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Causal Fermion Systems: A Primer for Lorentzian Geometers

Felix Finster

Fakultät für Mathematik, Universität Regensburg, D-93040 Regensburg, Germany

E-mail: finster@ur.de

Abstract. We give a brief introduction to causal fermion systems with a focus on the geometric structures in space-time.

In general relativity, space-time is described by a Lorentzian manifold. Although this description has been highly successful, it is generally believed that in order to reconcile general relativity with quantum theory, the mathematical structure of space-time should be modified on a microscopic scale (typically thought of as the Planck scale $\approx 10^{-35}$ meters). Apart from having far-reaching physical implications, such a modification is also of purely mathematical interest because going beyond the usual space-time continuum makes it possible to model non-smooth structures in space-time and of space-time itself which cannot be described (or at least are difficult to describe) on a Lorentzian manifold.

Causal fermion systems are a recent physical theory based on a novel mathematical model of space-time. The present paper is an introduction to causal fermion systems with a focus on the resulting geometric structures. It is addressed to readers who are familiar with Lorentzian geometry and the Dirac equation. More basic introductions in Minkowski space are given in the textbooks [5, Chapter 1] or [12]. Also, we do not explain the broader physical picture but refer instead to the non-technical survey [10]. Moreover, in [9] some of the concepts are introduced starting from Riemannian geometry. For the physical applications, we refer to [5, Chapters 3-5] and [4]. The relation to other approaches is explained in [5, §3.4.6].

The paper is organized as follows. We first explain how one gets from Lorentzian geometry to the setting of causal fermion systems (Section 1). Then a few structures of an abstract causal fermion system are reviewed (Section 2). In Section 3 we turn attention to the geometric structures and explain how Lorentzian geometry is recovered as a limiting case. In Section 4 it is explained why and specified how the geometric structures of a causal fermion system go beyond Lorentzian geometry. We focus on two aspects: First, space-time need not be smooth and the structures therein need not be regular (Section 4.1). Second, causal fermion systems allow for the description of collections of many different space-times which mutually interact with each other; this is what we mean by “quantum space-time” and “quantum geometry” (Section 4.2). Finally, in Section 5 a few more structures of a causal fermion system are worked out which clarify the connection to other approaches to non-regular space-time geometry (Lorentzian length spaces, causal sets, lattices) or might be useful for the geometric understanding and the future analysis of causal fermion systems.



1. From Lorentzian Geometry to Causal Fermion Systems

A basic concept behind causal fermion systems is to encode the geometry in a measure on linear operators on a Hilbert space. We now explain how to get into this setting starting from a usual four-dimensional space-time (for other dimensions see [13, Section 1.4], [14, Section 4] and [9]). More precisely, let (\mathcal{M}, g) be a smooth, globally hyperbolic, time-oriented Lorentzian spin manifold of dimension four. For the signature of the metric we use the convention $(+, -, -, -)$. We denote the corresponding spinor bundle by $S\mathcal{M}$. Its fibers $S_p\mathcal{M}$ are endowed with an inner product $\prec \cdot | \succ_p$ of signature $(2, 2)$. Clifford multiplication is described by a mapping γ which satisfies the anti-commutation relations,

$$\gamma : T_p\mathcal{M} \rightarrow L(S_p\mathcal{M}) \quad \text{with} \quad \gamma(u)\gamma(v) + \gamma(v)\gamma(u) = 2g(u, v) \mathbf{1}_{S_p(\mathcal{M})}. \quad (1)$$

We also write Clifford multiplication in components with the Dirac matrices γ^j . The metric connections on the tangent bundle and the spinor bundle are denoted by ∇ . The sections of the spinor bundle are also referred to as wave functions.

We denote the k -times continuously differentiable sections of the spinor bundle by $C^k(\mathcal{M}, S\mathcal{M})$. The Dirac operator \mathcal{D} is defined by

$$\mathcal{D} := i\gamma^j \nabla_j : C^\infty(\mathcal{M}, S\mathcal{M}) \rightarrow C^\infty(\mathcal{M}, S\mathcal{M}).$$

Given a real parameter $m \in \mathbb{R}$ (the “mass”), the Dirac equation reads

$$(\mathcal{D} - m)\psi = 0.$$

We mainly consider solutions in the class $C_{\text{sc}}^\infty(\mathcal{M}, S\mathcal{M})$ of smooth sections with spatially compact support. On such solutions, one has the scalar product

$$(\psi|\phi)_m = 2\pi \int_{\mathcal{N}} \prec \psi | \gamma(\nu) \phi \succ_p d\mu_{\mathcal{N}}(p),$$

where \mathcal{N} denotes any Cauchy surface and ν its future-directed normal (due to current conservation, the scalar product is in fact independent of the choice of \mathcal{N} ; for details see [14, Section 2]). Forming the completion gives the Hilbert space $(\mathcal{H}_m, (\cdot|\cdot)_m)$.

Next, we choose a closed subspace $\mathcal{H} \subset \mathcal{H}_m$ of the solution space of the Dirac equation. The induced scalar product on \mathcal{H} is denoted by $\langle \cdot | \cdot \rangle_{\mathcal{H}}$. There is the technical difficulty that the wave functions in \mathcal{H} are in general not continuous, making it impossible to evaluate them pointwise. For this reason, we need to introduce an *ultraviolet regularization* on the length scale ε , described mathematically by a linear

$$\text{regularization operator} \quad \mathfrak{R}_\varepsilon : \mathcal{H} \rightarrow C^0(\mathcal{M}, S\mathcal{M}).$$

In the simplest case, the regularization can be realized by a convolution on a Cauchy surface or in space-time (for details see [14, Section 4] or [5, Section §1.1.2]). For us, the regularization is not just a technical tool, but it realizes the concept already mentioned at the beginning of this paper that we want to change the geometric structures on the microscopic scale. With this in mind, we always consider the regularized quantities as those having mathematical and physical significance. Different choices of regularization operators realize different microscopic space-time structures.

