\right)\omega_3 \end{aligned} \tag{4.99}$$

where we have introduced new functions $u_1, u_2, u_3$ to simplify the form of these expressions. Upon substituting (4.94) and (4.99) into the identities

$$d\alpha_{ij} = -\alpha_{ik} \wedge \alpha_{kj} + \beta_{ik} \wedge \beta_{kj} + \omega_i \wedge \omega_j$$

we find

$$\begin{aligned} u_2 &= \tfrac{1}{2}\left(-2rt_1^2 + rt_2^2 - 3st_2t_3 + rt_3^2\right) - su_1 - r \\ u_3 &= \tfrac{1}{2}\left(2st_1^2 - st_2^2 + 3rt_2t_3 - st_3^2\right) + ru_1 + s - \tfrac{1}{s}. \end{aligned} \tag{4.100}$$

Expanding the relations $d^2t_i = 0$ for $i = 1, 2, 3$ gives the following expression for the exterior derivative of $u_1$ in terms of known quantities

$$\begin{aligned} du_1 = &- \frac{-2t_1[s^2(-t_1^2 + 2t_2^2 + 2t_3^2 - 1) + 2 + 3rsu_1]}{s}\omega_1 \\ &- (u_1(rt_2 + st_3) - 3(t_1^2 + 1)(st_2 + rt_3)\omega_2 \\ &+ (u_1(st_2 + rt_3) - 3(t_1^2 + 1)(rt_2 + st_3)\omega_3. \end{aligned} \tag{4.101}$$

It is also easy to check that $d(d(u_1)) = 0$ is an identity.

We therefore have arrived at the following: If $L \subset \mathbb{CP}^3$ is a minimal Lagrangian whose cubic form $C$ has a $\mathbb{Z}_2$-stabilizer, we can construct a $\mathbb{Z}_2$-subbundle of SU(4) over $L$ on which we have a coframe $\omega_1, \omega_2, \omega_3$ together with six functions $r, s, t_1, t_2, t_3, u_1$ satisfying the above structure equations. Since $d^2 = 0$ is an identity on all of these quantities, Corollary 2.13 tells us that for any six constants $\bar{r}, \bar{s}, \bar{t}_1, \bar{t}_2, \bar{t}_3, \bar{u}_1$ there is an an open neighborhood of $0 \in \mathbb{R}^3$, endowed with three real-analytic linearly independent 1-forms $\omega_i$ and real-analytic functions $r, s, t_1, t_2, t_3, u_1$ satisfying (4.94), (4.99), (4.100), (4.101) as well as

$$r(0) = \bar{r}, \quad s(0) = \bar{s}, \quad t_1(0) = \bar{t}_1, \quad t_2(0) = \bar{t}_2, \quad t_3(0) = \bar{t}_3, \quad u_1(0) = \bar{u}_1,$$

and that these are unique up to real-analytic diffeomorphisms that fix 0.

Now, note that the equation

$$d\omega_1 = st_3\omega_1 \wedge \omega_2 + st_2\omega_3 \wedge \omega_1 \tag{4.102}$$

implies that the differential system $\{\omega_1 = 0\}$ is integrable, so it gives rise to a foliation of two-dimensional leaves that we shall call $\Gamma_1$. Modulo $\omega_1$, the Maurer-Cartan form reads

$$\omega \equiv$$

$$\begin{bmatrix} 0 & 0 & -\omega_2 & -\omega_3 \\ 0 & 0 & (t_1 - \mathrm{i})(r\omega_2 - s\omega_3) & (t_1 - \mathrm{i})(r\omega_3 - s\omega_2) \\ \omega_2 & -(t_1 + \mathrm{i})(r\omega_2 - s\omega_3) & 0 & \gamma \\ \omega_3 & -(t_1 + \mathrm{i})(r\omega_3 - s\omega_2) & -\gamma & 0 \end{bmatrix} \mod \omega_1 \tag{4.103}$$

