Quasi-Newtonian scalar-tensor cosmologies

Heba Sami and Amare Abebe
Center for Space Research, North-West University, Mafikeng 2735, South Africa
E-mail: hebasami.abdulrahman@gmail.com

Abstract. In this contribution, classes of shear-free cosmological dust models with irrotational fluid flows will be investigated in the context of scalar-tensor theories. In particular, the integrability conditions describing a consistent evolution of the linearised field equations of quasi-Newtonian universes are presented.

1. Introduction

Although general relativity theory (GR) is a generalization of Newtonian gravity in the presence of strong gravitational fields, it has no properly defined Newtonian limit in cosmological scales. Newtonian cosmologies are an extension of the Newtonian theory of gravity and are usually referred to as *quasi-Newtonian*, rather than strictly Newtonian formulations [1, 2, 3]. The importance of investigating the Newtonian limit for general relativity in cosmological contexts is that, there is a viewpoint that cosmological studies can be done using Newtonian physics, with the relativistic theory only needed for examination of some observational relations [1]. General relativistic quasi-Newtonian cosmologies have been studied in the context of large-scale structure formation and non-linear gravitational collapse in the late-time universe. This despite the general covariant inconsistency of these cosmological models except in some special cases such as the spatially homogeneous and isotropic, spherically symmetric, expanding (FLRW) spacetimes. Higher-order or modified gravitational theories of gravity such as $f(R)$ theories of gravity have been shown to exhibit more shared properties with Newtonian gravitation than does general relativity.

In [1], a covariant approach to cold matter universes in quasi-Newtonian cosmologies has been developed and it has been applied and extended in [2] in order to derive and solve the equations governing density and velocity perturbations. This approach revealed the existence of integrability conditions in GR. The purpose of the current study is two-fold: to apply the lineaized covariant consistency analysis and study the existence of quasi-Newtonian cosmological space-times in scalar-tensor theories of gravitation. A direct result of our analysis will be presented in the form of the integrability conditions we derive.

1.1. $f(R)$ and scalar-tensor models of gravitation

The so-called $f(R)$ theories of gravity are among the simplest modification of Einstein's GR. These theories come about by a straightforward generalisation of the Lagrangian in the Einstein-Hilbert action [4, 5] as

$$S_{f(R)} = \frac{1}{2}\int d^4x\sqrt{-g}\Big(f(R) + 2\mathcal{L}_m\Big)\ , \tag{1}$$

where $\mathcal{L}_m$ is the matter Lagrangian and g is the determinant of the metric tensor $g_{\mu\nu}$. Another modified theory of gravity is the scalar-tensor theory of gravitation. This is a broad class of gravitational models that tries to explain the gravitational interaction through both a scalar field and a tensor field. A sub-class of this theory, known as Brans-Dicke (BD) theory, has an action of the form

$$S_{BD} = \frac{1}{2}\int d^4x\sqrt{-g}\left[\phi R - \frac{\omega}{\phi}\nabla_\mu\phi\nabla^\mu\phi + 2\mathcal{L}_m\right] , \tag{2}$$

where ϕ is the scalar field and ω is a coupling constant considered to be independent of the scalar field ϕ. An interesting aspect of $f(R)$ theories of gravity is their proven equivalence with the BD theory of gravity [5, 6] with $\omega = 0$. If we define the $f(R)$ extra degree of freedom [1] as

$$\phi \equiv f' - 1 , \tag{3}$$

then the actions (1) and (2) become dynamically equivalent.
In a FLRW background universe, the resulting non-trivial field equations lead to the following Raychaudhuri and Friedmann equations that govern the expansion history of the Universe [7]:

$$\dot{\Theta} + \frac{1}{3}\Theta^2 = -\frac{1}{2(\phi+1)}\left[\mu_m + 3p_m + f - R(\phi+1) + \Theta\dot{\phi} + 3\phi''\left(\frac{\dot{\phi}^2}{\phi'^2}\right) + 3\ddot{\phi} - 3\frac{\dot{\phi}\dot{\phi}'}{\phi'}\right] , \tag{4}$$