Given \mathfrak{R}_ε , for any space-time point $p \in \mathcal{M}$ we consider the bilinear form

$$b_p : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}, \quad b_p(\psi, \phi) = -\prec (\mathfrak{R}_\varepsilon \psi)(p) | (\mathfrak{R}_\varepsilon \phi)(p) \succ_p.$$

This bilinear form is well-defined and bounded because \mathfrak{R}_ε maps to the continuous wave functions and because evaluation at p gives a linear operator of finite rank. Thus for any $\phi \in \mathcal{H}$, the anti-linear form $b_p(\cdot, \phi) : \mathcal{H} \rightarrow \mathbb{C}$ is continuous. By the Fréchet-Riesz theorem, there is a unique $\chi \in \mathcal{H}$ such that $b_p(\psi, \phi) = \langle \psi | \chi \rangle_{\mathcal{H}}$ for all $\psi \in \mathcal{H}$. The mapping $\phi \mapsto \chi$ is linear and bounded. We thus obtain a unique bounded linear operator $F^\varepsilon(p)$ on \mathcal{H} which is characterized by the relation

$$(\psi | F^\varepsilon(p) \phi) = -\prec(\mathfrak{R}_\varepsilon \psi)(p) | (\mathfrak{R}_\varepsilon \phi)(p) \succ_p \quad \text{for all } \psi, \phi \in \mathcal{H}. \quad (2)$$

Taking into account that the inner product on the Dirac spinors at p has signature $(2, 2)$, the local correlation operator $F^\varepsilon(p)$ is a symmetric operator on \mathcal{H} of rank at most four, which (counting multiplicities) has at most two positive and at most two negative eigenvalues. Varying the space-time point, we obtain a mapping

$$F^\varepsilon : \mathcal{M} \rightarrow \mathcal{F} \subset \mathcal{L}(\mathcal{H}),$$

where \mathcal{F} denotes all symmetric operators of rank at most four with at most two positive and at most two negative eigenvalues. Finally, we introduce the

$$\text{universal measure} \quad d\rho := (F^\varepsilon)_* d\mu_{\mathcal{M}}$$

as the push-forward of the volume measure on \mathcal{M} under the mapping F^ε (thus $\rho(\Omega) := \mu_{\mathcal{M}}((F^\varepsilon)^{-1}(\Omega))$).

In this way, we obtain a measure ρ on the set $\mathcal{F} \subset \mathcal{L}(\mathcal{H})$ of linear operators on a Hilbert space \mathcal{H} . The basic concept is to work exclusively with these objects, but to drop all other structures (like the Lorentzian metric g , the structure of the spinor bundle $S\mathcal{M}$, the manifold structure of \mathcal{M} , and even the structure of \mathcal{M} being a point set). This leads us to the structure of a causal fermion system of spin dimension two, as will be defined abstractly at the beginning of the next section.

Before turning attention to the abstract setting, we make a few comments on the underlying physical picture. The vectors in the subspace $\mathcal{H} \subset \mathcal{H}_m$ have the interpretation as those Dirac wave functions which are realized in the physical system under consideration. If we describe for example a system of one electron, then the wave function of the electron is contained in \mathcal{H} . Moreover, \mathcal{H} includes all the wave functions which form the so-called Dirac sea (for an explanation of this point see for example [3]). We refer to the vectors in \mathcal{H} as the *physical wave functions*. The name causal *fermion* system is motivated by the fact that Dirac particles are fermions. According to (2), the local correlation operator $F^\varepsilon(p)$ describes densities and correlations of the physical wave functions at the space-time point p . Working exclusively with the local correlation operators and the corresponding push-forward measure ρ means in particular that the geometric structures are encoded in and must be retrieved from the physical wave functions. Since the physical wave functions describe the distribution of matter in space-time, one can summarize this concept by saying that *matter encodes geometry*.

2. The Abstract Setting and a Few Inherent Structures

2.1. Basic Definition of a Causal Fermion System

We now give the abstract definition of a causal fermion system:

Definition 2.1 Given a separable complex Hilbert space \mathcal{H} with scalar product $\langle \cdot | \cdot \rangle_{\mathcal{H}}$ and a parameter $n \in \mathbb{N}$ (the “*spin dimension*”), we let $\mathcal{F} \subset \mathcal{L}(\mathcal{H})$ be the set of all self-adjoint operators on \mathcal{H} of finite rank, which (counting multiplicities) have at most n positive and at most n negative eigenvalues. On \mathcal{F} we are given a measure ρ (defined on a σ -algebra of subsets of \mathcal{F}), the so-called *universal measure*. We refer to $(\mathcal{H}, \mathcal{F}, \rho)$ as a *causal fermion system*.

This definition, which was first given in [8, Section 1.2], no longer involves the usual geometric structures (like a manifold \mathcal{M} , a Lorentzian metric g or a spinor bundle $S\mathcal{M}$). Therefore, it is far from obvious why and how a causal fermion system should provide a setting for geometry in space-time. All we can say right away is that a causal fermion system encodes a lot of information. For example, parts of this information can be retrieved by taking traces of products of operators in \mathcal{F} and integrating,

$$\int_{\mathcal{F}} \text{tr}(x) d\rho(x), \quad \int_{\mathcal{F}} d\rho(x) \int_{\mathcal{F}} d\rho(y) \text{tr}(x \cdot y), \quad \dots$$

The point is that by describing all this information with useful and convenient notions, one recovers a space-time as well as geometric structures therein. All these structures are *inherent* in the sense that they are no additional input, but they only give information which is already encoded in the causal fermion system an apposite name, thereby providing a better intuitive understanding of the causal fermion system.

Working with the inherent structures, causal fermion systems provide a general mathematical framework in which there are many analytic, geometric and topological structures. In particular, it becomes possible to generalize notions of differential geometry to the non-smooth setting. From the physical point of view, causal fermion systems are a proposal for quantum geometry and an approach to quantum gravity. Giving quantum mechanics, general relativity and quantum field theory as limiting cases, they are a candidate for a unified physical theory. For the physical applications, the key point is that the physical equations can be formulated within the setting of causal fermion systems in terms of a variational principle called the *causal action principle* (see [5, Section 1.1]). The causal action principle is the analytic core of the theory. For brevity we cannot introduce the causal action principle here, but instead refer the interested reader to the mathematical introduction in the recent paper [11] and to the references therein.