where $\gamma = \frac{1}{2}(st_2 - rt_3)\omega_2 + \frac{1}{2}(st_3 - rt_2)\omega_3$. The structure equations (4.94) imply

$$dt_1 \equiv 0 \mod \omega_1\,, \tag{4.104}$$

so $t_1$ is constant on the leaves of $\Gamma_1$. Next, recall we have the $\mathbb{C}^4$-valued functions $\mathbf{e}_a : B_4 \to \mathbb{C}^4$ that satisfy the structure equations

$$d\mathbf{e}_a = \mathbf{e}_b \omega_a^b$$

where $\omega_b^a$ are the components of the Maurer-Cartan form of SU(4) pulled back to $B_4$. Consider the 4-plane $\hat{E}_4 : B_4 \to \Lambda^4_{\mathbb{R}}\mathbb{C}^4$ in the Grassmannian of real 4-planes in $\mathbb{C}^4$ given by

$$E_4 = \mathbf{e}_0 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge (t_1 - \mathrm{i})\mathbf{e}_1.$$

Examining the structure equations arising from the reduced Maurer-Cartan form (4.103), along with (4.104), we see that

$$\begin{aligned}
d(E_4) &\equiv d(\mathbf{e}_0 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge (t_1 - \mathrm{i})\mathbf{e}_1) \\
&\equiv (\mathbf{e}_2\omega_2 + \mathbf{e}_3\omega_3) \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge (t_1 - \mathrm{i})\mathbf{e}_1 \\
&+ \mathbf{e}_0 \wedge (-\mathbf{e}_0\omega_2 + (t_1 - \mathrm{i})\mathbf{e}_1(r\omega_2 - s\omega_3) - \mathbf{e}_3\gamma) \wedge \mathbf{e}_3 \wedge (t_1 - \mathrm{i})\mathbf{e}_1 \\
&+ \mathbf{e}_0 \wedge \mathbf{e}_2 \wedge (-\mathbf{e}_0\omega_3 + (t_1 - \mathrm{i})(r\omega_3 - s\omega_2)\mathbf{e}_1 + \mathbf{e}_2\gamma) \wedge (t_1 - \mathrm{i})\mathbf{e}_1 \\
&+ \mathbf{e}_0 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge (-(t_1^2 + 1)(r\omega_2 - s\omega_3)\mathbf{e}_2 - (t_1^2 + 1)(r\omega_3 - s\omega_2)\mathbf{e}_3) \\
&\equiv 0 \mod \omega_1.
\end{aligned}$$

Thus, the 4-plane $\hat{E}_4$ is constant along each leaf of $\Gamma_1$. Furthermore, since $E_4 \perp \mathrm{i}E_4$, we see that this 4-plane is actually a *totally real*[1] 4-plane in $\mathbb{C}^4$. Thus, its projectivization, which we shall denote $[\hat{E}_4]$, is an $\mathbb{RP}^3$, thus these examples are foliated in codimension 1 by $\mathbb{RP}^3$s.

Motivated by this, we see we can modify the vectors $\mathbf{e}_a$ adapted to this 4-plane. Let $\mathbf{f}_a : B_4 \to \mathbb{C}^4$ be defined by

$$\mathbf{f}_1 = \frac{t_1 - \mathrm{i}}{\sqrt{t_1^2 + 1}}\mathbf{e}_1 \tag{4.105}$$

$$\mathbf{f}_0 = \mathbf{e}_0, \quad \mathbf{f}_2 = \mathbf{e}_2, \quad \mathbf{f}_3 = \mathbf{e}_3. \tag{4.106}$$

[1] A real $k$-plane $E$ in $\mathbb{C}^4$ is called totally real if does not contain any complex subspaces. That is, if $v \in E$ then $\mathrm{i}v \notin E$.