$$\Theta^2 = \frac{3}{(\phi+1)}\left[\mu_m + \frac{R(\phi+1)-f}{2} + \Theta\dot{\phi}\right] , \tag{5}$$

where $\Theta \equiv 3H = 3\frac{\dot{a}}{a}$, H being the Hubble parameter, $a(t)$ is the scale factor, and μ_m and p_m are the energy density and isotropic pressure of standard matter, respectively.
The linearised thermodynamic quantities for the scalar field are the energy density μ_ϕ, the pressure p_ϕ, the energy flux q^ϕ_a and the anisotropic pressure π^ϕ_{ab}, respectively given by

$$\mu_\phi = \frac{1}{(\phi+1)}\left[\frac{1}{2}\Big(R(\phi+1) - f\Big) - \Theta\dot{\phi} + \tilde{\nabla}^2\phi\right] , \tag{6}$$

$$p_\phi = \frac{1}{(\phi+1)}\left[\frac{1}{2}\Big(f - R(\phi+1)\Big) + \ddot{\phi} - \frac{\dot{\phi}\dot{\phi}'}{\phi'} + \frac{\phi''\dot{\phi}^2}{\phi'^2} + \frac{2}{3}(\Theta\dot{\phi} - \tilde{\nabla}^2\phi)\right] , \tag{7}$$

$$q^\phi_a = -\frac{1}{(\phi+1)}\left[\frac{\dot{\phi}'}{\phi'} - \frac{1}{3}\Theta\right]\tilde{\nabla}_a\phi , \tag{8}$$

$$\pi^\phi_{ab} = \frac{\phi'}{(\phi+1)}\left[\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}R - \sigma_{ab}\left(\frac{\dot{\phi}}{\phi'}\right)\right] . \tag{9}$$

The total (*effective*) energy density, isotropic pressure, anisotropic pressure and heat flux of standard matter and scalar field combination are given by

$$\mu \equiv \frac{\mu_m}{(\phi+1)} + \mu_\phi, \quad p \equiv \frac{p_m}{(\phi+1)} + p_\phi, \quad \pi_{ab} \equiv \frac{\pi^m_{ab}}{(\phi+1)} + \pi^\phi_{ab}, \quad q_a \equiv \frac{q^m_a}{(\phi+1)} + q^\phi_a . \tag{10}$$

1.2. Covariant equations
Given a choice of 4-velocity field u^a in the Ehlers-Ellis covariant approach [8, 9], the FLRW background is characterised by the equations [2, 3]

$$\tilde{\nabla}_a\mu_m = 0 = \tilde{\nabla}_a p_m = \tilde{\nabla}_a\Theta , \quad q^m_a = 0 = A_a = \omega_a ,$$
$$\pi^m_{ab} = \sigma_{ab} = E_{ab} = 0 = H_{ab} , \tag{11}$$

[1] f' , f'', etc. are the first, second, etc. derivatives of f w.r.t. the Ricci scalar R.

where Θ, A_a, ω^a, and σ_{ab} are the expansion, acceleration, vorticity and the shear terms. E_{ab} and H_{ab} are the "gravito-electric" and "gravito-magnetic" components of the Weyl tensor C_{abcd} defined from the Riemann tensor R^a_{bcd} as

$$C^{ab}{}_{cd} = R^{ab}{}_{cd} - 2g^{[a}{}_{[c}R^{b]}{}_{d]} + \frac{R}{3}g^{[a}{}_{[c}g^{b]}{}_{d]} \ , \tag{12}$$

$$E_{ab} \equiv C_{agbh}u^g u^h, \qquad H_{ab} \equiv \tfrac{1}{2}\eta_{ae}{}^{gh}C_{ghbd}u^e u^d \ . \tag{13}$$