2.2. Space-Time and Causal Structure

We now introduce a few inherent structures of a causal fermion system $(\mathcal{H}, \mathcal{F}, \rho)$. *Space-time* M is defined as the support of the universal measure¹,

$$M := \text{supp } \rho \subset \mathcal{F}. \quad (3)$$

On M we consider the topology induced by \mathcal{F} (generated by the sup-norm on $L(\mathcal{H})$). Typically, M is a low-dimensional subset of \mathcal{F} which can have smooth but also discrete or non-regular components (see Figure 1).

The next definition is the reason for the name *causal* fermion system.

Definition 2.2 (*causal structure*) For any $x, y \in M$, the product xy is an operator of rank at most $2n$. We denote its non-trivial eigenvalues (counting algebraic multiplicities) by $\lambda_1^{xy}, \dots, \lambda_{2n}^{xy}$. The points x and y are called *spacelike* separated if all the λ_j^{xy} have the same absolute value. They are said to be *timelike* separated if the λ_j^{xy} are all real and do not all have the same absolute value. In all other cases (i.e. if the λ_j^{xy} are not all real and do not all have the same absolute value), the points x and y are said to be *lightlike* separated.

We remark that this definition is closely linked to the structure of the causal action (see [5, §1.1.2]).

¹ The *support* of a measure is defined as the complement of the largest open set of measure zero, i.e.

$$\text{supp } \rho := \mathcal{F} \setminus \bigcup \{ \Omega \subset \mathcal{F} \mid \Omega \text{ is open and } \rho(\Omega) = 0 \}.$$

It is by definition a closed set.

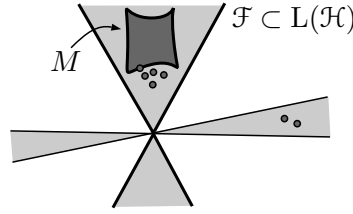


Figure 1. Space-time M of a causal fermion system

A causal fermion system also distinguishes a *direction of time*. To this end, we let π_x be the orthogonal projection in \mathcal{H} on the subspace $x(\mathcal{H}) \subset \mathcal{H}$ and introduce the functional

$$\mathcal{C} : M \times M \rightarrow \mathbb{R}, \quad \mathcal{C}(x, y) := i \operatorname{tr} (y x \pi_y \pi_x - x y \pi_x \pi_y) \quad (4)$$

(this functional was first stated in [9, Section 8.5], motivated by constructions in [7, Section 3.5]). Obviously, this functional is anti-symmetric in its two arguments. This makes it possible to introduce the notions

$$\begin{cases} y \text{ lies in the } \textit{future} \text{ of } x & \text{if } \mathcal{C}(x, y) > 0 \\ y \text{ lies in the } \textit{past} \text{ of } x & \text{if } \mathcal{C}(x, y) < 0. \end{cases} \quad (5)$$

We remark that in suitable limiting cases, the above notions of causality indeed agree with the causal structure of a Lorentzian space-time (for details see [5, §1.2.5] or [7, Sections 4 and 5]).

2.3. Spinors and Physical Wave Functions

For every $x \in \mathcal{F}$ we define the *spin space* $S_x M$ by $S_x M = x(\mathcal{H})$; it is a subspace of \mathcal{H} of dimension at most $2n$. On the spin space $S_x M$ the *spin scalar product* $\prec \cdot | \cdot \succ_x$ is defined by

$$\prec \cdot | \cdot \succ_x : S_x M \times S_x M \rightarrow \mathbb{C}, \quad \prec u | v \succ_x = -\langle u | x u \rangle_{\mathcal{H}}, \quad (6)$$

making the spin space $(S_x M, \prec \cdot | \cdot \succ_x)$ to an indefinite inner product of signature (p, q) with $p, q \leq n$.

A *wave function* ψ is defined as a function which to every $x \in M$ associates a vector of the corresponding spin space,

$$\psi : M \rightarrow \mathcal{H} \quad \text{with} \quad \psi(x) \in S_x M \quad \text{for all } x \in M. \quad (7)$$

To every vector $u \in \mathcal{H}$ we can associate a wave function ψ^u obtained by taking the orthogonal projections to the corresponding spin spaces,

$$\psi^u : M \rightarrow \mathcal{H}, \quad \psi^u(x) = \pi_x u \in S_x M$$

(with π_x as defined before (4)). We refer to ψ^u as the *physical wave function* of $u \in \mathcal{H}$ (we remark that, for the causal fermion systems obtained by the construction in Section 1, after suitable identifications made, the physical wave function ψ^u indeed coincides with the regularized Dirac wave function $\mathfrak{R}_\epsilon u$; for details see [5, §1.2.4]).

In this way, every vector of \mathcal{H} can be represented by a wave function in space-time. The resulting ensemble of all physical wave functions is described most conveniently by the *kernel of the fermionic projector* $P(x, y)$ defined by

$$P(x, y) = \pi_x y |_{S_y M} : S_y M \rightarrow S_x M. \quad (8)$$

Due to the factor x in the definition of the spin scalar product (6), the kernel of the fermionic projector is symmetric in the sense that $P(x, y)^* = P(y, x)$, as is immediately verified by the computation

$$\begin{aligned} \prec u | P(x, y) v \succ_x &= -\langle u | x P(x, y) v \rangle_{\mathcal{H}} = -\langle u | xy v \rangle_{\mathcal{H}} \\ &= -\langle \pi_y x u | y v \rangle_{\mathcal{H}} = \prec P(y, x) u | v \succ_y \quad (\text{for } u \in S_x M, v \in S_y M). \end{aligned}$$

3. Geometric Structures of a Causal Fermion System

In [7] the geometric structures of a causal fermion system were worked out abstractly, and it was shown that in a certain limiting case one recovers Lorentzian spin geometry. We now outline a few constructions and results from this paper.

3.1. Regular Causal Fermion Systems

According to Definition 2.1, the operators in $M \subset \mathcal{F}$ have at most n positive and at most n negative eigenvalues. In most situations of physical interest and most examples, the number of positive and negative eigenvalues equals n . This motivates the following definition:

Definition 3.1 A space-time point $x \in M$ is said to be *regular* if x has the maximal possible rank, i.e. $\dim x(\mathcal{H}) = 2n$. Otherwise, the space-time point is called *singular*. A causal fermion system is *regular* if all its space-time points are regular.