then, modulo $\omega_1$, the $\mathbf{f}_a$ satisfy the relations

$$d\begin{bmatrix}\mathbf{f}_0 & \mathbf{f}_1 & \mathbf{f}_2 & \mathbf{f}_3\end{bmatrix}$$

$$\equiv \begin{bmatrix}\mathbf{f}_0 & \mathbf{f}_1 & \mathbf{f}_2 & \mathbf{f}_3\end{bmatrix} \quad \begin{bmatrix} 0 & 0 & -\omega_2 & -\omega_3 \\ 0 & 0 & \sqrt{t_1^2+1}(r\omega_2 - s\omega_3) & \sqrt{t_1^2+1}(r\omega_3 - s\omega_2) \\ \omega_2 & -\sqrt{t_1^2+1}(r\omega_2 - s\omega_3) & 0 & \gamma \\ \omega_3 & -\sqrt{t_1^2+1}(r\omega_3 - s\omega_2) & -\gamma & 0. \end{bmatrix} \tag{4.107}$$

where, as before, $\gamma = \frac{1}{2}(st_2 - rt_3)\omega_2 + \frac{1}{2}(st_3 - rt_2)\omega_3$. Note that the projection into $L$ given by $[\mathbf{f}_0] = [\mathbf{e}_0] \in \mathbb{CP}^3$ still makes sense and the reduced Maurer-Cartan matrix in the right hand side in (4.107) is real and takes values in $\mathfrak{so}(4)$. The 4-plane $E_4 = \mathbf{f}_0 \wedge \mathbf{f}_1 \wedge \mathbf{f}_2 \wedge \mathbf{f}_3$ is also real and differs from $\hat{E}_4$ only by a real scalar multiple.

We claim that, on each leaf of $\Gamma_1$, $\mathbf{f}_0$ actually lies in a quadric hypersurface in the 4-plane $E_4$. To see this, we look at the map $\mathbf{f}_0^2 : B_5 \to \mathrm{Sym}^2(E_4)$, which is 10-dimensional and has basis $\mathbf{f}_a\mathbf{f}_b = \mathbf{f}_b\mathbf{f}_a$. Using the structure equations (4.107) on the components of the successive derivatives of $\mathbf{f}_0^2$, modulo $\omega_1$, we find that these derivatives span a 9-dimensional subspace generated by the elements

$$\begin{aligned}
\mathbf{x}_1 &= \mathbf{f}_0^2 \\
\mathbf{x}_2 &= \mathbf{f}_0\mathbf{f}_2 \\
\mathbf{x}_3 &= \mathbf{f}_0\mathbf{f}_3 \\
\mathbf{x}_4 &= \mathbf{f}_2^2 + r\mathbf{f}_0\mathbf{f}_1\sqrt{t_1^2+1} \\
\mathbf{x}_5 &= \mathbf{f}_2\mathbf{f}_3 - s\mathbf{f}_0\mathbf{f}_1\sqrt{t_1^2+1} \\
\mathbf{x}_6 &= \mathbf{f}_3^2 + r\mathbf{f}_0\mathbf{f}_1\sqrt{t_1^2+1} \\
\mathbf{x}_7 &= st_3\mathbf{f}_0\mathbf{f}_1 + \mathbf{f}_1\mathbf{f}_2 \\
\mathbf{x}_8 &= -st_2\mathbf{f}_0\mathbf{f}_1 + \mathbf{f}_1\mathbf{f}_3 \\
\mathbf{x}_9 &= f_1^2 + \frac{(rt_1^2 - su_1 + r)}{\sqrt{t_1^2+1}}\mathbf{f}_0\mathbf{f}_1.
\end{aligned}$$