The covariant linearised evolution equations in the general case are given by [2, 3, 10]

$$\dot{\Theta} = -\frac{1}{3}\Theta^2 - \frac{1}{2}(\mu + 3p) + \tilde{\nabla}_a A^a \ , \tag{14}$$

$$\dot{\mu}_m = -\mu_m\Theta - \tilde{\nabla}^a q^m_a \ , \tag{15}$$

$$\dot{q}^m_a = -\frac{4}{3}\Theta q^m_a - \mu_m A_a \ , \tag{16}$$

$$\dot{\omega}^{\langle a\rangle} = -\frac{2}{3}\Theta\omega^a - \frac{1}{2}\eta^{abc}\tilde{\nabla}_b A_c \ , \tag{17}$$

$$\dot{\sigma}_{ab} = -\frac{2}{3}\Theta\sigma_{ab} - E_{ab} + \frac{1}{2}\pi_{ab} + \tilde{\nabla}_{\langle a}A_{b\rangle} \ , \tag{18}$$

$$\dot{E}^{\langle ab\rangle} = \eta^{cd\langle a}\tilde{\nabla}_c H^{\rangle b}_d - \Theta E^{ab} - \frac{1}{2}\dot{\pi}^{ab} - \frac{1}{2}\tilde{\nabla}^{\langle a}q^{b\rangle} - \frac{1}{6}\Theta\pi^{ab} \ , \tag{19}$$

$$\dot{H}^{\langle ab\rangle} = -\Theta H^{ab} - \eta^{cd\langle a}\tilde{\nabla}_c E^{\rangle b}_d + \frac{1}{2}\eta^{cd\langle a}\tilde{\nabla}_c \pi^{\rangle b}_d \ , \tag{20}$$

and the linearised constraint equations are given by

$$C^{ab}_0 \equiv E^{ab} - \tilde{\nabla}^{\langle a}A^{b\rangle} - \frac{1}{2}\pi^{ab} = 0 \ , \tag{21}$$

$$C^a_1 \equiv \tilde{\nabla}_b\sigma^{ab} - \eta^{abc}\tilde{\nabla}_b\omega_c - \frac{2}{3}\tilde{\nabla}^a\Theta + q^a = 0 \ , \tag{22}$$

$$C_2 \equiv \tilde{\nabla}^a\omega_a = 0 \ , \tag{23}$$

$$C^{ab}_3 \equiv \eta_{cd(}\tilde{\nabla}^c\sigma^d_{b)} + \tilde{\nabla}^{\langle a}\omega^{b\rangle} - H^{ab} = 0 \ , \tag{24}$$

$$C^a_5 \equiv \tilde{\nabla}_b E^{ab} + \frac{1}{2}\tilde{\nabla}_b\pi^{ab} - \frac{1}{3}\tilde{\nabla}^a\eta + \frac{1}{3}\Theta q^a = 0 \ , \tag{25}$$

$$C^a_b \equiv \tilde{\nabla}_b H^{ab} + (\mu + p)\omega^a + \frac{1}{2}\eta^{abc}\tilde{\nabla}_b q_a = 0 \ . \tag{26}$$

2. Quasi-Newtonian spacetimes

If a comoving 4-velocity $\tilde{u}^a$ is chosen such that, in the linearised form

$$\tilde{u}^a = u^a + v^a, \quad v_a u^a = 0, \quad v_a v^a << 1 \ , \tag{27}$$

the dynamics, kinematics and gravito-electromagnetics quantities (11) undergo transformation. Here v^a is the relative velocity of the comoving frame with respect to the observers in the quasi-Newtonian frame, defined such that it vanishes in the background. In other words, it is a non-relativistic peculiar velocity. Quasi-Newtonian cosmological models are irrotational, shear-free dust spacetimes characterised by [2, 3]:

$$p_m = 0 \ , \quad q^m_a = \mu_m v_a \ , \quad \pi^m_{ab} = 0 \ , \omega_a = 0 \ , \quad \sigma_{ab} = 0 \ . \tag{28}$$