For a regular causal fermion system, the set

$$SM := \bigcup_{x \in M} S_x M$$

has the structure of a topological vector bundle with base M and fibers $S_x M$ (for details and all topological issues see [9]). The fibers are endowed with an inner product $\prec \cdot | \cdot \succ_p$ of signature (n, n) . A wave function (7) is a section of this bundle (for the notion of continuity of such wave functions and the space $C^0(M, SM)$ we refer to [5, §1.1.4]). As a specific feature, all the fibers are subspaces of the same Hilbert space $(\mathcal{H}, \langle \cdot | \cdot \rangle_{\mathcal{H}})$.

3.2. Clifford Subspaces

The structure of wave functions (7) taking values in the spin spaces is reminiscent of sections of a vector bundle. However, one important structure is missing: we have no Dirac matrices and no notion of Clifford multiplication. The following definition is a step towards introducing these additional structures.

Definition 3.2 (*Clifford subspace*) We denote the space of symmetric linear operators on $(S_x M, \prec \cdot | \cdot \succ_x)$ by $\text{Sym}(S_x M) \subset L(S_x M)$. A subspace $K \subset \text{Sym}(S_x M)$ is called a *Clifford subspace* of signature (r, s) at the point x (with $r, s \in \mathbb{N}_0$) if the following conditions hold:

- (i) For any $u, v \in K$, the anti-commutator $\{u, v\} \equiv uv + vu$ is a multiple of the identity on $S_x M$.
- (ii) The bilinear form $\langle \cdot, \cdot \rangle$ on K defined by

$$\frac{1}{2} \{u, v\} = \langle u, v \rangle \mathbb{1} \quad \text{for all } u, v \in K \quad (9)$$

is non-degenerate and has signature (r, s) .

In view of the anti-commutation relations (9), a Clifford subspace can be regarded as a generalization of the space spanned by the usual Dirac matrices. However, the above definition has two shortcomings: First, there are many different Clifford subspaces, so that there is no unique notion of Clifford multiplication. Second, we are missing the structure of tangent vectors. We now explain how to overcome these shortcomings.

3.3. Connection and Curvature

The kernel of the kernel of the fermionic projector (8) is a mapping from one spin space to another and thus induces relations between space-time points. The idea is to use these relations for the construction of a spin connection $D_{x,y}$, being a unitary mapping between the corresponding spin spaces,

$$D_{x,y} : S_y M \rightarrow S_x M$$

(we consistently use the notation that the subscript xy denotes an object at the point x , whereas the additional comma x,y denotes an operator which maps an object at y to an object at x). The simplest idea for the construction of the spin connection would be to form a polar decomposition $P(x,y) = S U$ with a symmetric operator $S : S_x M \rightarrow S_x M$ and a unitary operator $U : S_y M \rightarrow S_x M$ and to introduce the spin connection as the unitary part, $D_{x,y} = U$. However, this method is too naive, because we want the spin connection to be compatible with a corresponding metric connection $\nabla_{x,y}$ which should map Clifford subspaces at x and y (see Definition 3.2 above) isometrically to each other. Another complication is that, as explained after Definition 3.2, the Clifford subspaces at x and y are not unique. The method to bypass these problems is to work with several Clifford subspaces and to use so-called splice maps, as we now briefly explain.

First, it is useful to restrict the freedom in choosing the Clifford subspaces with the following construction. Recall that for any $x \in M$, the operator $(-x)$ on \mathcal{H} has at most n positive and at most n negative eigenvalues. We denote its positive and negative spectral subspaces by $S_x^+ M$ and $S_x^- M$, respectively. In view of (6), these subspaces are also orthogonal with respect to the spin scalar product,

$$S_x M = S_x^+ M \oplus S_x^- M.$$

We introduce the *Euclidean sign operator* s_x as a symmetric operator on $S_x M$ whose eigenspaces corresponding to the eigenvalues ± 1 are the spaces $S_x^+ M$ and $S_x^- M$, respectively. Since $s_x^2 = \mathbf{1}$, the span of the Euclidean sign operator is a one-dimensional Clifford subspace of signature $(1, 0)$. The idea is to extend s_x to obtain higher-dimensional Clifford subspaces. We thus define a *Clifford extension* as a Clifford subspace which contains s_x . By restricting attention to Clifford extensions, we have reduced the freedom in choosing Clifford subspaces. However, still there is not a unique Clifford extension, even for fixed dimension and signature. But one can define the *tangent space* T_x as an equivalence class of Clifford extensions; for details see [7, Section 3.1]. Choosing $r = 1$, the bilinear form $\langle \cdot, \cdot \rangle$ in (9) induces a Lorentzian metric on the tangent space.

In the remainder of this section, we assume that the causal fermion system is regular (see Definition 3.1). Moreover, we need the following stronger version of timelike separation:

Definition 3.3 The space-time point $x \in M$ is *properly timelike separated* from $y \in M$ if the closed chain A_{xy} defined by

$$A_{xy} = P(x,y) P(y,x) : S_x M \rightarrow S_x M$$

has a strictly positive spectrum and if all eigenspaces are (either positive or negative) definite subspaces of $(S_x M, \prec \cdot | \cdot \succ_x)$.

The two following observations explain why the last definition makes sense (for the proof see [5, §1.1.6]):

- Properly timelike separation implies timelike separation (see Definition 2.2).
- The notion is symmetric in x and y .

So far, the construction of the spin connection has been worked out only in the case of spin dimension $n = 2$. Then for two properly timelike separated points $x, y \in M$, the spin space $S_x M$ can be decomposed uniquely into an orthogonal direct sum $S_x M = I^+ \oplus I^-$ of a two-dimensional

positive definite subspace I^+ and a two-dimensional negative definite subspace I^- of A_{xy} . We define the *directional sign operator* v_{xy} of A_{xy} as the unique operator on $S_x M$ such that the eigenspaces corresponding to the eigenvalues ± 1 are the subspaces I^\pm .