The derivatives of each of the $\mathbf{x}_\rho$ for $\rho = 1 \dots 9$ lie in the span of the $\mathbf{x}_\rho$. This tells us that $\mathbf{f}_0^2$ lies in a hyperplane in $\mathrm{Sym}^2(E_4)$, and so it must be annihilated by some quadratic form $Q \in \mathrm{Sym}^2(E_4^*)$. We see then that the image of $\mathbf{f}_0 = \mathbf{e}_0$ in the real 4-plane $E_4 = \mathbf{f}_0 \wedge \mathbf{f}_1 \wedge \mathbf{f}_2 \wedge \mathbf{f}_3$ is a quadric hypersurface and so the projectivization $[\mathbf{e}_0] \in L$ lies in the $\mathbb{RP}^3$ given by $[E_4] = [\mathbf{f}_0 \wedge \mathbf{f}_1 \wedge \mathbf{f}_2 \wedge \mathbf{f}_3]$ and intersects $[E_4]$ in a quadric surface. □

These are the projective analogues of the examples of the Lawlor-Harvey examples of special Lagrangians in $\mathbb{C}^3$. In these examples, one starts with a compact 2-dimensional ellipsoid, or more generally, quadratic hypersurface in a Lagrangian (but not special Lagrangian) 3-plane of $\mathbb{C}^3$. The special Lagrangian thickening of this ellipsoid has the property that it is foliated in codimension 1 by 3-plane sections, and these sections are ellipsoids. In [Bry00], Bryant showed that any special Lagrangian 3-fold $L \subset \mathbb{C}^3$ whose cubic has an $S_3$ symmetry are exactly the Lawlor-Harvey-Joyce examples.

## 4.6 $\mathbb{Z}_3$-stabilizer

**Theorem 4.7.** Minimal Lagrangian submanifolds $L \subset \mathbb{CP}^3$ whose fundamental cubic form has a $\mathbb{Z}_3$ symmetry at every point exist locally, and depend on 2 functions of 1 variable.

*Proof.* Assume $L \subset \mathbb{CP}^3$ is a minimal Lagrangian submanifold whose fundamental cubic has a $\mathbb{Z}_3$-stabilizer at every point. We assume, without loss of generality, that $C$ is nowhere vanishing on $L$. By Proposition 4.1, there exist positive real-analytic functions $r, s : L \to \mathbb{R}^+$ with $r \neq s$ with $s \neq r\sqrt{2}$, such that the equation

$$C = r\omega_1(2\omega_1^2 - 3\omega_2^2 - 3\omega_3^2) + s(\omega_2^3 - 3\omega_2\omega_3^2) \tag{4.108}$$

defines a $\mathbb{Z}_3$-subbundle $B_5 \subset B_L$ of the $L$-adapted frame bundle $B_L \subset \mathrm{SU}(4) \to L$. On this subbundle, we have

$$\begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{bmatrix} = \begin{bmatrix} 2r\omega_1 & -r\omega_2 & -r\omega_3 \\ -r\omega_2 & -r\omega_1 + s\omega_2 & -s\omega_3 \\ -r\omega_3 & -s\omega_3 & -r\omega_1 - s\omega_2 \end{bmatrix}. \tag{4.109}$$

Since $B_5$ is an $\mathbb{Z}_3$-bundle, the relations $\alpha_{12} \equiv \alpha_{31} \equiv \alpha_{23} \equiv 0 \mod \{\omega_1, \omega_2, \omega_3\}$ hold on $B_5$, so the $\alpha_{ij}$ can be written

$$\begin{aligned} \alpha_{12} &= t_{31}\omega_1 + t_{32}\omega_2 + t_{33}\omega_3 \\ \alpha_{31} &= t_{21}\omega_1 + t_{22}\omega_2 + t_{23}\omega_3 \\ \alpha_{23} &= t_{11}\omega_1 + t_{12}\omega_2 + t_{13}\omega_3 \end{aligned} \tag{4.110}$$

for some functions $t_{ij}$ on $B_5$., and there also exist, for $i = 1, 2, 3$, functions $r_i$ and $s_i$ on $B_5$ so that

