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Eberly College of Science

**NEW APPLICATIONS OF ON-SHELL METHODS IN QUANTUM
FIELD THEORY**

A Dissertation in
Physics
by
Oluf Tang Englund

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The dissertation of Oluf Tang Engelund was reviewed and approved* by the following:

Radu Roiban
Professor of Physics
Dissertation Advisor, Chair of Committee

Murat Günaydin
Professor of Physics

Jainendra Jain
Professor of Physics

Nigel Higson
Professor of Mathematics

Rick Robinett
Professor of Physics
Chair of Graduate Program

*Signatures are on file in the Graduate School.

Abstract

This thesis incorporates 4 years of work: it gives a small introduction to the field of scattering amplitudes and especially into the method of generalized unitarity then discuss 4 different projects all in the field of scattering amplitudes.

First we will look at a duality between correlation functions in a special light-like limit and Wilson loops in $\mathcal{N} = 4$ Super-Yang-Mills. The duality, originally suggested by Alday, Eden, Korchemsky, Maldacena and Sokatchev, was part of an effort to put a firmer footing on the duality between scattering amplitudes and Wilson loops.

The duality between correlation functions and Wilson loops does not have any regularization issues (like the other duality) as both have infrared divergences in the specific limits considered. We show how the duality works vertex-by-vertex using just Feynman rules. The method is sufficiently general to allow for extensions of the original duality including operators not taking part in the special light-like limit, other types of operators as well as other theories than $\mathcal{N} = 4$ Super-Yang-Mills.

After that we look at how to use generalized unitarity for correlation functions with some examples from $\mathcal{N} = 4$ Super-Yang-Mills. For computations one needs quantities known as form factors which have both asymptotic states like scattering amplitudes and local operators like correlation functions. We compute several form factors using modern methods from scattering amplitudes.

Thirdly, we study how to use generalized unitarity for two-dimensional integrable systems. Two-dimensional systems have their own unique set of challenges but generalized unitarity can be adapted to them and we show how one can carry out tests of integrability which would otherwise be difficult.

Finally, we look at the 3-dimensional theory known as ABJ(M). Its tree-level amplitudes can be incorporated into a single formula very reminiscent of a result in $\mathcal{N} = 4$ Super-Yang-Mills. Since the result from $\mathcal{N} = 4$ Super-Yang-Mills follow directly from a twistor string theory it is natural to guess that something similar could be true for ABJ(M). We construct a twistor string theory that after a certain set of projections give us the ABJ(M) formula.

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Chapter 1 —

Introduction

Scattering amplitudes represent the connection between theory and experiment. Textbooks methods suggest they can be computed from Lagrangians via Feynman rules: as such they can appear tedious to calculate and giving unilluminating results that typically exhibit little structure. In this thesis we will look at scattering amplitudes from a different approach: the amplitudes (and other perturbative quantities) we will compute will often be simpler than similar results obtained via Feynman rules and the methods for computing them will be simple as well.

Part of the simplicity will arise from the theories we choose to look at: $\mathcal{N} = 4$ Super-Yang-Mills, ABJ(M) and integrable two-dimensional theories. These theories have very large Lagrangians when written out in terms of components but the theories have a lot of symmetries, some appearing already at the level of the Lagrangian while others emerging dynamically. An example of the former is supersymmetry, an example of the latter is dual conformal symmetry - conformal symmetry in momentum space.

The simplicity described will however not just be a result of convenient choices of theories. Some of the methods described in chapters 2 and 3 can be applied to more realistic theories (see for instance [13, 39]). We will also see in chapter 4 that the duality is quite robust and can be extended to a lot of other theories.

1.1 A Duality Between Correlation Functions and Wilson Loops

In chapter 4 we discuss a duality between correlation functions in a special light-like limit and polygonic Wilson loops. This relation emerged from an attempt to gain a better understanding of the duality between scattering amplitudes and Wilson loops. The duality is related to dual conformal symmetry as the dual conformal symmetry of scattering amplitudes corresponds to the normal conformal symmetry of Wilson loops.

The duality between scattering amplitudes and Wilson loops can be complicated to handle because they have divergences in two different regimes so regularization breaks the duality so it was proposed to be part of a triality where correlation in the special light-like limit is dual to both scattering amplitudes and polygonic Wilson loops; these two extra dualities do not have regularization issues as Wilson loops and correlation functions in the specific light-like limits have divergences in the same regime and the duality between correlation function and scattering amplitudes is at the level of the integrand meaning that no regularization is necessary.

We show how the duality works in a way that allows for generalizations to other theories, other dimensions and other operators appearing in the correlation functions.

1.2 Generalized Unitarity and Correlation Functions

Generalized unitarity is a method that has been very successful in computing scattering amplitudes and so it is natural to try to extend its use to other perturbative quantities.

In chapter 5 we will discuss how to use generalized unitarity to compute correlation functions. Some of the points will be general but all of the examples are computed in $\mathcal{N} = 4$ Super-Yang-Mills. Even though correlation functions naturally exist in real space while generalized unitarity is in momentum space we will see that the method can be quite effective in fact most of the calculations are quite similar to using generalized unitarity to compute scattering amplitudes only we

need to calculate an integral in real space in the end.

1.3 Generalized Unitarity for Integrable Systems

In chapter 6 we look at integrable systems. All scattering in the integrable systems can be described in terms of the 2 goes to 2 worldsheet S-matrix and the worldsheet S-matrix can also be used to write down the Bethe equations¹ so the worldsheet S-matrix is an important object in integrable systems even if it is strictly speaking not an observable.

Generalized unitarity is in a way a natural fit for two-dimensional integrable systems because these theories have large Lagrangians that continue to infinite order in the coupling constant while the S-matrices have compact expressions. The two-dimensional kinematics is a challenge in the context of generalized unitarity but nonetheless we show how to get useful results out of generalized unitarity with simple tests of integrability.

1.4 Twistor String Theory for ABJ(M)

In chapter 7 we explore the tree-level scattering amplitudes of ABJ(M) and show how they can be computed as a projection from a twistor string theory. It is natural to expect such a string theory to be present as the tree-level amplitudes of ABJ(M) can be written in a way very reminiscent of a formula in $\mathcal{N} = 4$ Super-Yang-Mills which follow more or less directly from the twistor string theories constructed by Witten and Berkovits [53, 14].

The construction allows for modifications and it leads to the question whether any of these modifications can be given an interpretation in terms of a Lagrangian description of a theory. Another open question is whether the theory of the enlarged twistor space can be given a meaning.

¹which in turn can be used to find the spectrum of the dual theory

Chapter 2 —

Random Facts about Scattering Amplitudes

Scattering Amplitudes serve as connections between theories and experiments: computed from Lagrangians via Feynman rules and predicting probabilities for the experiments to measure. However scattering amplitudes are interesting in their own right having beautiful patterns and they may be computed from a lot simpler methods than standard Feynman rules.

The modern methods for computing scattering amplitudes are in a way a continuation of the old S-matrix program however whereas the S-matrix program tried to disregard Feynman rules entirely a lot of the modern methods rely on the existence of a Feynman diagram representation even though they avoid using Feynman rules directly.

In this chapter we will discuss some of the features of scattering amplitudes that we are going to need in later chapters. A lot of the calculations will focus on $\mathcal{N} = 4$ Super-Yang-Mills however many of the techniques may be applicable in other theories as well. We will also only deal with massless particles though with sufficient care massive states can also be incorporated.

2.1 Spinors and Color-Ordered Amplitudes

It is possible to separate the gauge group from the kinematics of Yang-Mills theory such that the full amplitude is a sum of products of traces times kinematical functions:

$$A = \sum_{\sigma \in S_n/Z_n} \text{Tr}(T^{a_{\sigma_1}} \dots T^{a_{\sigma_n}}) A(\sigma(1) \dots \sigma(n)) + \text{double traces} + \text{triple traces}. \quad (2.1)$$

Here the sum is over all permutations excluding cyclic ones. The term with only a single trace is going to be the only contribution if one considers a special limit where the size of the gauge group generators become infinite¹ this is called the planar part. At tree-level only the planar part is there and $A(\sigma(1) \dots \sigma(n))$ is referred to as the color-ordered amplitude.

The color-ordered amplitudes can be incredibly simple if written for specific helicity configurations. In order to do that we need to introduce the following spinors: start with the solutions to the Dirac equation:

$$\not{p}u(p) = \bar{u}(p)\not{p} = 0. \quad (2.2)$$

Divide these into negative helicity spinors:

$$|p\rangle = \frac{1}{2}(1 + \gamma_5)u(p), \quad (2.3)$$

$$\langle p| = \bar{u}(p)\frac{1}{2}(1 + \gamma_5), \quad (2.4)$$

and positive helicity spinors:

$$|p] = \frac{1}{2}(1 - \gamma_5)u(p), \quad (2.5)$$

$$[p| = \bar{u}(p)\frac{1}{2}(1 - \gamma_5). \quad (2.6)$$

The following spinor products vanish:

$$\langle ij \rangle = 0, \quad [ij] = 0, \quad (2.7)$$

¹So if the gauge group is $SU(N)$: $N \rightarrow \infty$

The other products $\langle ij \rangle$ and $[ij]$ are antisymmetric in i and j .

The spinors can be related to the momentum by:

$$\langle i | \gamma^\mu | i \rangle = 2p_i^\mu, \quad (2.8)$$

and using that:

$$\langle i | \gamma_\mu | j \rangle \langle k | \gamma^\mu | l \rangle = 2 \langle ik \rangle [lj], \quad (2.9)$$

we can relate the spinor products to the Mandelstam variables:

$$[ij] \langle ji \rangle = 2p_i \cdot p_j = (p_i + p_j)^2. \quad (2.10)$$

One can also introduce the spinors by using the 4-dimensional Pauli matrices:

$$p_\mu (\sigma^\mu)^{a\dot{a}} = \lambda^a \tilde{\lambda}^{\dot{a}}, \quad (2.11)$$

where:

$$\langle ij \rangle = \epsilon_{ab} \lambda_i^a \lambda_j^b, \quad (2.12)$$

$$[ij] = \epsilon_{\dot{a}\dot{b}} \tilde{\lambda}_i^{\dot{a}} \tilde{\lambda}_j^{\dot{b}}. \quad (2.13)$$

Written in this way one can also see that the Schouten identity gives us:

$$0 = \langle ij \rangle \langle kl \rangle + \langle ik \rangle \langle lj \rangle + \langle il \rangle \langle jk \rangle \quad (2.14)$$

The spinors are not just used for fermions as one might guess from the way I defined them based on (2.2) however for spin-1 particles² one needs to pick a reference momentum which can be arbitrary except it needs to satisfy:

²and spin-3/2 and spin-2

$$k^2 = 0, \quad k \cdot p \neq 0. \quad (2.15)$$

With this one can define polarization vectors for the spin-1 states by:

$$\varepsilon_\mu^+ = \frac{[p|\gamma_\mu|k\rangle}{\sqrt{2}\langle pk\rangle}, \quad \varepsilon_\mu^- = -\frac{\langle p|\gamma_\mu|k\rangle}{\sqrt{2}[pk]}. \quad (2.16)$$

With this notation the color-ordered amplitudes³ with only two negative helicity gluons and the rest being positive helicity gluons is given by the famous Parke-Taylor formula [50]:

$$A(g_1^+ \cdots g_a^- \cdots g_b^- \cdots g_n^+) = \frac{\langle ab \rangle^4}{\prod_{i=1}^n \langle ii+1 \rangle}. \quad (2.17)$$

Here a and b are the locations of the negative helicity gluons and in the denominator $n+1 \equiv 1$.

What about the amplitudes with only one negative helicity gluon or the ones with no negative helicity gluons at all? Well, they are zero and so the amplitudes (2.17) are referred to as the Maximal-Helicity-Violating (MHV) amplitudes because they are the amplitudes with the largest total helicities that still give non-trivial results. The amplitudes with one more negative helicity gluon than the MHV amplitudes are called Next-to-MHV (NMHV) amplitudes and the amplitude with k more negative helicity gluons are called N^kMHV amplitudes.

In $\mathcal{N} = 4$ Super-Yang-Mills the equivalent result is:

$$A_{MHV} = \frac{\delta^8(\sum_{i=1}^n \eta_i \lambda_i)}{\prod_{i=1}^n \langle ii+1 \rangle}, \quad (2.18)$$

where η_{iA} are Grassmann variables and A runs from 1 to 4. Zero Grassmann variables correspond to a positive helicity gluon and four correspond to a negative

³We are really not interested in the full amplitudes as they merely are sums of the same thing with different orderings so we will use the term 'amplitude' we really mean 'color-ordered amplitude'

helicity gluon.

Of course negative helicity is not special compared to positive helicity so one can also define the $\overline{\text{MHV}}$ amplitude:

$$A_{\overline{\text{MHV}}} = \frac{\delta^8(\sum_{i=1}^n \bar{\eta}_i \tilde{\lambda}_i)}{\prod_{i=1}^n [i\tilde{i} + 1]}, \quad (2.19)$$

The variables $\bar{\eta}_i^A$ are conjugate of η_{iA} . In the same way as before amplitude with zero or only one positive helicity state vanish.

2.2 The BCFW On-Shell Recursion Relations

The formula (2.18) can actually be proven for any number of external legs. This seems impossible to do with Feynman rules as there are going to be more and more diagrams but there is another method for computing tree-level amplitudes that makes the calculation a lot simpler which is called BCFW recursion [24, 25].

In BCFW one treats λ and $\tilde{\lambda}$ as separate even though strictly speaking they should be conjugate of each other, this allows one to define a 3-point MHV amplitude:

$$A_{MHV} = \frac{\delta^8(\sum_{i=1}^3 \eta_i \lambda_i)}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle}, \quad (2.20)$$

which really only have 1 positive helicity state so we would expect it to be zero. From a kinematical point of view there is a very serious with this amplitude as all Mandelstam variables must be zero:

$$(p_1 + p_2)^2 = p_3^2 = 0, \quad (2.21)$$

$$(p_2 + p_3)^2 = p_1^2 = 0, \quad (2.22)$$

$$(p_3 + p_1)^2 = p_2^2 = 0. \quad (2.23)$$

This would normally mean that all of the spinor products would also be zero however when λ and $\tilde{\lambda}$ are no longer conjugate it is enough to pick:

$$[12] = 0, \quad [23] = 0, \quad [31] = 0. \quad (2.24)$$

In the same way we can define a 3-point $\overline{\text{MHV}}$ amplitude:

$$A_{\overline{\text{MHV}}} = \frac{\delta^8(\sum_{i=1}^3 \bar{\eta}_i \tilde{\lambda}_i)}{[12][23][31]}, \quad (2.25)$$

which comes with the conditions that:

$$\langle 12 \rangle = 0, \quad \langle 23 \rangle = 0, \quad \langle 31 \rangle = 0. \quad (2.26)$$

Having λ and $\tilde{\lambda}$ not being conjugate means allowing for complex momenta making the amplitudes complex functions and that is in fact what we are going to exploit.

Imagine that you introduce some complex parameter, z , into the amplitude and let $z = 0$ give the amplitude you are interested in. Then you could write down the following integral:

$$\oint \frac{A(z)}{z} dz. \quad (2.27)$$

It has a simple pole at $z = 0$ which is that amplitude you are interested in and as long as there is no pole at $z \rightarrow \infty$ this can then be expressed in terms of poles of the amplitude itself.

To be more specific let us look at an example from pure Yang-Mills theory⁴. We will consider an MHV amplitude of n particles where the negative helicity gluons are placed on sites 2 and 3. We then shift two of the spinors:

$$|\hat{1}\rangle = |1\rangle + z|2\rangle, \quad (2.28)$$

$$|\hat{2}\rangle = |2\rangle - z|1\rangle. \quad (2.29)$$

⁴Meaning a theory of only gluons

This shift clearly has the property that when $z \rightarrow 0$ we are left with the amplitude we are interested in. It also maintains momentum conservation:

$$\sum_{i=1}^n p_i^\mu(z) = \sum_{i=3}^n p_i^\mu + \frac{1}{2}z[1|\gamma^\mu|\hat{1}\rangle + \frac{1}{2}z[\hat{2}|\gamma^\mu|2\rangle] \quad (2.30)$$

$$= \sum_{i=1}^n p_i^\mu + \frac{1}{2}z[1|\gamma^\mu|2\rangle - \frac{1}{2}z[1|\gamma^\mu|2\rangle] \quad (2.31)$$

$$=0, \quad (2.32)$$

meaning that the function $A(z)$ is still an amplitude for other values of z . Finally it has the property that $A(z) \rightarrow 0$ as $z \rightarrow \infty$ so the integral (2.27) has no boundary term at infinity; to see this notice that the polarization vectors (2.16) will contribute with z^{-2} while vertices could contribute with z for each vertex there are on the way from particle 1 to particle 2 however all of those vertices will be connected by a propagator contributing with z^{-1} so all in all propagators and vertices can at most contribute with z and hence there is no boundary term at infinity. This means that $A(0)$ can be written in terms of the other poles of $A(z)$ and there is only one other source of poles, the propagators.

For a propagator to give rise to a pole the internal momentum must be dependent on z meaning that it must depend on either p_1 or p_2 but not both which in turn means it can be written as:

$$P^\mu(z) = P^\mu(0) + \frac{1}{2}z[1|\gamma^\mu|2\rangle]. \quad (2.33)$$

The inverse propagator of this momentum is given by:

$$P^2(z) = P^2(0) + z[1|\not{P}(0)|2\rangle]. \quad (2.34)$$

The propagator then gives rise to a pole when:

$$z^\star = \frac{-P^2(0)}{[1|\not{P}(0)|2\rangle].} \quad (2.35)$$

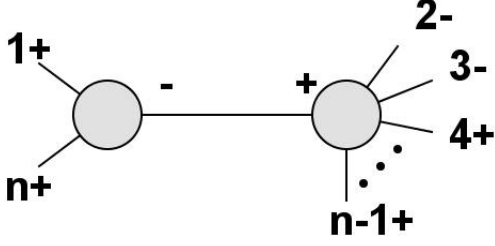


Figure 2.1. The only non-zero BCFW diagram

Notice that there was no z^2 -term in (2.34) so this is a simple pole so the residue of the propagator and the $1/z$ from the integral becomes:

$$\text{Res} \left(\frac{1}{z} \frac{1}{P^2(z)} \right) = \frac{1}{P^2(0)}, \quad (2.36)$$

which is the propagator of the unshifted momenta.

What about the other parts of the amplitude? Since the momentum (2.33) has become on-shell they split into two lower point amplitudes. In the example at hand the residue is given by the amplitudes in figure 2.1. Of course there are other poles but all of their residues involve vanishing amplitudes⁵, some of these residues are shown in figure 2.2.

This means that all in all the amplitude is given by:

$$A(0) = - A(-P(z^*)^-, n^+, \hat{1}^+) \frac{1}{P^2(0)} A(P(z^*)^+, \hat{2}^-, 3^-, 4^+, \dots, n-1^+), \quad (2.37)$$

$$= \frac{[n1]^4}{[-P(z^*)n][n1][1-P(z^*)]} \frac{-1}{[n1]\langle 1n \rangle \langle P(z^*)2 \rangle \langle 23 \rangle \dots \langle n-1P(z^*) \rangle} \langle 23 \rangle^4 \quad (2.38)$$

$$= \frac{\langle 23 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \dots \langle n-1n \rangle}. \quad (2.39)$$

Here we used the convention that $p \rightarrow -p$ means $|p\rangle \rightarrow -|p\rangle$ and $|p] \rightarrow |p]$. In the last step we used momentum conservation, the trick is to collect the $P(z^*)$ spinors in such a way that one of the spinors they are multiplied by removes the z^* dependent part:

⁵Remember that we are dealing with color-ordered amplitudes so this diagrams should respect the planar ordering of the external legs

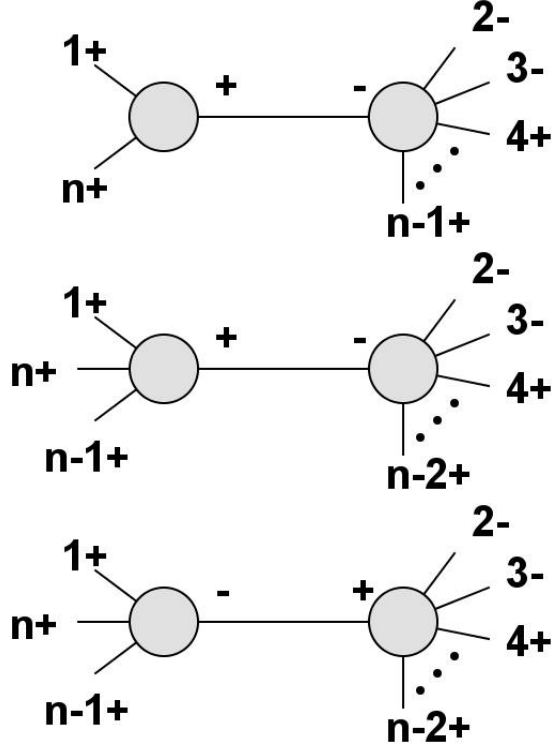


Figure 2.2. Some of the vanishing BCFW diagrams

$$[1P(z^*)]\langle P(z^*)b \rangle = [1P(0)]\langle P(0)b \rangle + z^*[11]\langle 2b \rangle. \quad (2.40)$$

As mentioned the spinor products are antisymmetric so the second term vanishes.

So this way we have demonstrated how the MHV formula can be shown recursively to hold for any number of external legs.

2.3 Twistors

Interest in scattering amplitudes rose when Witten wrote a paper about amplitudes of $\mathcal{N} = 4$ Super-Yang-Mills being amplitudes of strings in twistor space [53]. We will deal in more detail with twistor string theory in a later chapter but currently let us just see how they relate to the spinors mentioned in this chapter. In 4 dimensions one creates twistors by keeping one of the two spinors and exchanging the other

with a derivative, to be more specific one makes the following substitution:

$$\tilde{\lambda}_{\dot{a}} \rightarrow i \frac{\partial}{\partial \mu^{\dot{a}}}, \quad (2.41)$$

$$\frac{\partial}{\partial \tilde{\lambda}_{\dot{a}}} \rightarrow i \mu^{\dot{a}}, \quad (2.42)$$

then write the amplitudes in terms of the pairs:

$$Z = \begin{pmatrix} \lambda_a \\ \mu_{\dot{a}} \end{pmatrix}. \quad (2.43)$$

These are the twistors⁶ and it turns out that amplitudes become surprisingly simple when written in twistor space.

⁶Had we substituted λ instead it would have given us the conjugate twistors

Chapter 3 —

Generalized Unitarity

Generalized unitarity is a method that compute scattering amplitudes at loop order by multiplying together tree-level amplitudes or amplitudes of lower loop order. The name is perhaps a bit confusing as the method as such do not use unitarity. The name has historical reasons as the method was developed from earlier methods which used that the S-matrix is unitary. The modern method generalized unitarity is however based purely on the existence of a Feynman diagram representation.

3.1 Using Unitarity to Compute the S-matrix

So how can you compute higher loop order by using that the S matrix is unitary? In order to see this use the usual definition of the T matrix:

$$S = \mathbb{1} + iT, \tag{3.1}$$

For the S matrix to be unitary the T matrix must satisfy:

$$\mathbb{1} = \mathbb{1} + i(T - T^\dagger) + TT^\dagger, \tag{3.2}$$

or written differently:

$$-i\text{Im}[T] = TT^\dagger. \tag{3.3}$$

$$\text{Im} \text{ (one-loop bubble) } = \sum \text{ (two-loop diagram) }$$

Figure 3.1. The sum is over internal states and the ordering of the external states

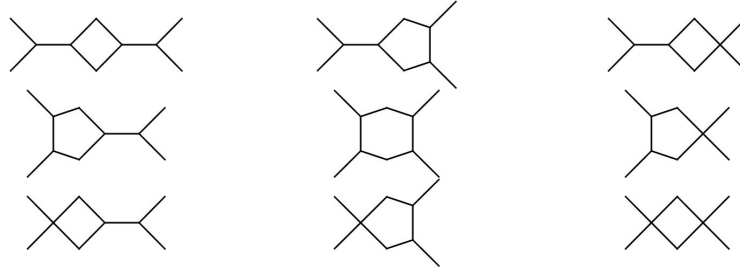


Figure 3.2. Feynman diagrams with two particular loop momenta present (all the diagrams have a bend where we will set the loop momentum on-shell but for these diagrams are just regular Feynman diagrams)

By paying attention to the number of coupling constants on both sides we will for instance see that if we want to compute the imaginary part of the 1-loop T matrix, it is given by two tree-level T matrices so this formula relates loop order amplitudes to products of lower order amplitudes. Pictorially it can be written as in figure 3.1 where the sum is over the internal states and the positioning of the external states.

3.2 Generalized Unitarity

Generalized unitarity is different in that it does not use that the S-matrix is unitarity. Instead it employs what we know about the structure of the amplitudes based on their description in terms of Feynman diagrams so the method does not use Feynman rules directly just like the type of unitarity methods described above but use the existence of an underlying Feynman diagram representation.

In order to explain the method let us begin by considering a fairly generic four-point 1-loop amplitude in a theory with three- and four-point vertices and let us choose to consider all the Feynman diagrams in which two particular loop momenta are present; the diagrams are shown in figure 3.2. A quick look at the

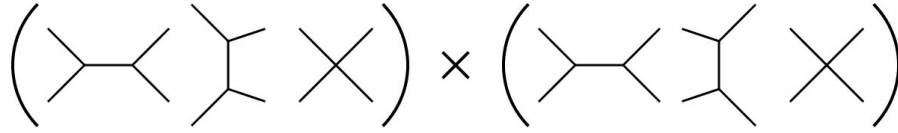


Figure 3.3. The Feynman diagrams from figure 3.2 ordered as a product

diagrams will show that they can be ordered like in figure 3.3 which look sort of like two tree-level amplitudes multiplied together except the internal lines are off-shell propagators and not on-shell like the external lines.

Let us now replace the two internal propagators with external legs formally by replacing the propagators with delta functions:

$$\frac{1}{\mathbf{p}^2 - m^2} \longrightarrow \delta(\mathbf{p}^2 - m^2). \quad (3.4)$$

Then the product in figure 3.3 actually becomes a product of two tree-level amplitudes save for some potential difference in normalization. Of course making the replacement in (3.4) removes some information from the diagrams. First of all we remove everything that is proportional to $\mathbf{p}^2 - m^2$ which is not too surprising since those terms would cancel one of the propagators and so can be combined with the diagrams where the propagators are not present. Secondly we removed the loop integral; this we will reinsert after we are done simplifying the product of tree-level amplitudes. This way we can reconstruct the part of the amplitude where these two particular propagators are present.

Making the replacement in (3.4) is referred to as 'cutting' the internal propagators and the resulting product of lower-order amplitudes is called a 'cut'. Several cuts may be necessary to compute the entire amplitude. Notice that since generalized unitarity relies on the underlying representation in terms of Feynman diagrams and not on the S matrix being unitary we do not need to cut exactly two propagators but can cut as many as we consider convenient. Naturally the fewer internal propagator we cut the more information we get out of the individual cut, however this may also make the expression more complicated. What cuts to make will depend on the theory and on the quantity studied.

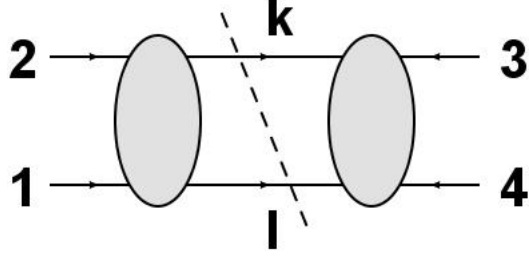


Figure 3.4. The cut of the 1-loop 4-point amplitude in $\mathcal{N} = 4$ Super-Yang-Mills

3.3 Example

Let us now look at a simple example, the four-point MHV amplitude in planar $\mathcal{N} = 4$ Super Yang-Mills. We will consider the cut shown in figure 3.4. Note that in $\mathcal{N} = 4$ SYM the sum over the internal states is particularly simple as one simply integrates over the Grassmann variables, η .

$$\begin{aligned}
C &= \int d^4\eta_k d^4\eta_l \frac{\delta^8(\eta_1\lambda_1 + \eta_2\lambda_2 - \eta_k\lambda_k - \eta_l\lambda_l)}{\langle 12 \rangle \langle 2k \rangle \langle kl \rangle \langle l1 \rangle} \\
&\quad \times \frac{\delta^8(\eta_k\lambda_k + \eta_l\lambda_l + \eta_3\lambda_3 + \eta_4\lambda_4)}{\langle lk \rangle \langle k3 \rangle \langle 34 \rangle \langle 4l \rangle} \\
&= - \frac{\delta^8(\eta_1\lambda_1 + \eta_2\lambda_2 + \eta_3\lambda_3 + \eta_4\lambda_4) \langle kl \rangle^2}{\langle 12 \rangle \langle 2k \rangle \langle l1 \rangle \langle k3 \rangle \langle 34 \rangle \langle 4l \rangle} \\
&= \frac{\delta^8(\eta_1\lambda_1 + \eta_2\lambda_2 + \eta_3\lambda_3 + \eta_4\lambda_4) [12][34]}{\langle 2k \rangle [kl] \langle l1 \rangle \langle 4l \rangle [lk] \langle k3 \rangle} \\
&= - \frac{\delta^8(\eta_1\lambda_1 + \eta_2\lambda_2 + \eta_3\lambda_3 + \eta_4\lambda_4) [12] \langle 23 \rangle [34] \langle 41 \rangle}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle (p_1 - l)^2 (p_4 + l)^2} \\
&= \frac{\delta^8(\eta_1\lambda_1 + \eta_2\lambda_2 + \eta_3\lambda_3 + \eta_4\lambda_4)}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \frac{s_{12}s_{14}}{(p_1 - l)^2 (p_4 + l)^2}
\end{aligned} \tag{3.5}$$

One then puts in the two propagators that were cut:

$$\frac{1}{l^2} \frac{1}{(l - p_1 - p_2)^2},$$

and reintroduce the integral:

$$\frac{\delta^8(\eta_1\lambda_1 + \eta_2\lambda_2 + \eta_3\lambda_3 + \eta_4\lambda_4)}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \int d^4l \frac{s_{12}s_{14}}{l^2(l-p_1)^2(l-p_1-p_2)^2(l+p_4)^2}. \quad (3.6)$$

To get the other 2-particle cut one simply has to rotate this result. We notice that they all contain the same information and in fact it would have sufficed to do a 4-particle cut. The result of the calculation can be written as:

$$A_{1-loop}^{MHV} = A_{tree}^{MHV} \int d^4l \frac{s_{12}s_{14}}{l^2(l-p_1)^2(l-p_1-p_2)^2(l+p_4)^2}. \quad (3.7)$$

The fastidious might complain that the integral in (3.7) is divergent and indeed it is necessary to regularize this integral like changing the dimensions to $4 - 2\epsilon$; however the spinor-helicity formalism that we are employing are particular to 4 dimensions so the regularization scheme we will use is one where the cut calculations are performed in 4 dimensions and the dimensions of the integrals are only changed afterward.

What is interesting about this calculation is that the ratio of the 1-loop result to the tree-level result matches the expression for a Wilson loop in a special set of coordinates; this duality seems to hold also to high loop orders and in chapter 4 we will return to look at a related duality.

Chapter 4 —

A Duality between Correlation Functions and Wilson Loops

As mentioned in a previous chapter the ratio of MHV loop amplitudes to the corresponding tree amplitude is in planar $\mathcal{N} = 4$ SYM related to Wilson loops up to as high loop orders as have been computed. In order to see the relation one must change from momentum space to what is known as dual space. Dual space is defined such that for an n -point amplitude there are n points in the dual space and their differences are given by the external momenta of the amplitude:

$$p_i = x_i - x_{i+1}. \quad (4.1)$$

Here $x_n + 1$ is defined to be x_1 . Notice how momentum conservation is automatically ensured with this definition:

$$\sum_{i=1}^n p_i = \sum_{i=1}^n x_i - \sum_{i=1}^n x_i. \quad (4.2)$$

What complicates this duality is that scattering amplitudes and Wilson loops have divergences in different regimes so the duality is somewhat broken by the regularization, for instance if one uses dimensional regularization with $d = 4 - 2\epsilon$ it will be necessary to change the sign of ϵ .

In order to make the duality more clear it was proposed in a series of papers [31, 7] to make it part of a triality including correlation function in a special light-

like limit. Correlation functions would then be directly related to Wilson loops without any need for going to dual space and they both have the same type of divergences so this duality is still valid after regularization have been introduced while the duality between correlation functions and amplitudes are at the level of the integrand so there is no need for a regulator.

In this chapter we will consider the duality between correlation functions and Wilson loops and an expansion of the conjecture made in [6]. We will stick to standard Feynman rules for these calculations as they will prove sufficient. The chapter is based on the work done in [33]. The original conjecture was proven in [2] and the expanded conjecture was also discussed in [1].

4.1 The Duality

Consider a set of n scalar operators of the type:

$$\mathcal{O} = \text{Tr}(\phi_{AB}\phi_{CD}) - \frac{1}{12}\epsilon_{ABCD}\text{Tr}(\bar{\phi}^{EF}\phi_{EF}), \quad (4.3)$$

where the latin letters denote the $SU(4)$ R-symmetry index. We place the operators at distinct points in space, x_i , and compute their correlation function and make the distance between adjacent points become light-like ($x_{i,i+1} \equiv x_{i+1} - x_i$). If we just consider the tree-level of the correlation function then this will clearly be divergent as the propagators will be proportional to $x_{i,i+1}^{-2}$:

$$\lim_{x_{i,i+1}^2 \rightarrow 0} \langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) \rangle \propto \frac{1}{\prod_{i=1}^n x_{i,i+1}^2} + \text{less divergent} \quad (4.4)$$

As we shall see the loop orders will have a divergence of the same order so the ratio of the correlation function to the tree-level will have a well-defined limit and this limit will be dual to a polygonic Wilson loop with corners at the locations of the operators:

$$\lim_{x_{i,i+1}^2 \rightarrow 0} \frac{\langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) \rangle}{\langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) \rangle_{tree}} = \langle W_n \rangle_{adj}. \quad (4.5)$$



Figure 4.1. Two scalar operators connected via a scalar-gluon-scalar vertex

The duality can be shown to work by using standard Feynman rules and consider each side of the polygon individually. The advantage of this approach is that we do not need to specify the dimension we work in or the numbers of supercharges of the theory (if any) so the results can easily be transferred to other theories than $\mathcal{N} = 4$ SYM.

It will also become apparent that one do not necessarily need scalar operators at the points, x_i , and we will consider not only the case with a scalar frame¹ but also a fermionic and a gluonic frame.

4.2 The Scalar Frame

Let us consider the case where a scalar at the point x_1 , a scalar at the point x_2 and a gluon connect to the same three-point vertex as shown in figure 4.1. The gluon is of course going to be connected to the rest of the diagram but the specific details of this are irrelevant for our purposes as we want to show that in the light-like limit $x_{i,i+1}^2 \rightarrow 0$ the vertex itself is going to look like a Wilson loop vertex.

Let us start by just considering a propagator so that we have something to compare to:

$$\begin{aligned}
& \int d^d p_1 d^d p_2 \frac{e^{ip_1 \cdot x_1 + ip_2 \cdot x_2}}{p_1^2 + i0} \delta^{(d)}(p_1 + p_2), \\
&= -i \int_0^\infty d\lambda \int d^d p_1 e^{ip_1 \cdot (x_1 - x_2) + i\lambda p_1^2 - \lambda 0}, \\
&= -i \int_0^\infty d\lambda \frac{\pi^{2-\epsilon}}{(-i\lambda)^{2-\epsilon}} e^{-i \frac{(x_1 - x_2)^2}{4\lambda} - 0/\lambda}.
\end{aligned} \tag{4.6}$$

Now we can move on to the three-point vertex with two scalars and one gluon, it is going to give us:

¹The polygon created by the light-like propagator will be referred to as the frame

$$\begin{aligned}
& \int d^d p_1 d^d p_2 (p_1 - p_2)_\mu \frac{e^{ip_1 \cdot x_1 + ip_2 \cdot x_2}}{(p_1^2 + i0)(p_2^2 + i0)} \delta^d(p_1 + p_2 + k) \\
&= \int d^d p_1 (2p_1 + k)_\mu \frac{e^{ip_1 \cdot (x_1 - x_2) - ik \cdot x_2}}{(p_1^2 + i0)((p_1 + k)^2 + i0)} \\
&= (-i)^2 \left(-2i \frac{\partial}{\partial x_1^\mu} + k_\mu \right) \int_0^\infty d\alpha_1 d\alpha_2 \int d^d p_1 e^{i\alpha_1 p_1^2 + i\alpha_2 (p_1 + k)^2 + i(\alpha_1 + \alpha_2)0} \\
&\quad e^{ip_1 \cdot (x_1 - x_2) - ik \cdot x_2} \\
&= (-i)^2 \left(-2i \frac{\partial}{\partial x_1^\mu} + k_\mu \right) \int_0^\infty d\alpha_1 d\alpha_2 e^{-ik \cdot x_2} \frac{\pi^{d/2}}{(-i(\alpha_1 + \alpha_2))^{d/2}} \\
&\quad \exp \left[i \frac{\alpha_1 \alpha_2 k^2}{\alpha_1 + \alpha_2} - i \frac{(x_1 - x_2)^2}{4(\alpha_1 + \alpha_2)} - i \frac{\alpha_2 k \cdot (x_1 - x_2)}{\alpha_1 + \alpha_2} - (\alpha_1 + \alpha_2)0 \right].
\end{aligned} \tag{4.7}$$

We now switch variables to $\alpha_1 = \lambda t$ and $\alpha_2 = \lambda(1 - t)$ and get:

$$(-i)^2 \left(-2i \frac{\partial}{\partial x_1^\mu} + k_\mu \right) \int_0^1 dt \int_0^\infty d\lambda \lambda \frac{\pi^{2-\epsilon} e^{-ik \cdot x_2}}{(-i\lambda)^{2-\epsilon}} e^{i\lambda(t-t^2)k^2 - i \frac{(x_1 - x_2)^2}{4\lambda} - i(1-t)k \cdot (x_1 - x_2) - \lambda 0 - 0/\lambda}. \tag{4.8}$$

The "0" has been rearranged to make the integral convergent. This is valid because the rearrangement does not change the sign of the "0". Notice that the derivative will bring down a factor of λ . That is important and distinguishes this vertex from all the other scalar vertices.

$$\begin{aligned}
& (-i)^2 \int_0^1 dt \int_0^\infty d\lambda \lambda \frac{\pi^{2-\epsilon} e^{-ik \cdot x_2}}{(-i\lambda)^{2-\epsilon}} \left(-\frac{(x_1 - x_2)_\mu}{\lambda} - (1 - 2t)k_\mu \right) \\
& \times e^{i\lambda(t-t^2)k^2 - i \frac{(x_1 - x_2)^2}{4\lambda} - i(1-t)k \cdot (x_1 - x_2) - \lambda 0 - 0/\lambda}.
\end{aligned} \tag{4.9}$$

The integrals here and the ones on the subsequent pages are of the type:

$$\begin{aligned}
& \int d\lambda \lambda^{m-2+\epsilon} e^{i\lambda f - i \frac{z}{\lambda} - o\lambda - 0/\lambda}, \\
& = 2(-if)^{\frac{1-m-\epsilon}{2}} (iz)^{\frac{m+\epsilon-1}{2}} K_{1-m-\epsilon} K_{1-m-\epsilon} \left(2\sqrt{fz} \right).
\end{aligned} \tag{4.10}$$

Here $K_\nu(z)$ is the modified Bessel function of the second kind. This function has the following limiting behaviour:

$$\lim_{z \rightarrow 0} K_\nu(z) = \frac{\Gamma(\nu)2^{\nu-1}}{z^\nu}. \quad (4.11)$$

This is valid for $\nu > 0$ if ν is negative one can use the identity $K_{-\nu}(z) = K_\nu(z)$. Taking the limit $z \rightarrow 0$ of the result in (4.10) gives us:

$$\begin{aligned} & \lim_{z \rightarrow 0} \int d\lambda \lambda^{m-2+\epsilon} e^{i\lambda f - i\frac{z}{\lambda} - o\lambda - 0/\lambda}, \\ &= \begin{cases} (-1)^{\frac{1-m-\epsilon}{2}} \Gamma(1-m-\epsilon) z^{m+\epsilon-1} & \text{if } m \leq 0 \\ (-1)^{\frac{1-m-\epsilon}{2}} \Gamma(m+\epsilon-1) f^{1-m-\epsilon} & \text{if } m \geq 1 \end{cases}. \end{aligned} \quad (4.12)$$

We can now go back to (4.9) and conclude that the second term in the parenthesis will drop out all together and the first term in the exponential will become irrelevant in the light-like limit. What is then left is an integral similar to the propagator (4.6) multiplied by a Wilson line vertex.