Having the Euclidean sign operator s_x and the directional sign operator v_{xy} to our disposal, under generic assumptions one can distinguish two Clifford subspaces at the point x : a Clifford subspace K_{xy} containing v_{xy} and a Clifford extension $K_x^{(y)}$ (for details see [7, Lemma 3.12]). Similarly, at the point y we have a distinguished Clifford subspace K_{yx} (which contains v_{yx}) and a distinguished Clifford extension $K_y^{(x)}$. For the construction of the *spin connection* $D_{x,y} : S_y M \rightarrow S_x M$ one works with the Clifford subspaces K_{xy} and K_{yx} and demands that these are mapped to each other. More precisely, the spin connection is uniquely characterized by the following properties (see [7, Theorem 3.20]):

(i) $D_{x,y}$ is of the form

$$D_{x,y} = e^{i\varphi_{xy} v_{xy}} A_{xy}^{-\frac{1}{2}} P(x,y) \quad \text{with} \quad \varphi_{xy} \in \left(-\frac{3\pi}{4}, -\frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{4}\right).$$

(ii) The spin connection maps the Clifford subspaces K_{xy} and K_{yx} to each other, i.e.

$$D_{y,x} K_{xy} D_{x,y} = K_{yx}.$$

The spin connection has the properties

$$D_{y,x} = (D_{x,y})^{-1} = (D_{x,y})^* \quad \text{and} \quad A_{xy} = D_{x,y} A_{yx} D_{y,x}.$$

All the assumptions needed for the construction of the spin connection are combined in the notion that x and y must be *spin-connectable* (see [7, Definition 3.17]).

By composing the spin connection along a “discrete path” of space-time points, one obtains a “parallel transport” of spinors. When doing so, it is important to keep track of the different Clifford subspaces and to carefully transform them to each other. In order to illustrate in an example how this works, suppose that we want to compose the spin connection $D_{y,z}$ with $D_{z,x}$. As mentioned above, the spin connection $D_{z,x}$ at the point z is constructed using the Clifford subspace K_{zx} . The spin connection $D_{y,z}$, however, takes at the same space-time point z the Clifford subspace K_{zy} as reference. This entails that before applying $D_{y,z}$ we must transform from the Clifford subspace K_{zx} to the Clifford subspace K_{zy} . This is accomplished by the *splice map* $U_z^{(y|x)}$, being a uniquely defined unitary transformation of $S_x M$ with the property that

$$K_{zy} = U_z^{(y|x)} K_{zx} (U_z^{(y|x)})^*.$$

The splice map must be sandwiched between the spin connections in combinations like

$$D_{y,z} U_z^{(y|x)} D_{z,x}.$$

In order to construct a corresponding metric connection $\nabla_{x,y}$, one uses a similar procedure to relate the Clifford subspaces to corresponding Clifford extensions. More precisely, one first unitarily transforms the Clifford extension $K_y^{(x)}$ to the Clifford subspace K_{yx} . Unitarily transforming with the spin connection D_{xy} gives the Clifford subspace K_{xy} . Finally, one unitarily transforms to the Clifford extension $K_x^{(y)}$. Since the Clifford extensions at the beginning and end are representatives of the corresponding tangent spaces, we thus obtain an isometry

$$\nabla_{x,y} : T_y \rightarrow T_x$$

between the tangent spaces (for details see [7, Section 3.4]).

In this setting, *curvature* is defined as usual as the holonomy of the connection. Thus the curvature of the spin connection is given by

$$\mathfrak{R}(x,y,z) = U_x^{(z|y)} D_{x,y} U_y^{(x|z)} D_{y,z} U_z^{(y|x)} D_{z,x} : S_x M \rightarrow S_x M,$$

and similarly for the metric connection.

3.4. Correspondence to Lorentzian Geometry

In [7, Sections 4 and 5] it is proven that the above notions reduce to the metric connections and the Riemannian curvature on a globally hyperbolic Lorentzian manifold if the causal fermion system is constructed by regularizing solutions of the Dirac equation (similar as explained in Section 1) and removing the regularization in a suitable way. These results show that the notions of connection and curvature defined above indeed generalize the corresponding notions in Lorentzian spin geometry.

As in Section 1, we begin with a globally hyperbolic space-time (\mathcal{M}, g) . In order to keep the setting as simple as possible, in [7] it was assumed that (\mathcal{M}, g) is isometric to Minkowski space in the past of a Cauchy-hypersurface \mathcal{N} . This has the technical advantage that one can work with simple and explicit regularization operators \mathcal{R}_ε defined in Minkowski space (more precisely, one works with $i\varepsilon$ -regularization of a Dirac see structure; see also [5, §2.4.1]). Due to the ultraviolet regularization, the causal fermion system only describes the “coarse geometry” down to the length scale ε . Therefore, in order to recover Lorentzian spin geometry, we need to analyze the limit $\varepsilon \searrow 0$.

For sufficiently small ε , one can identify objects of Lorentzian spin geometry with corresponding objects of the causal fermion system in the following way:

Lorentzian geometry	causal fermion system
space-time manifold \mathcal{M}	space-time $M := \text{supp } \rho \subset \mathcal{F} \subset L(\mathcal{H})$
space-time point $p \in \mathcal{M}$	space-time point $x := F^\varepsilon(p) \in M$
spinor space $S_p \mathcal{M}$	spin space $S_x M := x(\mathcal{H}) \subset \mathcal{H}$
wave function $u \in \mathcal{H} \subset \mathcal{H}_m$	physical wave function $\psi^u : M \rightarrow \mathcal{H}$
spinor $u(p) \in S_p \mathcal{M}$	spinor $\psi^u(x) \in S_x M$

Assume that $\gamma : [0, T] \rightarrow \mathcal{M}$ is a future-directed timelike curve joining the points $q := \gamma(0)$ with $p := \gamma(T)$, for simplicity parametrized by arc length. In order to avoid confusion with the different connections, we now denote the metric connection on the tangent bundle $T\mathcal{M}$ (i.e. the Levi-Civita connection) by ∇^{LC} . Likewise, the metric connection on the spinor bundle $S\mathcal{M}$ is denoted by D^{LC} .

In order to compare the parallel transports $D_{p,q}^{\text{LC}}$ and ∇^{LC} along γ with the corresponding connections D and ∇ of the causal fermion system, given $N \in \mathbb{N}$ we choose intermediate points on the curve by

$$p_n := \gamma\left(\frac{nT}{N}\right), \quad n = 0, \dots, N. \quad (10)$$

The following results show that in the limit $\varepsilon \searrow 0$ and $N \rightarrow \infty$, the connections agree, up to higher orders in the curvature tensor (for the proof see [7, Theorem 5.12 and Corollary 5.13]).