$$dr = r_i\omega_i, \qquad ds = s_i\omega_i. \tag{4.111}$$

Substituting (4.109), (4.110), and (4.111) into the structure equations

$$d\beta_{ij} = -\alpha_{ik} \wedge \beta_{kj} - \beta_{ik} \wedge \alpha_{kj} \tag{4.112}$$

together with the identities $d\omega_i = -\alpha_{ij} \wedge \omega_j$ give

$$\begin{aligned} \alpha_{12} &= -t_1\omega_2 \\ \alpha_{31} &= t_1\omega_3 \\ \alpha_{23} &= -t_2\omega_2 - t_3\omega_3 \end{aligned}$$

so that the following relations hold

$$\begin{aligned} d\omega_1 &= 0 \\ d\omega_2 &= t_1\omega_1 \wedge \omega_2 + t_2\omega_2 \wedge \omega_3 \\ d\omega_3 &= -t_1\omega_3 \wedge \omega_1 + t_3\omega_2 \wedge \omega_3 \\ dr &= -4rt_1\omega_1 \\ ds &= -s(t_1\omega_1 + 3t_3\omega_2 - 3t_2\omega_3). \end{aligned} \tag{4.113}$$

As usual, we have introduced the following new variables to simplify notation:

$$t_1 = -t_{32} \tag{4.114}$$

$$t_2 = -t_{12} \tag{4.115}$$

$$t_3 = -t_{13}. \tag{4.116}$$

There exist functions $u_{ij}$ on $B_5$ so that

$$\begin{aligned} dt_1 &= u_{11}\omega_1 + u_{12}\omega_2 + u_{13}\omega_3 \\ dt_2 &= u_{21}\omega_1 + u_{22}\omega_2 + u_{23}\omega_3 \\ dt_3 &= u_{31}\omega_1 + u_{32}\omega_2 + u_{33}\omega_3. \end{aligned} \tag{4.117}$$

Substituting (4.113) and (4.117) into the identities

$$0 = d^2(\omega_i) = d^2(r) = d^2(s)$$

and solving the resulting polynomial equations in $u_{ij}, r, s$, and $t_i$ yields

$$\begin{aligned} dt_1 &= (3r^2 - t_1^2 - 1)\omega_1 \\ dt_2 &= -t_1 t_2 \omega_1 + u_2\omega_2 + (u_3 - v)\omega_3 \\ dt_3 &= -t_1 t_3 \omega_1 + (u_3 + v)\omega_2 - u_2\omega_3. \end{aligned} \tag{4.118}$$

where we have renamed the remaining $u_{ij}$ as $u_2, u_3$, and $v$ is the quantity

$$v = s^2 - \tfrac{1}{2}(r^2 + t_1^2 + t_2^2 + t_3^2 + 1). \tag{4.119}$$

At this stage we have the following: A $\mathbb{Z}_3$ bundle $B_5 \to L$ endowed with a coframe $\omega_1, \omega_2, \omega_3$ and functions $s, r, t_1, t_2, t_3, u_2, u_3$ for which the structure equations (4.113) and (4.118) hold. The functions $s, r, t_1, t_2, t_3$ are the primary invariants and $u_2, u_3$ are the free derivatives of this augmented coframing. We do not show it here, but it is easy to check that the equations on the covariant derivatives of the $u_i$ arising from ensuring that $d^2 t_i = 0$ be an identity are solvable. Further, the tableau of free derivatives $A(u, v) : \mathbb{R}^5 \times \mathbb{R}^2 \to \mathrm{Hom}(\mathbb{R}^3, \mathbb{R}^5)$ is the subspace

$$A(u, v) = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & a & b \\ 0 & b & -a \end{bmatrix} \mid a, b \in \mathbb{R} \right\}. \tag{4.120}$$

This tableau is 2-dimensional and is involutive, with Cartan characters $(s_0, s_1, s_2, s_3) = (5, 2, 0, 0)$. Thus, solutions depend locally on 2 functions of 1 variable. □

# 5

# Conclusion

We have proven existence of families of minimal Lagrangian submanifolds of $\mathbb{CP}^3$ whose fundamental cubic is stabilized by any one of the following five possible subgroups of SO(3): SO(2), $A_4, S_3, \mathbb{Z}_2, \mathbb{Z}_3$. In certain cases, we were able to integrate the resulting structure equations and produce explicit examples. Future work includes fully integrating the remaining examples and also carrying out a similar analysis for the negatively curved complex hyperbolic 3-space.