This pattern will be similar for all the latter calculations: each additional propagator between the two operators will give an additional factor λ which will either lower the divergence or ruin it all together, the only way to prevent this is if a derivative bring down a factor of λ^{-1} and the only term in the exponential proportional to that is the $(x_1 - x_2)^2$.

Before turning to the general case let us study 2 one-gluon vertices. The momenta going to the two endpoint are denoted p_1 and p_2 while the momenta going out from the vertices are denoted by k_1 and k_2 and the momentum going between the vertices is denoted by q .

$$\int d^d p_1 \int d^d p_2 e^{ip_1 \cdot x_1 + ip_2 \cdot x_2} \int d^d q_1 \int d^d q_2 \frac{(p_1 - q)_\mu (-q - p_2)_\nu}{(p_1^2 + i0)(q^2 + i0)(p_2^2 + i0)} \delta^{(d)}(k_1 + q + p_1) \quad (4.13)$$

$$\delta^{(d)}(k_2 - q + p_2).$$

Again we use momentum conservation to get rid of all the integrals except for

the p_1 integral:

$$\int d^d p_1 e^{ip_1 \cdot (x_1 - x_2) - i(k_1 + k_2) \cdot x_2} \frac{(2p_1 + k_1)_\mu (2p_1 + 2k_1 + k_2)_\nu}{(p_1^2 + i0)((p_1 + k_1)^2 + i0)((p_1 + k_1 + k_2)^2 + i0)}. \quad (4.14)$$

Using derivatives we can reformulate the vectors such that we can pull them outside of the integral. The denominators are replaced by integrals:

$$\begin{aligned} & (-i)^3 \left(-2i \frac{\partial}{\partial x_1^\mu} + k_{1\mu} \right) \left(-2i \frac{\partial}{\partial x_1^\nu} + 2k_{1\nu} + k_{2\nu} \right) \left(\prod_{i=1}^3 \int_0^\infty d\alpha_i \right) \int d^d p_1 e^{-i(k_1 + k_2) \cdot x_2} \\ & \times e^{ip_1 \cdot (x_1 - x_2) + i\alpha_1 p_1^2 + i\alpha_2 (p_1 + k_1)^2 + i\alpha_3 (p_1 + k_1 + k_2)^2 - 0(\alpha_1 + \alpha_2 + \alpha_3)} \\ & = (-i)^3 \left(-2i \frac{\partial}{\partial x_1^\mu} + k_{1\mu} \right) \left(-2i \frac{\partial}{\partial x_1^\nu} + 2k_{1\nu} + k_{2\nu} \right) \left(\prod_{i=1}^3 \int_0^\infty d\alpha_i \right) \int d^d p_1 e^{-i(k_1 + k_2) \cdot x_2} \\ & \times \exp \left(i(\alpha_1 + \alpha_2 + \alpha_3) p_1^2 + i f(\alpha_i, k_i) - i \frac{((\alpha_2 + \alpha_3)k_1 + \alpha_3 k_2) \cdot (x_1 - x_2)}{\alpha_1 + \alpha_2 + \alpha_3} \right) \\ & \times \exp \left(-i \frac{(x_1 - x_2)^2}{4(\alpha_1 + \alpha_2 + \alpha_3)} - 0(\alpha_1 + \alpha_2 + \alpha_3) \right). \end{aligned} \quad (4.15)$$

In the last step p_1 was shifted to get an actual Gaussian integral which we can perform. f is some function which after the change in variables will depend linearly on λ but otherwise we do not care about its specific form. We now replace the α_i integrals with one integral over λ multiplied by three integrals that all have intervals between 0 and 1. The Jacobian for this transformation is λ^2 :

$$\prod_{i=1}^3 \int_0^\infty d\alpha_i = \int_0^\infty d\lambda \lambda^2 \left(\prod_{i=1}^3 \int_0^1 ds_i \right) \delta(1 - s_1 - s_2 - s_3). \quad (4.16)$$

We are now going to shift the s integrals such that $s_1 = t_1$, $s_2 = t_2 - t_1$, and $s_3 = t_3$:

$$\left(\prod_{i=1}^3 \int_0^1 ds_i \right) \delta(1 - s_1 - s_2 - s_3) = \int_0^1 dt_1 \int_{t_1}^{1+t_1} dt_2 \int_0^1 dt_3 \delta(1 - t_2 - t_3). \quad (4.17)$$



Figure 4.2. n one-gluon vertices on a side of the frame

Performing the t_3 integral replaces t_3 with $1-t_2$ but only as long as this function lies in the interval between 0 and 1. This is only satisfied when t_2 is less than 1 so the upper limit on the t_2 integral is lowered.

$$\begin{aligned}
& (-i)^3 \left(-2i \frac{\partial}{\partial x_1^\mu} + k_{1\mu} \right) \left(-2i \frac{\partial}{\partial x_1^\nu} + 2k_{1\nu} + k_{2\nu} \right) \int_0^\infty d\lambda \lambda^2 e^{-i(k_1+k_2) \cdot x_2} \frac{\pi^{2-\epsilon}}{(-i\lambda)^{2-\epsilon}} \\
& \times \int_0^1 dt_1 \int_{t_1}^1 dt_2 e^{i\lambda \tilde{f}(t_i, k_i) - i((1-t_1)k_1 + (1-t_2)k_2) \cdot (x_1 - x_2) - i \frac{(x_1 - x_2)^2}{4\lambda} - 0\lambda + 0/\lambda}. \quad (4.18)
\end{aligned}$$

We can now drop all the terms in the parenthesis not proportional to $x_1 - x_2$ and ignore the \tilde{f} function in the exponential. We are thus left with the integral from the propagator (4.6) and two correctly ordered Wilson line vertices.

Finally, we consider the general case with n one-gluon vertices as shown in figure 4.2. The momenta going to the two endpoint are denoted p_1 and p_2 while the momenta going out from the vertices are denoted by k_i and the momenta going between the vertices are denoted by q_i . The starting expression is thus:

$$\begin{aligned}
& \int d^d p_1 \int d^d p_2 e^{ip_1 \cdot x_1 + ip_2 \cdot x_2} \int d^d q_1 \cdots \int d^d q_{n-1} \\
& \times \frac{(p_1 - q_1)_{\sigma_1} (-q_1 - q_2)_{\sigma_2} \cdots (-q_{n-2} - q_{n-1})_{\sigma_{n-1}} (-q_{n-1} - p_2)_{\sigma_n}}{(p_1^2 + i0)(q_1^2 + i0) \cdots (q_{n-1}^2 + i0)(p_2^2 + i0)} \quad (4.19) \\
& \times \delta^{(d)}(k_1 + q_1 + p_1) \delta^{(d)}(k_2 - q_1 + q_2) \cdots \delta^{(d)}(k_{n-1} - q_{n-2} + q_{n-1}) \delta^{(d)}(k_n - q_{n-1} + p_2)
\end{aligned}$$

By performing all the integrals except the p_1 integral, we get:

$$\begin{aligned}
& \int d^d p_1 e^{ip_1 \cdot (x_1 - x_2) - i \sum_i k_i \cdot x_2} \frac{\prod_{i=1}^n (2p_1 + 2 \sum_{j=1}^{i-1} k_j + k_i)_{\sigma_i}}{(p_1^2 + i0) \prod_{i=1}^n ((p_1 + \sum_{j=1}^i k_j)^2 + i0)} \\
& = (-i)^{(n+1)} \prod_{i=1}^n \left(-2i \frac{\partial}{\partial x_1} + 2 \sum_{j=1}^{i-1} k_j + k_i \right)_{\sigma_i} \left(\prod_{i=1}^{n+1} \int_0^\infty d\alpha_i \right) e^{-i \sum_i k_i \cdot x_2} \\
& \times \int d^d p_1 e^{ip_1 \cdot (x_1 - x_2) + i \sum_{i=1}^{n+1} \alpha_i (p_1 + \sum_{j=1}^{i-1} k_j)^2 - 0 \sum \alpha_i} \quad (4.20)
\end{aligned}$$

We are now changing the integration variables such that $\alpha_i = \lambda s_i$, where the s_i can lie between 0 and 1 as long as they sum up to one while λ can go from zero to infinity. The Jacobian for this transformation is λ^n .

The integral now takes the form (f and \tilde{f} are again some functions whose specific forms are irrelevant):

$$\begin{aligned}
& (-i)^{(n+1)} \prod_{i=1}^n \left(-2i \frac{\partial}{\partial x_1} + 2 \sum_{j=1}^{i-1} k_j + k_i \right)_{\sigma_i} \left(\prod_{i=1}^{n+1} \int_0^1 ds_i \right) e^{-i \sum_i k_i \cdot x_2} \int_0^\infty d\lambda \lambda^n \int d^d p_1 \\
& \times \exp \left(i\lambda f(s_i, k_i) + ip_1 \cdot (x_1 - x_2) + i\lambda s_1 p_1^2 + 2i\lambda \sum_{i=1}^n \left(p_1 \cdot k_i \sum_{j=i+1}^{n+1} s_j \right) - 0\lambda \right) \\
& \times \delta(1 - \sum_i s_i) \quad (4.21)
\end{aligned}$$

We now shift the integration variables such that $s_1 = t_1$, $s_2 = t_2 - t_1$, $s_3 = t_3 - t_2$, and so on until $s_n = t_n - t_{n-1}$. The integration intervals is shifted correspondingly:

$$\begin{aligned}
& (-i)^{(n+1)} \prod_{i=1}^n \left(-2i \frac{\partial}{\partial x_1} + 2 \sum_{j=1}^{i-1} k_j + k_i \right)_{\sigma_i} \left(\prod_{i=1}^n \int_{t_{i-1}}^{1+t_{i-1}} dt_i \right) \int_0^1 ds_{n+1} \delta(1 - t_n - s_{n+1}) e^{-i \sum_i k_i \cdot x_2} \\
& \times \int_0^\infty d\lambda \lambda^n \exp \left(i\lambda \tilde{f}(t_i, k_i) + i\lambda \left(p_1 + \sum_{i=1}^n k_i (-t_i + t_n + s_{n+1}) + \frac{(x_1 - x_2)}{2\lambda} \right)^2 - i \frac{(x_1 - x_2)^2}{4\lambda} \right. \\
& \left. - i \sum_{i=1}^n k_i \cdot (x_1 - x_2) (-t_i + t_n + s_{n+1}) - 0\lambda \right) \quad (4.22)
\end{aligned}$$

We may now perform the s_{n+1} integral. Because of the delta function this

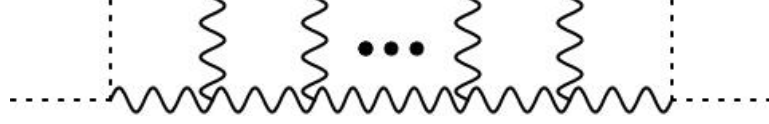


Figure 4.3. A competing type of vertices

replaces all occurrences of s_{n+1} with $1 - t_n$ but only as long as this value lies between 0 and 1. This makes 1 the upper limit for all the t_i integrals and we get:

$$\begin{aligned}
 & (-i)^{(n+1)} \prod_{i=1}^n \left(-2i \frac{\partial}{\partial x_1} + 2 \sum_{j=1}^{i-1} k_j + k_i \right)_{\sigma_i} \left(\prod_{i=1}^n \int_{t_{i-1}}^1 dt_i \right) e^{-i \sum_i k_i \cdot (x_2 t_i - x_1 (1-t_i))} \\
 & \int_0^\infty d\lambda \lambda^n \frac{\pi^{2-\epsilon}}{(-i\lambda)^{2-\epsilon}} e^{i\lambda \tilde{f}(t_i, k_i) - i \frac{(x_1 - x_2)^2}{4\lambda} - 0\lambda - 0/\lambda} \quad (4.23)
 \end{aligned}$$

The derivative in front will bring down exactly the terms we want together with λ^{-n} so this is the only thing that survives in the light-like limit and we get the same integral as for the scalar propagator (4.6) multiplied by n correctly ordered Wilson line vertices.

So the scalar-gluon vertices give us the right result but we now need to show that other vertices do not ruin the duality somehow. This is not too complicated as it is mainly a matter of counting: propagators lower the divergence, derivatives in the vertex could increase the divergence.

Since none of the other scalar vertices has a derivative, they will be removed by the light-like limit.

One could imagine a contribution from terms as shown in figure 4.3. If we denote the number of vertices n then there are $n+1$ propagators. This means that one could possibly get a term that diverge correctly in the $(x_1 - x_2)^2 \rightarrow 0$ limit if all vertices are proportional to $(x_1 - x_2)^\mu$. Notice however that there are only $n-2$ outgoing gluon lines so in Feynman gauge two of the vector indices from the vertices have to be contracted, this will bring about an extra factor of $(x_1 - x_2)^2$ so this still goes away in the light-like limit. Going to other gauges than Feynman gauge does not help even though one gets more possibilities for contracting the different vectors as it also brings about an extra factor of q^{-2} .

4.3 The Fermionic Frame

Let us now turn the case of two fermion operators. The special thing about fermions is that the gluon vertex does not contain a derivative. However the \not{p} in the fermion propagator is going to take on the role of the derivative. We begin by showing that without any vertices the propagator is more divergent than in the case of the scalars.

$$\begin{aligned} & \int d^d p_1 d^d p_2 \frac{\not{p}_2 e^{ip_1 \cdot x_1 + ip_2 \cdot x_2}}{p_1^2 + i0} \delta^{(d)}(p_1 + p_2), \\ &= i \not{\partial}_1 \int d^d p_1 e^{ip_1 \cdot (x_1 - x_2)} \frac{1}{p_1^2 + i0}. \end{aligned} \quad (4.24)$$

Here $\partial_1^\mu = \partial / \partial x_{1\mu}$. This becomes:

$$\begin{aligned} & i \not{\partial}_1 \int_0^\infty d\lambda \int d^d p_1 e^{ip_1 \cdot (x_1 - x_2) + i\lambda p_1^2 - \lambda 0}, \\ &= i \not{\partial}_1 \int_0^\infty d\lambda \int d^d p_1 \exp \left[i\lambda \left(p_1 + \frac{x_1 - x_2}{2\lambda} \right)^2 - i \frac{(x_1 - x_2)^2}{4\lambda} - \lambda 0 \right], \\ &= i \not{\partial}_1 \int_0^\infty d\lambda \frac{\pi^{d/2}}{(-i\lambda)^{d/2}} \exp \left[-i \frac{(x_1 - x_2)^2}{4\lambda} - 0/\lambda \right], \\ &= -i \not{\partial}_1 (i\pi)^{2-\epsilon} \left(\frac{4}{(x_1 - x_2)^2} \right)^{1-\epsilon} \int_0^\infty \frac{d\tau}{\tau^\epsilon} e^{-i\tau - 0\tau}, \\ &= - (1 - \epsilon) (\not{x}_1 - \not{x}_2) \pi^{2-\epsilon} 2^{3-2\epsilon} \Gamma(1 - \epsilon) \left(\frac{1}{(x_1 - x_2)^2} \right)^{2-\epsilon}. \end{aligned} \quad (4.25)$$

We see that the propagator is more divergent than in the case of the scalars.

We now jump straight to the case of n gluon vertices:

$$\begin{aligned} & \int d^d p_1 \int d^d p_2 e^{ip_1 \cdot x_1 + ip_2 \cdot x_2} \int d^d q_1 \cdots \int d^d q_{n-1} \frac{\not{p}_2 \gamma^{\sigma_n} \not{q}_{n-1} \cdots \not{q}_1 \gamma^{\sigma_1} \not{p}_1}{(p_1^2 + i0)(q_1^2 + i0) \cdots (q_{n-1}^2 + i0)(p_2^2 + i0)} \\ & \times \delta^{(d)}(k_1 + q_1 + p_1) \delta^{(d)}(k_2 - q_1 + q_2) \cdots \delta^{(d)}(k_{n-1} - q_{n-2} + q_{n-1}) \delta^{(d)}(k_n - q_{n-1} + p_2) \end{aligned} \quad (4.26)$$

By doing similar steps as before in order to do the p_1 integral, we end up with:

$$\begin{aligned}
& (-i)^{(n+1)}(-1)^n \gamma^\nu \left(-i \frac{\partial}{\partial x_1^\nu} \right) \prod_{i=1}^n \gamma^{\sigma_i} \gamma^{\mu_i} \left(-i \frac{\partial}{\partial x_1^{\mu_i}} + \sum_{j=1}^i k_{j\mu_i} \right) \\
& \times \left(\prod_{i=1}^{n+1} \int_0^1 ds_i \right) e^{-i \sum_i k_i \cdot x_2} \int_0^\infty d\lambda \lambda^n \int d^d p_1 \delta(1 - \sum_i s_i) \\
& \times \exp \left(i\lambda f(s_i, k_i) + i p_1 \cdot (x_1 - x_2) + i\lambda s_1 p_1^2 + 2i\lambda \sum_{i=1}^n \left(p_1 \cdot k_i \sum_{j=i+1}^{n+1} s_j \right) - 0\lambda \right), \\
& = (-i)^{(n+1)}(-1)^n \gamma^\nu \left(-i \frac{\partial}{\partial x_1^\nu} \right) \prod_{i=1}^n \gamma^{\sigma_i} \gamma^{\mu_i} \left(-i \frac{\partial}{\partial x_1^{\mu_i}} + \sum_{j=1}^i k_{j\mu_i} \right) \tag{4.27} \\
& \times \left(\prod_{i=1}^n \int_{t_{i-1}}^{1+t_{i-1}} dt_i \right) \int_0^1 ds_{n+1} \delta(1 - t_n - s_{n+1}) e^{-i \sum_i k_i \cdot x_2} \int_0^\infty d\lambda \lambda^n \\
& \times \exp \left(i\lambda \tilde{f}(t_i, k_i) + i\lambda \left(p_1 + \sum_{i=1}^n k_i(-t_i + t_n + s_{n+1}) + \frac{(x_1 - x_2)}{2\lambda} \right)^2 - i \frac{(x_1 - x_2)^2}{4\lambda} \right. \\
& \left. - i \sum_{i=1}^n k_i \cdot (x_1 - x_2)(-t_i + t_n + s_{n+1}) - 0\lambda \right), \\
& = (-i)^{(n+1)}(-1)^n \gamma^\nu \left(-i \frac{\partial}{\partial x_1^\nu} \right) \prod_{i=1}^n \gamma^{\sigma_i} \gamma^{\mu_i} \left(-i \frac{\partial}{\partial x_1^{\mu_i}} + \sum_{j=1}^i k_{j\mu_i} \right) \left(\prod_{i=1}^n \int_{t_{i-1}}^1 dt_i \right) \\
& \times e^{-i \sum_i k_i \cdot (x_2 t_i - x_1(1-t_i))} \int_0^\infty d\lambda \lambda^n \frac{\pi^{2-\epsilon}}{(-i\lambda)^{2-\epsilon}} e^{i\lambda \tilde{f}(t_i, k_i) - i \frac{(x_1 - x_2)^2}{4\lambda} - 0\lambda - 0/\lambda}.
\end{aligned}$$

Just as for the scalars, the derivatives will give the most divergent term and thus the only term that survives the limit. Using gamma matrix algebra (basically using that $(x_1 - x_2) \gamma^\sigma (x_1 - x_2) = 2(x_1 - x_2)^\sigma (x_1 - x_2) - \gamma^\sigma (x_1 - x_2)^2$ and discarding the second term), we get the same as for the propagator multiplied by:

$$i^{n+1} \left(\prod_{i=1}^n (x_1 - x_2)^{\sigma_i} \right) \left(\prod_{i=1}^n \int_{t_{i-1}}^1 dt_i \right) e^{-i \sum_i k_i \cdot (x_2 t_i - x_1(1-t_i))}$$

So just as for the scalar, the fermion frame acts like a Wilson line in the light-like limit.

Again we must make sure that other vertices do not ruin the duality. Let us begin with the vertex with two fermions and one scalar. This vertex is proportional to γ_5 and since



Figure 4.4. A competing type of vertices

$$(\not{x}_1 - \not{x}_2)\gamma_5(\not{x}_1 - \not{x}_2) = -(x_1 - x_2)^2\gamma_5,$$

this vertex will not contribute.

The other vertices that one might worry about are as shown in figure 4.4. Keeping the gauge of the gluon propagator general this diagram becomes

$$\begin{aligned} & \frac{1}{p_1^2 + i0} (\not{p}_1 \gamma^\mu) \frac{1}{q_1^2 + i0} \left(\eta_{\mu\alpha} + \xi \frac{q_{1\mu} q_{q\alpha}}{q_1^2 + i0} \right) ((q_1 - k_2)^\beta \eta^{\alpha\sigma} + (k_2 - q_2)^\alpha \eta^{\sigma\beta} + (q_2 - q_1)^\sigma \eta^{\alpha\beta}) \\ & \times \frac{1}{q_2^2 + i0} \left(\eta_{\beta\nu} + \xi \frac{q_{2\beta} q_{2\nu}}{q_2^2 + i0} \right) (\gamma^\nu \not{p}_2) \frac{1}{p_2^2 + i0}. \end{aligned} \quad (4.28)$$

The terms of order ξ^0 clearly drops out as they can have a maximum of 3 factors of p_1 and they need 4 in order to compete with the divergence of the fermion propagator. The terms of order ξ^1 seem to have the right order of divergence since they can have a maximum of 5 factors of p_1 and that is exactly what they need. However, these terms will have a factor of $\not{p}_1 \not{p}_1$ which lowers the divergence. The terms of order ξ^2 have exactly the same problem, they seem to be even more divergent than the propagator but have two factors of $\not{p}_1 \not{p}_1$ which makes them drop out as well.

So also for the fermionic frame the other vertices do not ruin the result.

4.4 The Gluonic Frame

Let us look at the propagator between two field strengths, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g[A_\mu, A_\nu]$. Since the derivatives are going to make the expression more divergent and we only are interested in the most divergent, we can already now throw away the commutator. We thus get:

$$\begin{aligned}
& \left(-i \frac{\partial}{\partial x_1^\mu} \eta_{\nu\kappa} + i \frac{\partial}{\partial x_1^\nu} \eta_{\mu\kappa} \right) \eta^{\kappa\lambda} \left(-i \frac{\partial}{\partial x_2^\alpha} \eta_{\beta\lambda} + i \frac{\partial}{\partial x_2^\beta} \eta_{\alpha\lambda} \right) \int d^d p_1 \frac{e^{ip_1 \cdot (x_1 - x_2)}}{p_1^2 + i0}, \\
&= \left(-i \frac{\partial}{\partial x_1^\mu} \eta_{\nu\kappa} + i \frac{\partial}{\partial x_1^\nu} \eta_{\mu\kappa} \right) \eta^{\kappa\lambda} \left(-i \frac{\partial}{\partial x_2^\alpha} \eta_{\beta\lambda} + i \frac{\partial}{\partial x_2^\beta} \eta_{\alpha\lambda} \right) \int_0^\infty d\lambda \int d^d p_1 e^{ip_1 \cdot (x_1 - x_2) + i\lambda p_1^2 - 0\lambda}, \\
&= \left(-i \frac{\partial}{\partial x_1^\mu} \eta_{\nu\kappa} + i \frac{\partial}{\partial x_1^\nu} \eta_{\mu\kappa} \right) \eta^{\kappa\lambda} \left(-i \frac{\partial}{\partial x_2^\alpha} \eta_{\beta\lambda} + i \frac{\partial}{\partial x_2^\beta} \eta_{\alpha\lambda} \right) \int_0^\infty d\lambda \frac{\pi^{2-\epsilon}}{(-i\lambda)^{2-\epsilon}} e^{-i \frac{(x_1 - x_2)^2}{4\lambda} - 0/\lambda}, \\
\end{aligned} \tag{4.29}$$

$$\begin{aligned}
&= \frac{1}{4} \int_0^\infty d\lambda \frac{\pi^{2-\epsilon}}{\lambda^2 (-i\lambda)^{2-\epsilon}} ((x_1 - x_2)_\mu \eta_{\nu\kappa} - (x_1 - x_2)_\nu \eta_{\mu\kappa}) \eta^{\kappa\lambda} \\
&\quad \times ((x_2 - x_1)_\alpha \eta_{\beta\lambda} - (x_2 - x_1)_\beta \eta_{\alpha\lambda}) e^{-i \frac{(x_1 - x_2)^2}{4\lambda} - 0/\lambda}.
\end{aligned}$$

We see that this is proportional to $(x_1 - x_2)^{-3+\epsilon}$. It is now straightforward to see that if the frame with vertices is going to compete with this divergence, every single derivative must be proportional to $x_1 - x_2$ and because of the antisymmetry of the field strengths these factors can only have the Lorentz indices of the outgoing gluon legs.

In short the derivatives makes both the denominator and the numerator in (4.5) more divergent canceling out in the ratio and the antisymmetry protects the momenta from going the wrong way. Other than that the calculation more or less follows the one done in the scalar section.

4.5 Expanding the Conjecture

The duality can be expanded in many different ways. In [6] it was suggested that to add additional operators at generic points not light-like separated from the rest would give the duality:

$$\lim_{x_{i,i+1}^2 \rightarrow 0} \frac{\langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) \mathcal{O}(a_1) \cdots \mathcal{O}(a_n) \rangle}{\langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) \rangle_{tree}} = \langle W_n \mathcal{O}(a_1) \cdots \mathcal{O}(a_n) \rangle_{adj}. \tag{4.30}$$

As the computations in the previous sections were done at the level of the individual sides of the frame not relying on what they were connected to it is not hard to expand (4.5) to (4.30). The additional operators cannot in any way

become part of the frame as they would ruin the divergence not being light-like separated from any other operators and so they must just interact with the Wilson loop coming from the limit.

In [33] we also expanded the duality to scalar operators with derivatives, this is not difficult either as just like in the case of the gluonic frame the derivatives give extra divergences both in the numerator and the denominator of (4.5) thus canceling when taking the ratio.

Finally since our calculations did not depend on the number of dimensions or supercharges of the theory the duality can also easily be expanded to other theories than $\mathcal{N} = 4$ Super-Yang-Mills like ABJ(M) or pure Yang-Mills².

²Though including gravitons could very well ruin the duality

Chapter 5 —

Correlation Functions computed through Generalized Unitarity

In this chapter we will discuss how to use generalized unitarity to compute correlation functions. Some of the points will be general but all of the examples are computed in $\mathcal{N} = 4$ SYM. Even though correlation functions naturally exist in real space while generalized unitarity is in momentum space we will see that the method can be quite effective in fact most of the calculations are quite similar to using generalized unitarity to compute scattering amplitudes only we need to calculate an integral in real space in the end¹. The work presented in this chapter is based on [34].

5.1 Correlation Functions

A correlation function can be defined in the following way:

$$\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle = \int [D\Phi] \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) e^{-S_E[\Phi]} , \quad (5.1)$$

where the x_i are points in space-time, Φ are the fields of the theory, S_E is the Euclidean action and \mathcal{O} are some gauge-invariant combinations of fundamental fields.

The correlation functions can be put into a generating function by introducing local sources for all the relevant local operators:

¹In some cases this integral can be quite difficult

$$Z[\mathcal{J}_1, \mathcal{J}_2, \dots] = \int [D\Phi] e^{-S_E[\Phi] - \int d^d x \sum_i \mathcal{J}_i(x) \mathcal{O}_i(x)} . \quad (5.2)$$

The correlation functions are then found by differentiating with respect to the sources and subsequently setting them to zero:

$$\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle = \frac{\delta^n}{\delta \mathcal{J}_1(x_1) \dots \delta \mathcal{J}_n(x_n)} Z[\mathcal{J}_1, \dots] \Big|_{\mathcal{J}_i \rightarrow 0} . \quad (5.3)$$

This is not very different from the definition of scattering amplitudes and of course correlation functions can be computed with Feynman rules just like amplitudes so we can use generalized unitarity as this method is based on the existence of a Feynman diagram representation. However cutting will lead to the appearance of asymptotic states so we will need other quantities known as form factors which contains both gauge-invariant local operators as well as asymptotic states

$$\langle \mathcal{O} | \Phi_1 \dots \Phi_m \rangle , \quad (5.4)$$

Generalized unitarity have been used to compute form factors in [23, 22].

In general one will need not just form factors with a single local operator as in (5.4) but form factors with several local operators in order to correctly capture the terms where the propagator connecting the local operators are canceled by some numerator factors. Form factors with multiple local operators can be avoided if one assumes that all of the operators are at distinct points in space-time or if one from a careful analysis know that the propagator connecting the operators is not going to be canceled by numerator factors.

5.2 Using Generalized Unitarity

As mentioned earlier generalized unitarity is naturally defined in momentum space so we are first going to compute the momentum space correlation functions:

$$\langle \tilde{\mathcal{O}}_1(q_1) \dots \tilde{\mathcal{O}}_n(q_n) \rangle, \quad (5.5)$$

and then subsequently transform back to real space. The quantities $\tilde{\mathcal{O}}_i(q_i)$ are related to the local operators in real space by a Fourier transform:

$$\tilde{\mathcal{O}}(q) = \int \frac{d^d q}{(2\pi)^d} e^{iq \cdot x} \mathcal{O}(x). \quad (5.6)$$

Being written in momentum space could hide some of the useful features of the correlation function such as conformal invariance. Another consequence of the local operators not naturally being defined in momentum space is that the momenta of the operators in (5.5) are completely arbitrary not satisfying any on-shellness conditions as the asymptotic states do.

5.3 Form Factors

We are now going to compute some of the form factors we will need. The methods we will use for computing them is BCFW recursion which in all cases can be applied in one form or another.

The form factors are all MHV not in the sense that they have the same asymptotic states as the MHV amplitudes but in the sense that these are the form factors with the largest differences in the helicities of the asymptotic states that give a non-trivial result *i.e.* they have the lowest number of the Grassmann variables, η , but that number may be different from 8 as it is for scattering amplitudes. As with amplitudes adding a positive helicity gluon to an MHV form factor gives another MHV form factor. Some of the form factors had already been calculated before [34] such as the chiral part of the stress tensor multiplet:

$$\mathcal{T} = \text{Tr}(\phi^{++}\phi^{++}) + 2\sqrt{2}i\theta_\alpha^{+a}\text{Tr}(\psi_a^{+\alpha}\phi^{++}) + \mathcal{O}(\theta^2). \quad (5.7)$$

Here θ_A is the Grassmann part of the super-space. The fields and the θ 's have their $SU(4)$ indices contracted with harmonic variables the conventions of which

be found in appendix A:

$$\phi^{++} = -\frac{1}{2}u_A^{+a}\epsilon_{ab}u_B^{+b}\bar{\phi}^{AB}, \quad \psi_\alpha^{+a} = u_A^{+a}\psi_\alpha^A. \quad (5.8)$$

The reason for the harmonic variables is that the supersymmetry of these chiral fields closes off-shell. This operator is BPS meaning that it commutes with a number of supercharges implying that it has a definite scaling dimension also at the quantum level. Commuting with supercharges translates into symmetries of the form factor so we should expect the form factor for this operator to be particularly simple.

The corresponding form factor was found in [22, 18]:

$$\mathcal{F}_{\mathcal{T}}^{\text{MHV}} = \prod_{i=1}^n \frac{1}{\langle i, i+1 \rangle} \delta^{(4)}(q - \sum_i \lambda_i \tilde{\lambda}_i) \delta^{(4)}(\gamma_+ - \sum_i \lambda_i \eta_{+,i}) \delta^{(4)}(\sum_i \lambda_i \eta_{-,i}) \quad (5.9)$$

Two of the delta functions can be combined to give:

$$\mathcal{F}_{\mathcal{T}}^{\text{MHV}} = \prod_{i=1}^n \frac{1}{\langle i, i+1 \rangle} \delta^{(4)}(q - \sum_i \lambda_i \tilde{\lambda}_i) \delta^{(8)}(\sum_{i=1}^n \eta_{Ai} \lambda_i^\alpha - 1_{Aa}^+ \gamma_{1+}^{a\alpha}). \quad (5.10)$$

We may also need the conjugate of this form factor:

$$\begin{aligned} \mathcal{F}_{\mathcal{T}}^{\text{N}^{\text{max}}\text{MHV}} = & \delta^4(q - \sum_i \lambda_i \tilde{\lambda}_i) \delta^{(4)}(\gamma_+ - \sum_i \lambda_i \eta_{+,i}) \prod_{i=1}^n \frac{1}{[i, i+1]} \\ & \int \prod_i d^4 \tilde{\eta}_i e^{\eta_{Ai} \tilde{\eta}_i^A} \delta^{(4)}(\sum_i \tilde{\lambda}_i \tilde{\eta}_i^A u_{+,A}^a) \end{aligned} \quad (5.11)$$

We now move on to the operator $\mathcal{O}_{A_1 B_1 \dots A_k B_k} = \text{Tr}(\phi_{A_1 B_1} \dots \phi_{A_k B_k})$. This operator is not BPS so it will have to be mixed with other operators to create an operator with a definite scaling dimension.

Still there are closed sectors where the mixing occurs between a fairly restricted class of operators [10]. For this particular operator if one chooses only two different types of scalars to appear in the trace and they are not conjugate to each other

this type of operator only mixes with itself.

The form factor for this operator (regardless of the choice of $SU(4)$ indices) is given by:

$$\langle \tilde{\mathcal{O}}_{A_1 B_1 \dots A_k B_k}(q) | 1 \dots n \rangle = \frac{\delta^4(q - \sum_{l=1}^n p_l)}{\prod_{m=1} \langle m, m+1 \rangle} \sum_{\{a_1, b_1, \dots, a_k, b_k\}} \left(\prod_{i=1}^k H_{a_i A_i b_i B_i} \right) \text{Sp} \left(\prod_{j=1}^k \Sigma_{a_j b_j} \right), \quad (5.12)$$

where n is the number of external fields, q is the momentum associated with the local operator and the sum runs over all of the sets $a_1, b_1, \dots, a_k, b_k$ with $a_1 \leq b_1 < a_2 \leq b_2 \dots b_{k_1} < a_k \leq b_k$ and its cyclic permutations while Sp stands for the trace over the spinor indices, *i.e.* the greek letters, of the Σ matrices. We have defined the quantities:

$$H_{aAbB} = \eta_{Aa}\eta_{Bb} - \eta_{Ba}\eta_{Ab} + \delta_{ab}\eta_{Ba}\eta_{Ab}, \quad (5.13)$$

$$(\Sigma_{a_1 b_1})^\alpha{}_\gamma = \lambda_{a_1}^\alpha \lambda_{b_1}^\beta \varepsilon_{\beta\gamma}. \quad (5.14)$$

Let us now show how to find this formula. First consider the case where the external states exactly match the fields in the trace in that case (5.12) yields just 1 times a momentum conserving delta function; this matches our expectations from the Feynman diagram perspective. Let us now add a positive helicity gluon² the scalar legs on either side of the gluon will be denoted i and j . We then perform the following shift:

$$|\hat{i}\rangle = |i\rangle + z|j\rangle, \quad (5.15)$$

$$|\hat{j}\rangle = |j\rangle - z|i\rangle. \quad (5.16)$$

This shift does not have a boundary term as the gluon will have to be attached to either of the two scalar legs leading to a propagator dependent on z and the

²One may notice that negative helicity gluons cannot appear in (5.12) for these one needs the NMHV form factor

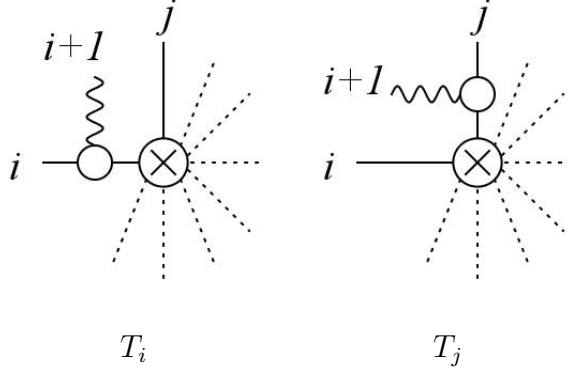


Figure 5.1. BCFW diagrams with only one gluon

form factor will go to 0 as $z \rightarrow \infty$ ³.