Theorem 3.4 *There is a subset of curves which is dense in the C^∞ -topology with the following property: Choosing $N \in \mathbb{N}$ sufficiently large and $\varepsilon > 0$ sufficiently small, the points p_n and p_{n-1} are spin-connectable for all $n = 1, \dots, N$, and every point p_n lies in the future of p_{n-1} (in the*

sense (5)). Moreover,

$$\begin{aligned}
& \lim_{N \rightarrow \infty} \lim_{\varepsilon \searrow 0} D_{(p_N, p_{N-1})} D_{(p_{N-1}, p_{N-2})} \cdots D_{(p_1, p_0)} \\
&= D_{p,q}^{\text{lc}} \text{Texp} \left(\frac{1}{6} \int_{\gamma} \left(m^2 - \frac{\text{scal}}{12} \right)^{-1} \right. \\
&\quad \left. \times D_{q, \gamma(t)}^{\text{lc}} \left[\epsilon_j (\nabla_{e_j} R)(\dot{\gamma}(t), e_j) \dot{\gamma}(t) \right] \cdot \dot{\gamma}(t) \cdot D_{\gamma(t), q}^{\text{lc}} dt \right) \\
& \lim_{N \rightarrow \infty} \lim_{\varepsilon \searrow 0} \nabla_{p_N, p_{N-1}} \nabla_{p_{N-1}, p_{N-2}} \cdots \nabla_{p_1, p_0} \\
&= \nabla_{q,p}^{\text{lc}} + \mathcal{O} \left(L(\gamma) \frac{\|\nabla R\|}{m^2} \right) \left(1 + \mathcal{O} \left(\frac{\text{scal}}{m^2} \right) \right),
\end{aligned}$$

where Texp is the time-ordered exponential, and $L(\gamma)$ is the length of the curve γ .

4. Going Beyond Lorentzian Geometry

4.1. Non-Smooth Geometries

The construction of connection and curvature in Section 3.3 did not rely on a smooth space-time structure. Indeed, these constructions even apply in non-regular or discrete space-times (for many examples see [9, Section 9]). Therefore, causal fermion systems are a framework for non-smooth geometries. We now explain in words how this works and what to keep in mind. Suppose that a causal fermion system $(\mathcal{H}, \mathcal{F}, \rho)$ is given. If space-time $M := \text{supp } \rho$ has the structure of a smooth manifold, then we are in the smooth setting. Otherwise, M merely is a topological space, with additional structures induced by the fact that the space-time points are linear operators in \mathcal{F} . The singular points (see Definition 3.1) must be treated separately. Removing them from M (by multiplying ρ with the characteristic function of the regular points), one gets a regular causal fermion system. Then the constructions in Sections 3.2 and 3.3 apply, giving a spin connection $D_{x,y}$, a metric connection $\nabla_{x,y}$ as well as corresponding curvatures.

A subtle point to keep in mind is that, since M has no manifold structure, there is no notion of tangent vectors. This also means that there is no analog of Clifford multiplication (1). Instead, the Clifford subspace (see Definition 3.2) merely is a vector space of linear operators acting on the spin space endowed with an inner product $\langle \cdot, \cdot \rangle$, but the vectors in the Clifford subspace are not related to usual tangent vectors defined as equivalence classes of curves or as derivations.

One method for getting along without tangent vectors is to work with *tangent cone measures*, as we now briefly outline (another method, which will be explained in Section 5.1 below, is to construct a corresponding Lorentzian length space). In the simplest version of this construction, one first chooses a mapping \mathcal{A} from space-time to the vector space $\text{Symm}(S_x)$ of all symmetric linear operators on the spin space,

$$\mathcal{A} : M \rightarrow \text{Symm}(S_x) \quad \text{with} \quad \mathcal{A}(u) = \pi_x(y - x)x|_{S_x}. \quad (11)$$

Then the push-forward $\mathcal{A}_*(\rho)$ is a measure on $\text{Symm}(S_x)$. A *conical set* is a subset of $\text{Symm}(S_x)$ with $\mathbb{R}^+ A = A$. The intuitive idea is to define the tangent cone measure μ_{con} at the space-time point x as a measure on conical sets obtained by restriction to smaller and smaller neighborhoods of x and rescaling, i.e.

$$\mu_{\text{con}}(A) := \liminf_{\delta \searrow 0} \frac{1}{\rho(B_\delta(x))} \rho(\mathcal{A}^{-1}(A_k) \cap B_\delta(x))$$

(where $B_\delta(x) \subset L(\mathcal{H})$ is the Banach space ball). In order to get σ -additivity, this definition must be modified and complemented by measure-theoretic constructions (for details see [9, Section 6]).

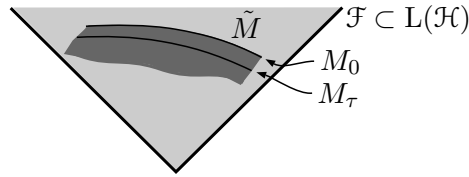


Figure 2. A quantum space-time

Such tangent cone measures make it possible to analyze the local structure of space-time in a neighborhood of a point $x \in M$, again without any differentiability assumptions. In particular, the tangent cone measures can be used to distinguish a specific Clifford subspace $\mathcal{C}\ell_x$ and to relate $\mathcal{C}\ell_x$ to neighboring space-time points (for details see again [9, Section 6]).

4.2. Quantum Geometries

The general definition of causal fermion systems (see Definition 2.1) covers many situation in which space-time $M := \text{supp } \rho$ does not have the structure of a manifold or a space-time lattice and cannot be thought of as a classical space-time. Since in our setting the geometric structure are encoded in the quantum matter, we subsume the general situation under the notion *quantum geometry*. This name is also motivated by the fact that these more general space-times are indeed relevant for the applications to quantum field theory (see the mechanism of *microscopic mixing* in [4] or the related *fragmentation of space-times* in [6]).