The gravito-magnetic constraint equation (24) and the shear-free and irrotational condition (28) show that the gravito-magnetic component of the Weyl tensor automatically vanishes:

$$H^{ab} = 0 \ . \tag{29}$$

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The vanishing of this quantity implies no gravitational radiation in quasi-Newtonian cosmologies, and equation (26) together with equation (28) show that q_a^m is irrotational and thus so is v_a:

$$\eta^{abc}\tilde{\nabla}_b q_a = 0 = \eta^{abc}\tilde{\nabla}_b v_a \ . \tag{30}$$

Since the vorticity vanishes, there exists a velocity potential such that

$$v_a = \tilde{\nabla}_a \Phi \ . \tag{31}$$

3. Integrability conditions

It has been shown that the non-linear models are generally inconsistent if the silent constraint (29) is imposed, but that the linear models are consistent [2, 3]. Thus, a simple approach to the integrability conditions for quasi-Newtonian cosmologies follows from showing that these models are in fact a sub-class of the linearised silent models. This can happen by using the transformation between the quasi-Newtonian and comoving frames. The transformed linearised kinematics, dynamics and gravito-electromagnetic quantities from the quasi-Newtonian frame to the comoving frame are given as follows:

$$\tilde{\Theta} = \Theta + \tilde{\nabla}^a v_a \ , \tag{32}$$

$$\tilde{A}_a = A_a + \dot{v}_a + \frac{1}{3}\Theta v_a \ , \tag{33}$$

$$\tilde{\omega}_a = \omega_a - \frac{1}{2}\eta_{abc}\tilde{\nabla}^b v^c \ , \tag{34}$$

$$\tilde{\sigma}_{ab} = \sigma_{ab} + \tilde{\nabla}_{\langle a} v_{b\rangle} \ , \tag{35}$$

$$\tilde{\mu} = \mu, \quad \tilde{p} = p, \quad \tilde{\pi}_{ab} = \pi_{ab}, \quad \tilde{q}_a^{\phi} = q_a^{\phi} \tag{36}$$

$$\tilde{q}_a^m = q_a^m - (\mu_m + p_m)v_a \ , \tag{37}$$

$$\tilde{E}_{ab} = E_{ab}, \quad \tilde{H}_{ab} = H_{ab} \ . \tag{38}$$

It follows from the above transformation equations that

$$\tilde{p}_m = 0 \ , \quad \tilde{q}_a^m = 0 = \tilde{A}_a = \tilde{\omega}_a \ , \quad \tilde{\pi}_{ab}^m = 0 = \tilde{H}_{ab} \ , \quad \tilde{\sigma}_{ab} = \tilde{\nabla}_{\langle a} v_{b\rangle} \ , \quad \tilde{E}_{ab} = E_{ab} \ . \tag{39}$$

These equations describe the linearised silent universe except that the restriction on the shear in equation (39) results in the integrability conditions for the quasi-Newtonian models. Due to the vanishing of the shear in the quasi-Newtonian frame, equation (18) is turned into a new constraint

$$E_{ab} - \frac{1}{2}\pi_{ab}^{\phi} - \tilde{\nabla}_{\langle a} A_{b\rangle} = 0 \ . \tag{40}$$

This can be simplified by using equation (17) and the identity for any scalar φ:

$$\eta^{abc}\tilde{\nabla}_a A_c = 0 \Rightarrow A_a = \tilde{\nabla}_a \varphi \ . \tag{41}$$

In this case φ is the covariant relativistic generalisation of the Newtonian potential.