There are going to be two BCFW diagrams, shown in figure 5.1⁴, they give the contributions:

$$T_i = \frac{[\hat{i}i+1][i+1\hat{P}_{ii+1}]}{[\hat{i}\hat{P}_{ii+1}]} \frac{1}{s_{ii+1}} = \frac{-[i+1i]\langle ij \rangle}{[ii+1]\langle i+1j \rangle \langle ii+1 \rangle} = \frac{\langle ij \rangle}{\langle ii+1 \rangle \langle i+1j \rangle}, \quad (5.17)$$

$$T_j = \frac{[\hat{P}_{i+1j}i+1][i+1\hat{j}]}{[\hat{P}_{i+1j}\hat{j}]} \frac{1}{s_{i+1j}} = 0. \quad (5.18)$$

The second diagram vanishes because $|\hat{j}\rangle = |j\rangle$ and the existence of the appropriate 3-pt. amplitude requires $\langle j+1\hat{j} \rangle = 0$ which is not satisfied for a generic kinematical configuration. The result is then given by (5.17) which matches the formula (5.12).

To see how this works more generically let us do the following: we call the scalar line on the other side of i for a and make the ansatz:

$$F = \frac{\langle ai \rangle \langle ij \rangle}{\prod_{k=a}^{j-1} \langle kk+1 \rangle} C \quad (5.19)$$

³Actually this argument is based on the gluon-scalar vertex not giving a factor proportional to z to get this one needs to set the reference momentum of the gluon to be p_j ; other choices will, at least superficially, lead to factors proportional to z which then cancel between the different diagrams

⁴In this and all other diagrams in this chapter a circle with a cross means a form factor and a circle without a cross means an amplitude

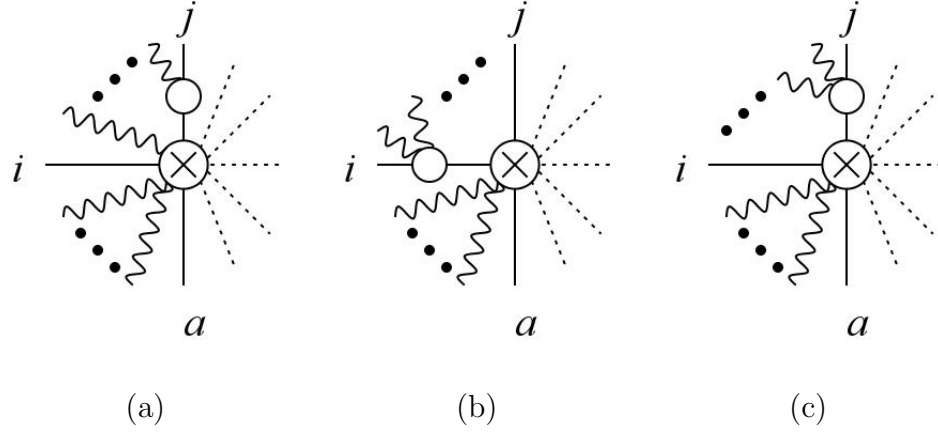


Figure 5.2. BCFW diagrams that all vanish

Here C denotes whatever is on the other side of j and a notice that the scalar lines effectively shields this part off from the piece we are interested in this is because at tree-level there are only planar diagrams, at loop level it would not be so.

We assume that there are at least 2 gluons between the scalar legs i and j and also at least 2 between the scalar legs i and a (the simpler configurations must be dealt separately but the following calculation can be tweaked pretty easily to accommodate them). The shift is going to be the same as before and so are the considerations about boundary contributions.

There are 3 types of diagrams that vanish identically these are shown in figure 5.2. The diagrams in 5.2(b) and 5.2(c) vanish because the amplitude with two scalars and two or more positive helicity gluons is zero (remember that since we are considering the MHV form factor all of the external gluons have positive helicity). The diagram in 5.2(a) vanishes just like the contribution in (5.18) because it requires external legs to have vanishing spinor products inconsistent with a generic kinematical configuration. This leaves us with the only two non-zero diagrams in figure 5.3:

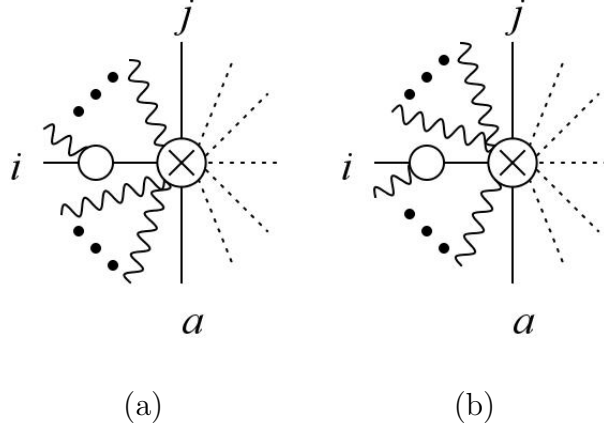


Figure 5.3. BCFW diagrams that contribute to the final result

$$\begin{aligned}
T_1^{\text{fig. 5.3(a)}} &= \frac{[\hat{i}i+1][i+1\hat{P}_{ii+1}]}{[\hat{i}\hat{P}_{ii+1}]} \frac{1}{s_{ii+1}} \frac{\langle a-\hat{P}_{ii+1} \rangle \langle -\hat{P}_{ii+1}\hat{j} \rangle}{\langle i-1-\hat{P}_{ii+1} \rangle \langle -\hat{P}_{ii+1}i+2 \rangle} \frac{C}{\prod_{k=a}^{i-2} \langle kk+1 \rangle \prod_{l=i+2}^{j-1} \langle ll+1 \rangle} \\
&= \frac{C \langle ij \rangle}{\prod_{k=a}^{j-1} \langle kk+1 \rangle} \frac{\langle i-1i \rangle \langle ai+1 \rangle}{\langle i-1i+1 \rangle} \quad (5.20)
\end{aligned}$$

$$\begin{aligned}
T_2^{\text{fig. 5.3(b)}} &= \frac{[\hat{P}_{i-1i}i-1][i-1\hat{i}]}{[\hat{P}_{i-1i}\hat{i}]} \frac{1}{s_{i-1i}} \frac{\langle a-\hat{P}_{i-1i} \rangle \langle -\hat{P}_{i-1i}\hat{j} \rangle}{\langle i-2-\hat{P}_{i-1i} \rangle \langle -\hat{P}_{i-1i}i+1 \rangle} \frac{C}{\prod_{k=a}^{i-3} \langle kk+1 \rangle \prod_{l=i+1}^{j-1} \langle ll+1 \rangle} \\
&= \frac{C \langle ij \rangle}{\prod_{k=a}^{j-1} \langle kk+1 \rangle} \frac{\langle ai-1 \rangle \langle ii+1 \rangle}{\langle i-1i+1 \rangle} \quad (5.21)
\end{aligned}$$

Adding these two contributions leads to

$$F = \frac{\langle ai \rangle \langle ij \rangle}{\prod_{k=a}^{j-1} \langle kk+1 \rangle} C \quad (5.22)$$

which is consistent with (5.19)

Let us now move on to the twist-2 spin- S operators. These operators are linear combinations of

$$\mathcal{O}_{2,S,x}^{AB,CD} = \text{Tr}(D_+^x \phi^{AB} D_+^{S-x} \phi^{CD}). \quad (5.23)$$

for some fixed S where $+$ denotes some light-like direction:

$$\mathcal{O}_{2,S}^{AB,CD} = \sum_{n=0}^S c_{S,n}(\lambda) \mathcal{O}_{2,S,n}^{AB,CD}, \quad (5.24)$$

The coefficients are determined order by order by requiring a definite scaling dimension. At 1-loop the coefficients are given by

$$c_{S,n} = (-1)^n \binom{S}{n}^2, \quad (5.25)$$

while the 2-loop coefficients may be found in [12].

The MHV form factor for the operator (5.23) is given by:

$$\begin{aligned} & \frac{1}{\prod_{m=1}^n \langle mm+1 \rangle} \sum_{\{a,b,c,d\}} H_{aAbB} H_{cCdD} \sum_{k=b}^{c-1} \sum_{l=d}^{a-1} \left(\sum_{r=l+1}^k p_r^- \right)^x \left(\sum_{s=k+1}^l p_s^- \right)^{S-x} \\ & \left(\frac{\langle b|\sigma^- \not{p}_k|c\rangle}{2p_k^-} + \frac{\langle b|\not{p}_{k+1}\sigma^-|c\rangle}{2p_{k+1}^-} - \langle bc \rangle \right) \left(\frac{\langle d|\sigma^- \not{p}_l|a\rangle}{2p_l^-} + \frac{\langle d|\not{p}_{l+1}\sigma^-|a\rangle}{2p_{l+1}^-} - \langle da \rangle \right). \end{aligned} \quad (5.26)$$

where again q is the momentum associated with the local operator and H is given in (5.13). The sum in the beginning runs over all sets $\{a, b, c, d\}$ where $a \leq b < c \leq d$ or $d < a \leq b < c$ etc. The indices a and b are associated with the scalar ϕ^{AB} and the external leg/legs that carries away its $SU(4)$ indices while the indices c and d are associated with the scalar ϕ^{CD} and the external leg/legs that carries away its $SU(4)$ indices see figure 5.4.

In order to show equation (5.26) we are going to use a BCFW shift (sort of). For simplicity we start with the cases without any external fermions.

From this we can get the form factor with the two scalars and a single positive helicity gluon. We want to do this using a BCFW shift but we need to be careful to avoid a boundary term as $z \rightarrow \infty$, this can be accomplished by the following shift that involves the momentum of a gluon and of the operator itself:

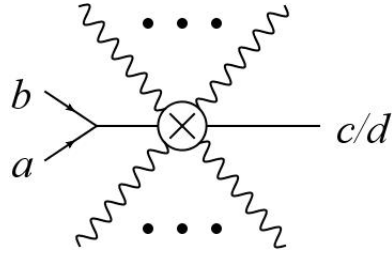


Figure 5.4. An example of the twist-2 form factor showing the conventions of the labels

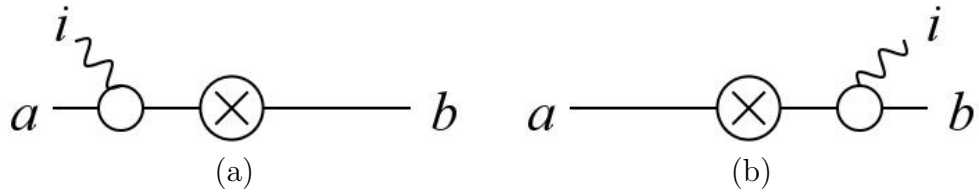


Figure 5.5. BCFW diagrams for the twist-2 operator with only one gluon

$$|\hat{i}\rangle = |i\rangle + z|\phi\rangle, \quad (5.27)$$

$$\hat{q}^\mu = q^\mu + \frac{1}{2}z[i|\sigma^\mu|\phi\rangle, \quad (5.28)$$

where $|\phi\rangle$ satisfy $[i|\sigma^-|\phi\rangle = 0$ which is always possible. With this shift there are two contributions that needs to be calculated shown in figure 5.5:

$$\begin{aligned}
T_1^{\text{fig. 5.5(a)}} &= \frac{[\hat{a}\hat{i}][\hat{i}-\hat{P}_{ia}]}{[a-\hat{P}_{ia}]} \frac{1}{P_{ia}^2} \left(\hat{P}_{ia}^-\right)^x (p_b^-)^{S-x} \\
&= \frac{\langle a\phi \rangle}{\langle ai \rangle \langle i\phi \rangle} (p_a^- + p_i^-)^x (p_b^-)^{S-x} \\
&= \frac{\langle a|\sigma^-|i \rangle}{2p_i^- \langle ai \rangle} (p_a^- + p_i^-)^x (p_b^-)^{S-x} \\
&= \frac{\langle a|\sigma^- \not{p}_i|b \rangle}{2p_i^- \langle ai \rangle \langle ib \rangle} (p_a^- + p_i^-)^x (p_b^-)^{S-x}
\end{aligned} \tag{5.29}$$

$$\begin{aligned}
T_2^{\text{fig. 5.5(b)}} &= \frac{[-\hat{P}_{ib}\hat{i}][\hat{i}b]}{[-\hat{P}_{ib}b]} \frac{1}{P_{ib}^2} (p_a^-)^x \left(\hat{P}_{ib}^-\right)^{S-x} \\
&= \frac{\langle \phi b \rangle}{\langle ib \rangle \langle \phi i \rangle} (p_a^-)^x (p_i^- + p_b^-)^{S-x} \\
&= \frac{\langle b|\sigma^-|i \rangle}{2p_i^- \langle ib \rangle} (p_a^-)^x (p_i^- + p_b^-)^{S-x} \\
&= \frac{\langle a|\not{p}_i\sigma^-|b \rangle}{2p_i^- \langle ai \rangle \langle ib \rangle} (p_a^-)^x (p_i^- + p_b^-)^{S-x}
\end{aligned} \tag{5.30}$$

Here we used that

$$\frac{\langle a\phi \rangle}{\langle i\phi \rangle} = \frac{\langle i|\sigma^-|i \rangle \langle a\phi \rangle}{2p_i^- \langle i\phi \rangle} \tag{5.31}$$

$$= \frac{\langle a|\sigma^-|i \rangle}{2p_i^-} \tag{5.32}$$

which follows from the Schouten identity and $[i|\sigma^-|\phi] = 0$.

Adding the results from the two diagrams we see that this indeed is consistent with the formula in (5.26). We now jump to a more general case shown in figure 5.6:

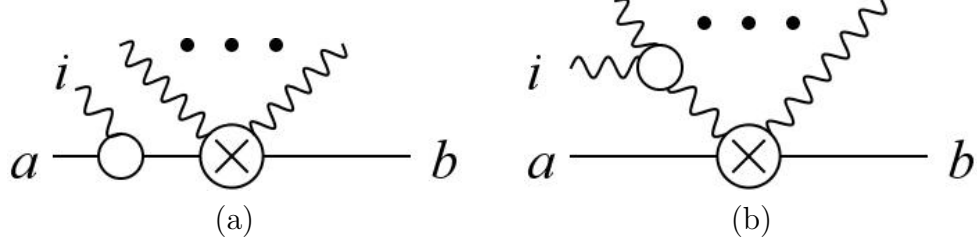


Figure 5.6. BCFW diagrams for the twist-2 operator with many gluons

$$\begin{aligned}
& T_1^{\text{fig. 5.6(a)}} \quad (5.33) \\
& = \frac{[a\hat{i}][\hat{i}-\hat{P}_{ia}]}{[a-\hat{P}_{ia}]} \frac{1}{P_{ia}^2} \frac{1}{\prod_{m=i+1}^b \langle mm+1 \rangle} \frac{1}{\langle \hat{P}_{ia}i+1 \rangle} \left[\sum_{k=i+1}^{b-1} \left(\hat{P}_{ia}^- + \sum_{r=i+1}^k p_r^- \right)^x \left(\sum_{s=k+1}^b p_s^- \right)^{S-x} \right. \\
& \quad \left. \left(\frac{\langle \hat{P}_{ia} | \sigma^- \not{p}_k | b \rangle}{2p_k^-} + \frac{\langle \hat{P}_{ia} \not{p}_{k+1} \sigma^- | b \rangle}{2p_{k+1}^-} - \langle \hat{P}_{ia} b \rangle \right) + \left(\hat{P}_{ia}^- \right)^x \left(\sum_{s=i+1}^b p_s^- \right)^{S-x} \frac{\langle \hat{P}_{ia} | \not{p}_{i+1} \sigma^- | b \rangle}{2p_{i+1}^-} \right] \\
& = \frac{-1}{\langle ia \rangle \prod_{m=i+1}^{b-1} \langle mm+1 \rangle} \frac{\langle a\phi \rangle}{\langle i\phi \rangle \langle ai+1 \rangle} \left[\sum_{k=i+1}^{b-1} \left(\sum_{r=a}^k p_r^- \right)^x \left(\sum_{s=k+1}^b p_s^- \right)^{S-x} \left(\frac{\langle a | \sigma^- \not{p}_k | b \rangle}{2p_k^-} \right. \right. \\
& \quad \left. \left. + \frac{\langle a | \not{p}_{k+1} \sigma^- | b \rangle}{2p_{k+1}^-} - \langle ab \rangle \right) + (p_i^- + p_a^-)^x \left(\sum_{s=k+1}^b p_m^- \right)^{S-x} \frac{\langle a | \not{p}_{i+1} \sigma^- | b \rangle}{2p_{i+1}^-} \right] \quad (5.34) \\
& = \frac{1}{\prod_{m=a}^{b-1} \langle mm+1 \rangle} (p_a^- + p_i^-)^x \left(\sum_{s=i+1}^b p_s^- \right)^{S-x} \left(\frac{\langle a | \sigma^- \not{p}_i | b \rangle}{2p_i^-} + \frac{\langle a | \not{p}_{i+1} \sigma^- | b \rangle}{2p_{i+1}^-} - \langle ab \rangle \right) \\
& \quad + \frac{1}{\prod_{m=a}^{b-1} \langle mm+1 \rangle} \frac{\langle a | \sigma^- | i \rangle \langle ii+1 \rangle}{2p_i^- \langle ai+1 \rangle} \sum_{k=i+1}^{b-1} \left(\sum_{r=a}^k p_r^- \right)^x \left(\sum_{s=k+1}^b p_s^- \right)^{S-x} \\
& \quad \left(\frac{\langle a | \sigma^- \not{p}_k | b \rangle}{2p_k^-} + \frac{\langle a | \not{p}_{k+1} \sigma^- | b \rangle}{2p_{k+1}^-} - \langle ab \rangle \right)
\end{aligned}$$

$$\begin{aligned}
& T_2^{\text{fig. 5.6(b)}} \\
&= \frac{[\hat{i}i+1]^4}{[-\hat{P}_{ii+1}\hat{i}][\hat{i}i+1][i+1-\hat{P}_{ii+1}]} \frac{1}{P_{ii+1}^2} \frac{1}{\prod_{m=i+2}^{b-1} \langle mm+1 \rangle} \frac{1}{\langle a\hat{P}_{ii+1} \rangle \langle \hat{P}_{ii+1}i+2 \rangle} \\
& \left[(p_a^-)^x \left(\hat{P}_{ii+1}^- + \sum_{s=i+2}^b p_s^- \right)^{S-x} \frac{\langle a|\hat{\mathcal{P}}_{ii+1}\sigma^-|b \rangle}{2\hat{P}_{ii+1}^-} + (p_a^- + \hat{P}_{ii+1}^-)^x \left(\sum_{s=i+2}^b p_s^- \right)^{S-x} \right. \\
& \left(\frac{\langle a|\sigma^- \hat{\mathcal{P}}_{ii+1}|b \rangle}{2\hat{P}_{ii+1}^-} + \frac{\langle a|\not{p}_{i+2}\sigma^-|b \rangle}{2p_{i+2}^-} - \langle ab \rangle \right) + \sum_{k=i+2}^{b-1} \left(p_a^- + \hat{P}_{ii+1}^- + \sum_{r=i+2}^k p_r^- \right)^x \left(\sum_{s=k+1}^b p_s^- \right)^{S-x} \\
& \left. \left(\frac{\langle a|\sigma^- \not{p}_k|b \rangle}{2p_k^-} + \frac{\langle a|\not{p}_{k+1}\sigma^-|b \rangle}{2p_{k+1}^-} - \langle ab \rangle \right) \right] \tag{5.35} \\
&= \frac{1}{\prod_{m=i}^{b-1} \langle mm+1 \rangle} \left[(p_a^-)^x \left(\sum_{s=i}^b p_s^- \right)^{S-x} \frac{\langle \phi|(\not{p}_i + \not{p}_{i+1})\sigma^-|b \rangle}{2\langle \phi i \rangle (p_i^- + p_{i+1}^-)} + \frac{\langle \phi i+1 \rangle}{\langle ai+1 \rangle \langle \phi i \rangle} \right. \\
& \sum_{k=i+2}^{b-1} \left(\sum_{r=a}^k p_r^- \right)^x \left(\sum_{s=k+1}^b p_s^- \right)^{S-x} \left(\frac{\langle a|\sigma^- \not{p}_k|b \rangle}{2p_k^-} + \frac{\langle a|\not{p}_{k+1}\sigma^-|b \rangle}{2p_{k+1}^-} - \langle ab \rangle \right) \\
& \left. + (p_a^- + p_i^- + p_{i+1}^-)^x \left(\sum_{s=i+2}^b p_s^- \right)^{S-x} \left(\frac{-\langle \phi|(\not{p}_i + \not{p}_{i+1})\sigma^-|b \rangle}{2\langle \phi i \rangle (p_i^- + p_{i+1}^-)} + \frac{\langle \phi i+1 \rangle}{\langle ai+1 \rangle \langle \phi i \rangle} \frac{\langle a|\not{p}_{i+2}\sigma^-|b \rangle}{2p_{i+2}^-} \right) \right] \\
&= \frac{1}{\prod_{m=a}^{b-1} \langle mm+1 \rangle} (p_a^-)^x \left(\sum_{s=i}^b p_s^- \right)^{S-x} \frac{\langle a|\not{p}_i\sigma^-|b \rangle}{2p_i^-} + \frac{1}{\prod_{m=i}^{b-1} \langle mm+1 \rangle} \frac{\langle i+1|\sigma^-|i \rangle}{2p_i^- \langle ai+1 \rangle} \sum_{k=i+1}^{b-1} \left(\sum_{r=a}^k p_r^- \right)^x \\
& \left(\sum_{s=k+1}^b p_s^- \right)^{S-x} \left(\frac{\langle a|\sigma^- \not{p}_k|b \rangle}{2p_k^-} + \frac{\langle a|\not{p}_{k+1}\sigma^-|b \rangle}{2p_{k+1}^-} - \langle ab \rangle \right)
\end{aligned}$$

Adding the two contributions together we get exactly what is in equation (5.26). Of course so far we have only been adding gluon on one side but the calculations are very similar. The same can be said about the form factor with fermions.

Finally let us consider the form factor with a single stress energy tensor. A stress energy tensor basically measures the momentum and energy on some internal line in the form factor. It is given by terms like:

$$\mathcal{T}^{\mu\nu} = \text{Tr}(D^\mu \bar{\phi}^{AB} D^\nu \phi_{AB}) + \dots \tag{5.36}$$

One might think that it would simply follow from the expression in (5.10) as

it is in the same multiplet however that is not the case. The stress energy tensor has the properties that it is conserved and traceless and the expression one would get from (5.10) is not. Instead the MHV form factor is given by:

$$\mathcal{F}^{\mu\nu} = \frac{\delta^4(q - \sum_i p_i)}{\prod_{j=1}^n \langle jj+1 \rangle} \frac{1}{4!} \sum_{a=1}^n \sum_{b=1}^n \sum_{c=1}^n \sum_{d=1}^n \epsilon^{ABCD} \eta_{aA} \langle a|q\bar{\sigma}^\mu|b\rangle \eta_{bB} \eta_{cC} \langle c|q\bar{\sigma}^\nu|d\rangle \eta_{dD}. \quad (5.37)$$

For some simple form factors with a low number of external legs this formula has to be shown by Feynman rules but with a couple of fundamental form factors in order, we can find the rest using BCFW shifts. It might seem at first hand that all shifts will have a problem when z goes to infinity as some of the terms in the operator have two derivatives ∂^μ and ∂^ν , however because of the form of the operator cancellations appear that give a good behaviour at infinity. The key is to find shifts that would behave nicely as long as neither ∂^μ nor ∂^ν carries away the shifted momentum. Let us make a shift that if it was not for the derivatives would have a nice behaviour at infinity using arguments like in the example in chapter 2:

$$|\hat{i}\rangle = |i\rangle + z|j\rangle, \quad (5.38)$$

$$|\hat{j}\rangle = |j\rangle - z|i\rangle, \quad (5.39)$$

we can write down a general formula:

$$\mathcal{F}^{\mu\nu} = z\langle j|\sigma^\mu|i\rangle\langle j|\sigma^\nu|i\rangle f_1 + \langle j|\sigma^\mu|i\rangle f_2^\nu + \langle j|\sigma^\nu|i\rangle f_2^\mu + \mathcal{O}(1/z). \quad (5.40)$$

We can now use conservation to find the functions f_1 and f_2^μ :

$$\begin{aligned} 0 &= q_\mu q_\nu \mathcal{F}^{\mu\nu}, \\ &= z\langle j|q|i\rangle\langle j|q|i\rangle f_1 + 2\langle j|q|i\rangle q \cdot f_2 + \mathcal{O}(1/z). \end{aligned} \quad (5.41)$$

From this we conclude that $f_1 = 0$ and $q \cdot f_2 = 0$. Now only contracted one of

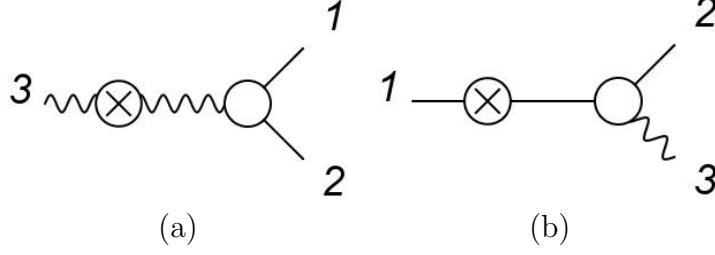


Figure 5.7. The legs 1 and 3 are shifted

the Lorentz indices we get:

$$\begin{aligned}
 0 &= q_\mu \mathcal{F}^{\mu\nu}, \\
 &= \langle j|q|i\rangle f_2^\nu + \mathcal{O}(1/z),
 \end{aligned} \tag{5.42}$$

and we can conclude that $f_2^\nu = 0$. In conclusion, even though the number of derivatives in this operator may be cause for the concern, the specific form of the operator means that the form factor has to be well behaved for $z \rightarrow \infty$ in order to be conserved.

So let us now consider the case with two scalars (particle 1 and 2) and one gluon (particle 3) and choose $j = 1$ and $i = 3$. There are 2 diagrams as shown in figure 5.7. Diagram (a) drops out because as with some of the diagrams in previous calculations it is not consistent with a generic kinematical configuration. For the second diagram the 3-point amplitude and the propagator gives:

$$T^{\text{fig. 5.7(b)}} = \mathcal{F}^{\mu\nu}(\hat{1}, \hat{P}_{23}) \frac{-1}{s_{23}} \frac{[-\hat{P}_{23}\hat{3}]^2 [\hat{3}2]^2}{[2\hat{3}][\hat{3}-\hat{P}_{23}][-\hat{P}_{23}2]} \tag{5.43}$$

$$\begin{aligned}
 &= \mathcal{F}^{\mu\nu}(\hat{1}, \hat{P}_{23}) \frac{[\hat{P}_{23}\hat{3}]^2 \langle \hat{P}_{23}1 \rangle \langle 1\hat{P}_{23} \rangle}{\langle 23 \rangle [32] \langle 21 \rangle \langle 13 \rangle [32]}, \\
 &= \mathcal{F}^{\mu\nu}(\hat{1}, \hat{P}_{23}) \frac{[\hat{P}_{23}\hat{3}]^2 \langle \hat{P}_{23}1 \rangle \langle 1\hat{P}_{23} \rangle}{[2\hat{3}]^2 \langle 12 \rangle \langle 23 \rangle \langle 31 \rangle}.
 \end{aligned} \tag{5.44}$$

We did not write this in full detail since the expression (5.37) is rather long and it might seem more complicated than it is. The first factor with the spinor products $[\hat{P}_{23}\hat{3}]$ is simply going to change the spinors $|\hat{P}_{23}\rangle$ in the numerator of

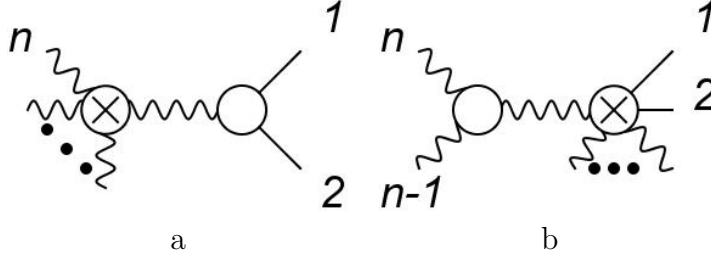


Figure 5.8. The legs 1 and n are shifted

which there are two into spinors of the type $|2\rangle$ (see equation (2.40)). The second factor is going to replace the numerator of the form factor with the correct one for the 3-point form factor so we see that:

$$T^{\text{fig. 5.7(b)}} = \mathcal{F}^{\mu\nu}(1, 2, 3). \quad (5.45)$$

In order to generalize this note that the BCFW shift can only generate diagrams with a 3-point $\overline{\text{MHV}}$ amplitude. This can be seen by counting the available Grassmann variables: there are 4 η 's in the MHV form factor, 4 more from the internal propagator, since the MHV form factor in the BCFW diagram also needs 4 η 's that leaves 4 for the amplitude which can only be accomplished this way. So for more external legs there are still just two BCFW diagrams shown in figure 5.8. Again the first diagram disappears while for the other diagram we get:

$$\frac{1}{s_{n-1n}} \frac{[n-1\hat{n}]^4}{[n-1\hat{n}][\hat{n}-\hat{P}_{n-1n}][-\hat{P}_{n-1n}n-1]} = \frac{\langle \hat{P}_{n-1n}n-2 \rangle \langle \hat{1}\hat{P}_{n-1n} \rangle}{\langle n-2n-1 \rangle \langle n-1n \rangle \langle n1 \rangle}, \quad (5.46)$$

which makes the MHV denominator correct. Adding positive helicity gluons to the other form factors work in a similar way.

The calculations work the same way for the other external fields one needs to compute the the lower-point form factors with Feynman rules recognize it is given by (5.40) and then use BCFW recursion to add positive helicity gluons.

The result (5.37) can in fact be written in terms of a delta function but it requires a slightly weird way of picking it out. If we write down a generating function:

$$\mathcal{G} = \frac{\delta^4(q - \sum_i p_i) \delta^8(\gamma - \sum_{i=1}^n \eta_i \lambda_i)}{\prod_{j=1}^n \langle jj + 1 \rangle}, \quad (5.47)$$

where γ has both an $SU(4)$ index and a spinor index, then the form factor of the energy momentum tensor can be picked out the following way:

$$\mathcal{F}^{\mu\nu} = \frac{1}{4!} \int d^8 \gamma \epsilon^{ABCD} \gamma_A q \bar{\sigma}^\mu \gamma_B \gamma_C q \bar{\sigma}^\nu \gamma_D \mathcal{G}. \quad (5.48)$$

Notice that (5.47) does incorporate (5.10) and that (5.48) manifestly respects that the stress-energy tensor is conserved and traceless.

5.4 Correlation Functions

Let us now turn towards correlation functions and how to compute them using generalized unitarity. We will start by reproducing a simple example that can be found in the literature to high loop order[30, 29]: that of four scalar BPS operators which we choose to be $\mathcal{O}_1 = \text{Tr}(\phi_{34}\phi_{14})$, $\mathcal{O}_2 = \text{Tr}(\phi_{23}\phi_{13})$, $\mathcal{O}_3 = \text{Tr}(\phi_{24}\phi_{24})$ and $\mathcal{O}_4 = \text{Tr}(\phi_{13}\phi_{12})$. This particular choice was made to avoid disconnected diagrams. At 1-loop the calculation is rather simple: as there are no derivatives in the operators themselves there is no way to construct numerators that cancel any propagators so the 4-particle cut will give us all we need to know and the 4-particle cut is just 1.

This means that the correlation function is just given by the propagators:

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle^{(1)} = \frac{1}{x_{1,2}^2 x_{2,3}^2 x_{3,4}^2 x_{4,1}^2}. \quad (5.49)$$

At 2-loop we use that the operators are BPS: this should ensure the correlation function is UV-finite and should be possible to calculate from the 4 cuts shown in figure 5.9. The cuts can be found to be:

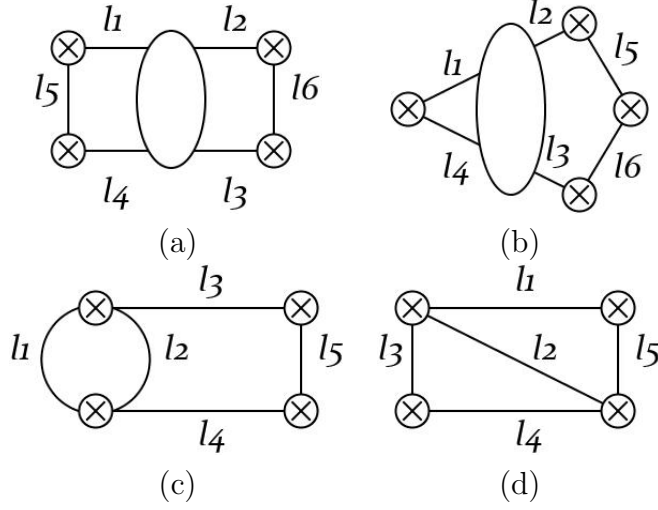


Figure 5.9. The 2-loop cuts for 4 BPS operators

$$\mathcal{C}_{5.9(a)} = \frac{\langle l_1 l_4 \rangle \langle l_2 l_3 \rangle}{\langle l_1 l_3 \rangle \langle l_4 l_2 \rangle} = \frac{(q_1 + q_4)^2}{(l_1 + l_3)^2}, \quad \mathcal{C}_{5.9(b)} = \frac{\langle l_1 l_4 \rangle \langle l_2 l_3 \rangle}{\langle l_1 l_3 \rangle \langle l_4 l_2 \rangle} = \frac{q_1^2}{(l_1 + l_3)^2} \quad (5.50)$$

$$\mathcal{C}_{5.9(c)} = \frac{1}{2l_1 \cdot l_2} \left[\frac{\langle l_1 l_3 \rangle [l_1 l_4]}{\langle l_2 l_3 \rangle [l_2 l_4]} + \frac{\langle l_2 l_3 \rangle [l_2 l_4]}{\langle l_3 l_1 \rangle [l_4 l_1]} + 2 \right] \quad \mathcal{C}_{5.9(d)} = \frac{\langle l_1 l_3 \rangle [l_4 l_5]}{\langle l_1 l_2 \rangle \langle l_2 l_3 \rangle [l_2 l_5] [l_4 l_2]}. \quad (5.51)$$

Using the integrals defined in figure 5.10 this can be written as follows:

$$\begin{aligned} & \langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle^{(2)} \\ &= \delta^{(4)} \left(\sum_{i=1}^4 q_i \right) \left[(q_1 + q_2)^2 \text{DB}(1, 2|3, 4) + (q_1 + q_4)^2 \text{DB}(4, 1|2, 3) \right. \\ & \quad + q_1^2 \text{TriP}(1|2, 3, 4) + q_2^2 \text{TriP}(2|3, 4, 1) + q_3^2 \text{TriP}(3|4, 1, 2) + q_4^2 \text{TriP}(4|1, 2, 3) \\ & \quad - \text{TriB}(1|2|3, 4) - \text{TriB}(2|1|3, 4) - \text{TriB}(2|3|4, 1) - \text{TriB}(3|2|4, 1) \\ & \quad \left. - \text{TriB}(3|4|1, 2) - \text{TriB}(4|3|1, 2) - \text{TriB}(4|1|2, 3) - \text{TriB}(1|4|2, 3) \right]. \quad (5.53) \end{aligned}$$

Fourier-transforming back to real space this gives the expected result originally found in [28]. The integrals are however not that easy to do and the transformation

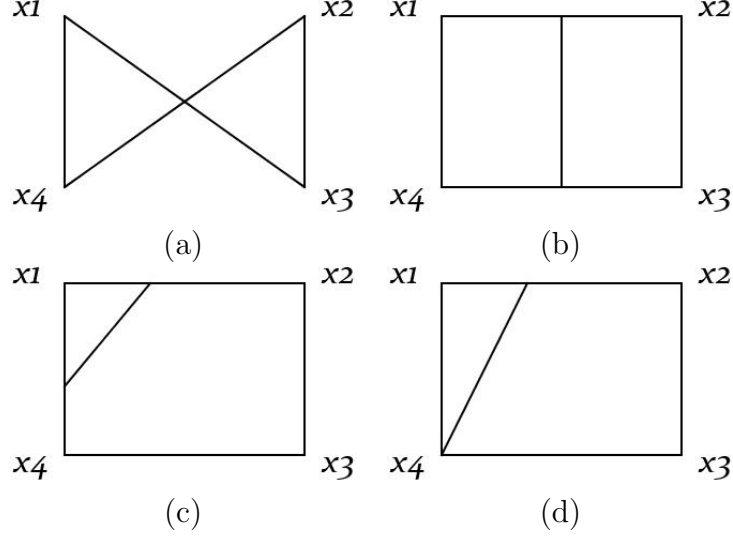


Figure 5.10. 2-loop integrals: BTie(4, 1|2, 3), DB(4, 1|2, 3), TriP(4, 1|2, 3) and TriB(4|1|2, 3)

will not be shown here in fact the integral DB is not known in general however the way it appears in (5.53) does lead to a doable integral. We went over this example fairly quickly as it merely reproduces a well-known result and the complicated part about this correlation function is doing the Fourier-transform which is not the focus of this chapter (nor of any of the other chapters of this thesis).