# Bibliography


[Bry82] Robert L Bryant. Conformal and minimal immersions of compact surfaces into the 4-sphere. *Journal of Differential Geometry*, 17(3):455–473, 1982.

[Bry87] Robert L. Bryant. Minimal lagrangian submanifolds of kähler-einstein manifolds. In Chaohao Gu, Marcel Berger, and Robert L. Bryant, editors, *Differential Geometry and Differential Equations*, pages 1–12, Berlin, Heidelberg, 1987. Springer Berlin Heidelberg.

[Bry00] Robert L. Bryant. Second order families of special Lagrangian 3-folds. *arXiv Mathematics e-prints*, page math/0007128, Jul 2000.

[Bry14] Robert L. Bryant. Notes on exterior differential systems. *arXiv e-prints*, page arXiv:1405.3116, May 2014.

[Car35] Elie Cartan. La mthode du repere mobile, la theorie des groupes continus et les espaces gnraliss. by e. cartan. pp. 66. 12 fr. 1935. actualits scientifiques et industrielles, 194; exposs de gomtrie. *Exposs de Gomtrie*, (5), 1935.

[DL05] Yuxin Dong and Guozhen Lu. On second order minimal lagrangian sub manifolds in complex space forms. *Science in China Series A: Mathematics*, 48(11):1505–1516, 2005.

[Gri74] Phillip Griffiths. On cartans method of lie groups and moving frames as applied to uniqueness and existence questions in differential geometry. *Duke Mathematical Journal*, 41(4):775–814, 1974.

[Har90] F Reese Harvey. *Spinors and calibrations*. Elsevier, 1990.

[HL82] Reese Harvey and H Blaine Lawson. Calibrated geometries. *Acta Mathematica*, 148(1):47–157, 1982.

[Ion03] Marianty Ionel. Second order families of special lagrangian submanifolds in 4. *Journal of Differential Geometry*, 65(2):211–272, 2003.

[JMN16] Gary R Jensen, Emilio Musso, and Lorenzo Nicolodi. *Surfaces in classical geometries: a treatment by moving frames*. Springer, 2016.

[Joy01a] Dominic Joyce. 0.5 cm constructing special lagrangian m-folds in c^m by evolving quadrics. *Mathematische Annalen*, 320(4):757–797, 2001.

[Joy01b] Dominic Joyce. Special lagrangian 3-folds and integrable systems. *arXiv preprint math/0101249*, 2001.

[Joy02] Dominic Joyce. Special lagrangian m-folds in m with symmetries. *Duke Mathematical Journal*, 115(1):1–51, 2002.

[Law89] Gary Lawlor. The angle criterion. *Inventiones mathematicae*, 95(2):437–446, 1989.

[War13] Frank W Warner. *Foundations of differentiable manifolds and Lie groups*, volume 94. Springer Science & Business Media, 2013.

# Biography

Michael Bell earned a BS in Applied and Computational Mathematics in 2008 from Rochester Institute of Technology, and in 2012 earned a MS in Applied Mathematics from the same instutition. In September 2019, Michael obtained his PhD in mathematics from Duke University where he studied differential geometry under the supervision of Robert Bryant. During his time at Duke, Michael won the Captain L.P & Barbara Smith Award for Teaching Excellence in 2017 and 2018. He was also a graduate student member of the Simons Collaboration on Special Holonomy in Geometry, Analysis, and Physics.