3.1. First integrability condition

Since equation (40) is a new constraint, we need to ensure its consistent propagation at all epochs and in all spatial hypersurfaces. Differentiating it with respect to cosmic time t and by using equations (9), (19) and (22), one obtains

$$\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}\left[\dot{\varphi} + \frac{1}{3}\Theta + \frac{\dot{\phi}}{(\phi+1)}\right] + \left[\dot{\varphi} + \frac{1}{3}\Theta + \frac{\dot{\phi}}{(\phi+1)}\right]\tilde{\nabla}_a\tilde{\nabla}_b\varphi = 0 \ , \tag{42}$$

which is the first integrability condition for quasi-Newtonian cosmologies in scalar-tensor theory of gravitation and it is a generalisation of the one obtained in [2], *i.e.*, (42) reduces to an identity for the generalized van Elst-Ellis condition [1, 2, 3]

$$\dot{\varphi}+\frac{1}{3}\Theta=-\frac{\dot{\phi}}{(\phi+1)}\;. \tag{43}$$

Using equation (14) with the time evolution of the modified van Elst-Ellis condition, we obtain the covariant modified Poisson equation in scalar-tensor gravity as follows:

$$\tilde{\nabla}^2\varphi=\frac{\mu_m}{2(\phi+1)}-(3\ddot{\varphi}+\Theta\dot{\varphi})+\frac{1}{2(\phi+1)}\Big[f-R(\phi+1)+\Big(\frac{3\dot{\phi}'}{\phi'}-\Theta\Big)\dot{\phi}-3\ddot{\phi}+3\Big(\frac{2}{(\phi+1)}-\frac{\phi''}{\phi'^2}\Big)\dot{\phi}^2-\tilde{\nabla}^2\phi\Big]\;. \tag{44}$$

The evolution equation of the 4-acceleration A_a can be shown, using equations (43) and (22), to be

$$\dot{A}_a+\Big[\frac{2}{3}\Theta+\frac{\dot{\phi}}{(1+\phi)}\Big]A_a=-\frac{1}{2(1+\phi)}\Big[\mu_m v_a+\Big(\frac{1}{3}\Theta+\frac{\dot{\phi}'}{\phi'}-\frac{2\dot{\phi}}{(1+\phi)}\Big)\tilde{\nabla}_a\phi\Big]\;. \tag{45}$$

3.2. Second integrability condition

There is a second integrability condition arising by checking for the consistency of the constraint (40) on any spatial hyper-surface of constant time t. By taking the divergence of (40) and by using the following identity:

$$\tilde{\nabla}^b\tilde{\nabla}_{\langle a}A_{b\rangle}=\frac{1}{2}\tilde{\nabla}^2A_a+\frac{1}{6}\tilde{\nabla}_a(\tilde{\nabla}^cA_c)+\frac{1}{3}(\mu-\frac{1}{3}\Theta^2)A_a\;, \tag{46}$$

which holds for any projected vector A_a, and by using equation (41) it follows that:

$$\tilde{\nabla}^b\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}\varphi=\frac{2}{3}\tilde{\nabla}_a(\tilde{\nabla}^2\varphi)+\frac{2}{3}(\mu-\frac{1}{3}\Theta^2)\tilde{\nabla}_a\varphi\;. \tag{47}$$

By using equations (47), (22) and (25), one obtains:

$$\begin{aligned}&\tilde{\nabla}_a\mu_m-\Big[\dot{\phi}+\frac{2}{3}(\phi+1)\Theta\Big]\tilde{\nabla}_a\Theta+\frac{1}{(\phi+1)}\Big[\frac{f}{2}-\mu_m+\Theta\dot{\phi}-\frac{\Theta\dot{\phi}'(\phi+1)}{\phi'}\Big]\tilde{\nabla}_a\phi\\&-2(\phi+1)\tilde{\nabla}^2(\tilde{\nabla}_a\varphi)-2\Big[\mu_m+\frac{R(\phi+1)}{2}-\frac{f}{2}-\Theta\dot{\phi}-\frac{\Theta^2(\phi+1)}{3}\Big]\tilde{\nabla}_a\varphi\\&-\tilde{\nabla}^2(\tilde{\nabla}_a\phi)=0\;,\end{aligned} \tag{48}$$

which is the second integrability condition and in general it appears to be independent of the first integrability condition (42). By taking the gradient of equation (43) and using equation (22), one can obtain the peculiar velocity:

$$v_a=-\frac{1}{\mu_m}\Big[2(\phi+1)\tilde{\nabla}_a\dot{\varphi}+\Big(\frac{\dot{\phi}'}{\phi'}-\dot{\varphi}-\frac{3\dot{\phi}}{(\phi+1)}\Big)\tilde{\nabla}_a\phi\Big]\;. \tag{49}$$