The second example is that of two BPS operators and a single twist-2 operator⁵:

$$\mathcal{O}_1 = \text{Tr}(\phi^{++}\phi^{++}), \quad \mathcal{O}_2 = \text{Tr}(\phi^{++}\phi^{++}), \quad \mathcal{O}_3 = \text{Tr}(D_+^x \phi_{AB} D_+^{S-x} \phi_{AB}). \quad (5.54)$$

The harmonic variables depend on the position so we will use the notation $\eta[j]_{i-}^a$ to indicate $\bar{u}_-^{Aa} \eta_{iA}$ at the position x_j . It is also convenient to define:

$$S(a, b, x) = (a^-)^{S-x} (b^-)^x + (b^-)^{S-x} (a^-)^x, \quad (5.55)$$

where $-$ denotes a light-like direction⁶. To leading order the momentum space

⁵or rather an operator that when summed with others of the same type give the twist-2 operator

⁶same as $+$ in (5.54) the index has simply been raised

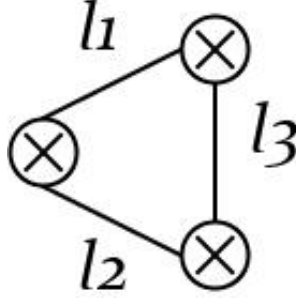


Figure 5.11. The only 1-loop cut for the correlation function with one twist-2 operator

result is determined entirely by the 3-particle cut shown in figure 5.11. All of the cut diagrams will be drawn in the following way: the insertion point is the top right corner is x_1 (related to incoming momentum q_1), the point in the lower right corner is x_2 (related to incoming momentum q_2) while the left point is x_3 (related to incoming momentum q_3). The 1-loop cut gives us:

$$\begin{aligned}
\mathcal{C}_{5.11} &= S(l_1, l_2, x) \int d^4\eta_1 d^4\eta_2 d^4\eta_3 \eta_{1A} \eta_{1B} \eta_{2A} \eta_{2B} \delta^4(\eta[1]_1 - l_1 - \eta[1]_3 - l_3) \\
&\quad \delta^4(\eta[2]_2 - l_2 + \eta[2]_3 - l_3), \\
&= (\mathbf{12}) \epsilon^{ab} 1_{Aa}^+ 1_{Bb}^+ \epsilon^{cd} 2_{Ac}^+ 2_{Bd}^+ S(l_1, l_2, x).
\end{aligned} \tag{5.56}$$

With this we can reconstruct the 1-loop correlation function:

$$(\mathbf{12}) \epsilon^{ab} 1_{Aa}^+ 1_{Bb}^+ \epsilon^{cd} 2_{Ac}^+ 2_{Bd}^+ S(l_1, l_2, x) \text{---} \triangle . \tag{5.57}$$

The 2-loop case is more complicated. We start by looking at the cut shown in figure 5.12:

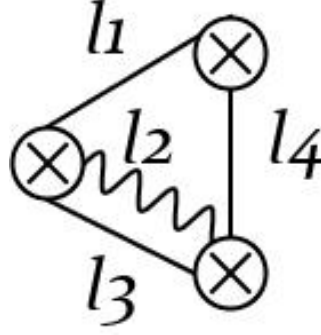


Figure 5.12. Cut number 1

$$\begin{aligned}
\mathcal{C}_{5.12} &= \frac{1}{\langle -l_1 l_4 \rangle \langle l_4 - l_1 \rangle} \frac{1}{[-l_4 - l_3][l_3 - l_2][l_2 - l_4]} \frac{\langle l_3 l_1 \rangle}{\langle l_1 l_2 \rangle \langle l_2 l_3 \rangle \langle l_3 l_1 \rangle} \\
&\quad \left(S(l_1, l_2 + l_3, x) \frac{\langle l_1 | l_2 \bar{\sigma}^- | l_3 \rangle}{2l_2^-} + S(l_1 + l_2, l_3, x) \frac{\langle l_1 | \sigma^- \bar{l}_2 | l_3 \rangle}{2l_2^-} \right) \\
&\quad \int d^4 \eta_1 d^4 \eta_2 d^4 \eta_3 d^4 \eta_4 \int d^4 \tilde{\eta}_2 d^4 \tilde{\eta}_3 d^4 \tilde{\eta}_4 \eta_{1A} \eta_{1B} \eta_{3A} \eta_{3B} \delta^4(\eta[1]_{1-} l_1 - \eta[1]_{4-} l_4) \\
&\quad e^{\eta_{2E} \tilde{\eta}_2^E} e^{\eta_{3E} \tilde{\eta}_3^E} e^{\eta_{4E} \tilde{\eta}_4^E} \delta^4(\tilde{\eta}[2]_2^+ \tilde{l}_2 + \tilde{\eta}[2]_3^+ \tilde{l}_3 + \tilde{\eta}[2]_4^+ \tilde{l}_4) \\
&= \frac{\langle l_1 l_4 \rangle^2 [l_3 l_4]^2 \langle l_3 l_1 \rangle}{\langle l_1 l_4 \rangle \langle l_4 l_1 \rangle [l_4 l_3] [l_3 l_2] [l_2 l_4] \langle l_1 l_2 \rangle \langle l_2 l_3 \rangle \langle l_3 l_1 \rangle} \left(S(l_1, l_2 + l_3, x) \frac{\langle l_1 | l_2 \bar{\sigma}^- | l_3 \rangle}{2l_2^-} \right. \\
&\quad \left. + S(l_1 + l_2, l_3, x) \frac{\langle l_1 | \sigma^- \bar{l}_2 | l_3 \rangle}{2l_2^-} \right) \int d^4 \eta_1 d^4 \eta_2 d^4 \eta_3 d^4 \eta_4 \int d^4 \tilde{\eta}_2 d^4 \tilde{\eta}_3 d^4 \tilde{\eta}_4 \\
&\quad \eta_{1A} \eta_{1B} \eta_{3A} \eta_{3B} \eta[1]_{1-}^1 \eta[1]_{4-}^1 \eta[1]_{1-}^2 \eta[1]_{4-}^2 - \frac{1}{4!} (\eta_{2E} \tilde{\eta}_2^E)^4 \frac{1}{2} (\eta_{3E} \tilde{\eta}_3^E)^2 \frac{1}{2} (\eta_{4E} \tilde{\eta}_4^E)^2 \\
&\quad \tilde{\eta}[2]_{31}^+ \tilde{\eta}[2]_{41}^+ \tilde{\eta}[2]_{32}^+ \tilde{\eta}[2]_{42}^+ \\
&= \frac{[l_4 l_3]}{[l_3 l_2] [l_2 l_4] \langle l_1 l_2 \rangle \langle l_2 l_3 \rangle} \left(S(l_1, l_2 + l_3, x) \frac{\langle l_1 | l_2 \bar{\sigma}^- | l_3 \rangle}{2l_2^-} + S(l_1 + l_2, l_3, x) \right. \\
&\quad \left. \frac{\langle l_1 | \sigma^- \bar{l}_2 | l_3 \rangle}{2l_2^-} \right) \int d^4 \eta_1 \eta_{1A} \eta_{1B} \eta[1]_{1-}^1 \eta[1]_{1-}^2 - \int d^4 \eta_3 \eta_{3A} \eta_{3B} \frac{1}{2} (\eta_{3E} \tilde{\eta}_3^E)^2 \tilde{\eta}[2]_{31}^+ \tilde{\eta}[2]_{32}^+ \\
&\quad \int d^4 \eta_4 \eta[1]_{1-}^1 \eta[1]_{1-}^2 - \frac{1}{2} (\eta_{4E} \tilde{\eta}_4^E)^2 \tilde{\eta}[2]_{41}^+ \tilde{\eta}[2]_{42}^+ \\
&= \frac{[l_4 l_3]}{[l_3 l_2] [l_2 l_4] \langle l_1 l_2 \rangle \langle l_2 l_3 \rangle} \left(S(l_1, l_2 + l_3, x) \frac{\langle l_1 | l_2 \bar{\sigma}^- | l_3 \rangle}{2l_2^-} + S(l_1 + l_2, l_3, x) \right. \\
&\quad \left. \frac{\langle l_1 | \sigma^- \bar{l}_2 | l_3 \rangle}{2l_2^-} \right) (\mathbf{12}) \epsilon^{ab} 1_{Aa}^+ 1_{Bb}^+ \epsilon^{cd} 2_{Ac}^+ 2_{Bd}^+ \\
\end{aligned} \tag{5.58}$$

(5.59)

If we add this to the “twin” diagram where which part is MHV and which is $\overline{\text{MHV}}$ has been switched, we get:

$$\frac{(\mathbf{12})\epsilon^{ab}1_{Aa}^+1_{Bb}^+\epsilon^{cd}2_{Ac}^+2_{Bd}^+}{2l_2^-(l_2+l_3)^2(l_2+l_4)^2(l_1+l_2)^2}\left(S(l_1,l_2+l_3,x)(l_1+l_2)^2\left(2l_2^-(l_3+l_4)^2+\right.\right. \\ \left.2l_3^-(l_2+l_4)^2-l_4^-(l_2+l_3)^2\right)+S(l_1+l_2,l_3,x)\left(2l_2^-(l_1+l_3)^2(l_4+l_2)^2+\right. \\ \left.2l_2^-(l_1-l_4)^2(l_2+l_3)^2-2l_3^-(l_1+l_2)^2(l_2+l_4)^2+2l_4^-(l_1+l_2)^2(l_2+l_3)^2\right)\right) \quad (5.60)$$

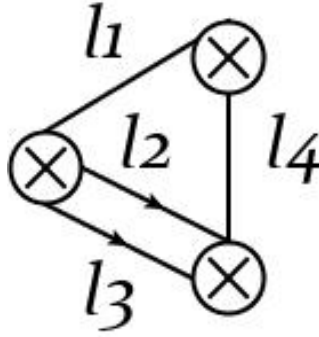


Figure 5.13. Cut number 2

Let us now look at the same thing but with fermions as shown in figure 5.13

$$\mathcal{C}_{5.13} = \frac{1}{\langle -l_1 l_4 \rangle \langle l_4 - l_1 \rangle} \frac{1}{[-l_4 - l_3][-l_3 - l_2][-l_2 - l_4]} \frac{S(l_1, l_2 + l_3, x)}{\langle l_2 l_3 \rangle} \quad (5.61)$$

$$\int d^4\eta_1 d^4\eta_2 d^4\eta_3 d^4\eta_4 \int d^4\tilde{\eta}_2 d^4\tilde{\eta}_3 d^4\tilde{\eta}_4 \eta_{1A} \eta_{1B} \eta_{2A} \eta_{3B} \delta^4(\eta[1]_1 - l_1 - \eta[1]_4 - l_4) \\ e^{\eta_{2E} \tilde{\eta}_2^E} e^{\eta_{3E} \tilde{\eta}_3^E} e^{\eta_{4E} \tilde{\eta}_4^E} \delta^4(\tilde{\eta}[2]_2^+ \tilde{l}_2 + \tilde{\eta}[2]_3^+ \tilde{l}_3 + \tilde{\eta}[2]_4^+ \tilde{l}_4) \\ = (\mathbf{12})\epsilon^{ab}1_{Aa}^+1_{Bb}^+\epsilon^{cd}2_{Ac}^+2_{Bd}^+ \frac{1}{(l_2+l_3)^2} S(l_1, l_2 + l_3, x) \quad (5.62)$$

This cut of course also has a similar one where MHV and $\overline{\text{MHV}}$ have been switched, which gives the same thing.

Adding cut 1 and cut 2 we get:

$$\begin{aligned}
(12) & \epsilon^{ab} 1_{Aa}^+ 1_{Bb}^+ \epsilon^{cd} 2_{Ac}^+ 2_{Bd}^+ \left(\frac{l_3^-}{l_2^-} \frac{1}{(l_2 + l_3)^2} \left(S(l_1, l_2 + l_3, x) - S(l_1 + l_2, l_3, x) \right) \right. \\
& + \frac{l_4^-}{l_2^-} \frac{1}{(l_2 + l_3)^2} \left(S(l_1 + l_2, l_3, x) - S(l_1, l_2 + l_3, x) \right) + S(l_1 + l_2, l_3, x) \\
& \times \left(\frac{q_3^2}{(l_2 + l_3)^2 (l_1 + l_2)^2} - \frac{1}{(l_1 + l_2)^2} - \frac{1}{(l_2 + l_3)^2} + \frac{q_1^2}{(l_2 + l_4)^2 (l_1 + l_2)^2} \right) \\
& \left. + S(l_1, l_2 + l_3, x) \left(\frac{q_2^2}{(l_2 + l_3)^2 (l_2 + l_4)^2} - \frac{1}{(l_2 + l_4)^2} + \frac{1}{(l_2 + l_3)^2} \right) \right)
\end{aligned}$$

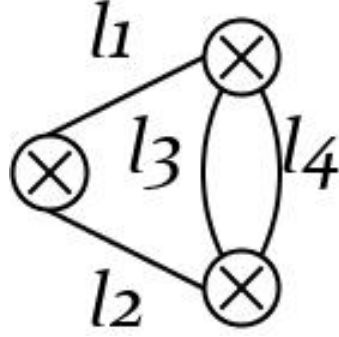


Figure 5.14. Cut number 3

Let us consider the cut in figure 5.14

$$\begin{aligned}
\mathcal{C}_{5.14} &= \frac{1}{\langle -l_1 l_4 \rangle \langle l_4 l_3 \rangle \langle l_3 - l_1 \rangle} \frac{1}{[-l_2 - l_3][-l_3 - l_4][-l_4 - l_2]} S(l_1, l_2, x) \\
&\quad \int d^4 \eta_1 d^4 \eta_2 d^4 \eta_3 d^4 \eta_4 \int d^4 \tilde{\eta}_2 d^4 \tilde{\eta}_3 d^4 \tilde{\eta}_4 \eta_{1A} \eta_{1B} \eta_{2A} \eta_{2B} e^{i\eta_{2E} \tilde{\eta}_2^E} \\
&\quad e^{i\eta_{3E} \tilde{\eta}_3^E} e^{i\eta_{4E} \tilde{\eta}_4^E} \delta^4(-\eta[1]_{1-} l_1 + \eta[1]_{3-} l_3 + \eta[1]_{4-} l_4) \\
&\quad \delta^4(\tilde{\eta}[2]_2^+ \tilde{l}_2 + \tilde{\eta}[2]_3^+ \tilde{l}_3 + \tilde{\eta}[2]_4^+ \tilde{l}_4) \\
&= \frac{1}{\langle l_1 l_4 \rangle \langle l_4 l_3 \rangle \langle l_3 l_1 \rangle} \frac{S(l_1, l_2, x)}{[l_2 l_3][l_3 l_4][l_4 l_2]} \\
&\quad \int d^4 \eta_1 d^4 \eta_2 d^4 \eta_3 d^4 \eta_4 \int d^4 \tilde{\eta}_2 d^4 \tilde{\eta}_3 d^4 \tilde{\eta}_4 \eta_{1A} \eta_{1B} \eta_{2A} \eta_{2B} e^{\eta_{2E} \tilde{\eta}_2^E} \\
&\quad e^{\eta_{3E} \tilde{\eta}_3^E} e^{\eta_{4E} \tilde{\eta}_4^E} \eta[1]_{1-}^1 \eta[1]_{1-}^2 (\langle l_1 l_3 \rangle \eta[1]_{3-}^1 + \langle l_1 l_4 \rangle \eta[1]_{4-}^1) \\
&\quad (\langle l_1 l_3 \rangle \eta[1]_{3-}^2 + \langle l_1 l_4 \rangle \eta[1]_{4-}^2) \tilde{\eta}[2]_{21}^+ \tilde{\eta}[2]_{22}^+ ([l_2 l_3] \tilde{\eta}[2]_{31}^+ + [l_2 l_4] \tilde{\eta}_{41}^+) \\
&\quad ([l_2 l_3] \tilde{\eta}[2]_{32}^+ + [l_2 l_4] \tilde{\eta}_{42}^+) \\
&= -(\mathbf{12}) \epsilon^{ab} 1_{Aa}^+ 1_{Bb}^+ \epsilon^{cd} 2_{Ac}^+ 2_{Bd}^+ S(l_1, l_2, x) \left(\frac{[l_1 l_4] \langle l_4 l_2 \rangle [l_2 l_3] \langle l_3 l_1 \rangle}{(l_3 + l_4)^2 (l_1 - l_4)^2 (l_2 + l_4)^2} \right. \\
&\quad \left. + \frac{[l_3 l_1] \langle l_1 l_4 \rangle [l_4 l_2] \langle l_2 l_3 \rangle}{(l_3 + l_4)^2 (l_1 - l_3)^2 (l_2 + l_3)^2} - \frac{2}{(l_3 + l_4)^2} \right)
\end{aligned} \tag{5.63}$$

By adding the “twin” diagram we get:

$$\begin{aligned}
& -(\mathbf{12}) \epsilon^{ab} 1_{Aa}^+ 1_{Bb}^+ \epsilon^{cd} 2_{Ac}^+ 2_{Bd}^+ S(l_1, l_2, x) \left(\frac{-(l_2 + l_3)^2}{(l_3 + l_4)^2 (l_2 + l_4)^2} \right. \\
& - \frac{(l_1 - l_3)^2}{(l_3 + l_4)^2 (l_1 - l_4)^2} - \frac{(l_1 + l_2)^2}{(l_2 + l_4)^2 (l_1 - l_4)^2} - \frac{(l_2 + l_4)^2}{(l_3 + l_4)^2 (l_2 + l_3)^2} \\
& \left. - \frac{(l_1 - l_4)^2}{(l_3 + l_4)^2 (l_1 - l_3)^2} - \frac{(l_1 + l_2)^2}{(l_1 - l_3)^2 (l_2 + l_3)^2} - \frac{4}{(l_3 + l_4)^2} \right) \\
& = (\mathbf{12}) \epsilon^{ab} 1_{Aa}^+ 1_{Bb}^+ \epsilon^{cd} 2_{Ac}^+ 2_{Bd}^+ S(l_1, l_2, x) \left(\frac{q_2^2}{(l_3 + l_4)^2 (l_2 + l_4)^2} \right. \\
& + \frac{q_2^2}{(l_3 + l_4)^2 (l_2 + l_3)^2} + \frac{q_1^2}{(l_3 + l_4)^2 (l_1 + l_3)^2} + \frac{q_1^2}{(l_3 + l_4)^2 (l_1 - l_4)^2} \\
& + \frac{q_3^2}{(l_2 + l_4)^2 (l_1 - l_4)^2} + \frac{q_3^2}{(l_2 + l_3)^2 (l_1 - l_3)^2} - \frac{1}{(l_2 + l_4)^2} \\
& \left. - \frac{1}{(l_2 + l_3)^2} - \frac{1}{(l_1 + l_3)^2} - \frac{1}{(l_1 - l_4)^2} \right)
\end{aligned} \tag{5.64}$$

We are over-counting in this cut because $l_3 \leftrightarrow l_4$ gives exactly the same thing.

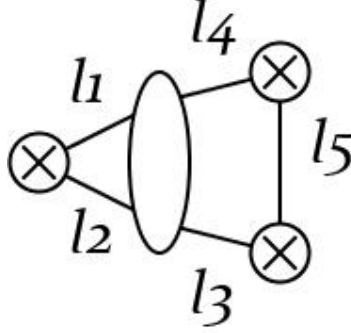


Figure 5.15. Cut number 4

Let us now look at the cut in figure 5.15:

$$\begin{aligned}
\mathcal{C}_{5.15} &= - \frac{S(l_1, l_2, x)}{\langle l_4 - l_5 \rangle \langle -l_5 l_4 \rangle \langle l_5 l_3 \rangle \langle l_3 l_5 \rangle \langle -l_3 - l_2 \rangle \langle -l_2 - l_1 \rangle \langle -l_1 - l_4 \rangle \langle -l_4 - l_3 \rangle} \\
&\quad \int d^4 \eta_1 d^4 \eta_2 d^4 \eta_3 d^4 \eta_4 d^4 \eta_5 \eta_{1A} \eta_{1B} \eta_{2A} \eta_{2B} \delta^4(\eta[1]_{4-} l_4 - \eta[1]_{5-} l_5) \\
&\quad \delta^4(\eta[2]_{3-} l_3 + \eta[2]_{5-} l_5) \delta^8(\eta_1 l_1 + \eta_2 l_2 + \eta_3 l_3 + \eta_4 l_4) \tag{5.65} \\
&= - \frac{S(l_1, l_2, x)}{\langle l_4 l_5 \rangle^2 \langle l_3 l_5 \rangle^2 \langle l_1 l_2 \rangle \langle l_2 l_3 \rangle \langle l_3 l_4 \rangle \langle l_4 l_1 \rangle} \int d^4 \eta_1 d^4 \eta_2 d^4 \eta_3 d^4 \eta_4 d^4 \eta_5 \eta_{1A} \eta_{1B} \\
&\quad \eta_{2A} \eta_{2B} \eta[1]_{4-}^1 \eta[1]_{4-}^2 \langle l_4 l_5 \rangle^2 \eta[1]_{5-}^1 \eta[1]_{5-}^2 \eta[2]_{3-}^1 \eta[2]_{3-}^2 \langle l_3 l_5 \rangle^2 \eta[2]_{5-}^1 \eta[2]_{5-}^2 \\
&\quad \delta^8(\eta_1 l_1 + \eta_2 l_2 + \eta_3 l_3 + \eta_4 l_4) \\
&= - \frac{S(l_1, l_2, x)}{\langle l_2 l_3 \rangle \langle l_4 l_1 \rangle} \int d^4 \eta_3 d^4 \eta_4 d^4 \eta_5 \eta[1]_{4-}^1 \eta[1]_{4-}^2 \eta[1]_{5-}^1 \eta[1]_{5-}^2 \eta[2]_{3-}^1 \eta[2]_{3-}^2 \\
&\quad \eta[2]_{5-}^1 \eta[2]_{5-}^2 \eta_{3A} \eta_{3B} \eta_{4A} \eta_{4B} \langle l_3 l_4 \rangle \langle l_1 l_2 \rangle \\
&= - (\mathbf{12}) \epsilon^{ab} 1_{Aa}^+ 1_{Bb}^+ \epsilon^{cd} 2_{Ac}^+ 2_{Bd}^+ \frac{S(l_1, l_2, x) \langle l_1 l_2 \rangle \langle l_3 l_4 \rangle}{\langle l_2 l_3 \rangle \langle l_4 l_1 \rangle} \\
&= (\mathbf{12}) \epsilon^{ab} 1_{Aa}^+ 1_{Bb}^+ \epsilon^{cd} 2_{Ac}^+ 2_{Bd}^+ S(l_1, l_2, x) \frac{q_3^2}{(l_2 + l_3)^2}
\end{aligned}$$

Finally, we consider the cut shown in figure 5.16:

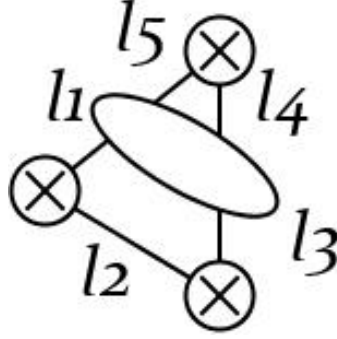


Figure 5.16. Cut number 5

$$\begin{aligned}
\mathcal{C}_{5.16} &= - \frac{S(l_1, l_2, x)}{\langle -l_2 l_3 \rangle \langle l_3 - l_2 \rangle \langle -l_1 - l_5 \rangle \langle -l_5 - l_4 \rangle \langle -l_4 - l_3 \rangle \langle -l_3 - l_1 \rangle \langle l_4 l_5 \rangle \langle l_5 l_4 \rangle} \\
&\quad \int d^4 \eta_1 d^4 \eta_2 d^4 \eta_3 d^4 \eta_4 d^4 \eta_5 \eta_{1A} \eta_{1B} \eta_{2A} \eta_{2B} \delta^4(\eta[1]_{4-l_4} + \eta[1]_{5-l_5}) \\
&\quad \delta^4(\eta[2]_{2-l_2} - \eta[2]_{3-l_3}) \delta^8(\eta_1 l_1 + \eta_3 l_3 + \eta_4 l_4 + \eta_5 l_5) \\
&= - \frac{S(l_1, l_2, x)}{\langle l_2 l_3 \rangle^2 \langle l_4 l_5 \rangle^2 \langle l_1 l_5 \rangle \langle l_5 l_4 \rangle \langle l_4 l_3 \rangle \langle l_3 l_1 \rangle} \frac{1}{(\mathbf{12})^2} \int d^4 \eta_1 d^4 \eta_2 d^4 \eta_3 d^4 \eta_4 d^4 \eta_5 \\
&\quad \eta_{1A} \eta_{1B} \eta_{2A} \eta_{2B} \delta^4(\eta[1]_{4-l_4} + \eta[1]_{5-l_5}) \delta^4(\eta[2]_{2-l_2} - \eta[2]_{3-l_3}) \\
&\quad \delta^4(\eta[1]_{1-l_1} + \eta[1]_{3-l_3}) \delta^4(\eta[2]_{1-l_1} + \eta[2]_{3-l_3} + \eta[2]_{4-l_4} + \eta[2]_{5-l_5}) \\
&= - \frac{S(l_1, l_2, x) \langle l_3 l_1 \rangle \langle l_5 l_4 \rangle}{\langle l_1 l_5 \rangle \langle l_4 l_3 \rangle} \frac{1}{(\mathbf{12})^2} \int d^4 \eta_1 d^4 \eta_2 d^4 \eta_3 d^4 \eta_4 d^4 \eta_5 \eta_{1A} \eta_{1B} \eta_{2A} \eta_{2B} \\
&\quad \eta[1]_4^1 - \eta[1]_4^2 - \eta[1]_5^1 - \eta[1]_5^2 - \eta[2]_2^1 - \eta[2]_2^2 - \eta[2]_3^1 - \eta[2]_3^2 - \eta[1]_1^1 - \eta[1]_1^2 - \\
&\quad \eta[1]_3^1 - \eta[1]_3^2 - \eta[2]_4^1 - \eta[2]_4^2 - \eta[2]_5^1 - \eta[2]_5^2 - \\
&= - (\mathbf{12}) \epsilon^{ab} 1_{Aa}^+ 1_{Bb}^+ \epsilon^{cd} 2_{Ac}^+ 2_{Bd}^+ \frac{S(l_1, l_2, x) \langle l_3 l_1 \rangle \langle l_5 l_4 \rangle}{\langle l_1 l_5 \rangle \langle l_4 l_3 \rangle} \\
&= (\mathbf{12}) \epsilon^{ab} 1_{Aa}^+ 1_{Bb}^+ \epsilon^{cd} 2_{Ac}^+ 2_{Bd}^+ S(l_1, l_2, x) \frac{q_1^2}{(l_1 + l_5)^2} \tag{5.66}
\end{aligned}$$

From the above cuts we can reconstruct the correlation function:

$$\begin{aligned}
& \left[S(l_1, l_2, x) \left(q_3^2 \rightarrow \text{triangle}(l_1, l_2, q_3) + q_1^2 \rightarrow \text{triangle}(l_1, l_2, q_1) + q_2^2 \rightarrow \text{triangle}(l_1, l_2, q_2) - \text{triangle}(l_1, l_2, q_1) - \text{triangle}(l_1, l_2, q_2) \right) \right. \\
& + \left(\frac{l_4^-}{l_2^-} S(l_1 + l_2, l_3, x) + \frac{l_5^-}{l_2^-} S(l_1, l_2 + l_3) \right) \rightarrow \text{triangle}(l_1, l_2, l_3) \text{ with } l_4, l_5 \text{ on } l_2 \\
& + \frac{l_2^- + l_3^-}{l_2^-} \left(S(l_1, l_2 + l_3, x) - S(l_1 + l_2, l_3, x) \right) \rightarrow \text{triangle}(l_1, l_2, l_3) \text{ with } l_2, l_3 \text{ on } l_2 \\
& \left. + \frac{l_1^- + l_2^-}{l_2^-} \left(S(l_1 + l_2, l_3, x) - S(l_1, l_2 + l_3, x) \right) \rightarrow \text{triangle}(l_1, l_2, l_3) \text{ with } l_1, l_2 \text{ on } l_2 \right] (\mathbf{12}) \epsilon^{ab} 1_{Aa}^+ 1_{Bb}^+ \epsilon^{cd} 2_{Ac}^+ 2_{Bd}^+
\end{aligned}$$

All left to be done is Fourier-transform this expression.

We have seen in this chapter that generalized unitarity can be used effectively to compute the momentum space correlation functions. Fourier-transforming the expressions back to real space remains a hurdle. It should however be noted that this is not a problem unique to generalized unitarity as with conventional methods one will often get the correlation functions with derivatives of some integrals that can be hard to solve in general as was the case with the four scalar operators at 2-loops.

Chapter 6 —

Using Generalized Unitarity on 2D Integrable Systems

In this chapter we will consider how to use generalized unitarity on integrable worldsheet S-matrices. We will meet a specific set of challenges due to the 2-dimensional kinematics. However, we will also see the advantages of generalized unitarity over Feynman rules with features of integrability appearing in a manifest way at loop level as well as some simple tests to see if the systems are quantum integrable. The work in this chapter is based on [32] which came out around the same time as [17] which has many similar results.

6.1 Worldsheet Scattering

In regular quantum field theory as a particle travels through space it creates a 1-dimensional object in space-time called a worldline. Similarly in string theory a string traveling through space creates a 2-dimensional object in space-time called the worldsheet. Normally when considering scattering processes in string theory, one deals with worldsheets merging or breaking apart creating something that looks like normal Feynman diagrams except worldlines have been replaced by worldsheets. Worldsheet scattering however turns things on its head and instead of the worldsheet being some object in space-time it is treated as the space in which the scattering takes place. One begins with a particular solution then quantizes the fluctuations around that solution, this means that the directions of space-time is treated as the fields that scatter on the worldsheet.

Worldsheet scattering become interesting when the space is dual to an inte-

grable field theory then all scattering processes can be written in terms of the $2 \rightarrow 2$ S-matrix which is a process that due to 2-dimensional kinematics can be written as a function of two 1-dimensional momenta and the process itself can be written in terms of the scattering of excited states in the spin-chain corresponding to the dual quantum field theory.

Using Feynman rules the actual computations are not that easy and the Lagrangians consist of an infinite number of terms¹. and for $AdS_5 \times S^5$ which is dual to $\mathcal{N} = 4$ Super-Yang-Mills has been found with Feynman rules only fully at the classical level [43] and in the Maldacena-Swanson limit [47] for 1- and 2-loop [42].

Most of the theories we will consider will have a factorized symmetry group and the S-matrix will in turn be factorized:

$$\mathbb{S} = \mathbf{S} \otimes \mathbf{S} , \quad (6.1)$$

As an example of such a factorization consider $AdS_5 \times S^5$ it has 8 bosonic fields, 4 related to the AdS_5 , z_μ , and 4 related to S^5 , y_m , using the Pauli matrices the fields can be written in a two-index notation:

$$Y^{a\dot{a}} = (\sigma_m)^{a\dot{a}} y^m \quad a = 1, 2 \quad Z^{\alpha\dot{\alpha}} = (\sigma_\mu) z^\mu \quad \alpha = 3, 4 \quad (6.2)$$

The fermions that appear will be $\Upsilon^{\alpha\dot{a}}$ and $\Psi^{a\dot{\alpha}}$. The S-matrix then factorizes into one S-matrix for the dotted indices and one S-matrix for the undotted indices. From the spin-chain perspective the dotted and undotted indices will correspond to excited states in two separate spin-chains. This factorization is a rather non-trivial feature as it relates the scattering of completely different fields to each other.

The worldsheet S-matrix can be expanded in terms of the coupling constant g defining a T-matrix:

$$\mathbb{S} = \mathbb{1} + \frac{1}{\hat{g}} i\mathbb{T}^{(0)} + \frac{1}{\hat{g}^2} i\mathbb{T}^{(1)} + \mathcal{O}\left(\frac{1}{\hat{g}^3}\right) \equiv \mathbb{1} + i\mathbb{T} , \quad (6.3)$$

¹Of course only a finite number at each order in the coupling constant but the higher loop order S-matrix one computes the more Lagrangian terms one has to deal with

but the same thing can be done for the individual factors in (6.1):

$$\mathbf{S} = \mathbb{1} + \frac{1}{\hat{g}} i\mathbf{T}^{(0)} + \frac{1}{\hat{g}^2} i\mathbf{T}^{(1)} + \mathcal{O}\left(\frac{1}{\hat{g}^3}\right) \equiv \mathbb{1} + i\mathbf{T} , \quad (6.4)$$

and it is actually those we will focus on because all of the interesting features of the calculations using (6.3) is captured by just focusing on (6.4). We can write the full T-matrix in a factorized way:

$$i\mathbf{T}^{(L)} = \sum_{l=0}^{L+1} (i\mathbf{T}^{(l-1)}) \otimes (i\mathbf{T}^{(L-l)}) \quad i\mathbf{T}^{(-1)} = \mathbb{1}. \quad (6.5)$$

Clearly the additional information in the full T-matrix are simply reiterations of the lower loop results. The cuts do correctly capture these terms as I will briefly touch upon at 1-loop; the calculation is fairly trivial as one might expect since generalized unitarity writes loop level amplitudes as products of amplitudes of lower loop orders and this part of the T-matrix is manifestly a product of lower loop amplitudes.

Before proceeding with more specific details about the S-matrices let us briefly touch upon two rather special properties not common to normal quantum field theories. First of all energy and momentum conservation become rather simple in 2 dimensions when the masses are all the same:

$$\delta^2(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4) \sim \delta(p_1 - p_3)\delta(p_2 - p_4) - \delta(p_1 - p_4)\delta(p_2 - p_3), \quad (6.6)$$

The outgoing momenta are going to be equal to the incoming momenta so there are only two independent momenta in the problem (which we will denote p and p' with the assumption that $p > p'$). This identification of the outgoing momenta with the incoming is quite essential and the S-matrices would not necessarily factorize without it later on we will however see that this identification also causes a problem when one tries to use generalized unitarity.

Equation (6.6) assumed that all masses were the same however we will also consider cases where 2 particles of different masses scatter, in these cases one of

the solutions in (6.6) is still valid, the one where the particles do not exchange momentum *i.e.* a state of momentum p and mass m scatter with a state of momentum p' and mass m' and out comes a state with momentum p and mass m and another state with momentum p' and mass m' . The other solution will become more complicated with the outgoing momenta becoming functions dependent on both momenta and both masses. The S-matrices we will consider with more than one mass will be reflectionless meaning that this complicated solution to energy-momentum conservation will correspond to a vanishing S-matrix element.

The rewriting of the delta function in (6.6) gives rise to a Jacobian which together with a normalization factor for the external states gives a factor that will appear so often in the computations we will define it to be:

$$J = (\sqrt{2\varepsilon}\sqrt{2\varepsilon'})^2 \left(\frac{d\varepsilon}{dp} - \frac{d\varepsilon'}{dp'} \right) ; \quad (6.7)$$

With a dispersion relation $\varepsilon = \sqrt{m^2 + p^2}$ this becomes:

$$(\sqrt{2\varepsilon}\sqrt{2\varepsilon'})^2 \frac{\varepsilon' p - \varepsilon p'}{\varepsilon \varepsilon'} = 4(\varepsilon' p - \varepsilon p') = \frac{2(m^2 p'^2 - m'^2 p^2)}{p - p'} . \quad (6.8)$$

We may from time to time refer to this as the Jacobian though strictly that is not quite accurate.

The second uncommon feature is that worldsheet scattering is not Lorentz invariant so that the S-matrices are not invariant under crossing transformations²:

$$\mathbb{S}^{\text{cross}} = \mathcal{C}^{-1} \mathbb{S}^{\text{st}} \mathcal{C} \quad \quad \mathbf{S}^{\text{cross}} = \mathcal{C}^{-1} \mathbf{S}^{\text{st}} \mathcal{C} . \quad (6.9)$$

Here \mathcal{C} are charge conjugation matrices and st means the supertranspose of the matrix:

$$(M^{\text{st}})_{AB} = (-)^{[A][B]+[B]} M_{BA} , \quad (6.10)$$

²Of course lack of Lorentz invariance may be a common feature in solid state physics but in high energy physics it is not

$[]$ is the grade of the argument meaning it is zero for bosonic indices and 1 for fermionic indices. Going back to the $AdS_5 \times S^5$ example around equation (6.2) the greek indices are fermionic and the latin indices are bosonic making Y, Z bosons³ while Υ and Ψ are fermions. These crossing relations are responsible for some signs in the cuts.

Let us finally go into a bit more details about the exact S-matrices found for the spin-chains. The spin-chain S-matrices can be found using the symmetries of the theory. As an example consider the $SU(2|2)$ S-matrix constructed by Beisert [11] this is the one relevant to $AdS_5 \times S^5$. The S-matrix acting on the spin-chain is defined by:

$$\mathbf{S}^B |\phi_a \phi'_b\rangle = \mathbf{A}^B |\phi'_{[a} \phi_{b]}\rangle + \mathbf{B}^B |\phi'_{[a} \phi_{b]}\rangle + \frac{1}{2} \mathbf{C}^B \epsilon_{ab} \epsilon^{\alpha\beta} |\mathcal{Z}^- \psi'_\alpha \psi_\beta\rangle , \quad (6.11)$$

$$\mathbf{S}^B |\psi_\alpha \psi'_\beta\rangle = \mathbf{D}^B |\psi'_{\{\alpha} \psi_{\beta\}}\rangle + \mathbf{E}^B |\psi'_{[\alpha} \psi_{\beta]}\rangle + \frac{1}{2} \mathbf{F}^B \epsilon_{\alpha\beta} \epsilon^{ab} |\mathcal{Z}^+ \phi'_a \phi_b\rangle , \quad (6.12)$$

$$\mathbf{S}^B |\phi_a \psi'_\beta\rangle = \mathbf{G}^B |\psi'_\beta \phi_a\rangle + \mathbf{H}^B |\phi'_a \psi_\beta\rangle , \quad (6.13)$$

$$\mathbf{S}^B |\psi_\alpha \phi'_b\rangle = \mathbf{K}^B |\psi'_\alpha \phi_b\rangle + \mathbf{L}^B |\phi'_b \psi_\alpha\rangle . \quad (6.14)$$

where ϕ are spin-chain bosons, ψ are spin-chain fermions and Z denotes the creation or destruction of a site on the spin-chain. The worldsheet states correspond to the products of two spin-chain states as implied by (6.1). The symmetry fixes the coefficients of the S-matrix to be:

³Since Z has two fermionic indices it is overall a boson

$$\begin{aligned}
\mathbf{A}^B &= S_{pp'}^0 \frac{x_{p'}^+ - x_p^-}{x_{p'}^- - x_p^+}, \\
\mathbf{B}^B &= S_{pp'}^0 \frac{x_{p'}^+ - x_p^-}{x_{p'}^- - x_p^+} \left(1 - 2 \frac{1 - \frac{1}{x_p^+ x_{p'}^-}}{1 - \frac{1}{x_p^+ x_{p'}^+}} \frac{x_{p'}^- - x_p^-}{x_{p'}^+ - x_p^-} \right), \\
\mathbf{C}^B &= S_{pp'}^0 \frac{2\gamma_p \gamma_{p'}}{x_p^+ x_{p'}^+} \frac{1}{1 - \frac{1}{x_p^+ x_{p'}^+}} \frac{x_{p'}^- - x_p^-}{x_{p'}^- - x_p^+}, \\
\mathbf{D}^B &= -S_{pp'}^0, \\
\mathbf{E}^B &= -S_{pp'}^0 \left(1 - 2 \frac{1 - \frac{1}{x_p^- x_{p'}^+}}{1 - \frac{1}{x_p^- x_{p'}^-}} \frac{x_{p'}^+ - x_p^+}{x_{p'}^- - x_p^+} \right), \\
\mathbf{F}^B &= -S_{pp'}^0 \frac{2}{\gamma_p \gamma_{p'} x_p^- x_{p'}^-} \frac{(x_p^+ - x_p^-)(x_{p'}^+ - x_{p'}^-)}{1 - \frac{1}{x_p^- x_{p'}^-}} \frac{x_{p'}^+ - x_p^+}{x_{p'}^- - x_p^+}, \\
\mathbf{G}^B &= S_{pp'}^0 \frac{x_{p'}^+ - x_p^+}{x_{p'}^- - x_p^+}, \mathbf{H}^B = S_{pp'}^0 \frac{\gamma_p}{\gamma_{p'}} \frac{x_{p'}^+ - x_{p'}^-}{x_{p'}^- - x_p^+}, \\
\mathbf{K}^B &= S_{pp'}^0 \frac{\gamma_{p'}}{\gamma_p} \frac{x_p^+ - x_p^-}{x_{p'}^- - x_p^+}, \mathbf{L}^B = S_{pp'}^0 \frac{x_{p'}^- - x_p^-}{x_{p'}^- - x_p^+},
\end{aligned} \tag{6.15}$$

where

$$\gamma_p = |x_p^- - x_p^+|^{1/2}, \tag{6.16}$$

and

$$x_p^\pm = \frac{\pi e^{\pm \frac{i}{2}p}}{\sqrt{\lambda} \sin \frac{p}{2}} \left(1 + \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p}{2}} \right). \tag{6.17}$$

The momenta appearing here is the spin-chain momentum which is related to the worldsheet momentum by

$$p_{\text{chain}} = \frac{2\pi}{\sqrt{\lambda}} p_{\text{ws}} = \frac{1}{\hat{g}} p_{\text{ws}}. \tag{6.18}$$

We would need to introduce some additional phases and redefine the functions

a bit to get something we can plug into (6.1) let us wait with that for a bit and focus on something else. Notice the factor $S_{pp'}^0$, which have not yet been defined appearing in all of the coefficients of the S-matrix, this is called the dressing phase and it cannot be determined by symmetries alone. The dressing phase is quite important and play a prominent role in our calculations later on. The reason for this can be found by a closer inspection of (6.15); there are simply not many different function that can appear in the expansion of the part determined by symmetries: polynomials of p and ε divided by other polynomials potentially with some square roots in there but not much else can appear⁴. This means that special functions such as logarithms must necessarily come from the dressing phase putting heavy constraints on how special functions can appear in the S-matrix. This is quite useful as it will turn out that special functions is something generalized unitarity have an especially good grip on where as there are potential issues with the rational terms.