We now illustrate in a simple example what a “quantum space-time” is about. To this end, we generalize the setting of Section 1 by considering on \mathcal{M} a family of Lorentzian metrics g_τ indexed by a parameter $\tau \in [0, 1]$ (for example, the family can be obtained by varying the metric inside a compact subset $K \subset \mathcal{M}$). We assume that for every τ , the manifold (\mathcal{M}, g_τ) is globally hyperbolic, time-oriented and spin. Moreover, we identify the Hilbert spaces \mathcal{H}_m for different values of τ (for example by identifying boundary values on a Cauchy surface which does not intersect K). Then the construction of Section 1 gives for every $\tau \in [0, 1]$ a measure ρ_τ on \mathcal{F} . The equation

$$\tilde{\rho}(\Omega) = \int_0^1 \rho_\tau(\Omega) d\tau$$

(where $d\tau$ is the Lebesgue measure) defines a measure on \mathcal{F} . The resulting space-time \tilde{M} as given by

$$\tilde{M} := \text{supp } \tilde{\rho} = \overline{\bigcup_{\tau \in (0,1)} M_\tau} \quad \text{with} \quad M_\tau := \text{supp } \rho_\tau$$

can no longer be identified with the original space-time manifold \mathcal{M} , but it can be thought of as a collection of the whole family of space-times $(\mathcal{M}, g_\tau)_{\tau \in [0,1]}$.

In other words, the space-time \tilde{M} is no longer described by a single metric, but by the whole family of metrics (g_τ) . The situation becomes even more interesting if an interaction (as described by the causal action principle) is taken into account. Then the different “copies” M_τ of space-time interact with each other (as made precise by the constructions in [6]), giving rise to a complicated measure which can no longer be described in terms of a family of classical metrics. But, as shown in [6], the resulting measure can be described in terms of quantum fields. This justifies the name “quantum space time.”

5. Connection to Other Approaches and Further Structures

5.1. Connection to Lorentzian Length Spaces

A recent approach to non-smooth Lorentzian geometry is provided by Lorentzian length spaces [15]. We now explain how a causal fermion system encodes the structure of a Lorentzian length space. Our construction involves the choice of two length scales ℓ_{\min} and ℓ_{\max} . This is a subtle point which we explain in detail. To this end, we return to the setting of a globally hyperbolic space-time (\mathcal{M}, g) and consider a “discrete timelike path” (10) of space-time points $p_0, \dots, p_N \in \mathcal{M}$, such that the Lorentzian distance of adjacent points is of the order $\ell > 0$. In order to get a good approximation of a continuous path, the points should be sufficiently close to each other. The relevant length scale ℓ_{\max} must be chosen so small that curvature effects are not relevant on this scale. Moreover, we choose ℓ_{\max} so small that the oscillations of the Dirac wave functions due to the rest mass do not come into play. Thus we choose ℓ_{\max} to be smaller than the Compton scale m^{-1} . The length scale ℓ_{\min} , on the other hand, must be chosen so large that the effects of the regularization on the scale ε do not yet come into play (where ε can again be thought of as the Planck length). We thus obtain the admissible range

$$\varepsilon \ll \ell_{\min} < \ell < \ll \ell_{\max} \ll m^{-1}, \quad (12)$$

leaving us with a lot of freedom to choose the parameters ℓ_{\min} and ℓ_{\max} (note that for electrons, $m\varepsilon \lesssim 10^{-23}$). If the Lorentzian geodesic distance ℓ is within the range (12), then the kernel of the fermionic projector of the corresponding causal fermion system is well-approximated by the unregularized massless kernel in Minkowski space (as computed in detail in [5, Section 1.2]). In particular, the eigenvalues of xy scale like ℓ^{-6} . Since these eigenvalues are defined in terms of objects of the corresponding causal fermion system, we can generalize the notion the Lorentzian distance as follows.

Given a causal fermion system $(\mathcal{H}, \mathcal{F}, \rho)$, we introduce the functional

$$\ell : M \times M \rightarrow \mathbb{R}_0^+, \quad \ell(x, y) = \begin{cases} |xy|^{-\frac{1}{6}} & \text{if } \ell_{\min} < |xy|^{-\frac{1}{6}} < \ell_{\max} \\ 0 & \text{otherwise.} \end{cases}$$

A finite sequence of points $(x_0, \dots, x_N) \in M$ is called a *causal chain* if for all $n = 1, \dots, N$ the following conditions hold:

- (i) x_n and x_{n-1} are timelike separated
- (ii) x_n lies in the future of x_{n-1}
- (iii) $\ell(x_n, x_{n+1}) > 0$

The condition (i) could be strengthened by demanding that x_n and x_{n+1} are spin-connectable (see [7, Definition 3.17]). The causal chain is said to *connect* the space-time point x with y if $x_0 = x$ and $x_N = y$. The *length* of the causal chain is defined by

$$L[x_0, \dots, x_N] = \sum_{n=1}^N \ell(x_{n-1}, x_n).$$

The *Lorentzian distance* of two points is defined by

$$d(x, y) = \sup \left\{ L[x_0, \dots, x_N] \mid (x_0, \dots, x_N) \text{ is a causal chain connecting } x \text{ with } y \right\}. \quad (13)$$

If the set on the right is empty, then the supremum is defined to be zero. We also point out that the last supremum could be infinite, even if the points x and y are close together. A typical

example when this happens is when there are closed causal chains (corresponding to closed timelike curves on a Lorentzian manifold).

With the above definitions, a causal fermion system indeed gives rise to a Lorentzian length space (our setting agrees precisely with [15], up to a few additional technical assumptions in [15] which would have to be verified in the applications).

5.2. Connection to Causal Sets and Lattices

Another approach to non-regular geometry and quantum geometry are *causal sets* (see for example [1]), where the main structure is a transitive partial order relation \leq with the interpretation of “lies in the future of.” Given a causal fermion system, such a partial order relation can be defined by

$$x \leq y \quad \text{if } x = y \text{ or } d(x, y) > 0$$

(where $d(x, y)$ is again the Lorentzian distance (13)). By construction, this relation is reflexive and transitive. However, it is not necessarily anti-symmetric in the sense that $x \leq y$ and $y \leq x$ implies $x = y$. Indeed, the above partial order might be trivial in the sense that it holds for all pairs $x, y \in M$. This is the case in particular if the causal fermion system is constructed starting from a Lorentzian manifold with closed timelike curves.