By virtue of equations (15) and (16), v_a evolves according to

$$\dot{v_a}+\frac{1}{3}\Theta v_a=-A_a\;. \tag{50}$$

The coupled evolution equations (45) and (50) decouple to produce the second-order propagation equation of the peculiar velocity v_a. By using equations (4) and (5) in equation (50) one obtains:

$$\ddot{v}_a + \Big[\Theta + \frac{\dot{\phi}}{(\phi+1)}\Big]\dot{v}_a + \Big[\frac{1}{9}\Theta^2 - \frac{1}{6(\phi+1)}(5\mu_m - f - 4\Theta\dot{\phi})\Big]v_a + \frac{1}{(\phi+1)}\Big[\frac{\dot{\phi}}{(\phi+1)} - \frac{\phi''}{2\phi'} - \frac{\Theta}{6} - \frac{\dot{\phi}'}{2\phi'} + \frac{\phi''\dot{\phi}}{2\phi'^2}\Big]\tilde{\nabla}_a\phi = 0 \,. \tag{51}$$

By substituting equation (49) into equation (50) one obtains

$$2(\phi+1)\tilde{\nabla}_a\ddot{\varphi} + 2\Big[\dot{\phi} + \Theta(\phi+1)\Big]\tilde{\nabla}_a\dot{\varphi} - \Big[\mu_m - 2(\phi+1)\ddot{\varphi} + \ddot{\phi} - \frac{\dot{\phi}\dot{\phi}'}{\phi'} - \frac{\dot{\phi}^2\phi''}{\phi'^2} + \dot{\varphi}\phi' + \frac{3\dot{\phi}^2}{(\phi+1)}\Big]\tilde{\nabla}_a\varphi + \Big[\frac{3\phi''\dot{\phi}\dot{\phi}'}{\phi'^3} + \frac{\ddot{\phi}'}{\phi'} - \frac{\dot{\phi}'^2}{\phi'^2} - \frac{\dot{\phi}\dot{\phi}''}{\phi'^2} + \frac{\Theta\dot{\phi}'}{\phi'} - \frac{\dot{\varphi}\dot{\phi}'}{\phi'} - \frac{2\phi''\dot{\phi}^2}{\phi'^4} - \frac{3\dot{\phi}^2\phi''}{\phi'^2(\phi+1)} - \Theta\dot{\varphi} - \ddot{\varphi} + \frac{3\dot{\phi}}{(\phi+1)^2} + \frac{\phi'''\dot{\phi}^2}{\phi'^3} - \frac{3\ddot{\phi}}{(\phi+1)} - \frac{3\Theta\dot{\phi}}{(\phi+1)}\Big]\tilde{\nabla}_a\phi = 0 \,. \tag{52}$$

By using equation (15) together with equation (43), one can show that the acceleration potential φ satisfies

$$\ddot{\varphi} = \frac{1}{9}\Theta^2 + \frac{1}{6(\phi+1)}\Big[\mu_m + f - R(\phi+1) - 3\ddot{\phi} + \Theta\dot{\phi} + \frac{3\dot{\phi}\dot{\phi}'}{\phi'} - \frac{3\dot{\phi}^2\phi''}{\phi'^2} + \frac{6\dot{\phi}^2}{(\phi+1)} - \tilde{\nabla}^2\phi\Big] - \frac{1}{3}\tilde{\nabla}^2\varphi \,. \tag{53}$$

4. Conclusion

In this work, we have demonstrated how imposing special restrictions to the linearized perturbations of FLRW universes in the quasi-Newtonian setting result in the integrability conditions that give us consistency relations for the evolution and constraint field equations in the scalar-tensor theories of gravity. In addition, we have derived the velocity and acceleration propagation equations in scalar-tensor theories.

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