To see the constraints this makes on the results we can potentially get let us write the dressing phase as an expansion:

$$\theta_{12} = \frac{1}{\hat{g}} \sum_{n=0}^{\infty} \frac{1}{\hat{g}^n} \hat{\theta}_{12}^{(n)} . \quad (6.19)$$

The dressing phase has to be at least of order \hat{g}^{-1} otherwise it would give some normalization to the process where no scattering occurs *i.e.* the $\mathbb{1}$ in (6.3) and (6.4) would be $\mathbb{1}$ times some function. So (6.19) is a generic expansion of the dressing phase. We now define the symmetry-determined part of the S-matrix with the tree-level dressing phase incorporated into it:

$$\mathbf{S} = e^{\frac{i}{2}(\theta_{12} - \frac{1}{\hat{g}}\hat{\theta}_{12}^{(0)})} \hat{\mathbf{S}} ; \quad (6.20)$$

This is done because the tree-level dressing phase is merely going to be some rational terms (at least in all the examples we will be considering here). If we now expand the S-matrix we get:

⁴These types of terms will be refer to as rational terms

$$\mathbf{S} = \mathbb{1} + \frac{1}{\hat{g}} i \mathbf{T}^{(0)} + \frac{1}{\hat{g}^2} i \left(\hat{\mathbf{T}}^{(1)} + \frac{1}{2} \hat{\theta}_{12}^{(1)} \mathbb{1} \right) + \frac{1}{\hat{g}^3} i \left(\hat{\mathbf{T}}^{(2)} + \frac{i}{2} \hat{\theta}_{12}^{(1)} \mathbf{T}^{(0)} + \frac{1}{2} \hat{\theta}_{12}^{(2)} \mathbb{1} \right) + \mathcal{O} \left(\frac{1}{\hat{g}^4} \right) . \quad (6.21)$$

From this we get that at 1-loop special functions can appear only in the diagonal terms and they must all be exactly the same. At 2-loop the special functions that appeared at 1-loop will show up again this time multiplied by the tree-level amplitude potentially with some additional special functions appearing in the diagonal terms.

The computations from generalized unitarity can be used in several different ways depending on what is known:

- If the tree-level S-matrix has been computed and an exact S-matrix is known one can use generalized unitarity to get higher loop orders and compare with the exact S-matrix and fix the dressing phase
- If the tree-level S-matrix has been computed but no exact S-matrix is known one can use generalized unitarity to calculate higher loop orders and determine whether the structure is such that an exact S-matrix might be possible
- If only an exact S-matrix is known one can expand it and use generalized unitarity to check if the loop-level amplitudes is consistent with the tree-level amplitudes⁵

In [32] generalized unitarity was used in all 3 ways.

6.2 Using Generalized Unitarity

At 1-loop there are 3 different integrals that can appear, shown in figure 6.1, and 3 different cuts to consider, shown in figure 6.2.

The s- and the u-channel are fairly easy to deal with as the cuts are just going to be functions of the external momenta because the cuts set the momenta of the

⁵Of course such a procedure will not determine whether the exact S-matrix is meaningful in of itself but only whether the expansion can be the result of the scattering in a worldsheet perturbation theory

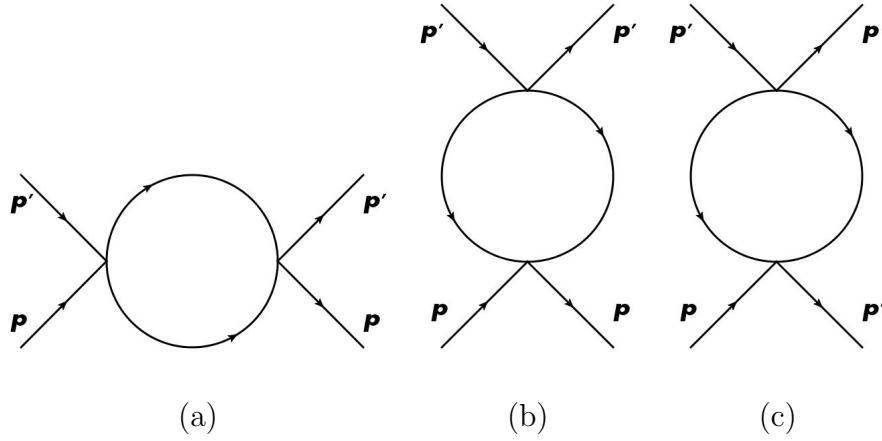


Figure 6.1. The 3 different integrals

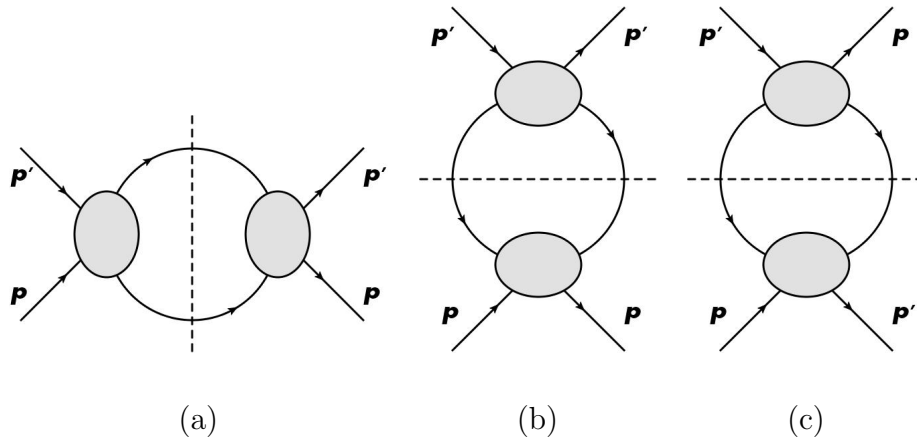


Figure 6.2. The 3 different cuts

cut lines equal to the external momenta. This way the cuts simply become the coefficients of scalar integrals. The t-channel is otherwise problematic as both the tree-level amplitudes in the cut sets the momentum of one of the internal lines equal to the momentum of the other giving rise to the square of a delta function.

In [17] the authors figured a procedure for getting this cut⁶ while in [32] it was shown how to get something meaningful out while ignoring the t-channel; here we will follow the approach of [32].

The key to getting something useful out of only the s- and the u-cut is to notice that of the 3 scalar integrals shown in appendix B only the s- and the u-channel

⁶Though it is not clear if the procedure is something that will work in general or requires certain properties like integrability to be satisfied for the procedure to work

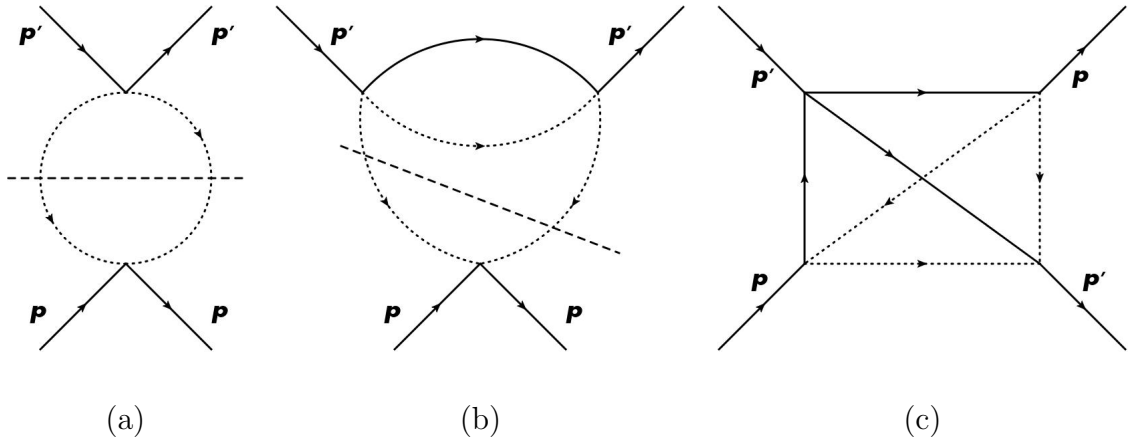


Figure 6.3. Integrals with fields being truncated away

integrals contain a logarithm and as discussed previously logarithms must come from the dressing phase putting heavy constraints on how they can appear. In other words we are lucky that the thing we control very well is also the thing whose appearance is strongly constrained by the form of the exact S-matrices.

Since we are only interested in the special functions this also allows us to look at theories where some of the states with different masses than the others have been truncated away. This can be justified at 1- and 2-loop simply by assuming that the scattering with these states is reflectionless⁷. States that scatter reflectionlessly will only contribute to cuts with no special functions at 1- and 2-loop as demonstrated in figure 6.3 here the dotted line represents fields to be truncated away and we see that in diagram (a) and (b) the assumption of no reflections ensures that there can be no momentum transfer across the cuts meaning that the cuts correspond to integrals producing only rational terms.

Diagram (c) is different here there are no clear cuts that ensure that the integral will not produce logarithms. These arguments ensure that under the assumption of no reflections at 1- and 2-loop we can safely truncate states with different masses away.

⁷If scattering between states of different masses is not reflectionless the outgoing momenta will be some complicated functions of the incoming momenta and it is hard to see how the S-matrix would be integrable in that case

6.3 The Cuts

Let us delve more deeply into the actual cuts. We will try to keep discussion at the generic level also incorporating theories with several different masses.

At 1-loop we can write the generic amplitude as:

$$iT^{(1)} = \frac{1}{2}C_s \tilde{I}_s + \frac{1}{2}C_u \tilde{I}_u + \frac{1}{2}C_t \tilde{I}_t + \text{rational} . \quad (6.22)$$

The integrals \tilde{I} are shown in appendix B. The factors of $\frac{1}{2}$ are included to counter some symmetry factors from the integrals. We introduce the following notation that the lower indices represent incoming states and the upper indices represent outgoing states such that:

$$\mathbb{T}|\Phi_{AA}\Phi'_{BB'}\rangle = (-)^{[\dot{A}][B]+[D]}|\Phi_{CA}\Phi'_{DB'}\rangle T_{AB}^{CD} + (-)^{[B][\dot{A}]+[\dot{C}]}|\Phi_{AC}\Phi'_{BD'}\rangle T_{\dot{A}\dot{B}}^{\dot{C}\dot{D}} , \quad (6.23)$$

This way the cuts can be written as

$$\begin{aligned} (C_s)^{CD'}_{AB'} &= (i)^2 J \sum_{E,F'} (iT^{(0)})^{CD'}_{EF'} (iT^{(0)})^{EF'}_{AB'} \\ (C_u)^{CD'}_{AB'} &= (i)^2 J \sum_{E,F'} (-)^{([B]+[F])([D]+[F])} (iT^{(0)})^{CF'}_{EB'} (iT^{(0)})^{ED'}_{AF'} . \end{aligned} \quad (6.24)$$

Here the sign function appearing in the u-cut are of a fermionic nature and the exact expression can be found by relating the u-cut to and s-cut via the crossing transformations mentioned earlier (6.9). The J 's appear in order to counter the normalization of the internal lines and the appearance of Jacobian factors related to the rewriting of the delta functions (6.6) on both sides of the cuts. Using the explicit forms of the 1-loop integrals we see that the factor in front of the logarithm is going to be⁸:

⁸We should of course note that even when we do not write the indices explicitly as we did in (6.24) the C 's remain tensors

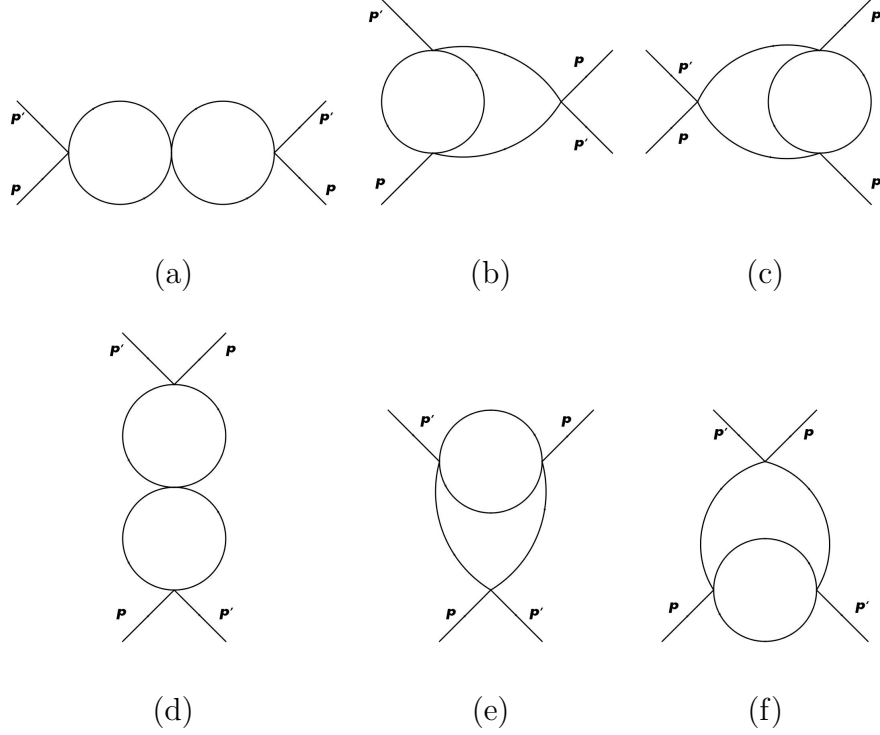


Figure 6.4. The integrals with non-singular maximal cuts

$$\frac{C_s}{J} - \frac{C_u}{J}. \quad (6.25)$$

\tilde{I}_s has a rational part while \tilde{I}_u does not so we can write the 1-loop amplitude as:

$$iT^{(1)} = \frac{1}{2} \frac{C_s}{J} (J\tilde{I}_s + 1) + \frac{1}{2} C_u \tilde{I}_u + i\tilde{T}^{(1)}, \quad (6.26)$$

with $\tilde{T}^{(1)}$ being defined as the rational part of the 1-loop amplitude.

At 2-loop we will need both cut diagrams with 2 cut propagators and with 4 cut propagators (also referred to as the maximal cut). In constructing the ansatz we are going to need the integrals shown in figure 6.4 which are all those that have non-singular maximal cuts, in addition to these we need to include some integrals that do not have non-singular maximal cuts but do contribute to the 2-particle

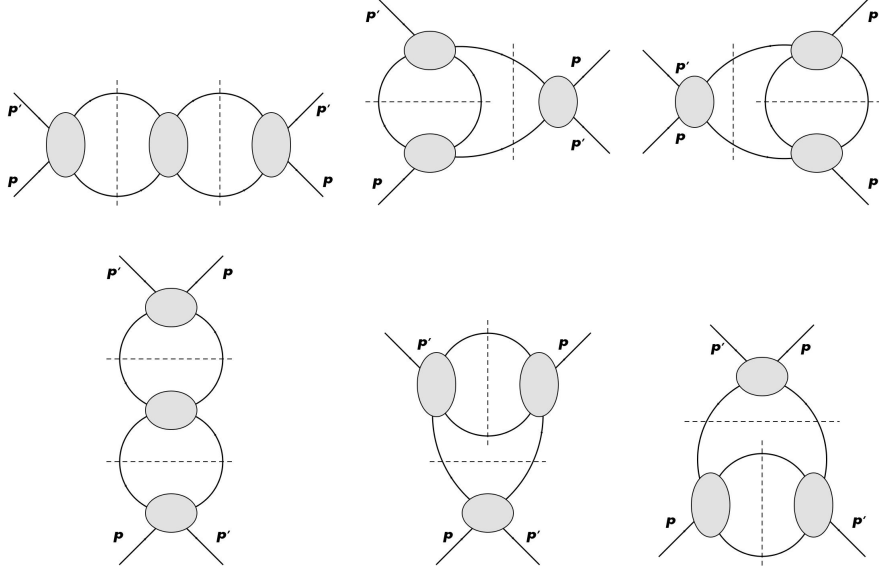


Figure 6.5. The maximal cuts

cuts. At 2-loop we are just considering the case when all masses are the same:

$$\begin{aligned}
 iT^{(2)} = & \frac{1}{4}C_a\tilde{I}_a + \frac{1}{2}C_b\tilde{I}_b + \frac{1}{2}C_c\tilde{I}_c + \frac{1}{4}C_d\tilde{I}_d + \frac{1}{2}C_e\tilde{I}_e + \frac{1}{2}C_f\tilde{I}_f \\
 & + \frac{1}{2}C_{s,\text{extra}}\tilde{I}_s + \frac{1}{2}C_{u,\text{extra}}\tilde{I}_u \\
 & + \text{rational} ,
 \end{aligned} \tag{6.27}$$

Again factors have been included to cancel symmetry factors of the integrals. We can now compute the maximal cuts in figure 6.5 they are given by:

$$\begin{aligned}
(C_a)_{AB'}^{CD'} &= (i)^2 J \sum_{G,H'} (iT^{(0)})_{GH'}^{CD'} (C_s)_{AB'}^{GH'} \\
(C_b)_{AB'}^{CD'} &= (i)^2 J \sum_{G,H'} (iT^{(0)})_{GH'}^{CD'} (C_u)_{AB'}^{GH'} \\
(C_c)_{AB'}^{CD'} &= (i)^2 J \sum_{G,H'} (C_s)_{GH'}^{CD'} (iT^{(0)})_{AB'}^{GH'} \\
(C_d)_{AB'}^{CD'} &= (i)^2 J \sum_{G,H'} (-)^{([B]+[H])([D]+[H])} (iT^{(0)})_{GB'}^{CH'} (C_u)_{AH'}^{GD'} \\
(C_e)_{AB'}^{CD'} &= (i)^2 J \sum_{G,H'} (-)^{([B]+[H])([D]+[H])} (iT^{(0)})_{GB'}^{CH'} (C_s)_{AH'}^{GD'} \\
(C_f)_{AB'}^{CD'} &= (i)^2 J \sum_{G,H'} (-)^{([B]+[H])([D]+[H])} (C_s)_{GB'}^{CH'} (iT^{(0)})_{AH'}^{GD'}
\end{aligned} \tag{6.28}$$

To compute the remaining two coefficients we need to compare the 2-particle cut of the amplitude with that of the ansatz. The former is given by:

$$iT^{(2)}_{AB'}^{CD'} \Big|_{s\text{-cut}} = (i)^2 J \sum_{G,H'} \left((iT^{(0)})_{GH'}^{CD'} (iT^{(1)})_{AB'}^{GH'} + (iT^{(1)})_{GH'}^{CD'} (iT^{(0)})_{AB'}^{GH'} \right) \tag{6.29}$$

$$\begin{aligned}
iT^{(2)}_{AB'}^{CD'} \Big|_{s\text{-cut}} &= (i)^2 J \sum_{G,H'} (-)^{([B]+[H])([D]+[H])} \left((iT^{(0)})_{GB'}^{CH'} (iT^{(1)})_{AH'}^{GD'} \right. \\
&\quad \left. + (iT^{(1)})_{GB'}^{CH'} (iT^{(0)})_{AH'}^{GD'} \right),
\end{aligned} \tag{6.30}$$

while the latter is given by:

$$iT^{(2)} \Big|_{s\text{-cut}}^{\text{eq. 6.27}} = \frac{C_a}{J^2} ((J\tilde{I}_s + 1) - 1) + \frac{1}{2} \left(\frac{C_b}{J^2} + \frac{C_c}{J^2} \right) J\tilde{I}_u \tag{6.31}$$

$$\begin{aligned}
&+ \frac{1}{2} \left(\frac{C_b}{J} + \frac{C_c}{J} \right) \tilde{I}_t + \frac{C_{s,\text{extra}}}{J}, \\
iT^{(2)} \Big|_{u\text{-cut}}^{\text{eq. 6.27}} &= \frac{C_d}{J^2} J\tilde{I}_u + \frac{1}{2} \left(\frac{C_e}{J^2} + \frac{C_f}{J^2} \right) ((J\tilde{I}_s + 1) - 1) \\
&+ \frac{1}{2} \left(\frac{C_e}{J} + \frac{C_f}{J} \right) \tilde{I}_t + \frac{C_{u,\text{extra}}}{J}.
\end{aligned} \tag{6.32}$$

One can use (6.28) to show that the double logarithmic terms in these expressions are the same as indeed they should be for generalized unitarity to be consistent. Comparing (6.31) and (6.32) with (6.29) and (6.30) we get the following expressions for the remaining coefficients:

$$\begin{aligned} \frac{1}{J}(C_{s,\text{extra}})_{AB'}^{CD'} = & (i)^2 J \sum_{G,H'} \left((iT^{(0)})_{GH'}^{CD'} (i\tilde{T}^{(1)})_{AB'}^{GH'} + (i\tilde{T}^{(1)})_{GH'}^{CD'} (iT^{(0)})_{AB'}^{GH'} \right) \\ & + \frac{(C_a)_{AB'}^{CD'}}{J^2} - \frac{1}{2} \left(\frac{(C_e)_{AB'}^{CD'}}{J} + \frac{(C_f)_{AB'}^{CD'}}{J} \right) I_t, \end{aligned} \quad (6.33)$$

$$\begin{aligned} \frac{1}{J}(C_{u,\text{extra}})_{AB'}^{CD'} = & (i)^2 J \sum_{G,H'} (-)^{([B]+[H])([D]+[H])} \left((iT^{(0)})_{GB'}^{CH'} (i\tilde{T}^{(1)})_{AH'}^{GD'} \right. \\ & + (i\tilde{T}^{(1)})_{GB'}^{CH'} (iT^{(0)})_{AH'}^{GD'} \left. \right) + \frac{1}{2} \left(\frac{(C_e)_{AB'}^{CD'}}{J^2} + \frac{(C_f)_{AB'}^{CD'}}{J^2} \right) \\ & - \frac{1}{2} \left(\frac{(C_e)_{AB'}^{CD'}}{J} + \frac{(C_f)_{AB'}^{CD'}}{J} \right) I_t. \end{aligned} \quad (6.34)$$

Here the tildes mean the rational parts of those coefficients as defined back in (6.26). Using the explicit formulas for the 2-loop integrals shown in appendix B we get that the coefficient of the double logarithm is going to be:

$$C_{\ln^2} = \frac{1}{8\pi^2 J^2} (-2C_a + C_b + C_c - 2C_d + C_e + C_f), \quad (6.35)$$

while for the single logarithm it is going to be:

$$\begin{aligned} C_{\ln^1} = & \frac{i}{2\pi} \left[\frac{1}{2J^2} (2C_a - C_b - C_c) \right. \\ & \left. - \frac{1}{J} (C_{s,\text{extra}} - C_{u,\text{extra}}) - \frac{i}{8\pi J} (C_b + C_c - C_e - C_f) \right]. \end{aligned} \quad (6.36)$$

Having done the calculations for some generic theories let us now look at some explicit examples.

6.4 $AdS_5 \times S^5$

We have already looked a bit at this example. The S-matrix is given by two copies of (6.15) (written slightly differently which we will get back to). The T-matrix is parametrized as follows⁹:

$$\begin{aligned}
T_{ab}^{cd} &= A \delta_a^c \delta_b^d + B \delta_a^d \delta_b^c, & T_{ab}^{\gamma\delta} &= C \epsilon_{ab} \epsilon^{\gamma\delta}, \\
T_{\alpha\beta}^{\gamma\delta} &= D \delta_\alpha^\gamma \delta_\beta^\delta + E \delta_\alpha^\delta \delta_\beta^\gamma, & T_{\alpha\beta}^{cd} &= F \epsilon_{\alpha\beta} \epsilon^{cd}, \\
T_{a\beta}^{c\delta} &= G \delta_a^c \delta_\beta^\delta, & T_{\alpha b}^{\gamma d} &= L \delta_\alpha^\gamma \delta_b^d, \\
T_{a\beta}^{\gamma d} &= H \delta_a^d \delta_\beta^\gamma, & T_{\alpha b}^{c\delta} &= K \delta_\alpha^\delta \delta_b^c.
\end{aligned} \tag{6.37}$$

Here A, D, G and L are the diagonal elements while the rest are off-diagonal. In [43] the complete tree-level S-matrix was found to be:

$$\begin{aligned}
A^{(0)}(p, p') &= \frac{1}{4} \left[(1 - 2a) (\varepsilon' p - \varepsilon p') + \frac{(p - p')^2}{\varepsilon' p - \varepsilon p'} \right], \\
B^{(0)}(p, p') &= -E^{(0)}(p, p') = \frac{pp'}{\varepsilon' p - \varepsilon p'}, \\
C^{(0)}(p, p') &= F^{(0)}(p, p') = \frac{1}{2} \frac{\sqrt{(\varepsilon + 1)(\varepsilon' + 1)} (\varepsilon' p - \varepsilon p' + p' - p)}{\varepsilon' p - \varepsilon p'}, \\
D^{(0)}(p, p') &= \frac{1}{4} \left[(1 - 2a) (\varepsilon' p - \varepsilon p') - \frac{(p - p')^2}{\varepsilon' p - \varepsilon p'} \right], \\
G^{(0)}(p, p') &= -L^{(0)}(p', p) = \frac{1}{4} \left[(1 - 2a) (\varepsilon' p - \varepsilon p') - \frac{p^2 - p'^2}{\varepsilon' p - \varepsilon p'} \right], \\
H^{(0)}(p, p') &= K^{(0)}(p, p') = \frac{1}{2} \frac{pp'}{\varepsilon' p - \varepsilon p'} \frac{(\varepsilon + 1)(\varepsilon' + 1) - pp'}{\sqrt{(\varepsilon + 1)(\varepsilon' + 1)}}.
\end{aligned} \tag{6.38}$$

Here $\varepsilon = \sqrt{1 + p^2}$ and a is a gauge parameter. To relate this to (6.15) one first introduce some phases which has to do with the creation/destruction of lattice sites not having an equivalent in the worldsheet theory:

⁹By which we mean one of the two factors of the full T-matrix

$$\begin{aligned}
\hat{\mathbf{A}}^B &= \mathbf{A}^B e^{i(1-2a)(p-p')} & \hat{\mathbf{B}}^B &= \mathbf{B}^B e^{i(1-2a)(p-p')} \\
\hat{\mathbf{C}}^B &= \mathbf{C}^B e^{i((\frac{5}{4}+b-2a)p - (\frac{1}{4}-b-2a)p')} & \hat{\mathbf{D}}^B &= \mathbf{D}^B e^{i((\frac{1}{2}-2a)p - (\frac{1}{2}-2a)p')} \\
\hat{\mathbf{E}}^B &= \mathbf{E}^B e^{i((\frac{1}{2}-2a)p - (\frac{1}{2}-2a)p')} & \hat{\mathbf{F}}^B &= \mathbf{F}^B e^{i((\frac{1}{4}-b-2a)p - (\frac{5}{4}+b-2a)p')} \\
\hat{\mathbf{G}}^B &= \mathbf{G}^B e^{i(-\frac{1}{2}p + (1-2a)(p-p'))} & \hat{\mathbf{H}}^B &= \mathbf{H}^B e^{i((\frac{3}{4}+b-2a)p - (\frac{3}{4}+b-2a)p')} \\
\hat{\mathbf{K}}^B &= \mathbf{K}^B e^{i((\frac{3}{4}-b-2a)p - (\frac{3}{4}-b-2a)p')} & \hat{\mathbf{L}}^B &= \mathbf{L}^B e^{i(\frac{1}{2}p' + (1-2a)(p-p'))}.
\end{aligned} \tag{6.39}$$

These new functions are then related to the coefficients of the T-matrix by:

$$\begin{aligned}
\mathbf{A} &= \frac{1}{2\sqrt{\hat{\mathbf{A}}^B}}(\hat{\mathbf{A}}^B - \hat{\mathbf{B}}^B), & \mathbf{B} &= \frac{1}{2\sqrt{\hat{\mathbf{A}}^B}}(\hat{\mathbf{A}}^B + \hat{\mathbf{B}}^B), & \mathbf{C} &= \frac{i}{2\sqrt{\hat{\mathbf{A}}^B}}\hat{\mathbf{C}}^B, \\
\mathbf{D} &= \frac{1}{2\sqrt{\hat{\mathbf{A}}^B}}(-\hat{\mathbf{D}}^B + \hat{\mathbf{E}}^B), & \mathbf{E} &= \frac{1}{2\sqrt{\hat{\mathbf{A}}^B}}(-\hat{\mathbf{D}}^B - \hat{\mathbf{E}}^B), & \mathbf{F} &= \frac{i}{2\sqrt{\hat{\mathbf{A}}^B}}\hat{\mathbf{F}}^B, \\
\mathbf{G} &= \frac{1}{\sqrt{\hat{\mathbf{A}}^B}}\hat{\mathbf{G}}^B, & \mathbf{H} &= \frac{1}{\sqrt{\hat{\mathbf{A}}^B}}\hat{\mathbf{H}}^B, & & \\
\mathbf{L} &= \frac{1}{\sqrt{\hat{\mathbf{A}}^B}}\hat{\mathbf{L}}^B, & \mathbf{K} &= \frac{1}{\sqrt{\hat{\mathbf{A}}^B}}\hat{\mathbf{K}}^B. & &
\end{aligned} \tag{6.40}$$

Let us now compute the cuts for some specific values of the tensor indices¹⁰:

$$\frac{1}{J}(C_s)_{ab}^{cd} = (A^{(0)2} + B^{(0)2} + 2C^{(0)}F^{(0)})\delta_a^c\delta_b^d + 2(A^{(0)}B^{(0)} - C^{(0)}F^{(0)})\delta_a^d\delta_b^c, \tag{6.41}$$

$$\frac{1}{J}(C_u)_{ab}^{cd} = A^{(0)2}\delta_a^c\delta_b^d + 2(A^{(0)}B^{(0)} + B^{(0)2} - H^{(0)}K^{(0)})\delta_a^d\delta_b^c. \tag{6.42}$$

As mentioned it is only the difference between these two expressions that give us the factor in front of the logarithm. We notice that for the B -function this difference is zero as it should be since that is an off-diagonal element while for the A -function it is non-zero. Doing the cuts for the other S-matrix elements we get the simple formula:

$$\frac{C_s}{J} - \frac{C_u}{J} = + \frac{p^2 p'^2 (\varepsilon \varepsilon' - p p')}{(\varepsilon p' - \varepsilon' p)^2} \mathbb{1}, \tag{6.43}$$

¹⁰Remember the C 's are in fact tensors even though we out of laziness do not always display the tensor indices

This matches the form of (6.21):

$$iT^{(1)} = i \left(\frac{1}{2} \left(-\frac{1}{\pi} \frac{p^2 p'^2 (\varepsilon \varepsilon' - p p')}{(\varepsilon p' - \varepsilon' p)^2} \ln \left| \frac{p'_-}{p_-} \right| \right) \mathbb{1} + \text{rational} \right) = i \left(\frac{1}{2} \hat{\theta}_{12}^{(1)} \mathbb{1} + \text{rational} \right), \quad (6.44)$$

and we have identified the logarithmic part of the 1-loop dressing phase.

Let us now move on to the 2-loop calculation. Again just focusing on some specific values of the external indices the cuts are given by:

$$\begin{aligned} \frac{1}{J^2} (C_a)^{cd} &= (A_{s\text{-cut}}^{(1)} A^{(0)} + B_{s\text{-cut}}^{(1)} B^{(0)} + 2C_{s\text{-cut}}^{(1)} F^{(0)}) \delta_a^c \delta_b^d \\ &\quad + (A_{s\text{-cut}}^{(1)} B^{(0)} + B_{s\text{-cut}}^{(1)} A^{(0)} - 2C_{s\text{-cut}}^{(1)} F^{(0)}) \delta_a^d \delta_b^c \\ &= (A^{(0)} A_{s\text{-cut}}^{(1)} + B^{(0)} B_{s\text{-cut}}^{(1)} + 2C^{(0)} F_{s\text{-cut}}^{(1)}) \delta_a^c \delta_b^d \\ &\quad + (A^{(0)} B_{s\text{-cut}}^{(1)} + B^{(0)} A_{s\text{-cut}}^{(1)} - 2C^{(0)} F_{s\text{-cut}}^{(1)}) \delta_a^d \delta_b^c, \end{aligned} \quad (6.45)$$

$$\begin{aligned} \frac{1}{J^2} (C_d)^{cd} &= A_{u\text{-cut}}^{(1)} A^{(0)} \delta_a^c \delta_b^d \\ &\quad + (A_{u\text{-cut}}^{(1)} B^0 + B_{u\text{-cut}}^{(1)} A^{(0)} + 2B_{u\text{-cut}}^{(1)} B^{(0)} - 2H_{u\text{-cut}}^{(1)} K^{(0)}) \delta_a^d \delta_b^c \\ &= A^{(0)} A_{u\text{-cut}}^{(1)} \delta_a^c \delta_b^d \\ &\quad + (A^{(0)} B_{u\text{-cut}}^{(1)} + B^{(0)} A_{u\text{-cut}}^{(1)} + 2B^{(0)} B_{u\text{-cut}}^{(1)} - 2H^{(0)} K_{u\text{-cut}}^{(1)}) \delta_a^d \delta_b^c, \end{aligned} \quad (6.46)$$

$$\begin{aligned} \frac{1}{J^2} (C_b)^{cd} &= (A_{u\text{-cut}}^{(1)} A^{(0)} + B_{u\text{-cut}}^{(1)} B^{(0)} + 2C_{u\text{-cut}}^{(1)} F^{(0)}) \delta_a^c \delta_b^d \\ &\quad + (A_{u\text{-cut}}^{(1)} B^{(0)} + B_{u\text{-cut}}^{(1)} A^{(0)} - 2C_{u\text{-cut}}^{(1)} F^{(0)}) \delta_a^d \delta_b^c, \end{aligned} \quad (6.47)$$

$$\begin{aligned} \frac{1}{J^2} (C_e)^{cd} &= A_{s\text{-cut}}^{(1)} A^{(0)} \delta_a^c \delta_b^d \\ &\quad + (A_{s\text{-cut}}^{(1)} B^{(0)} + B_{s\text{-cut}}^{(1)} A^{(0)} + 2B_{s\text{-cut}}^{(1)} B^{(0)} - 2H_{s\text{-cut}}^{(1)} K^{(0)}) \delta_a^d \delta_b^c, \end{aligned} \quad (6.48)$$

$$\begin{aligned} \frac{1}{J^2} (C_c)^{cd} &= (A^{(0)} A_{u\text{-cut}}^{(1)} + B^{(0)} B_{u\text{-cut}}^{(1)} + 2C^{(0)} F_{u\text{-cut}}^{(1)}) \delta_a^c \delta_b^d \\ &\quad + (A^{(0)} B_{u\text{-cut}}^{(1)} + B^{(0)} A_{u\text{-cut}}^{(1)} - 2C^{(0)} F_{u\text{-cut}}^{(1)}) \delta_a^d \delta_b^c, \end{aligned} \quad (6.49)$$

$$\begin{aligned} \frac{1}{J^2} (C_f)^{cd} &= A^{(0)} A_{s\text{-cut}}^{(1)} \delta_a^c \delta_b^d \\ &\quad + (A^{(0)} B_{s\text{-cut}}^{(1)} + B^{(0)} A_{s\text{-cut}}^{(1)} + 2B^{(0)} B_{s\text{-cut}}^{(1)} - 2H^{(0)} K_{s\text{-cut}}^{(1)}) \delta_a^d \delta_b^c, \end{aligned} \quad (6.50)$$

where for convenience we use the notation:

$$iA_{s\text{-cut}}^{(1)}\delta_a^c\delta_b^d = \frac{1}{J}(C_s)_{ab}^{cd} \quad iA_{u\text{-cut}}^{(1)}\delta_a^c\delta_b^d = \frac{1}{J}(C_u)_{ab}^{cd}. \quad (6.51)$$

Using the tree-level amplitude (6.38) we get that:

$$-2C_a + C_b + C_c - 2C_d + C_e + C_f = 0, \quad (6.52)$$

so there are no double logarithms at 2-loop. This result is sensible, though not required by integrability, as double logarithms would have to come from the 2-loop dressing phase and the expected expression for the dressing phase [8] do not have such a term. The last two coefficients to be determined are:

$$\begin{aligned} \frac{C_{s,\text{extra}}^A}{J} = & 2i(i)^2(A^{(0)}(i\tilde{A}^{(1)}) + B^{(0)}(i\tilde{B}^{(1)}) + C^{(0)}(i\tilde{F}^{(1)}) + F^{(0)}(i\tilde{C}^{(1)})) \\ & + \frac{C_a^A}{J^2} - \frac{1}{2}\left(\frac{C_b^A}{J} + \frac{C_c^A}{J}\right)I_t, \end{aligned} \quad (6.53)$$

$$\frac{C_{u,\text{extra}}^A}{J} = 2(i)^2iA^{(0)}(i\tilde{A}^{(1)}) + \frac{1}{2}\left(\frac{C_e^A}{J^2} + \frac{C_f^A}{J^2}\right) - \frac{1}{2}\left(\frac{C_e^A}{J} + \frac{C_f^A}{J}\right)I_t. \quad (6.54)$$

Here we use the superscript A to indicate that these are the coefficients multiplying $\delta_a^c\delta_b^d$. It is important that it is only the difference that matters to us because when subtracting one from the other, the term with the factor $\tilde{A}^{(1)}$ drops out and this factor we could not determine fully from the exact S-matrix as it also depends on the dressing phase making it hard to say anything specific at this loop level.