Having the partial order relation, one can follow the constructions in [2] and introduce an *orthogonality relation* by

$$x \perp y \quad \text{if } x \not\leq y \text{ and } y \not\leq x$$

and setting

$$A^\perp = \{x \in M \mid x \perp a \text{ for all } a \in A\}.$$

Likewise, the sets of subsets of M

$$\mathcal{L}(M, \perp) := \{A \subset M \mid A^{\perp\perp} = A\}$$

forms a *lattice*.

5.3. A Riemannian Metric on \mathcal{F}

We finally introduce another inherent structure which has not been used so far but which might be helpful or of interest for geometers. To this end, let us again assume that our causal fermion system $(\mathcal{H}, \mathcal{F}, \rho)$ is regular (see Definition 3.1). Then we can also redefine \mathcal{F} as the set of all self-adjoint operators on \mathcal{H} which have exactly n positive and exactly n negative eigenvalues. This has the advantage that \mathcal{F} has a smooth structure. In order to keep the setting reasonably simple, we restrict attention to the case that \mathcal{H} is finite-dimensional, in which case \mathcal{F} is a smooth manifold (in the infinite-dimensional setting, \mathcal{F} would be a Banach manifold). Next, on \mathcal{F} the Hilbert-Schmidt norm gives a distance function

$$d : \mathcal{F} \times \mathcal{F} \rightarrow \mathbb{R}_0^+, \quad d(x, y) = \|x - y\|_{\text{HS}} := \sqrt{\text{tr}((x - y)^2)}$$

(note that the existence of the trace is not an issue even in the infinite-dimensional setting because all operators in \mathcal{F} have finite rank). The square of this distance function is smooth. Moreover, its first derivative vanishes on the diagonal, i.e. $D(d(x, \cdot)^2)|_x = 0$. Therefore, taking its quadratic Taylor expansion about a point $x \in M$ gives a scalar product on $T_x \mathcal{F}$, i.e.

$$h_x : T_x \mathcal{F} \times T_x \mathcal{F} \rightarrow \mathbb{R}, \quad h_x(u, v) = \text{tr}(uv).$$

Clearly, this mapping depends smoothly on x and thus defines a *Riemannian metric* on \mathcal{F} .

Since every manifold can be endowed with a Riemannian metric, the only point of this construction is the fact that the above Riemannian metric is *canonical* in the sense that there is a distinguished Riemannian metric. This Riemannian metric could be useful for different purposes. Just to give an example, we here mention that it can be used for a *gauge-fixing procedure*: In computations (like the perturbation expansion in [6]), one often needs to expand functions on \mathcal{F} in a Taylor expansion about a point $p \in \mathcal{F}$. Such an expansion is performed in a chart. The expansion coefficients of second and higher order depend in a complicated way on the choice of the chart. The freedom in choosing the charts includes the local gauge freedom in physics (for details see [6, Section 6.2]). Using the Riemannian metric h , the Taylor expansion can be performed in the distinguished chart given by the exponential map

$$\exp_x : U \subset T_x \mathcal{F} \rightarrow \mathcal{F}.$$

In these charts, the gauge freedom is fixed completely.

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References

- [1] L. Bombelli, J. Lee, D. Meyer, and R.D. Sorkin, *Space-time as a causal set*, Phys. Rev. Lett. **59** (1987), no. 5, 521–524.
- [2] W. Cegła, B. Jancewicz, and J. Florek, *Orthomodular lattice in Lorentzian globally hyperbolic space-time*, Rep. Math. Phys. **79** (2017), no. 2, 187–195.
- [3] F. Finster, *A formulation of quantum field theory realizing a sea of interacting Dirac particles*, arXiv:0911.2102 [hep-th], Lett. Math. Phys. **97** (2011), no. 2, 165–183.
- [4] ———, *Perturbative quantum field theory in the framework of the fermionic projector*, arXiv:1310.4121 [math-ph], J. Math. Phys. **55** (2014), no. 4, 042301.
- [5] ———, *The Continuum Limit of Causal Fermion Systems*, arXiv:1605.04742 [math-ph], Fundamental Theories of Physics, vol. 186, Springer, 2016.
- [6] ———, *Perturbation theory for critical points of causal variational principles*, arXiv:1703.05059 [math-ph] (2017).
- [7] F. Finster and A. Grotz, *A Lorentzian quantum geometry*, arXiv:1107.2026 [math-ph], Adv. Theor. Math. Phys. **16** (2012), no. 4, 1197–1290.
- [8] F. Finster, A. Grotz, and D. Schiefeneder, *Causal fermion systems: A quantum space-time emerging from an action principle*, arXiv:1102.2585 [math-ph], Quantum Field Theory and Gravity (F. Finster, O. Müller, M. Nardmann, J. Tolksdorf, and E. Zeidler, eds.), Birkhäuser Verlag, Basel, 2012, pp. 157–182.
- [9] F. Finster and N. Kamran, *Spinors on singular spaces and the topology of causal fermion systems*, arXiv:1403.7885 [math-ph], to appear in Mem. Amer. Math. Soc. (2019).
- [10] F. Finster and J. Kleiner, *Causal fermion systems as a candidate for a unified physical theory*, arXiv:1502.03587 [math-ph], J. Phys.: Conf. Ser. **626** (2015), 012020.
- [11] ———, *A Hamiltonian formulation of causal variational principles*, arXiv:1612.07192 [math-ph], Calc. Var. Partial Differential Equations **56:73** (2017), no. 3, 33.
- [12] F. Finster, J. Kleiner, and J.-H. Treude, *An Introduction to the Fermionic Projector and Causal Fermion Systems*, in preparation.
- [13] F. Finster and O. Müller, *Lorentzian spectral geometry for globally hyperbolic surfaces*, arXiv:1411.3578 [math-ph], Adv. Theor. Math. Phys. **20** (2016), no. 4, 751–820.
- [14] F. Finster and M. Reintjes, *A non-perturbative construction of the fermionic projector on globally hyperbolic manifolds I – Space-times of finite lifetime*, arXiv:1301.5420 [math-ph], Adv. Theor. Math. Phys. **19** (2015), no. 4, 761–803.
- [15] M. Kunzinger and C. Sämann, *Lorentzian length spaces*, arXiv:1711.08990 [math.DG] (2017).