In order to compute the terms with a single logarithm we do need the relations:

$$\begin{aligned} \hat{B}^{(1)} &= i(A^{(0)} + D^{(0)})B^{(0)} + \frac{i}{8}aJB^{(0)} \\ \hat{C}^{(1)} &= \frac{i}{2}(A^{(0)} + D^{(0)})C^{(0)} + ib(p + p')C^{(0)} \\ \hat{F}^{(1)} &= \frac{i}{2}(A^{(0)} + D^{(0)})F^{(0)} - ib(p + p')F^{(0)}, \end{aligned} \quad (6.55)$$

which we determined from the exact S-matrix. Note however that these relations

are independent of the dressing phase. With them the second line in equation (6.36) vanishes and we get that the factor in front of the single logarithm is given by:

$$-4\pi i \mathcal{C}_{\ln^1}^A = \frac{1}{J^2}(2C_a^A - C_b^A - C_c^A) = -2(iA^{(0)}) \left(\frac{C_s^A}{J} - \frac{C_u^A}{J} \right) \quad (6.56)$$

For the other choices of external indices we also get:

$$\frac{C_{s,\text{extra}}}{J} - \frac{C_{u,\text{extra}}}{J} + \frac{i}{8\pi J} (C_b + C_c - C_e - C_f) = 0, \quad (6.57)$$

and the remaining parts can be rewritten to something similar to (6.56) and we end up with the result:

$$iT^{(2)} = -\frac{1}{2} \left(-\frac{1}{\pi} \frac{p^2 p'^2 (\varepsilon \varepsilon' - p p')}{(\varepsilon' p - \varepsilon p')^2} \ln \frac{p'_-}{p_-} \right) T^{(0)} + \text{rational} = -\frac{1}{2} \hat{\theta}_{12}^{(1)} T^{(0)} + \text{rational}, \quad (6.58)$$

which is consistent with (6.21).

6.5 $AdS_4 \times CP^3$

Let us now turn toward a worldsheet theory which is fairly similar to $AdS_5 \times S^5$ so similar in fact that we will not need to do any new computations but can simply recycle what we did in the previous section. This theory is type IIA string theory on $AdS_4 \times CP^3$ it is dual to ABJ(M) and it appears to be quantum integrable.

The spectrum consists of 8 bosons and 8 fermions with 4 of each being light ($m^2 = 1/4$) and 4 being heavy ($m^2 = 1$). It has been argued that the heavy excitations are unstable decaying into two light excitations [54] and from the spin-chain perspective they are considered composite states so that the S-matrix only scatters the 8 light excitations. The light excitations are divided into two separate representations denoted A and B particles or multiplets. The bosonic worldsheet S-matrix was found in [41] while in [36] a proposal was made for the Bethe equations

with the dressing phase being similar to $AdS_5 \times S^5$ with only a factor of $1/2$ difference as the $AdS_5 \times S^5$ dressing phase gets 2 identical contributions.

Based on the Bethe equations an exact S-matrix was proposed in [4]. This S-matrix has four different sector corresponding to the four different ways one can choose the multiplets of the incoming particles, it is also reflectionless meaning that particles of different multiplets do not exchange momentum even though they have the same mass:

$$S^{BB}(p, p') = S^{AA}(p, p') = S_0(p, p')\widehat{S}(p, p'), \quad (6.59)$$

$$S^{AB}(p, p') = S^{BA}(p, p') = \widetilde{S}_0(p, p')\widehat{S}(p, p'). \quad (6.60)$$

Here the factors S_0 and \widetilde{S}_0 are two potentially different dressing phases and \widehat{S} is the same $SU(2|2)$ invariant S-matrix that we used for $AdS_5 \times S^5$, (6.40). Being reflectionless the S-matrix do not mix the different sectors in the s- and the u-channel cuts so we may consider the cuts for each sector individually and our computations from the previous section clearly show that through two loops each of the sectors will have a dressing phase that is half that of the $AdS_5 \times S^5$ dressing phase consistent with the proposal in [36].

An interesting alternative S-matrix was proposed in [5] where the authors ultimately rejected it based on a failure to match perturbative calculations done in [48]. The alternative S-matrix is based on the same $SU(2|2)$ invariant S-matrix but it is not reflectionless. We will show that this alternative S-matrix can also be rejected based on calculations done with generalized unitarity. For this purpose let us consider the scattering of two incoming scalars from the A -multiplet going to two fermions from the A -multiplet. This is an off-diagonal element and so should not have any logarithmic terms at 1-loop meaning that the s- and the u-channel cuts must cancel out. For a reflectionless S-matrix this is accomplished by the two cuts being equal however allowing for reflections means increasing the number of particles in the u-cut while the s-cut stays the same, the discrepancy in the number of particles ruins the cancellation leading to forbidden logarithmic terms. So we see that the alternative S-matrix with reflections is not consistent as a worldsheet perturbation theory at least not if truncated to the light fields.

6.6 $AdS_3 \times S^3 \times S^3 \times S^1$

The next theory to consider is the Green-Schwarz string in $AdS_3 \times S^3 \times S^3 \times S^1$. Unlike the theories considered so far it has asymptotic states of different masses. The theory has a free parameter α which is the ratio of the square of radii:

$$\alpha = \frac{R_{\text{AdS}}^2}{R_{S_1^3}^2} = 1 - \frac{R_{\text{AdS}}^2}{R_{S_2^3}^2}. \quad (6.61)$$

For α going to 1 or 0 the space becomes $AdS_3 \times S^3 \times T^4$, which we will deal with later, however this limit is non-trivial and one cannot simply take the limit of the S-matrix described here and get the S-matrix for the Green-Schwarz string in $AdS_3 \times S^3 \times T^4$. The spectrum for $AdS_3 \times S^3 \times S^3 \times S^1$ consists of:

- two bosons and two fermions of $m = 1$
- two bosons and two fermions of $m = 0$
- two bosons and two fermions of $m = \alpha$
- two bosons and two fermions of $m = 1 - \alpha$.

The α -dependent states can be divided into left (L), $|\phi\rangle$ and $|\psi\rangle$, and right (R), $|\bar{\phi}\rangle$ and $|\bar{\psi}\rangle$, excitations¹¹. The representations are not decoupled and the S-matrix will have non-trivial LL, LR, RL and RR scattering. Two S-matrices was proposed for this theory [20, 3] (which we will refer to by the abbreviations BOSS and AB respectively) and we will consider both. Tree-level calculations favors the BOSS S-matrix[51]; the 1-loop dressing phase was found in [9]. Both exact S-matrices are shown in appendix C

The S-matrices only deals with the α -dependent states; there are 16 different sectors corresponding to the representations and masses of the incoming states and scattering between states of different masses is reflectionless, in order to use generalized unitarity we assume that the scattering with the remaining states is also reflectionless. The scattering of states in different representations but with the same masses is in fact also reflectionless.

¹¹The terms are misleading as one cannot directly relate these to left- and right-movers however we choose to retain the notation of [19, 20]

In order to make the notation as close to $AdS_5 \times S^5$ as possible we define:

$$\begin{aligned}
T_{\phi\phi}^{\phi\phi} &= A_{LL}, & T_{\psi\psi}^{\psi\psi} &= D_{LL}, & T_{\phi\psi}^{\phi\psi} &= G_{LL}, \\
T_{\phi\psi}^{\psi\phi} &= H_{LL}, & T_{\psi\phi}^{\phi\psi} &= K_{LL}, & T_{\psi\phi}^{\psi\phi} &= L_{LL}, \\
T_{\phi\bar{\phi}}^{\phi\bar{\phi}} &= A_{LR}, & T_{\phi\bar{\phi}}^{\psi\bar{\psi}} &= C_{LR}, & T_{\psi\bar{\psi}}^{\psi\bar{\psi}} &= D_{LR}, \\
T_{\phi\bar{\psi}}^{\phi\bar{\psi}} &= G_{LR}, & T_{\psi\bar{\phi}}^{\phi\bar{\phi}} &= F_{LR}, & T_{\phi\bar{\psi}}^{\psi\bar{\phi}} &= H_{LR}, \\
T_{\psi\bar{\phi}}^{\phi\bar{\psi}} &= K_{LR}, & T_{\psi\bar{\phi}}^{\psi\bar{\phi}} &= L_{LR}.
\end{aligned} \tag{6.62}$$

The lower indices on the T-matrix denote incoming states and the upper indices denote outgoing states. If needed we could have introduced subscripts to indicate the masses of the states however this is not necessary and so to keep the expressions as clean as possible they have been left out. The RR and RL sectors have also been left out as they will be completely equivalent to the other sectors.

All of the components have an expansion in the coupling constant similar to (6.21):

$$A_{LL} = \frac{1}{\hat{g}} A_{LL}^{(0)} + \frac{1}{\hat{g}^2} A_{LL}^{(1)} + \dots \tag{6.63}$$

We do not need the explicit expressions for the diagonal elements we only need to know that they satisfy the relation:

$$A^{(0)} + D^{(0)} - G^{(0)} - L^{(0)} = 0, \tag{6.64}$$

which they do in both proposals for all sectors of the S-matrix. The off-diagonal elements of [20] are given by:

$$\begin{aligned}
H_{LL}^{(0)\text{BOSS}} &= K_{LL}^{(0)\text{BOSS}} = \frac{1}{2} \frac{pp'}{\varepsilon'p - p'\varepsilon} \frac{(\varepsilon + m)(\varepsilon' + m') - pp'}{\sqrt{(\varepsilon + m)(\varepsilon' + m')}}, \\
C_{LR}^{(0)\text{BOSS}} &= F_{LR}^{(0)\text{BOSS}} = \frac{1}{2} \frac{\sqrt{(\varepsilon + m)(\varepsilon' + m')}(\varepsilon'p - \varepsilon p' + p'm - pm')}{\varepsilon'p - p'\varepsilon}, \\
H_{LR}^{(0)\text{BOSS}} &= K_{LR}^{(0)\text{BOSS}} = 0.
\end{aligned} \tag{6.65}$$

where m is the mass of the state with momentum p and m' is the mass of the state with momentum p' both can be either α or $1 - \alpha$ ¹². For the S-matrix from [3] the off-diagonal elements are given by:

$$\begin{aligned}
H_{LL}^{(0)\text{AB}} &= H_{LR}^{(0)\text{AB}} = \frac{1}{2} \frac{pp'}{\varepsilon'p - p'\varepsilon} \frac{(\varepsilon + m)(\varepsilon' + m') - pp'}{(\varepsilon + m)}, \\
C_{LR}^{(0)\text{AB}} &= F_{LR}^{(0)\text{AB}} = 0, \\
K_{LL}^{(0)\text{AB}} &= K_{LR}^{(0)\text{AB}} = \frac{1}{2} \frac{pp'}{\varepsilon'p - p'\varepsilon} \frac{(\varepsilon + m)(\varepsilon' + m') - pp'}{(\varepsilon + m')}.
\end{aligned} \tag{6.66}$$

Note that although H_{LL} and K_{LL} are different for the two S-matrices their product remains the same which will mean that some the 1-loop results will be the same for the two proposals.

Writing out the cuts for the LL sectors we get for both S-matrices:

$$\begin{aligned}
\frac{1}{J}(C_{s,LL})_{\phi\phi} &= A_{LL}^{(0)2} & \frac{1}{J}(C_{u,LL})_{\phi\phi} &= A_{LL}^{(0)2} - H_{LL}^{(0)}K_{LL}^{(0)} \\
\frac{1}{J}(C_{s,LL})_{\psi\psi} &= D_{LL}^{(0)2} & \frac{1}{J}(C_{u,LL})_{\psi\psi} &= D_{LL}^{(0)2} - H_{LL}^{(0)}K_{LL}^{(0)} \\
\frac{1}{J}(C_{s,LL})_{\phi\psi} &= G_{LL}^{(0)2} + H_{LL}^{(0)}K_{LL}^{(0)} & \frac{1}{J}(C_{u,LL})_{\phi\psi} &= G_{LL}^{(0)2} \\
\frac{1}{J}(C_{s,LL})_{\psi\phi} &= G_{LL}^{(0)}H_{LL}^{(0)} + H_{LL}^{(0)}L_{LL}^{(0)} & \frac{1}{J}(C_{u,LL})_{\psi\phi} &= D_{LL}^{(0)}H_{LL}^{(0)} + H_{LL}^{(0)}A_{LL}^{(0)} \\
\frac{1}{J}(C_{s,LL})_{\phi\psi} &= L_{LL}^{(0)}K_{LL}^{(0)} + K_{LL}^{(0)}G_{LL}^{(0)} & \frac{1}{J}(C_{u,LL})_{\phi\psi} &= A_{LL}^{(0)}K_{LL}^{(0)} + K_{LL}^{(0)}D_{LL}^{(0)} \\
\frac{1}{J}(C_{s,LL})_{\psi\phi} &= L_{LL}^{(0)2} + H_{LL}^{(0)}K_{LL}^{(0)} & \frac{1}{J}(C_{u,LL})_{\psi\phi} &= L_{LL}^{(0)2}
\end{aligned} \tag{6.67}$$

¹²Though there is the assumption that $p/m > p'/m'$

Because of (6.64) the terms with diagonal elements drop out when subtracting the u- from the s-channel cuts. Thus the difference only depend on the product of H_{LL} and K_{LL} :

$$\frac{C_{s,LL}^{\text{BOSS}}}{J} - \frac{C_{u,LL}^{\text{BOSS}}}{J} = \frac{C_{s,LL}^{\text{AB}}}{J} - \frac{C_{u,LL}^{\text{AB}}}{J} = \frac{p^2(p')^2(\mathbf{p} \cdot \mathbf{p}' + mm')}{2(\varepsilon'p - p'\varepsilon)^2} \mathbb{1}, \quad (6.68)$$

Again we see that only the diagonal elements get logarithmic contributions at 1-loop:

$$iT_{LL}^{(1)} = i \left(\frac{1}{2} \left(-\frac{1}{\pi} \frac{p^2(p')^2(\mathbf{p} \cdot \mathbf{p}' + mm')}{2(\varepsilon'p - p'\varepsilon)^2} \left(\ln \left| \frac{p'_-}{p_-} \right| - \ln \left| \frac{m'}{m} \right| \right) \right) \mathbb{1} + \text{rational} \right). \quad (6.69)$$

Incidentally this may be recognized as the LL dressing phase computed for $AdS_3 \times S^3 \times T^4$ in [9] or rather a suitable generalization to the case of different masses. There is an extra factor of 1/2 in front reminiscent of the relationship between $AdS_5 \times S^5$ and $AdS_4 \times CP^3$. The calculation for the RR sectors are completely equivalent.

While the result is completely consistent with the BOSS S-matrix[20] it is not quite in line with the expectations of [3] as that S-matrix does not have a dressing phase for the scattering of states with different masses.

For the AB S-matrix the LR and RL sectors are similar to the LL and RR sectors so they are going to lead to the same result (6.69). For the BOSS S-matrix these sectors are however different and the cuts will be given by:

$$\begin{aligned}
\frac{1}{J}(C_{s,LR}^{\text{BOSS}})_{\phi\bar{\phi}}^{\phi\bar{\phi}} &= A_{LR}^{(0)2} + C_{LR}^{(0)}F_{LR}^{(0)} & \frac{1}{J}(C_{u,LR}^{\text{BOSS}})_{\phi\bar{\phi}}^{\phi\bar{\phi}} &= A_{LR}^{(0)2} \\
\frac{1}{J}(C_{s,LR}^{\text{BOSS}})_{\phi\bar{\phi}}^{\psi\bar{\psi}} &= A_{LR}^{(0)}C_{LR}^{(0)} + C_{LR}^{(0)}D_{LR}^{(0)} & \frac{1}{J}(C_{u,LR}^{\text{BOSS}})_{\phi\bar{\phi}}^{\psi\bar{\psi}} &= G_{LR}^{(0)}C_{LR}^{(0)} + C_{LR}^{(0)}L_{LR}^{(0)} \\
\frac{1}{J}(C_{s,LR}^{\text{BOSS}})_{\psi\bar{\psi}}^{\psi\bar{\psi}} &= D_{LR}^{(0)2} + C_{LR}^{(0)}F_{LR}^{(0)} & \frac{1}{J}(C_{u,LR}^{\text{BOSS}})_{\psi\bar{\psi}}^{\psi\bar{\psi}} &= D_{LR}^{(0)2} \\
\frac{1}{J}(C_{s,LR}^{\text{BOSS}})_{\psi\bar{\psi}}^{\phi\bar{\phi}} &= D_{LR}^{(0)}F_{LR}^{(0)} + F_{LR}^{(0)}A_{LR}^{(0)} & \frac{1}{J}(C_{u,LR}^{\text{BOSS}})_{\psi\bar{\psi}}^{\phi\bar{\phi}} &= L_{LR}^{(0)}F_{LR}^{(0)} + F_{LR}^{(0)}G_{LR}^{(0)} \\
\frac{1}{J}(C_{s,LR}^{\text{BOSS}})_{\phi\bar{\psi}}^{\phi\bar{\psi}} &= G_{LR}^{(0)2} & \frac{1}{J}(C_{u,LR}^{\text{BOSS}})_{\phi\bar{\psi}}^{\phi\bar{\psi}} &= G_{LR}^{(0)2} - C_{LR}^{(0)}F_{LR}^{(0)} \\
\frac{1}{J}(C_{s,LR}^{\text{BOSS}})_{\psi\bar{\phi}}^{\psi\bar{\phi}} &= L_{LR}^{(0)2} & \frac{1}{J}(C_{u,LR}^{\text{BOSS}})_{\psi\bar{\phi}}^{\psi\bar{\phi}} &= L_{LR}^{(0)2} - C_{LR}^{(0)}F_{LR}^{(0)}
\end{aligned} \tag{6.70}$$

The difference between the two cuts then becomes:

$$\frac{C_s^{\text{BOSS}}}{J} - \frac{C_u^{\text{BOSS}}}{J} = \frac{p^2(p')^2(\mathbf{p} \cdot \mathbf{p}' - mm')}{2(\varepsilon'p - p'\varepsilon)^2} \mathbb{1}, \tag{6.71}$$

which lead to the following 1-loop T-matrix:

$$iT_{LR}^{(1),\text{BOSS}} = i \left(\frac{1}{2} \left(-\frac{1}{\pi} \frac{p^2(p')^2(\mathbf{p} \cdot \mathbf{p}' - mm')}{2(\varepsilon'p - p'\varepsilon)^2} \left(\ln \left| \frac{p'_-}{p_-} \right| - \ln \left| \frac{m'}{m} \right| \right) \right) \mathbb{1} + \text{rational} \right). \tag{6.72}$$

Similarly to the LL sectors this can be identified with the LR dressing phase of [9] generalized to the case of different masses with an additional factor of $1/2$ in front.

6.7 $AdS_3 \times S^3 \times T^4$

We now turn towards the Green-Schwarz string for $AdS_3 \times S^3 \times T^4$ and we begin by considering the generic case with a mixture of RR and NSNS fluxes, the S-matrix was found for this scenario in [37]. We will find the 1-loop result with a mixed flux but for 2-loop with pure RR flux.

The massive spectrum of $AdS_3 \times S^3 \times T^4$ consists of 4 bosons and 4 fermions divided into two representations as in $AdS_3 \times S^3 \times S^3 \times S^1$ this time however the

S-matrix is factorized as in (6.1).

There are also massless excitations in the spectrum but as previously we will just assume that they scatter in a reflectionless way with the massive excitations and ignore them. The massive spectrum is factorized as follows:

$$\begin{aligned} |y_+\rangle &= |\phi\rangle \otimes |\phi\rangle & |z_+\rangle &= |\psi\rangle \otimes |\psi\rangle & |\zeta_+\rangle &= |\phi\rangle \otimes |\psi\rangle & |\chi_+\rangle &= |\psi\rangle \otimes |\phi\rangle, \\ |y_-\rangle &= |\bar{\phi}\rangle \otimes |\bar{\phi}\rangle & |z_-\rangle &= |\bar{\psi}\rangle \otimes |\bar{\psi}\rangle & |\zeta_-\rangle &= |\bar{\phi}\rangle \otimes |\bar{\psi}\rangle & |\chi_-\rangle &= |\bar{\psi}\rangle \otimes |\bar{\phi}\rangle. \end{aligned} \quad (6.73)$$

The notation have been chosen to make the similarities with $AdS_3 \times S^3 \times S^3 \times S^1$ as clear as possible. The T-matrix *i.e.* one of the two factors of the full T-matrix is also parametrized as before:

$$\begin{aligned} T_{\phi\phi}^{\phi\phi} &= A_{LL}, & T_{\psi\psi}^{\psi\psi} &= D_{LL}, & T_{\phi\psi}^{\phi\psi} &= G_{LL}, \\ T_{\phi\psi}^{\psi\phi} &= H_{LL}, & T_{\psi\phi}^{\phi\psi} &= K_{LL}, & T_{\psi\phi}^{\psi\phi} &= L_{LL}, \\ T_{\phi\bar{\phi}}^{\phi\bar{\phi}} &= A_{LR}, & T_{\phi\bar{\phi}}^{\psi\bar{\psi}} &= C_{LR}, & T_{\psi\bar{\psi}}^{\psi\bar{\psi}} &= D_{LR}, \\ T_{\phi\bar{\psi}}^{\phi\bar{\psi}} &= G_{LR}, & T_{\psi\bar{\psi}}^{\phi\bar{\phi}} &= F_{LR}, & T_{\psi\bar{\phi}}^{\psi\bar{\phi}} &= L_{LR}, \end{aligned} \quad (6.74)$$

From [37] we get the tree-level expressions:

$$A_{LL} = \frac{1}{2}(l_1 + c), \quad G_{LL} = \frac{1}{2}(l_3 + c), \quad H_{LL} = -l_5, \quad (6.75)$$

$$D_{LL} = \frac{1}{2}(-l_1 + c), \quad L_{LL} = \frac{1}{2}(-l_3 + c), \quad K_{LL} = -l_5, \quad (6.76)$$

$$A_{LR} = \frac{1}{2}(l_2 + c), \quad C_{LR} = l_4, \quad G_{LR} = \frac{1}{2}(l_3 + c), \quad (6.77)$$

$$D_{LR} = \frac{1}{2}(-l_2 + c), \quad F_{LR} = l_4, \quad L_{LR} = \frac{1}{2}(-l_3 + c). \quad (6.78)$$

It is clear that for both sectors the coefficients satisfy:

$$A^{(0)} + D^{(0)} - G^{(0)} - L^{(0)} = 0, \quad (6.79)$$

irrespectively of what l_1 , l_2 , l_3 and c are so we will not need these functions. All we will need are the following expressions:

$$l_4 = -\frac{pp'}{2(p+p')} \left(\sqrt{(\varepsilon_+ + p + q)(\varepsilon'_- + p' - q)} - \sqrt{(\varepsilon_+ - p - q)(\varepsilon'_- - p' + q)} \right), \quad (6.80)$$

$$l_5 = -\frac{pp'}{2(p-p')} \left(\sqrt{(\varepsilon_+ + p + q)(\varepsilon'_+ + p' + q)} + \sqrt{(\varepsilon_+ - p - q)(\varepsilon'_+ - p' - q)} \right), \quad (6.81)$$

$$\varepsilon_{\pm} = \sqrt{(p \pm q)^2 + 1 - q^2}. \quad (6.82)$$

Here q is the ratio of the NSNS flux over the RR flux, $q \rightarrow 0$ gives pure RR flux. The other two sectors can be found by changing ε_+ and ε_- as well as the signs on q . Note that taking the $q \rightarrow 0$ limit of this S-matrix gives the $\alpha \rightarrow 1$ limit of the BOSS S-matrix for both sectors [21].

The presence of the NSNS flux changes the dispersion relations and in turn the propagator which we will need to take into account. The integrals can be interpreted as regular equal-mass integrals with shifted momenta:

$$\begin{aligned} \tilde{I}(p, p')_{LL} &= I(p + q, p' + q), \\ \tilde{I}(p, p')_{LR} &= I(p + q, p' - q). \end{aligned} \quad (6.83)$$

The Jacobians are also modified:

$$J_{LL} = 4((p + q)\varepsilon'_+ - (p' + q)\varepsilon_+), \quad J_{LR} = 4((p + q)\varepsilon'_- - (p' - q)\varepsilon_+). \quad (6.84)$$

We are now ready to compute the cuts for the LL sector:

$$\begin{aligned}
\frac{1}{J_{s,LL}}(C_{s,LL})_{\phi\phi}^{\phi\phi} &= A_{LL}^{(0)2}, & \frac{1}{J_{u,LL}}(C_{u,LL})_{\phi\phi}^{\phi\phi} &= A_{LL}^{(0)2} - H_{LL}^{(0)}K_{LL}^{(0)}, \\
\frac{1}{J_{s,LL}}(C_{s,LL})_{\psi\psi}^{\psi\psi} &= D_{LL}^{(0)2}, & \frac{1}{J_{u,LL}}(C_{u,LL})_{\psi\psi}^{\psi\psi} &= D_{LL}^{(0)2} - H_{LL}^{(0)}K_{LL}^{(0)}, \\
\frac{1}{J_{s,LL}}(C_{s,LL})_{\phi\psi}^{\phi\psi} &= G_{LL}^{(0)2} + H_{LL}^{(0)}K_{LL}^{(0)}, & \frac{1}{J_{u,LL}}(C_{u,LL})_{\phi\psi}^{\phi\psi} &= G_{LL}^{(0)2}, \\
\frac{1}{J_{s,LL}}(C_{s,LL})_{\psi\phi}^{\phi\psi} &= G_{LL}^{(0)}H_{LL}^{(0)} + H_{LL}^{(0)}L_{LL}^{(0)}, & \frac{1}{J_{u,LL}}(C_{u,LL})_{\psi\phi}^{\phi\psi} &= D_{LL}^{(0)}H_{LL}^{(0)} + H_{LL}^{(0)}A_{LL}^{(0)}, \\
\frac{1}{J_{s,LL}}(C_{s,LL})_{\phi\psi}^{\psi\phi} &= L_{LL}^{(0)}K_{LL}^{(0)} + K_{LL}^{(0)}G_{LL}^{(0)}, & \frac{1}{J_{u,LL}}(C_{u,LL})_{\phi\psi}^{\psi\phi} &= A_{LL}^{(0)}K_{LL}^{(0)} + K_{LL}^{(0)}D_{LL}^{(0)}, \\
\frac{1}{J_{s,LL}}(C_{s,LL})_{\psi\phi}^{\psi\phi} &= L_{LL}^{(0)2} + H_{LL}^{(0)}K_{LL}^{(0)}, & \frac{1}{J_{u,LL}}(C_{u,LL})_{\psi\phi}^{\psi\phi} &= L_{LL}^{(0)2},
\end{aligned} \tag{6.85}$$

and the LR sector:

$$\begin{aligned}
\frac{1}{J_{s,LR}}(C_{s,LR})_{\phi\bar{\phi}}^{\phi\bar{\phi}} &= A_{LR}^{(0)2} + C_{LR}^{(0)}F_{LR}^{(0)}, & \frac{1}{J_{u,LR}}(C_{u,LR})_{\phi\bar{\phi}}^{\phi\bar{\phi}} &= A_{LR}^{(0)2}, \\
\frac{1}{J_{s,LR}}(C_{s,LR})_{\phi\bar{\phi}}^{\psi\bar{\psi}} &= A_{LR}^{(0)}C_{LR}^{(0)} + C_{LR}^{(0)}D_{LR}^{(0)}, & \frac{1}{J_{u,LR}}(C_{u,LR})_{\phi\bar{\phi}}^{\psi\bar{\psi}} &= G_{LR}^{(0)}C_{LR}^{(0)} + C_{LR}^{(0)}L_{LR}^{(0)}, \\
\frac{1}{J_{s,LR}}(C_{s,LR})_{\psi\bar{\psi}}^{\psi\bar{\psi}} &= D_{LR}^{(0)2} + C_{LR}^{(0)}F_{LR}^{(0)}, & \frac{1}{J_{u,LR}}(C_{u,LR})_{\psi\bar{\psi}}^{\psi\bar{\psi}} &= D_{LR}^{(0)2}, \\
\frac{1}{J_{s,LR}}(C_{s,LR})_{\psi\bar{\psi}}^{\phi\bar{\phi}} &= D_{LR}^{(0)}F_{LR}^{(0)} + F_{LR}^{(0)}A_{LR}^{(0)}, & \frac{1}{J_{u,LR}}(C_{u,LR})_{\psi\bar{\psi}}^{\phi\bar{\phi}} &= L_{LR}^{(0)}F_{LR}^{(0)} + F_{LR}^{(0)}G_{LR}^{(0)}, \\
\frac{1}{J_{s,LR}}(C_{s,LR})_{\phi\bar{\psi}}^{\phi\bar{\psi}} &= G_{LR}^{(0)2}, & \frac{1}{J_{u,LR}}(C_{u,LR})_{\phi\bar{\psi}}^{\phi\bar{\psi}} &= G_{LR}^{(0)2} - C_{LR}^{(0)}F_{LR}^{(0)}, \\
\frac{1}{J_{s,LR}}(C_{s,LR})_{\psi\bar{\phi}}^{\psi\bar{\phi}} &= L_{LR}^{(0)2}, & \frac{1}{J_{u,LR}}(C_{u,LR})_{\psi\bar{\phi}}^{\psi\bar{\phi}} &= L_{LR}^{(0)2} - C_{LR}^{(0)}F_{LR}^{(0)}.
\end{aligned} \tag{6.86}$$

The difference between the u- and the s-channel cuts then becomes:

$$\frac{C_{s,LL}}{J} - \frac{C_{u,LL}}{J} = \frac{p^2(p')^2}{2(p-p')^2} (\varepsilon_+ \varepsilon'_+ + (p+q)(p'+q) + (1-q^2)) \mathbb{1}, \tag{6.87}$$

for the LL sector and

$$\frac{C_{s,LR}}{J_{s,LR}} - \frac{C_{u,LR}}{J_{u,LR}} = \frac{p^2(p')^2}{2(p+p')^2} (\varepsilon_+ \varepsilon'_- + (p+q)(p'-q) - (1-q^2)) \mathbb{1}, \quad (6.88)$$

for the LR sector. Using the integrals in (6.83) this gives us the 1-loop T-matrix corrections:

$$iT_{LL}^{(1)} = i \left(\frac{1}{2} \left(- \frac{1}{\pi} \frac{p^2(p')^2 (\varepsilon_+ \varepsilon'_+ + (p+q)(p'+q) + (1-q^2))}{2(p-p')^2} \ln \left| \frac{\varepsilon'_+ - p' - q}{\varepsilon_+ - p - q} \right| \right) \mathbb{1} + \text{rat.} \right), \quad (6.89)$$

$$iT_{LR}^{(1)} = i \left(\frac{1}{2} \left(- \frac{1}{\pi} \frac{p^2(p')^2 (\varepsilon_+ \varepsilon'_- + (p+q)(p'-q) - (1-q^2))}{2(p+p')^2} \ln \left| \frac{\varepsilon'_- - p' + q}{\varepsilon_+ - p - q} \right| \right) \mathbb{1} + \text{rat.} \right). \quad (6.90)$$

The results are in line with the expectation that only diagonal elements get logarithmic contributions and the $q \rightarrow 0$ limit also matches the dressing phase found in [9].

For the 2-loop calculation we just consider the case of $q = 0$. The maximal cuts are given by:

$$\begin{aligned} \frac{1}{J^2} C_a^{ALL} &= A_{LL}^{(1)} \text{ } s\text{-cut} A_{LL}^{(0)}, \\ \frac{1}{J^2} C_d^{ALL} &= A_{LL}^{(1)} \text{ } u\text{-cut} A_{LL}^{(0)} - \frac{1}{2} H_{LL}^{(1)} \text{ } u\text{-cut} K_{LL}^{(0)} - \frac{1}{2} H_{LL}^{(0)} K_{LL}^{(1)} \text{ } u\text{-cut}, \\ \frac{1}{J^2} C_b^{ALL} &= A_{LL}^{(1)} \text{ } u\text{-cut} A_{LL}^{(0)}, \\ \frac{1}{J^2} C_e^{ALL} &= A_{LL}^{(1)} \text{ } s\text{-cut} A_{LL}^{(0)} - H_{LL}^{(1)} \text{ } s\text{-cut} K_{LL}^{(0)}, \\ \frac{1}{J^2} C_c^{ALL} &= A_{LL}^{(1)} \text{ } u\text{-cut} A_{LL}^{(0)}, \\ \frac{1}{J^2} C_f^{ALL} &= A_{LL}^{(1)} \text{ } s\text{-cut} A_{LL}^{(0)} - H_{LL}^{(0)} K_{LL}^{(1)} \text{ } s\text{-cut}. \end{aligned} \quad (6.91)$$

It is trivial to show that:

$$\mathcal{C}_{\ln^2}^{ALL} \propto -2C_a^{ALL} + C_b^{ALL} + C_c^{ALL} - 2C_d^{ALL} + C_e^{ALL} + C_f^{ALL} = 0, \quad (6.92)$$

so there are no double-logarithm for this coefficient this is also true for the other components of the S-matrix.

Let us now find the remaining two coefficients that allows us to find the terms with only a single logarithm. The 2-particle cuts give us:

$$\begin{aligned}\frac{C_{s,\text{extra}}^{\text{ALL}}}{J} &= -iA_{LL}^{(0)} \frac{C_s^{\text{ALL}}}{J} - \frac{1}{2} \left(\frac{C_b^{\text{ALL}}}{J^2} + \frac{C_c^{\text{ALL}}}{J^2} \right) JI_t + 2A_{LL}^{(0)} \tilde{A}_{LL}^{(1)}, \\ \frac{C_{u,\text{extra}}^{\text{ALL}}}{J} &= -iA_{LL}^{(0)} \frac{C_s^{\text{ALL}}}{J} + \frac{1}{2} iH_{LL}^{(0)} \frac{C_s^{\text{KLL}}}{J} + \frac{1}{2} iK_{LL}^{(0)} \frac{C_s^{\text{HLL}}}{J} - \frac{1}{2} \left(\frac{C_e^{\text{ALL}}}{J^2} + \frac{C_f^{\text{ALL}}}{J^2} \right) JI_t \\ &\quad + 2A_{LL}^{(0)} \tilde{A}_{LL}^{(1)} - H_{LL}^{(0)} \tilde{K}_{LL}^{(1)} - K_{LL}^{(0)} \tilde{H}_{LL}^{(1)},\end{aligned}\tag{6.93}$$

Subtracting these two expressions from each other gives us:

$$\begin{aligned}\frac{C_{s,\text{extra}}^{\text{ALL}}}{J} - \frac{C_{u,\text{extra}}^{\text{ALL}}}{J} &= \frac{1}{2} \left(\frac{C_e^{\text{ALL}}}{J^2} + \frac{C_f^{\text{ALL}}}{J^2} - \frac{C_b^{\text{ALL}}}{J^2} - \frac{C_c^{\text{ALL}}}{J^2} \right) JI_t \\ &\quad - \frac{1}{2} H_{LL}^{(0)} (2\tilde{K}_{LL}^{(1)} - (i)^2 K_{LL}^{(1)}{}_{s\text{-cut}}) \\ &\quad - \frac{1}{2} K_{LL}^{(0)} (2\tilde{H}_{LL}^{(1)} - (i)^2 H_{LL}^{(1)}{}_{s\text{-cut}}).\end{aligned}\tag{6.94}$$

Again we need to use some relations for the 1-loop rational terms:

$$\tilde{H}_{LL}^{(1)} = \hat{H}_{LL}^{(1)} = \frac{i}{2} \left(A_{LL}^{(0)} + D_{LL}^{(0)} \right) H_{LL}^{(0)} + \frac{i}{4} (1+4b)(p-p') H_{LL}^{(0)},\tag{6.95}$$

$$\tilde{K}_{LL}^{(1)} = \hat{K}_{LL}^{(1)} = \frac{i}{2} \left(A_{LL}^{(0)} + D_{LL}^{(0)} \right) K_{LL}^{(0)} - \frac{i}{4} (1+4b)(p-p') K_{LL}^{(0)}.\tag{6.96}$$

With these relations we find the 2-loop single logarithmic term to be proportional to:

$$-4\pi i \mathcal{C}_{\ln^1}^{\text{ALL}} = \frac{1}{J^2} (2C_a^{\text{ALL}} - C_b^{\text{ALL}} - C_c^{\text{ALL}}) = -2(iA_{LL}^{(0)}) \left(\frac{C_s^{\text{ALL}}}{J} - \frac{C_u^{\text{ALL}}}{J} \right)\tag{6.97}$$

The calculations for the other components are fairly similar and we get:

$$-4\pi i \mathcal{C}_{\ln^1}^{\text{A}_{LR}} = \frac{1}{2J^2} (2C_a^{\text{A}_{LR}} - C_b^{\text{A}_{LR}} - C_c^{\text{A}_{LR}}) = -2(iA_{LR}^{(0)}) \left(\frac{C_s^{\text{A}_{LR}}}{J} - \frac{C_u^{\text{A}_{LR}}}{J} \right). \quad (6.98)$$

So also for this sector we get something consistent with the structure from equation (6.21):

$$iT_{LR}^{(2)} = -\frac{1}{2}T^{(0)}\theta_{LR}^{(1)} + \text{rational}. \quad (6.99)$$

This concludes the chapter about using generalized unitarity for integrable worldsheet S-matrices. As we have seen generalized unitarity can be quite effective in terms of finding the logarithmic part of the dressing phase but also in testing whether an exact S-matrix could be consistent as a worldsheet perturbation theory.

Chapter 7 —

A Twistor String Theory for ABJ(M)

ABJ(M) is a 3-dimensional theory with a gauge field governed by a Chern-Simons Lagrangian plus some matter fields. It is interesting because it has several nice features similar to $\mathcal{N} = 4$ SYM such as integrability and Yangian invariance of amplitudes.

In this chapter we will look at another similarity, the formulation of the tree-level amplitudes as simple integrands localized on holomorphic curves in twistor space. For $\mathcal{N} = 4$ SYM this formulation of the scattering amplitudes follow directly from string theory in twistor space we will show that the same thing holds for ABJ(M).

I will attempt to make it not too technical focusing more on the motivation behind the venture and less on some of the details. The full details can be found in the paper on which this chapter is based [35].

7.1 ABJ(M) Theory

Because of the Chern-Simons Lagrangian there are no external gluons. There are two gauge groups and the gluons are adjoint in either of them while the matter fields are fundamental under one of the gauge groups and anti-fundamental under the other. The external fields consist of 4 bosons, 4 fermions and their conjugates and they can be grouped super-fields as follows:

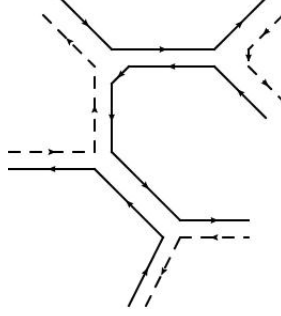


Figure 7.1. The dashed line and the full line represent the two different gauge groups

$$\hat{\Phi}(\lambda, \eta) = \phi(\lambda) + \psi_I(\lambda)\eta^I + \phi_{IJ}(\lambda)\eta^I\eta^J + \psi_{IJK}(\lambda)\eta^I\eta^J\eta^K, \quad (7.1)$$

$$\hat{\Psi}^{IJK}(\lambda, \eta) = \bar{\psi}^{IJK}(\lambda) + \eta^I\bar{\phi}^{JK}(\lambda) + \eta^I\eta^J\bar{\psi}^K(\lambda) + \eta^I\eta^J\eta^K\bar{\phi}(\lambda), \quad (7.2)$$

where the ϕ 's are scalars and the ψ 's are fermions and the capital latin letters are $SU(4)$ indices. For a Lagrangian expression see the appendix D though there the notation is slightly different.

This first of all means that scattering amplitudes has to have an even number of external fields otherwise they would transform under the gauge groups. Another consequence is that for planar amplitudes every second external field has to be of the type (7.1) and the rest of the type (7.2) as illustrated in figure 7.1 which is a double-line diagram similar to the ones introduced by 't Hooft [52] the dashed and the full line represent the two different gauge groups and the arrows show how the fundamental and anti-fundamental indices are matched up. This way the amplitudes become color-ordered in a way similar to other gauge theories:

$$\mathcal{A}(\bar{\Phi}_{a_1}^{\bar{a}_1}, \Phi_{\bar{b}_2}^{b_2}, \bar{\Phi}_{a_3}^{\bar{a}_3} \cdots \Phi_{\bar{b}_n}^{b_n}) = \sum_{\sigma} \mathcal{A}(\bar{\Lambda}_{\sigma_1}, \Lambda_{\sigma_2}, \bar{\Lambda}_{\sigma_3} \cdots \Lambda_{\sigma_n}) \delta_{a_{\sigma_1}}^{b_{\sigma_2}} \delta_{\bar{b}_{\sigma_2}}^{\bar{a}_{\sigma_3}} \delta_{a_{\sigma_3}}^{b_{\sigma_4}} \cdots \delta_{\bar{b}_{\sigma_n}}^{\bar{a}_{\sigma_1}}, \quad (7.3)$$

where the sum runs over the permutations of the odd and the even sites separately modulo cyclic permutations of all states.

7.2 Two Beautiful Amplitudes

All of the tree-level amplitudes of $\mathcal{N} = 4$ Super-Yang-Mills can be put into one beautiful formula:

$$A_{n,k}(\mathcal{W}) = \int \frac{d^{2 \times n} \sigma}{\text{vol}[GL(2)]} \frac{\prod_{m=1}^k \delta^{4|4}(C_{mi}[\sigma] \mathcal{W}_i)}{(12)(23) \cdots (n1)}, \quad (7.4)$$

where n is the number of external states, k denotes that it is the N^{k-2} MHV amplitude, the σ is a n cross 2 matrix with the entries denoted a_i and b_i and \mathcal{W} is the conjugate twistor variables *i.e.* the conjugate of λ , $\bar{\lambda}$ and $\bar{\eta}$. The C -matrices are given by:

$$C_{mi}[\sigma] = a_i^{k-m} b_i^{m-1}, \quad (7.5)$$

and the products in the denominator are:

$$(ij) = a_i b_j - a_j b_i. \quad (7.6)$$

The integral has a $GL(2)$ symmetry making it divergent so in order to make it sensible one has to divide by the volume of $GL(2)$ this corresponds to fixing four of the integration variables and multiply by a corresponding Jacobian.

The expression may look rather complicated what with $2n - 4$ integrals to be done however if we for a moment leave the Grassmann variables aside and focus on the bosonic variables, we can go back to the regular twistor variables using:

$$\begin{aligned} & \int d^n \mathcal{W} e^{-i \sum_i \mathcal{Z}_i \cdot \mathcal{W}_i} \prod_{m=1}^k \delta(C_{mi}[\sigma] \mathcal{W}_i) \\ &= \int d^k z \int d^n \mathcal{W} e^{i \sum_m \sum_i C_{mi} z_m \cdot \mathcal{W}_i - i \sum_i \mathcal{Z}_i \cdot \mathcal{W}_i} \\ &= \int d^k z \prod_{i=1}^n \delta \left(\mathcal{Z}_i - \sum_{m=1}^k z_{(m)} C_{mi} \right) \end{aligned} \quad (7.7)$$

What is that good for? Well $\tilde{\lambda}$ is part of the conjugate twistor variables while λ is part of the normal twistor variables so to get an expression of the regular spinors we need to transform half of the twistor variables this way we end up with $2n + 2k - 4$ integration variables and $2n + 2k$ delta functions so the integral is not really an integral but simply an expression with variables projected onto a specific point; the remaining four delta functions simply give us momentum conservation.

It is in fact quite easy to see that momentum conservation is incorporated into (7.4) once one have Fourier transformed half of the variables¹ and it is a exercise that will be useful later on. After Fourier transforming half the variables we get delta function that enforce the following equations:

$$0 = \lambda_i^a - \sum_{m=1}^k z_{(m)}^a C_{mi}, \quad (7.8)$$

$$0 = \sum_{i=1}^n C_{mi} \bar{\lambda}_i^{\dot{a}}. \quad (7.9)$$

We then write down the following:

$$0 = \sum_{m=1}^k z_{(m)}^a \sum_{i=1}^n C_{mi} \bar{\lambda}_i^{\dot{a}}. \quad (7.10)$$

This is definitely true since every single term in the sum is 0. However this can also be reformulated to give us:

$$0 = \sum_{i=1}^n \bar{\lambda}_i^{\dot{a}} \sum_{m=1}^k z_{(m)}^a C_{mi} = \sum_{i=1}^n \bar{\lambda}_i^{\dot{a}} \lambda_i^a. \quad (7.11)$$

So we see that momentum conservation is in fact incorporated into (7.4).

The tree-level amplitudes of ABJM can be put into a similar formula [45, 38]:

¹Though not that easy to pull the momentum conserving delta function outside

$$A_{2k}(\Lambda) = \int \frac{d^{2 \times n} \sigma}{\text{vol}[GL(2)]} \frac{J \Delta \prod_{m=1}^k \delta^{2|3}(C_{mi}[\sigma] \Lambda_i)}{(12)(23) \cdots (n1)}. \quad (7.12)$$

There are certain differences though: the Λ 's are the regular twistor *i.e.* a two component spinor λ and 3 Grassmann variables, η^2 not the conjugate twistors and one should note that k now has to be half of the number of external states meaning that only the 4-pt. amplitude is MHV. Finally, there are these additional functions:

$$\Delta = \prod_{j=1}^{2k-1} \delta \left(\sum_i a_i^{2k-1-j} b_i^{j-1} \right), \quad (7.13)$$

$$J = \frac{\text{Num}}{\text{Den}}, \quad (7.14)$$

$$\text{Num} = \det_{1 \leq i, j \leq 2k-1} (a_i^{2k-1-j} b_i^{j-1}) = \prod_{1 \leq i < j \leq 2k-1} (i, j), \quad (7.15)$$

$$\text{Den} = \det_{1 \leq i, j \leq k} (a_{2i-1}^{k-j} b_{2i-1}^{j-1}) = \prod_{1 \leq i < j \leq k} (2i-1, 2j-1). \quad (7.16)$$

Still the similarities between (7.12) and (7.4) are quite clear. Notice however that momentum conservation is not so easy to see as before and it depends on the factor Δ . It may be useful to note that even though Num only depend on $n-1$ of the external legs the particular leg is arbitrary due to the delta function (7.13).

7.3 Twistor String Theory

The formula (7.4) was shown to follow from a topological string theory in twistor space by Witten [53] and later it was formulated as an open string theory [14, 15]. It is the open string theory that our construction will mimic. Essential to the construction is the appearance of worldsheet instantons, the scattering with $k-1$ instantons will correspond to the N^{k-2} MHV amplitudes.

For $\mathcal{N} = 4$ Super-Yang-Mills the relevant twistor space is $CP^{3|4}$ where the worldsheet fields are given by:

²Strictly speaking it is what is called mini-twistor space

$$Z^I = (\lambda^a, \mu^{\dot{a}}, \psi^A), \quad Y_I = (\bar{\mu}_a, \bar{\lambda}_{\dot{a}}, \bar{\psi}_A). \quad (7.17)$$

Both comes in a right-moving and left-moving set which will be denoted by a subscript (L or R). The action is giving by:

$$S = \int d^2\rho (Y_{LI} \nabla_R Z_L^I + Y_{RI} \nabla_L Z_R^I) + S_G. \quad (7.18)$$

The theory has a $GL(1)$ symmetry:

$$Y \rightarrow t^{-1}Y, \quad Z \rightarrow tZ. \quad (7.19)$$

This symmetry has been gauged and the ∇ 's are covariant derivative that include the $GL(1)$ worldsheet gauge field; it is this gauge field that give rise to the instantons. S_G is the action of the current algebra; there are different ways to create a current algebra but for the purposes of [14] it is not necessary to specify a particular one what matters is that the correlation function of the currents is given by:

$$\langle J^{a_1}(\rho_1) \dots J^{a_n}(\rho_n) \rangle = \sum \frac{\text{Tr}[T^{a_1} T^{a_2} \dots T^{a_n}]}{\text{cyc}(1, \dots, n)} + \text{double traces}, \quad (7.20)$$

$$\text{cyc}(\rho_l, \dots, \rho_m) = (\rho_m - \rho_l) \prod_{i=l}^{m-1} (\rho_i - \rho_{i+1}),$$

where the sum runs over permutations up til cyclicity. The denominator in this expression is ultimately what will lead to the denominator in (7.4).

The string theory is an open string theory so the following boundary conditions are imposed:

$$Y_L = Y_R, \quad Z_L = Z_R, \quad (7.21)$$

so the vertices are only going to be functions of one set of variables. The vertices will also need to be invariant under the $GL(1)$ transformation such vertices are the ones below:

$$V = \int d\rho J^a(\rho) \int \frac{d\xi}{\xi} \delta^2(\lambda - \xi\lambda(\rho)) \exp(i\xi\bar{\lambda}_{\dot{a}}\mu^{\dot{a}}(\rho)) \delta^4(\eta^A - \xi\psi^A(\rho)). \quad (7.22)$$

Here the λ , $\bar{\lambda}$ and η not dependent on any variables are the regular spinors and Grassmann variables that the amplitude depend on while the others are the worldsheet fields. The ξ is merely introduced to make the formula appear nicer it is not a necessary variable but it makes the $GL(1)$ invariance easy to see as any rescaling of the worldsheet fields can be canceled by a rescaling of ξ . With a simple Fourier transform one can also see that (7.22) has the full $SU(4|4)$ symmetry.

The worldsheet coordinates become functions of the collective coordinates, $z_{(m)}$, of the instanton:

$$\lambda^a(\rho) = \sum_{m=1}^k z_{(m)}^a \rho^{m-1}, \quad \mu^{\dot{a}} = \sum_{m=1}^k z_{(m)}^{\dot{a}} \rho^{m-1}, \quad \psi^A(\rho) = \sum_{m=1}^k z_{(m)}^A \rho^{m-1}. \quad (7.23)$$

These are in fact all of the elements one need to find (7.4); one simply compute the correlation function of the vertices in (7.22):

$$\langle V(\rho_1)V(\rho_2)\cdots V(\rho_n) \rangle, \quad (7.24)$$

After integrating over the collective coordinates, one Fourier transform the variables into the conjugate twistor variables³ and make the identification:

$$a_i = \xi_i \quad b_i = \xi_i \rho_i, \quad (7.25)$$

and we end up with the desired result.

There are however other vertices that are invariant under the $GL(1)$ transfor-

³Note that inverting (7.7) is going to remove the z 's which are the collective coordinates

mation one could imagine vertices like:

$$V_f = f^I(Z(\rho))Y_I(\rho) \qquad V_g = g_I(Z(\rho))\partial Z^I(\rho). \quad (7.26)$$

These vertices were investigated in [16] where they were shown to represent conformal supergravity. Conformal supergravity is in fact already present even without these vertices, it is those fields that are responsible for the multiple trace structure that will arrive from (7.20).

7.4 Making it Work for ABJ(M)

Since (7.12) and (7.4) are so similar and the $\mathcal{N} = 4$ Super-Yang-Mills result follow from a twistor string it seems natural to try to come up with a twistor string theory for ABJ(M). First one may notice that simply reducing the space from $CP^{3|4}$ to $CP^{2|x}$ is not going to work no matter what x is because the bosonic part does satisfy momentum conservation; we need some kind of scheme to provide momentum conservation for us and in turn give these additional functions (7.15) and (7.16). So even though in the twistor Λ has fewer entries than \mathcal{W} and \mathcal{Z} we are actually going to start with a bigger space $CP^{4|5}$ which incidentally is also a Calabi-Yau space like $CP^{3|4}$; that is we are going to introduce 3 extra bosonic directions that we need and 2 extra fermionic, extra directions we are subsequently going to project away.

It is not difficult to see how we can now get momentum conservation: fix the fifth bosonic variable to some value, write some twistor string theory like for $\mathcal{N} = 4$ Super-Yang-Mills which will necessarily satisfy the relation⁴:

$$0 = \sum_i^n \mu^{\dot{a}} \bar{\mu}^b. \quad (7.27)$$

Then one simply imposes

⁴It may be slightly confusing but the regular twistors appear in the ABJ(M) in the way that the conjugate twistors appear in $\mathcal{N} = 4$ Super-Yang-Mills so we actually mean conservation of the conjugate variables

$$\mu_{\dot{1}} = \bar{\mu}_1 \qquad \mu_{\dot{2}} = \bar{\mu}_2 \ . \qquad (7.28)$$

This clearly leads to

$$0 = \sum_i^n \bar{\mu}^a \mu^b, \qquad (7.29)$$

but unfortunately it is clearly also a divergent procedure as two different conditions in (7.27) leads to two identical conditions in (7.29):

$$0 = \sum_i^n \bar{\mu}^1 \mu^2, \qquad (7.30)$$

giving us $\delta(0)$ so the procedure will have to be done with care.

The second projection we are going to need is also going to lead to a $\delta(0)$ albeit it will be a Grassmann delta function. We are going to apply the following projector:

$$\mathcal{P}_F(\bullet) = \int d\bar{\eta}^4 d\bar{\eta}^5 \delta(\mu_{\dot{1}} \bar{\eta}^4 + \mu_{\dot{2}} \bar{\eta}^5)(\bullet) \ . \qquad (7.31)$$

Just like the bosonic condition correspond to the vanishing of the momentum in a specific direction this imposes the vanishing of the super-momentum in a specific direction. It may appear that the projector breaks the symmetry between $\mu_{\dot{1}}$ and $\mu_{\dot{2}}$ however this is restored by the integration over the Grassmann variables.

7.5 The Enlarged Twistor String Theory

The worldsheet fields in our theory will be:

$$Z^I(\rho) = (\bar{\mu}^\alpha(\rho), \bar{\lambda}^{\dot{a}}(\rho), \bar{\psi}^A(\rho)) \qquad Y_I(\rho) = (\lambda_\alpha(\rho), \mu_{\dot{a}}(\rho), \psi_A(\rho)) \ , \qquad (7.32)$$

The fields transform in the fundamental representation of $SU(3, 2|4, 1)$ ($\alpha = 1, 2, 3, \dot{\alpha} = 1, 2, A = 1 \cdots 5$). The action is very similar to (7.18)⁵:

$$S = \int d^2\rho (Y_{LI}\nabla_R Z_L^I + Y_{RI}\nabla_L Z_R^I) + S_G . \quad (7.33)$$

Again ∇ is a covariant derivative that includes a $GL(1)$ gauge field. Unlike in [14] we have to more specific in picking a current action because we want the special feature of ABJ(M) that every second field is part of a different superfield. The action we need consists of N fermionic fields, Ψ_1 and M other fermionic fields Ψ_2 :

$$S_G = \int d^2\rho \sum_{i=1}^N (\bar{\Psi}_{1,L}^i \partial_R \Psi_{1,iL} + \bar{\Psi}_{1,R}^i \partial_L \Psi_{1,iR} + \sum_{j=1}^M \bar{\Psi}_{2,L}^j \partial_R \Psi_{2,jL} + \bar{\Psi}_{2,R}^j \partial_L \Psi_{2,jR}) . \quad (7.34)$$

Quantum mechanically there are restrictions on what M and N can be and unwise choices can make lead to anomalies at the loop level but classically (and we are so far only interested in the tree-level amplitudes) they can be whatever they want. We want to define the current:

$$J_F = q \left(\frac{1}{N} \bar{\Psi}_1 \Psi_1 - \frac{1}{M} \bar{\Psi}_2 \Psi_2 \right) . \quad (7.35)$$

This current have no anomalous term in the OPE with the stress tensor because the stress tensor basically counts the number of fields, with appropriate fermionic signs, and gives:

$$0 = \left(\frac{1}{N}(-N) - \frac{1}{M}(-M) \right) . \quad (7.36)$$

Having vanishing mixed anomaly means that non-vanishing correlation func-

⁵Notice again that regular variables and conjugate variables are mixed up compared to $\mathcal{N} = 4$ Super-Yang-Mills

tions must have zero charge under this current.

As for $\mathcal{N} = 4$ Super-Yang-Mills the string theory is an open string theory so the vertices will only depend on one set of variables. However unlike in that case we do not want the full symmetry of the twistor space one way that will break the symmetry is of course the projection mentioned earlier but already before then we want to break the symmetry from $SU(3, 2)$ to $SU(2, 2)$.

The breaking of the symmetry comes from treating the third component of $\bar{\mu}$ differently from the other two components. So we write down the vertex family⁶:

$$U_n(\bar{\mu}_a, \mu_{\dot{a}}, \bar{\eta}; \rho) = \int \frac{d\xi}{\xi} (\xi \bar{\mu}^3(\rho))^n \delta^2(\bar{\mu} - \xi \bar{\mu}(\rho)) e^{i\mu_{\dot{a}} \xi \bar{\lambda}^{\dot{a}}(\rho)} \delta^{0|5}(\bar{\eta}^A - \xi \bar{\psi}^A(\rho)). \quad (7.37)$$

These vertices will then be dressed with currents:

$$J_{ij}^a = \bar{\Psi}_i T^a \Psi_j, \quad (7.38)$$

to create the full vertices:

$$V_{ij} = J_{ij}^a U_n. \quad (7.39)$$

The fourth and the fifth fermionic directions are also extra dimensions to be treated differently from the others so it seems sensible to construct a current out of these extra directions:

$$J_E(\rho) = q_B \bar{\mu}^3 \lambda_3(\rho) + q_F \sum_{A=4}^5 \bar{\psi}^A \psi_A(\rho). \quad (7.40)$$

It has no mixed anomaly if

$$q_B = 2q_F. \quad (7.41)$$

⁶One could also imagine a vertex where μ^3 is localized by a delta function but it is more or less equivalent to U_{-1} as the delta functions, also for the $\mathcal{N} = 4$ Super-Yang-Mills twistor string theory, should be interpreted as simple poles with the integration encircling the singularity

After having used the fermionic projector (7.31) the vertices have the following charges under J_E (we denote the charge, q_E):

$$\begin{aligned} q_E(\mathcal{P}_F U_n) &= nq_B + q_F \\ &= (2n + 1)q_F. \end{aligned} \tag{7.42}$$

If we now make the following choice for the charge appearing in (7.35):

$$q = -\frac{NM}{N + M}q_F, \tag{7.43}$$

then the currents from the current algebra will have the charges:

$$q(J_{11}) = 0, \qquad q(J_{12}) = +q_F, \tag{7.44}$$

$$q(J_{21}) = -q_F \qquad q(J_{22}) = 0. \tag{7.45}$$

So if we now define a new current:

$$\mathbf{J} = J_E + J_F, \tag{7.46}$$

and enforce that all vertex operators have zero charge with respect to it we are left with only two vertices:

$$V_{-1,12}(\bar{\mu}_a, \mu_{\dot{a}}, \bar{\eta}; \rho) = J_{12} \mathcal{P}_F U_{-1}(\bar{\mu}_a, \mu_{\dot{a}}, \bar{\eta}; \rho) \tag{7.47}$$

$$V_{0,21}(\bar{\mu}_a, \mu_{\dot{a}}, \bar{\eta}; \rho) = J_{21} \mathcal{P}_F U_0(\bar{\mu}_a, \mu_{\dot{a}}, \bar{\eta}; \rho). \tag{7.48}$$

This is what we want as we shall see when we compute the scattering amplitudes; already now one may notice that due to the absence of both J_{11} and J_{22} every second external field will indeed be different as it should.

Whether this scheme is in fact sensible at the quantum level with no loop order anomalies is not quite clear but at the tree-level it is fully consistent to truncate

to the two vertex operators (7.47) and (7.48).

7.6 Computing the Amplitudes

After having defined the vertex operators we want, we may now compute the scattering amplitudes. Similarly to $\mathcal{N} = 4$ Super-Yang-Mills the worldsheet coordinates become functions of collective coordinates (or zero modes):

$$\bar{\mu}^\alpha(\rho_i) = \sum_{m=1}^k z_{(m)}^\alpha \rho_i^{m-1}, \quad \bar{\lambda}^{\dot{a}}(\rho_i) = \sum_{m=1}^k z_{(m)}^{\dot{a}} \rho_i^{m-1} \quad \bar{\psi}^A(\rho_i) = \sum_{m=1}^k z_{(m)}^A \rho_i^{m-1} \quad (7.49)$$

The amplitude with n external particles and $k - 1$ instantons then becomes⁷:

$$\mathcal{A}_{n,k} = \int \frac{\prod_{i=1}^n d\rho_i}{\text{vol}[GL(2)]} \prod_{m=1}^k d^{5|5} z_{(m)}^I \prod_{i \in \{V_{-1}\}} U_{-1}(\rho_i) \prod_{j \in \{V_0\}} U_0(\rho_j) \langle J_{12}(\rho_1) \dots J_{21}(\rho_{2n}) \rangle, \quad (7.50)$$

where $\{V_{-1}\} \equiv \{1, 3, 5, \dots\}$ and $\{V_0\} \equiv \{2, 4, 6, \dots\}$ while current correlator is given by something similar to (7.20) with the difference that only permutations within the two sets of currents are allowed. As mentioned the $\text{vol}[GL(2)]$ is there to remove a divergence as the integrand has a $GL(2)$ so we would be integrating over an infinite number of physically equivalent states, the $SL(2)$ is the usual divergence appearing in string amplitudes that can be removed by fixing 3 ρ 's [44], the $GL(1)$ symmetry is the one from (7.19).

Note that after the projecting:

$$\mathcal{P}_F \delta^{0|5}(\bar{\eta}^A - \xi \bar{\psi}^A(\rho)) = \delta^{0|3}(\bar{\eta}^A - \xi \bar{\psi}^A(\rho)) \delta(\xi \bar{\psi}^4(\rho) \mu_1 + \xi \bar{\psi}^5(\rho) \mu_2), \quad (7.51)$$

there will be exactly n of the z^4/z^5 -variables in the integrand meaning there also has to be that many integration variables setting:

⁷Or rather the only part that survives after the projections

$$k = \frac{n}{2}. \quad (7.52)$$

We want to deal with the projections and additional variables one at a time so we define:

$$\begin{aligned} \mathcal{P}_B \mathcal{P}_F \mathcal{A}_{n,k} = & \int \frac{\prod_{i=1}^n d\rho_i}{\text{vol}[\text{GL}(2)]} \frac{d\xi_i}{\xi_i} \prod_{m=1}^k d^{2|3} dz_{(m)}^I I_3 I_{\text{exp}} I_{4,5} \prod_{i=1}^n \delta^{0|3}(\bar{\eta}_i^A - \xi_i \bar{\psi}^A(\rho_i)) \\ & \langle J_{12}(\rho_1) \dots J_{21}(\rho_n) \rangle. \end{aligned} \quad (7.53)$$

Here I_3 is the $\bar{\mu}^3$ -part, $I_{4,5}$ contains the integral over the fourth and fifth fermionic directions and I_{exp} contains the integral over the $\bar{\lambda}$ zero modes being subjected to the constraints (7.28) (as indicated by the projector \mathcal{P}_B).

I_3 is rather simple because the relationship (7.52) makes it the determinant of $k \times k$ matrix:

$$I_3 = \int \prod_{m=1}^k dz_{(m)}^3 \prod_{i \in \{V_{-1}\}} \frac{1}{\xi_i \bar{\mu}^3(\rho_i)} = \frac{1}{\det_{i \in \{V_{-1}\}}(\xi_i \rho_i^{m-1})} \int \prod_{m=1}^k \frac{dz_{(m)}^3}{z_{(m)}^3}. \quad (7.54)$$

As mentioned both here and for $\mathcal{N} = 4$ Super-Yang-Mills the integration contours are always chosen to enclose the singularity:

$$I_3 = \frac{1}{\det_{i \in \{V_{-1}\}}(\xi_i \rho_i^{m-1})}. \quad (7.55)$$

I_{exp} is slightly more complicated, before employing the projection (7.28) we have:

$$I_{\text{exp}} = \mathcal{P}_B \int \prod_{m=1}^k d^2 z_{(m)}^{\dot{a}} \prod_{i=1}^n \delta^2(\bar{\mu}_i - \xi_i \bar{\mu}(\rho_i)) e^{i\mu_{i\dot{a}} \xi_i \bar{\lambda}^{\dot{a}}(\rho_i)}. \quad (7.56)$$

After applying the projection the integrand becomes invariant under the fol-

lowing shift:

$$\bar{\lambda}^{\dot{1}}(\rho) \rightarrow \bar{\lambda}^{\dot{1}}(\rho) + a\bar{\mu}^1(\rho) \quad \bar{\lambda}^{\dot{2}}(\rho) \rightarrow \bar{\lambda}^{\dot{2}}(\rho) + a\bar{\mu}^2(\rho), \quad (7.57)$$

or put differently:

$$z_{(m)}^{\dot{1}} \rightarrow z_{(m)}^{\dot{1}} + az_{(m)}^1, \quad z_{(m)}^{\dot{2}} \rightarrow z_{(m)}^{\dot{2}} + az_{(m)}^2. \quad (7.58)$$

This shift means that when integrating over the z -coordinates one integrates over an infinite number of equivalent configurations. To pull the part we want away from the problematic part we exchange the integral over one of the z 's (which is set to zero) for a collective coordinate a :

$$I_{\text{exp}} = \int \left(\prod_{m=1}^k \prod_{\dot{a}=1}^2 \right)' dz_{(m)}^{\dot{a}} \prod_{i=1}^n \delta^2(\bar{\mu}_i - \xi_i \bar{\mu}(\rho_i)) e^{i\mu_{i\dot{a}} \xi_i \hat{\lambda}^{\dot{a}}(\rho_i)} \int (z_{(1)}^1 da) e^{ia \sum_{i=1}^n \bar{\mu}_i^a \mu_{i\dot{a}} \delta_{\dot{a}}^a}. \quad (7.59)$$

Here $\hat{\lambda} = \bar{\lambda}|_{z_{(1)}^{\dot{1}}=0}$ and the primed measure does not include an integral over $z_{(1)}^{\dot{1}}$. In the first part of (7.59) the projection causes no problems:

$$\begin{aligned} & \prod_{i=1}^n \delta^2(\bar{\mu}_i - \xi_i \bar{\mu}(\rho_i)) e^{i\mu_{i\dot{a}} \xi_i \hat{\lambda}^{\dot{a}}(\rho_i)} \\ & \longrightarrow \prod_{i=1}^n \delta^2(\bar{\mu}_i - \xi_i \bar{\mu}(\rho_i)) e^{i\xi_i^2 \sum_{m,l=1}^k (z_{(m)1} \hat{z}_{(l)}^{\dot{1}} + z_{(m)2} \hat{z}_{(l)}^{\dot{2}}) \rho_i^{m+l-2}}. \end{aligned} \quad (7.60)$$

Again the $\hat{}$ signifies that the variables $z_{(1)}^{\dot{1}}$ has been removed. Looking at the exponent in the second line we see that there is a more useful set of variables:

$$\alpha_w = \sum_{m=1}^k \sum_{l=1}^k \delta_{\dot{a}}^a z_{(m)a} \hat{z}_{(l)}^{\dot{a}} \delta_{m+l-1,w} \quad (7.61)$$

Changing to these variables gives the Jacobian:

$$\det \frac{\partial \alpha_w}{\partial \hat{z}_{(l)}^{\dot{a}}} = \det \sum_{m=1}^k z_{(m)a} \delta_{m+l-1,w} = \det_{w \times (l,a)} (z_{(w-l+1)a}). \quad (7.62)$$

Here $z_{(w-l+1)a}$ is considered a $2k-1 \times 2k-1$ matrix with $w = 1 \cdots 2k-1$ while $l = 1 \cdots k$ and $a = 1, 2$ with the pair $(l, a) = (1, 1)$ being excluded. The α_w 's only appear linearly in the exponentials of (7.60) and so merely create delta functions:

$$I_{\text{exp}} = \frac{z_{(1)}^1}{\det_{w \times (l,a)} (z_{(w-l+1)a})} \prod_{w=1}^{2k-1} \delta \left(\sum_{i=1}^{2n} \xi_i^2 \rho_i^{w-1} \right) \delta \left(\sum_{i=1}^n \bar{\mu}_i^a \mu_{i\dot{a}} \delta_a^{\dot{a}} \right) \prod_{i=1}^n \delta^2(\bar{\mu}_i - \xi_i \bar{\mu}(\rho_i)). \quad (7.63)$$

We now turn to the final piece of the amplitude, $I_{4,5}$:

$$I_{\text{exp}} I_{4,5} = I_{\text{exp}} \int \prod_{l=1}^k dz_{(l)}^4 dz_{(l)}^5 \prod_{i=1}^n \delta(\xi_i \bar{\psi}^4(\rho_i) \mu_{i\dot{1}} + \xi_i \bar{\psi}^5(\rho_i) \mu_{i\dot{2}}) \quad (7.64)$$

The bosonic projection (7.28) is important for the following considerations so that is why I_{exp} is included in the expression.

Just like I_{exp} the expression turn out to have a shift symmetry:

$$\bar{\psi}^4 \rightarrow \bar{\psi}^4 + \eta \bar{\mu}^1, \quad \bar{\psi}^5 \rightarrow \bar{\psi}^5 + \eta \bar{\mu}^2, \quad (7.65)$$

where η is a Grassmann variable. Put differently:

$$z_{(m)}^4 \rightarrow z_{(m)}^4 + \eta z_{(1)}^1, \quad z_{(m)}^5 \rightarrow z_{(m)}^5 + \eta z_{(1)}^1, \quad (7.66)$$

and so as before we exchange an integral over one of the z 's for an integral over $\eta = z_{(1)}^4 / z_{(1)}^1$:

$$I_{\text{exp}} I_{4,5} = I_{\text{exp}} \det \frac{\partial(\xi_i(\mu_{i1} \sum_{m=2}^k z_{(m)}^4 \rho_i^{m-1} + \mu_{i2} \sum_{m=1}^k z_{(m)}^5 \rho_i^{m-1}))_{i=2,\dots,n}}{\partial z_{(m)}^A|_{A=4,5}} \int \frac{d\eta}{z_{(1)}^1} \delta(\eta \sum_{i=1}^n \bar{\mu}_i^a \mu_{i\dot{a}} \delta_a^{\dot{a}}) ,$$

The integral over the η -variable creates a regularized zero while the remaining z -integrations create a determinant:

$$I_{\text{exp}} I_{4,5} = I_{\text{exp}} \det(\xi_i \mu_{i1} \rho_i^{m-1}|_{m=2,\dots,k}; \xi_i \mu_{i2} \rho_i^{m-1}|_{m=1,\dots,k}) \frac{1}{z_{(1)}^1} \left(\sum_{i=1}^n \bar{\mu}_i^a \mu_{i\dot{a}} \delta_a^{\dot{a}} \right) . \quad (7.67)$$

Notice that i only take $n - 1$ values as one delta function was used by the η -projection⁸. Now that we have safely isolated the potential problem coming from the shift we can impose the projection and use the delta functions of I_{exp} to rewrite this as:

$$I_{\text{exp}} I_{4,5} = I_{\text{exp}} \det(\xi_i^2 \sum_{l=1}^k \rho_i^{m+l-2} z_{(l)}^1|_{m=2,\dots,k} \xi_i^2 \sum_{l=1}^k \rho_i^{m+l-2} z_{(l)}^2|_{m=1,\dots,k}) \frac{1}{z_{(1)}^1} \left(\sum_{i=1}^n \bar{\mu}_i^a \mu_{i\dot{a}} \delta_a^{\dot{a}} \right) . \quad (7.68)$$

Using the expression for I_{exp} from (7.63) we see that not only does the factor $z_{(1)}^1$ drop out but the z -dependence of the two determinants also cancel each other out and we end up with:

$$I_{\text{exp}} I_{4,5} = \det_{w \times i} (\xi_i^2 \rho_i^{w-1}) \prod_{w=1}^{2k-1} \delta \left(\sum_{i=1}^n \xi_i^2 \rho_i^{w-1} \right) \prod_{i=1}^n \delta^2(\bar{\mu}_i - \xi_i \bar{\mu}(\rho_i)) \quad (7.69)$$

$$\times \left(\sum_{i=1}^n \bar{\mu}_i^a \mu_{i\dot{a}} \delta_a^{\dot{a}} \right) \delta \left(\sum_{i=1}^n \bar{\mu}_i^a \mu_{i\dot{a}} \delta_a^{\dot{a}} \right)$$

For the determinant $i = 2 \cdots n$ and $w = 1 \cdots 2k - 1$. Putting all of these results back into (7.53) we get:

⁸Which one does not matter just like it did not matter which external leg was picked out of Num in (7.15)

$$\begin{aligned}
\mathcal{A}_{n,k} = & \int \frac{\prod_{i=1}^n d\rho_i}{\text{volGL}(2)} \frac{d\xi_i}{\xi_i} \prod_{m=1}^n d^{2|3} dz_{(m)}^I \frac{\det_{w \times i}(\xi_i^2 \rho_i^{w-1})}{\det_{(i \in \{V_{-1}\}) \times m}(\xi_i \rho_i^{m-1})} \prod_{w=1}^{2k-1} \delta \left(\sum_{i=1}^n \xi_i^2 \rho_i^{w-1} \right) \\
& \times \prod_{i=1}^{2n} \delta^{2|0}(\bar{\mu}_i - \xi_i \bar{\mu}(\rho_i)) \delta^{0|3}(\bar{\eta}_i^A - \xi_i \bar{\psi}^A(\rho_i)) \langle J_{12}(\rho_1) \dots J_{21}(\rho_n) \rangle \\
& \times \left(\sum_{i=1}^n \bar{\mu}_i^a \mu_{i\dot{a}} \delta_a^{\dot{a}} \right) \delta \left(\sum_{i=1}^n \bar{\mu}_i^a \mu_{i\dot{a}} \delta_a^{\dot{a}} \right).
\end{aligned} \tag{7.70}$$

The factors in the third are a divergence times a zero completely isolated from the rest of the integral so we drop them. Only focusing on the single trace part we make the change of variables $\xi_i = \tilde{\xi}_i^{k-1}$ and Fourier transform to regular twistor variables giving us (up to a numerical factor which we throw away):

$$\delta_{k, \frac{n}{2}} \int \frac{d^{2 \times n} \sigma}{\text{vol}[\text{GL}(2)]} J \Delta \langle J_{12}(\sigma_1) \dots J_{21}(\sigma_n) \rangle \prod_{m=1}^k \delta^{2|3}(C_{mi}[\sigma] \Lambda_i), \tag{7.71}$$

where we use:

$$\begin{aligned}
C_{mi}[\sigma] &= a_i^{k-m} b_i^{m-1}, & \sigma_i &= (a_i, b_i) = \tilde{\xi}_i(1, \rho_i), \\
(i, j) &= a_i b_j - a_j b_i, & \Delta &= \prod_{j=1}^{2k-1} \delta \left(\sum_i a_i^{2k-1-j} b_i^{j-1} \right), \\
\langle \Psi_p(\sigma_i) \bar{\Psi}_q(\sigma_j) \rangle &= \frac{\delta_{pq}}{(i, j)}, & J &= \frac{\text{Num}}{\text{Den}}, \\
\text{Den} &= \prod_{1 \leq i < j \leq k} (2i-1, 2j-1), & \text{Num} &= \prod_{1 \leq i < j \leq 2k-1} (i, j),
\end{aligned}$$

The expression (7.71) is the same as the tree-level amplitudes of ABJ(M) (7.12) so we have succeeded in constructing a twistor string theory that reproduces the scattering amplitudes of ABJ(M).

7.7 Conformal Supergravity

Let us briefly comment on conformal supergravity; one may notice that the vertices (7.26) do not satisfy the condition of having no charge under the current (7.46). This is as expected since 3-dimensional conformal supergravity do not have any asymptotic states [46]. This does not mean that it is not present instead it gives rise to the amplitudes with multiple traces.

To justify the claim that it is conformal supergravity that creates the amplitudes with multiple traces without having to do too many calculations we are going to identify double-trace amplitudes that give the same result as some single-trace amplitudes compare with a Lagrangian for conformal supergravity [27, 26, 49] (for convenience we have reproduced the Lagrangian from [49] in appendix D).

Let us consider two simple double-trace amplitudes (traces being of field 1/2 and 3/4) and pick out the following Grassmann variables from the super-amplitude:

$$\begin{aligned} & \int d\eta_1^1 d\eta_2^1 d\eta_3^2 d\eta_4^2 \frac{\delta(\sum_{i=1}^4 \xi_i \eta_i^1) \delta(\sum_{i=1}^4 \xi_i \rho_i \eta_i^1) \delta(\sum_{i=1}^4 \xi_i \eta_i^2) \delta(\sum_{i=1}^4 \xi_i \rho_i \eta_i^2)}{(\rho_1 - \rho_2)^2 (\rho_3 - \rho_4)^2} \\ & \sim \frac{\xi_1 \xi_2 \xi_3 \xi_4}{(\rho_1 - \rho_2)(\rho_3 - \rho_4)} \end{aligned} \quad (7.72)$$

we get the same as by picking these other Grassmann variables from a single-trace amplitude:

$$\begin{aligned} & \int d\eta_1^2 d\eta_2^1 d\eta_3^1 d\eta_4^2 \frac{\delta(\sum_{i=1}^4 \xi_i \eta_i^1) \delta(\sum_{i=1}^4 \xi_i \rho_i \eta_i^1) \delta(\sum_{i=1}^4 \xi_i \eta_i^2) \delta(\sum_{i=1}^4 \xi_i \rho_i \eta_i^2)}{(\rho_1 - \rho_2)(\rho_2 - \rho_3)(\rho_3 - \rho_4)(\rho_4 - \rho_1)} \\ & \sim \frac{\xi_1 \xi_2 \xi_3 \xi_4}{(\rho_1 - \rho_2)(\rho_3 - \rho_4)}. \end{aligned} \quad (7.73)$$

So if we look at the Lagrangian from [49] we should see that the gravity interactions correspond to the ABJM interaction for these amplitudes. It turns out that they do; underneath is a table with the different combinations of Grassmann variables than remain, what amplitudes they correspond to and what the interactions are:

$\int \eta_3^3 \eta_4^3$	
$\text{Tr}(\bar{\phi}_1(1)\psi_1(2)) \text{Tr}(\bar{\psi}^1(3)\phi^1(4))$	$\text{Tr}(\bar{\phi}_2(1)\psi_1(2)\bar{\psi}^2(3)\phi^1(4))$
Contact term	
$\int \eta_2^3 \eta_4^3$	
$\text{Tr}(\bar{\phi}_1(1)\phi^2(2)) \text{Tr}(\bar{\phi}_2(3)\phi^1(4))$	$\text{Tr}(\bar{\phi}_2(1)\phi^2(2)\bar{\phi}_1(3)\phi^1(4))$
Exchange of spin-1 Chern-Simons field with $q = p_1 + p_2$	
$\int \eta_1^3 \eta_4^3$	
$\text{Tr}(\bar{\psi}^2(1)\psi_1(2)) \text{Tr}(\bar{\phi}_2(3)\phi^1(4))$	$\text{Tr}(\bar{\psi}^1(1)\psi_1(2)\bar{\phi}_1(3)\phi^1(4))$
Contact term and exchange of spin-1 Chern-Simons field with $q = p_1 + p_2$	
$\int \eta_1^3 \eta_3^3$	
$\text{Tr}(\bar{\psi}^2(1)\psi_1(2)) \text{Tr}(\bar{\psi}^1(3)\psi_2(4))$	$\text{Tr}(\bar{\psi}^1(1)\psi_1(2)\bar{\psi}^2(3)\psi_2(4))$
Exchange of spin-1 Chern-Simons field with $q = p_1 + p_2$	

So at least in these cases the double-trace amplitudes correspond to the interactions found in the Lagrangian in [49].

We now turn towards higher-point amplitudes to see if there is a similar system. With a larger number of external legs and thus a larger number of k there is a possibility that switching 2 η 's will change more than what it did before because there is k external legs to be picked out for each $SU(3)$ index. The solution is to make all the other legs for the two η 's be the same. As an example let us choose i and j to be the particles that are adjacent in one of the two traces but not in the amplitude and let both of them have an η^1 but not an η^2 and let both $i - 1$ and $j + 1$ have an η^2 but not an η^1 . The other particles have to then either have both η^1 and η^2 or none of them; the double-trace amplitude should then be similar to the single-trace amplitude with the η^1 of j and the η^2 of $i - 1$ switched. In terms of fields this means that i is either $\bar{\Psi}^2$ or $\bar{\phi}_1$, j is either Ψ_1 or ϕ^2 , $i - 1$ is either Ψ_2 or ϕ^1 and $j + 1$ is either $\bar{\Psi}^1$ or $\bar{\phi}_2$ while all the other fields are either one of $\Psi_3/\Psi_4/\phi^3/\phi^4/\bar{\Psi}^3/\bar{\Psi}^4/\bar{\phi}_3/\bar{\phi}_4$ (also see figure 7.2).

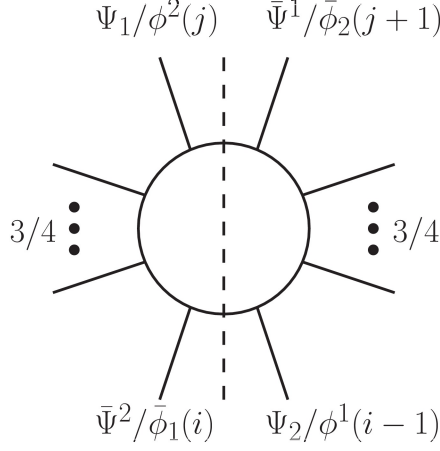


Figure 7.2. The double-trace amplitude

The $SU(4)$ indices on either side clearly do not match up so the field that connects the two traces must necessarily carry some $SU(4)$ indices, this excludes several fields including the graviton. Notice furthermore the gravitino (as well as some auxilliary fields) interact through vertices that are antisymmetric in two $SU(4)$ which is not possible here. All that is left are the spin-1 Chern-Simons field B_μ^{IJ} as well as some contact terms. If we now consider the situation for the corresponding single-trace amplitude shown in figure 7.3 we see that now the $SU(4)$ indices do match on each side so the interaction should not transfer any $SU(4)$ indices, this can be accomplished by the spin-1 Chern Simons field A_μ as well as by contact terms. So we see that the double-trace amplitudes and single-trace amplitudes that are similar also interact in same way as one would expect.

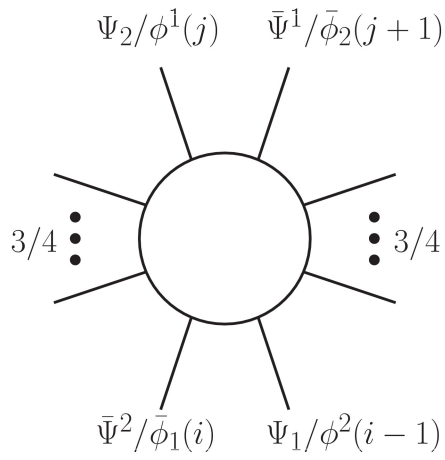


Figure 7.3. The single-trace amplitude

Chapter A —

Harmonic Variables

We split the $SU(4)$ group by introducing the harmonic variables u_{Aa}^+ and u_{Aa}^- that together form a 4×4 matrix. This matrix is unitary meaning that the harmonic variables have the following properties:

$$\bar{u}_+^{Aa} u_{Ab}^+ = \delta_b^a, \quad \bar{u}_-^{Aa'} u_{Ab'}^- = \delta_{b'}^{a'}, \quad \bar{u}_-^{Aa'} u_{Ab}^+ = \bar{u}_+^{Aa} u_{Ab'}^- = 0, \quad (\text{A.1})$$

which in turn leads to the relation:

$$u_{Aa}^+ \bar{u}_+^{Ba} + u_{Aa'}^- \bar{u}_-^{Ba'} = \delta_A^B. \quad (\text{A.2})$$

Being unitary also sets the following condition on the determinant of the matrix:

$$\frac{1}{4} \epsilon^{ABCD} u_{Aa}^+ \epsilon^{ab} u_{Bb}^+ u_{Cc'}^- \epsilon^{c'd'} u_{Dd'}^- = 1, \quad (\text{A.3})$$

where we use the convention:

$$\epsilon^{12} = \epsilon^{1'2'} = 1 = -\epsilon_{12} = -\epsilon_{1'2'} = \epsilon_1{}^2. \quad (\text{A.4})$$

(A.3) leads to the relation:

$$\frac{1}{2}\epsilon^{ABCD}u_{Aa}^+\epsilon^{ab}u_{Bb}^+ = -\bar{u}_-^{Cc'}\epsilon_{c'd'}\bar{u}_-^{Dd'}. \quad (\text{A.5})$$

When we are dealing with several insertion points it is convenient to denote u_{Aa}^+ at insertion point k by k_{Aa}^+ and introduce the notation:

$$(\mathbf{12}) = \frac{1}{4}\epsilon^{ABCD}k_{Aa}^+\epsilon^{ab}k_{Bb}^+l_{Cc}^+\epsilon^{cd}l_{Dd}^+. \quad (\text{A.6})$$

Chapter B —

One- and two-loop integrals

In this appendix we go through the integrals needed in chapter 6 as well as their cuts.

First of all we are going to need the following integral at 1-loop, it can be found using the techniques described in [40]:

$$\begin{aligned}
 I(p, p') &= \int \frac{d^2 q}{(2\pi)^2} \frac{1}{(q^2 + 1 + i\varepsilon)((q + p + p')^2 + 1 + i\varepsilon)} \\
 &= \frac{i}{2\pi m^2} \frac{p_- p'_-}{p_-^2 - p_-'^2} \ln \left| \frac{p'_-}{p_-} \right| - \frac{p_- p'_-}{4m^2(p_- + p'_-)|p_- - p'_-|} \left(\frac{p_-}{|p_-|} + \frac{p'_-}{|p'_-|} \right)
 \end{aligned} \tag{B.1}$$

or written differently

$$I(p, p') = \frac{i}{2\pi m^2} \frac{p_- p'_-}{p_-^2 - p_-'^2} \begin{cases} \ln \left(\frac{p'_-}{p_-} \right) - i\pi & \text{for } 0 < p_- < p'_- \text{ or } p'_- < p_- < 0 \\ \ln \left(-\frac{p'_-}{p_-} \right) & \text{for } p_- < 0 < p'_- \text{ or } p'_- < 0 < p_- \\ \ln \left(\frac{p'_-}{p_-} \right) + i\pi & \text{for } p_- < p'_- < 0 \text{ or } 0 < p'_- < p_- \end{cases} \tag{B.2}$$

where

$$p_{\pm} = \frac{1}{2}(\varepsilon \pm p) \qquad p_+ p_- = \frac{1}{4}. \tag{B.3}$$

In the computations done in chapter we will always assume that $p_{\pm}, p'_{\pm} > 0$ ¹

¹If it was the other way around we would have mixed up incoming with outgoing

and $p_- < p'_-$ the 1- and 2-loop integrals are going to be[42]:

$$\begin{aligned}
I_s &= \frac{1}{J_s} \left(-\frac{i}{\pi} \ln \frac{p'_-}{p_-} - 1 \right) \\
I_u &= \frac{1}{J_u} \left(+\frac{i}{\pi} \ln \frac{p'_-}{p_-} + 0 \right) \\
I_t &= \frac{i}{4\pi} \\
I_a &= \left(\frac{1}{J_s} \left(-\frac{i}{\pi} \ln \frac{p'_-}{p_-} - 1 \right) \right)^2 \\
I_d &= \left(\frac{1}{J_u} \left(+\frac{i}{\pi} \ln \frac{p'_-}{p_-} + 0 \right) \right)^2 \\
I_b &= \frac{1}{16\pi^2} \left(\frac{4}{J_u^2} \ln^2 \frac{p'_-}{p_-} + \left(-\frac{8i\pi}{J_u^2} + \frac{2}{J_u} \right) \ln \frac{p'_-}{p_-} + \text{rational} \right) \\
I_c &= \frac{1}{16\pi^2} \left(\frac{4}{J_u^2} \ln^2 \frac{p'_-}{p_-} + \left(-\frac{8i\pi}{J_u^2} + \frac{2}{J_u} \right) \ln \frac{p'_-}{p_-} + \text{rational} \right) \\
I_e &= \frac{1}{16\pi^2} \left(\frac{4}{J_u^2} \ln^2 \frac{p'_-}{p_-} - \frac{2}{J_u} \ln \frac{p'_-}{p_-} + \text{rational} \right) \\
I_f &= \frac{1}{16\pi^2} \left(\frac{4}{J_u^2} \ln^2 \frac{p'_-}{p_-} - \frac{2}{J_u} \ln \frac{p'_-}{p_-} + \text{rational} \right)
\end{aligned} \tag{B.4}$$

Some 1-loop calculation involve two different masses the integrals are then giving by (assuming that $\frac{p}{m} > \frac{p'}{m'}$):

$$\tilde{I}_s = \frac{-i}{4\pi(p\varepsilon' - p'\varepsilon)} \left(\ln \left| \frac{p'_-}{p_-} \right| - \ln \left| \frac{m'}{m} \right| - i\pi \right) \tag{B.5}$$

$$\tilde{I}_u = \frac{+i}{4\pi(p\varepsilon' - p'\varepsilon)} \left(\ln \left| \frac{p'_-}{p_-} \right| - \ln \left| \frac{m'}{m} \right| \right) \tag{B.6}$$

where

$$\varepsilon = \sqrt{p^2 + m^2} . \tag{B.7}$$

When computing the single logarithm contribution to the 2-loop results we need to find the 2-particle cut of the integrals that appear in the 2-loop ansatz.

²Which counter-intuitively correspond to $p > p'$

The different cuts are given by:

$$\begin{aligned}
I_s|_{s\text{-cut}} &= \frac{2}{4(p\varepsilon' - p'\varepsilon)} & I_s|_{u\text{-cut}} &= 0 \\
I_u|_{s\text{-cut}} &= 0 & I_u|_{u\text{-cut}} &= \frac{2}{4(p\varepsilon' - p'\varepsilon)} \\
I_a|_{s\text{-cut}} &= \frac{2 \times 2}{4(p\varepsilon' - p'\varepsilon)} I_s & I_a|_{u\text{-cut}} &= 0 \\
I_b|_{s\text{-cut}} &= \frac{1}{4(p\varepsilon' - p'\varepsilon)} (I_u + I_t) & I_b|_{u\text{-cut}} &= 0 \\
I_c|_{s\text{-cut}} &= \frac{1}{4(p\varepsilon' - p'\varepsilon)} (I_u + I_t) & I_c|_{u\text{-cut}} &= 0 \\
I_d|_{s\text{-cut}} &= 0 & I_d|_{u\text{-cut}} &= \frac{2 \times 2}{4(p\varepsilon' - p'\varepsilon)} I_u \\
I_e|_{s\text{-cut}} &= 0 & I_e|_{u\text{-cut}} &= \frac{1}{2(p\varepsilon' - p'\varepsilon)} (I_s + I_t) \\
I_f|_{s\text{-cut}} &= 0 & I_f|_{u\text{-cut}} &= \frac{1}{2(p\varepsilon' - p'\varepsilon)} (I_s + I_t)
\end{aligned} \tag{B.8}$$

Chapter C —

$AdS_3 \times S^3 \times S^3 \times S^1$

In $AdS_3 \times S^3 \times S^3 \times S^1$ there are two different mass scales, which we denote m_1 and m_2 ¹, as well as a left-moving representation, $|\phi\rangle$ and $|\psi\rangle$, and a right-moving representation, $\bar{\phi}$ and $\bar{\psi}$. There are also massless states which we will ignore for the present calculation. We will touch upon why this can be justified.

Consider some generic scattering process:

$$|\mathcal{X}_{p_{in}}^{(in)} \mathcal{X}_{p'_{in}}^{(in)}\rangle \rightarrow |\mathcal{X}_{p_{out}}^{(out)} \mathcal{X}_{p'_{out}}^{(out)}\rangle, \quad (C.1)$$

where it is implied by the ordering of the fields that $\frac{p_{in}}{m_{in}} > \frac{p'_{in}}{m'_{in}}$, $\frac{p_{out}}{m_{out}} > \frac{p'_{out}}{m'_{out}}$ ². For the considered S-matrices the individual masses are conserved. When all the masses are the same momentum and energy conservation leads to the solution $p_{in} = p_{out}$, $p'_{in} = p'_{out}$ plus another solution that does not satisfy the above condition.

In the cases where the masses are not the same one can still have a similar solution to the conservation equations, $p_{in} = p_{out}$, $p'_{in} = p'_{out}$, $m_{in} = m_{out} \neq m'_{in} = m'_{out}$ but there is also another solution where the outgoing momenta are not just equal to the incoming momenta. The proposed S-matrices we will consider do however not have any such scattering processes, they are reflectionless.

In scattering of states in different representations, the S-matrices are also reflectionless, *i.e.* when scattering a left-mover and right-mover they will not interchange

¹The notation m and m' will be used to denote the mass associated with the momentum p and p' respectively

²Note that this is slightly different from [19] however this is the condition that makes sense in terms of the Jacobian in string theory computation because it determines that $\varepsilon'p - p'\varepsilon$ is positive

momentum even if they have the same mass.

The reflectionless nature of the S-matrices and the conservation of the individual masses means that for the s - and the u -channel cuts there is no mixing of the different sectors of the S-matrices. In ignoring massless states we are basically assuming that this property remains valid for scattering processes involving massless excitations.

The S-matrix proposed by Borsato, Ohlson Sax and Sfondrini behaves differently depending on whether the scattered states are a left-mover and a right-mover or are two of the same kind. There are also differences depending on the masses of the two states but for the most part this is contained within the Zhukovsky variables and we can write down the S-matrix in terms of general coefficients without having to specify what the masses. For the LL sectors we have:

$$\mathbf{S}^{BOSS}|\phi\phi'\rangle = \mathbf{A}_{LL}^{BOSS}|\phi'\phi\rangle, \quad \mathbf{S}^{BOSS}|\phi\psi'\rangle = \mathbf{G}_{LL}^{BOSS}|\psi'\phi\rangle + \mathbf{H}_{LL}^{BOSS}|\phi'\psi\rangle, \quad (\text{C.2})$$

$$\mathbf{S}^{BOSS}|\psi\psi'\rangle = \mathbf{D}_{LL}^{BOSS}|\psi'\psi\rangle, \quad \mathbf{S}^{BOSS}|\psi\phi'\rangle = \mathbf{K}_{LL}^{BOSS}|\psi'\phi\rangle + \mathbf{L}_{LL}^{BOSS}|\phi'\psi\rangle, \quad (\text{C.3})$$

with the RR sectors behaving in a completely equivalent way. For the LR sectors the S-matrix acts like:

$$\mathbf{S}^{BOSS}|\phi\bar{\phi}'\rangle = \mathbf{A}_{LR}^{BOSS}|\bar{\phi}'\phi\rangle + \mathbf{C}_{LR}^{BOSS}|\bar{\psi}'\psi Z^-\rangle, \quad \mathbf{S}^{BOSS}|\phi\bar{\psi}'\rangle = \mathbf{G}_{LR}^{BOSS}|\bar{\psi}'\phi\rangle, \quad (\text{C.4})$$

$$\mathbf{S}^{BOSS}|\psi\bar{\psi}'\rangle = \mathbf{D}_{LR}^{BOSS}|\bar{\psi}'\psi\rangle + \mathbf{F}_{LR}^{BOSS}|\bar{\phi}'\phi Z^+\rangle, \quad \mathbf{S}^{BOSS}|\psi\bar{\phi}'\rangle = \mathbf{L}_{LR}^{BOSS}|\bar{\phi}'\psi\rangle, \quad (\text{C.5})$$

and again the RL sectors are similar to this.

For the L_1L_1 sector the coefficients are given by:

$$\mathbf{A}_{L_1 L_1}^{BOSS} = S_{L_1 L_1} \frac{x_{p'}^+ - x_p^-}{x_{p'}^- - x_p^+}, \quad \mathbf{D}_{L_1 L_1}^{BOSS} = -S_{L_1 L_1}, \quad (\text{C.6})$$

$$\mathbf{G}_{L_1 L_1}^{BOSS} = S_{L_1 L_1} \frac{x_{p'}^+ - x_p^+}{x_{p'}^- - x_p^+}, \quad \mathbf{H}_{L_1 L_1}^{BOSS} = S_{L_1 L_1} \frac{x_{p'}^+ - x_{p'}^-}{x_{p'}^- - x_p^+} \frac{\eta_p}{\eta_{p'}}, \quad (\text{C.7})$$

$$\mathbf{K}_{L_1 L_1}^{BOSS} = S_{L_1 L_1} \frac{x_p^+ - x_p^-}{x_{p'}^- - x_p^+} \frac{\eta_{p'}}{\eta_p}, \quad \mathbf{L}_{L_1 L_1}^{BOSS} = S_{L_1 L_1} \frac{x_{p'}^- - x_p^-}{x_{p'}^- - x_p^+}. \quad (\text{C.8})$$

The $R_1 R_1$ sector is exactly the same with $S_{L_1 L_1} = S_{R_1 R_1}$ and so are the $L_2 L_2 / R_2 R_2$ sectors except there the mass appearing in Zhukovsky variables are different and the phase factor could be different as well.

The coefficients of the $L_1 L_2$ sectors are given by:

$$\mathbf{A}_{L_1 L_2}^{BOSS} = S_{L_1 L_2}, \quad \mathbf{D}_{L_1 L_2}^{BOSS} = -S_{L_1 L_2} \frac{x_{p'}^- - x_p^+}{x_{p'}^+ - x_p^-}, \quad (\text{C.9})$$

$$\mathbf{G}_{L_1 L_2}^{BOSS} = S_{L_1 L_2} \frac{x_{p'}^+ - x_p^+}{x_{p'}^+ - x_p^-}, \quad \mathbf{H}_{L_1 L_2}^{BOSS} = S_{L_1 L_2} \frac{x_{p'}^+ - x_{p'}^-}{x_{p'}^+ - x_p^-} \frac{\eta_p}{\eta_{p'}}, \quad (\text{C.10})$$

$$\mathbf{K}_{L_1 L_2}^{BOSS} = S_{L_1 L_2} \frac{x_p^+ - x_p^-}{x_{p'}^+ - x_p^-} \frac{\eta_{p'}}{\eta_p}, \quad \mathbf{L}_{L_1 L_2}^{BOSS} = S_{L_1 L_2} \frac{x_{p'}^- - x_p^-}{x_{p'}^+ - x_p^-}. \quad (\text{C.11})$$

Again the $R_1 R_2$ sector is the same with $S_{L_1 L_2} = S_{R_1 R_2}$ and the $L_2 L_1 / R_2 R_1$ sectors only differ by change of masses and scalar factors.

In the $L_1 R_1$ sector the coefficients are:

$$\mathbf{A}_{L_1 R_1}^{BOSS} = S_{L_1 R_1} \frac{1 - \frac{1}{x_p^+ x_{p'}^-}}{1 - \frac{1}{x_p^- x_{p'}^+}}, \quad \mathbf{C}_{L_1 R_1}^{BOSS} = -S_{L_1 R_1} \frac{\eta_p \eta_{p'}}{x_p^- x_{p'}^-} \frac{1}{1 - \frac{1}{x_p^- x_{p'}^+}}, \quad (\text{C.12})$$

$$\mathbf{D}_{L_1 R_1}^{BOSS} = -S_{L_1 R_1} \frac{1 - \frac{1}{x_p^- x_{p'}^+}}{1 - \frac{1}{x_p^+ x_{p'}^-}}, \quad \mathbf{F}_{L_1 R_1}^{BOSS} = -S_{L_1 R_1} \frac{\eta_p \eta_{p'}}{x_p^+ x_{p'}^+} \frac{1}{1 - \frac{1}{x_p^+ x_{p'}^-}}, \quad (\text{C.13})$$

$$\mathbf{G}_{L_1 R_1}^{BOSS} = S_{L_1 R_1}, \quad \mathbf{L}_{L_1 R_1}^{BOSS} = S_{L_1 R_1} \frac{1 - \frac{1}{x_p^+ x_{p'}^+}}{1 - \frac{1}{x_p^- x_{p'}^-}}. \quad (\text{C.14})$$

The $L_2 R_1 / L_1 R_2 / L_2 R_2$ sectors behave in exactly the same way with the possi-

bility of different scalar factors.

The coefficients of the $R_1 L_1$ sector are given by:

$$\mathbf{A}_{R_1 L_1}^{BOSS} = S_{R_1 L_1} \frac{1 - \frac{1}{x_p^+ x_{p'}^-}}{1 - \frac{1}{x_p^+ x_{p'}^+}}, \quad \mathbf{C}_{R_1 L_1}^{BOSS} = -S_{R_1 L_1} \frac{\eta_p \eta_{p'}}{x_p^- x_{p'}^-} \frac{1}{1 - \frac{1}{x_p^+ x_{p'}^+}}, \quad (\text{C.15})$$

$$\mathbf{D}_{R_1 L_1}^{BOSS} = -S_{R_1 L_1} \frac{1 - \frac{1}{x_p^- x_{p'}^+}}{1 - \frac{1}{x_p^+ x_{p'}^+}}, \quad \mathbf{F}_{R_1 L_1}^{BOSS} = -S_{R_1 L_1} \frac{\eta_p \eta_{p'}}{x_p^+ x_{p'}^+} \frac{1}{1 - \frac{1}{x_p^+ x_{p'}^+}}, \quad (\text{C.16})$$

$$\mathbf{G}_{R_1 L_1}^{BOSS} = S_{R_1 L_1} \frac{1 - \frac{1}{x_p^- x_{p'}^-}}{1 - \frac{1}{x_p^+ x_{p'}^+}}, \quad \mathbf{L}_{R_1 L_1}^{BOSS} = S_{R_1 L_1}. \quad (\text{C.17})$$

The $R_2 L_1 / R_1 L_2 / R_2 L_2$ sectors are the same with the possibility of different phase factors.

There are further relations between the different scalar factors which we will not go into.

The S-matrix proposed in [3] is somewhat simpler in that it does not depend as much on what representations the scattered states are in.

The $L_1 L_1$ sector of the S-matrix can be written as

$$\mathbf{S}^{AB} |\phi \phi'\rangle = \mathbf{A}_{LL}^{AB} |\phi' \phi\rangle, \quad \mathbf{S}^{AB} |\phi \psi'\rangle = \mathbf{G}_{LL}^{AB} |\psi' \phi\rangle + \mathbf{H}_{LL}^{AB} |\phi' \psi\rangle, \quad (\text{C.18})$$

$$\mathbf{S}^{AB} |\psi \psi'\rangle = \mathbf{D}_{LL}^{AB} |\psi' \psi\rangle, \quad \mathbf{S}^{AB} |\psi \phi'\rangle = \mathbf{K}_{LL}^{AB} |\psi' \phi\rangle + \mathbf{L}_{LL}^{AB} |\phi' \psi\rangle, \quad (\text{C.19})$$

with the coefficients given by:

$$\mathbf{A}_{L_1 L_1}^{AB} = S_{L_1 L_1}, \quad \mathbf{D}_{L_1 L_1}^{AB} = -S_{L_1 L_1} \frac{x_{p'}^- - x_p^+}{x_{p'}^+ - x_p^-}, \quad (\text{C.20})$$

$$\mathbf{G}_{L_1 L_1}^{AB} = S_{L_1 L_1} \frac{x_{p'}^+ - x_p^+}{x_{p'}^+ - x_p^-}, \quad \mathbf{H}_{L_1 L_1}^{AB} = S_{L_1 L_1} \frac{x_{p'}^+ - x_{p'}^-}{x_{p'}^+ - x_p^-} \frac{\omega_p}{\omega_{p'}}, \quad (\text{C.21})$$

$$\mathbf{K}_{L_1 L_1}^{AB} = S_{L_1 L_1} \frac{x_p^+ - x_p^-}{x_{p'}^+ - x_p^-} \frac{\omega_{p'}}{\omega_p}, \quad \mathbf{L}_{L_1 L_1}^{AB} = S_{L_1 L_1} \frac{x_{p'}^- - x_p^-}{x_{p'}^+ - x_p^-}. \quad (\text{C.22})$$

where ω_p and $\omega_{p'}$ are chosen to be 1. The $L_2 L_2 / R_1 R_1 / R_2 R_2$ sectors behave exactly

the same as this with the appropriate changes of masses, the $L_1 R_1 / L_2 R_1 / R_1 L_2 / R_2 L_1$ also behave similarly but with a different scalar factor while the $L_1 L_2 / L_2 L_1 / R_1 R_2 / R_2 R_1$ and the $L_1 R_2 / L_2 R_1 / R_1 L_2 / R_2 L_1$ sectors also behave in this way except the scalar factor is set to be 1.

Chapter D —

ABJ(M) and CSG Lagrangian

In this appendix we write the Lagrangian of ABJ(M) coupled to $\mathcal{N} = 6$ conformal Supergravity from [49]:

$$\begin{aligned}
\mathcal{L}_{ABJM} = & -e D_\mu \bar{\phi}_A^i D^\mu \phi_i^A - \frac{1}{2} e (\bar{\psi}^{Ai} \gamma^\mu D_\mu \psi_{Ai} - D_\mu \bar{\psi}^{Ai}) - \frac{2}{3} e |Y_{Ai}^{BC}|^2 \\
& - \frac{1}{2} \epsilon^{\mu\nu\rho} \left(f^{ij}_{kl} A_\mu^k A_\nu^l \partial_\rho A_i^j + \frac{2}{3} f^{ik}_{lp} f^{pm}_{nj} A_\mu^j A_\nu^l A_\rho^m \right) \\
& - 2e f^{ij}_{kl} \bar{\psi}^{Ak} \psi_{Bi} \left(\bar{\phi}_A^l \phi_j^B - \frac{1}{2} \delta_A^B \bar{\phi}_C^l \phi_j^C \right) + \left[-\frac{1}{2} e \epsilon_{ABCD} f^{ij}_{kl} \bar{\psi}^{Ak} \psi^{Bl} \phi_i^C \phi_j^D \right. \\
& + \frac{1}{4\sqrt{2}} e \bar{\psi}^{Ai} \gamma^{\mu\nu} D_{[\mu} \chi_{\nu]}^I (\Sigma^I)_{AB} \phi_i^B - \frac{1}{2\sqrt{2}} e \bar{\psi}^{Ai} \gamma^\mu \gamma^\nu \chi_\mu^I (\Sigma^I)_{AB} D_\nu \phi_i^B \\
& + \frac{1}{2\sqrt{2}} e \bar{\psi}^{Ai} \gamma^\mu \chi_\mu^I Y_{Ai}^{BC} (\Sigma^I)_{BC} + \frac{1}{16} e \bar{\chi}_\mu^I \gamma^{\mu\nu\rho} \chi_\nu^I D_\rho \phi_k^A \bar{\phi}_B^k (\Sigma^{IJ})_A^B + c.c. \left. \right] \\
& - \frac{1}{8} e \left(R - \frac{1}{2} \bar{\chi}_\mu^I \gamma^{\mu\nu\rho} D_{[\nu} \chi_{\rho]}^I \right) \bar{\phi}_A^i \phi_i^A + \frac{1}{\sqrt{2}} i e D^{IJ} \bar{\phi}_B^i \phi_i^A (\Sigma^{IJ})_A^B \\
& - \frac{1}{16} e \bar{\chi}_\mu^I \gamma^{\mu\nu} \chi_\nu^J \phi_i^A \phi_j^B \bar{\phi}_C^k \bar{\phi}_D^l f^{ij}_{kl} [(\Sigma^I)_{AB} (\bar{\Sigma})^{CD} + \delta^{IJ} \delta_A^C \delta_B^D] \\
& - \frac{1}{16} e \bar{\chi}_\mu^I \gamma_\rho \chi_\nu^J \bar{\psi}^{Ai} (\gamma^{\mu\nu\rho} + g^{\mu\nu} \gamma^\rho) \psi_{Bi} (\Sigma^I \bar{\Sigma}^J)_A^B - \frac{1}{4} e E^{IJ} E^{IJ} \bar{\phi}_A^i \phi_i^A \\
& + \frac{1}{16} e \bar{\chi}_\mu^I \chi_\nu^J \bar{\psi}^{Ai} \gamma^\mu \gamma^\nu \psi_{Bi} (\Sigma^I \bar{\Sigma}^J)_A^B + \left[\frac{1}{12} e \bar{\lambda}^{IJK} \psi^{Ai} (\Sigma^{IJK})_{AB} \phi_i^B \right. \\
& + \frac{1}{2} i e \bar{\lambda} \psi^{Ai} (\Sigma^I)_{AB} \phi_i^B - \frac{1}{8} i e \bar{\psi}^{Ai} \gamma^\mu \chi_\mu^K \phi_i^B E^{IJ} (\Sigma^{IJ} \Sigma^K)_{AB} + c.c. \left. \right] \\
& + \frac{1}{\sqrt{2}} i e E^{IJ} \left(f^{ij}_{kl} \phi_i^A \phi_j^C \bar{\phi}_B^k \bar{\phi}_C^l + \frac{1}{2} \bar{\psi}^{Ai} \psi_{Bi} \right) (\Sigma^{IJ})_A^B \\
& + \frac{24\sqrt{2}}{e} \bar{\lambda}^{IJK} \gamma^\mu \chi_\mu^L \bar{\phi}_B^i \phi_i^A (\Sigma^{IJKL})_A^B - \frac{1}{4\sqrt{2}} i e \bar{\lambda}^I \gamma^\mu \chi_\mu^J \bar{\phi}_B^i \phi_i^A (\Sigma^{IJ})_A^B \\
& - \frac{1}{4\sqrt{2}} i e \bar{\chi}_\mu^I \gamma^{\mu\nu} \chi_\nu^J \bar{\phi}_B^i \phi_i^A \left[E^{KI} (\Sigma^{KJ})_A^B - \frac{1}{4} \delta^{IJ} E^{KL} (\Sigma^{KL})_A^B \right].
\end{aligned} \tag{D.1}$$

The first 3 lines incorporates the normal ABJ(M) Lagrangian while the rest is purely couplings to conformal Supergravity. The fields ϕ , ψ and A_μ are the ABJM fields note that the notation used in the amplitudes are a bit different as we want the $SU(3)$ symmetry of the amplitudes. The f 's are structure constants of the 3-algebra:

$$T^a \bar{T}^{\bar{b}} T^c - T^c \bar{T}^{\bar{b}} T^a = f^{ab\bar{c}}_d T^d. \quad (D.2)$$

Indices can be raised and lowered with the metric.

$$\text{Tr}(T^a \bar{T}^{\bar{b}}) = h^{a\bar{b}}. \quad (D.3)$$

The covariant derivatives necessary for our considerations are given by

$$D_\mu \phi_a^A = \left(\partial_\mu + \frac{1}{2} i B_\mu \right) \phi_i^A - \frac{1}{4} B_\mu^{IJ} \phi_i^B (\Sigma)_{AB} - \phi_j^A A_\mu^l{}_k f^{jk}_{li}, \quad (D.4)$$

$$D_\mu \psi_{Ai} = \left(\partial_\mu + \frac{1}{4} \hat{\omega}_{\mu ab} \gamma^{ab} + \frac{1}{2} i B_\mu \right) \psi_{Ai} + \frac{1}{4} B_\mu^{IJ} (\Sigma^{IJ})_A{}^B \psi_{Bi} - \psi_{aj} A_\mu^l{}_k f^{jk}_{li}. \quad (D.5)$$

The supergravity fields are e_μ^a , χ_μ^I , B_μ^{IJ} , B_μ , λ^{IJK} , λ^I , E^{IJ} and D^{IJ} where I runs from 1 to 6. The Σ 's are 4 by 4 antisymmetric matrices satisfying:

$$\bar{\Sigma}^I = (\Sigma^I)^\dagger, \quad (D.6)$$

$$\Sigma^{IJ} = \Sigma^{[I} \bar{\Sigma}^{J]}, \quad (D.7)$$

$$\Sigma^{IJK} = \Sigma^{[I} \bar{\Sigma}^{J} \Sigma^{K]}, \quad (D.8)$$

$$\Sigma^{IJKL} = \Sigma^{[I} \bar{\Sigma}^{J} \Sigma^K \bar{\Sigma}^{L]}. \quad (D.9)$$

The Lagrangian of the conformal supergravity itself is given by:

$$\begin{aligned}
\mathcal{L}_{CSG} = & \frac{1}{2}\epsilon^{\mu\nu\rho}\left(\hat{\omega}_\mu{}^a{}_b\partial_\nu\hat{\omega}_\rho{}^b{}_a + \frac{2}{3}\hat{\omega}_\mu{}^a{}_b\hat{\omega}_\nu{}^b{}_c\hat{\omega}_\rho{}^c{}_a\right) + \frac{1}{4}e\overline{D_{[\mu}\chi_{\nu]}}\gamma^{\rho\sigma}\gamma^{\mu\nu}D_\rho\chi_\sigma \\
& - \epsilon^{\mu\nu\rho}\left(B_\mu^{IJ}\partial_\nu B_\rho^{JI} + \frac{2}{3}B_\mu^{IJ}B_\nu^{JK}B_\rho^{KL}\right) - 2\epsilon^{\mu\nu\rho}B_\mu\partial_\nu B_\rho \\
& + \frac{1}{3}e\bar{\lambda}^{IJK}\lambda^{IJK} - 2e\bar{\lambda}\lambda - 8eD^{IJ}E^{IJ} + \frac{1}{3\sqrt{2}}\eta e\epsilon^{IJKLMN}E^{IJ}E^{KL}E^{MN} \\
& + \frac{1}{6}\eta e\epsilon^{IJKLMN}\bar{\lambda}^{IJK}\gamma^\mu\chi_\mu^LE^{MN} + 2e\bar{\lambda}^I\gamma^\mu\chi_\mu^JE^{IJ} \\
& + e\bar{\chi}_\mu^I\gamma^{\mu\nu}\chi_n^Ju\left(E^{IK}E^{JK} - \frac{1}{4}\delta^{IJ}E^{KL}E^{KL}\right).
\end{aligned} \tag{D.10}$$

Here $\hat{\omega}_\mu{}^a{}_b$ is spin-connection including torsion from the fermions:

$$\hat{\omega}_\mu{}^a{}_b = \omega_\mu{}^a{}_b(e) + \frac{1}{8}\left(\bar{\chi}_a^I\gamma_\mu\chi_b^I + \bar{\chi}_\mu^I\gamma_a\chi_b^I - \bar{\chi}_\mu^I\gamma_b\chi_a^I\right). \tag{D.11}$$

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Vita

Education

- **Ph.D. in Physics** Pennsylvania State University **2010-2014**
- **M.Sc. in Physics** Copenhagen University **2007-2009**
- **B.Sc. in Physics** Copenhagen University **2004-2007**

Publications

- O. T. Engelund and R. Roiban, JHEP **1406**, 088 (2014)
- O. T. Engelund, R. W. McKeown and R. Roiban, JHEP **1308**, 023 (2013)
- O. T. Engelund and R. Roiban, JHEP **1303**, 172 (2013)
- O. T. Engelund and R. Roiban, JHEP **1205**, 158 (2012)
- N. E. J. Bjerrum-Bohr and O. T. Engelund, Phys. Rev. D **81**, 105009 (2010)

Awards and Honors

- W. Donald Miller Fellowship **2013**
- David C. Duncan Fellowship **2012**
- David H. Rank Memorial Physics Award **2011**