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TIME-DEPENDENT SYSTEMS AND CHAOS IN STRING THEORY

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TIME-DEPENDENT SYSTEMS AND CHAOS IN STRING THEORY

DISSERTATION

A dissertation submitted in partial fulfillment of the
requirements for the degree of Doctor of Philosophy in the
College of Arts and Sciences
at the University of Kentucky

By

Archisman Ghosh

Lexington, Kentucky

Director: Dr. Sumit R. Das, Professor of Department of Physics and Astronomy

Lexington, Kentucky

2012

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ABSTRACT OF DISSERTATION

TIME-DEPENDENT SYSTEMS AND CHAOS IN STRING THEORY

One of the phenomenal results emerging from string theory is the AdS/CFT correspondence or gauge-gravity duality: In certain cases a theory of gravity is equivalent to a “dual” gauge theory, very similar to the one describing non-gravitational interactions of fundamental subatomic particles. A difficult problem on one side can be mapped to a simpler and solvable problem on the other side using this correspondence. Thus one of the theories can be understood better using the other.

The mapping between theories of gravity and gauge theories has led to new approaches to building models of particle physics from string theory. One of the important features to model is the phenomenon of confinement present in strong interaction of particle physics. This feature is not present in the gauge theory arising in the simplest of the examples of the duality. However this $\mathcal{N} = 4$ supersymmetric Yang-Mills gauge theory enjoys the property of being integrable, i.e. it can be exactly solved in terms of conserved charges. It is expected that if a more realistic theory turns out to be integrable, solvability of the theory would lead to simple analytical expressions for quantities like masses of the hadrons in the theory. In this thesis we show that the existing models of confinement are all *nonintegrable* – such simple analytic expressions cannot be obtained.

We moreover show that these nonintegrable systems also exhibit features of chaotic dynamical systems, namely, sensitivity to initial conditions and a typical route of transition to chaos. We proceed to study the quantum mechanics of these systems and check whether their properties match those of chaotic quantum systems. Interestingly, the distribution of the spacing of meson excitations measured in the laboratory have been found to match with level-spacing distribution of typical quantum chaotic systems. We find agreement of this distribution with models of confining strong interactions, confirming these as viable models of particle physics arising from string theory.

KEYWORDS: String Theory, AdS/CFT Correspondence,
Integrable Systems, Chaos, Quantum Chaos.

Archisman Ghosh

2 August, 2012

TIME-DEPENDENT SYSTEMS AND CHAOS IN STRING THEORY

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Dedicated to the fond memories of my brother Atri (Bhutu), whose untimely demise left me with a changed perspective of life. A lot of my curiosity, that eventually drove me to pursue physics, was developed as a kid with my brother, when we would go around the house taking apart household gadgets and appliances, mixing together stuff that came in various bottles and looking for things to set on fire.

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Chapter 1

Overview

String theory is regarded as one of the most challenging frontiers of modern theoretical physics. In this theory the fundamental constituents of matter are open and closed strings rather than particles, and these strings describe both matter and the forces between them. Formulating a consistent quantum mechanical theory of gravitation has been one of most important problems of the twentieth century. String theory is a candidate solution – at low energies the equations governing the dynamics of closed strings reduce to Einstein’s equations of general relativity which describe all aspects of classical gravitational physics. In addition string theory is also a candidate for being a unified theory describing all matter and forces in the universe.

One of the most important results that has emerged from String Theory in the last decade is the AdS/CFT correspondence, also known as the gauge-gravity duality. The statement of the correspondence is that in certain cases a string theory is equivalent to a “dual” gauge theory very similar to the one describing non-gravitational interactions of fundamental particles. The gauge theory is in one lower dimension than the gravity theory and since it captures all the physics of the gravity theory, the duality is called *holographic* in analogy to how a two-dimensional hologram can reproduce a three-dimensional image. When the interactions in the gauge theory are strong enough, the dual string theory can be replaced by classical general relativity. An important point here is that the strongly interacting gauge theory, which is difficult to analyze by itself, is mapped to a much simpler problem in gravity. There are also regimes where gravity problem is no longer classical and does not admit a direct analysis, but is mapped to a simpler problem in a weakly interacting gauge theory – the duality can thus be used both ways.

The greatest triumph of gauge-gravity duality has been in the qualitative understanding of the quantum properties of black holes. Hawking had shown that black holes emit radiation like any other hot object and are characterized by thermodynamic quantities like temperature and entropy. General Relativity does not provide an underlying microscopic structure to calculate these quantities. Here gauge-gravity duality has been able to provide a microscopic description of a large class of black holes and has been able to reproduce their entropy and other thermodynamic quantities.

Applied in the other direction, gauge gravity has been able to make important predictions in QCD. Quantum Chromodynamics or QCD is the theory of strong interactions – the force that binds the nucleus of an atom together. At sufficiently high temperatures nuclear matter undergoes a phase transition into a state where quarks and gluons which are normally confined within the nucleus are freed up. This “quark-gluon plasma” resembles a fluid that flows with a certain viscosity and supports sound waves. Properties of this fluid are difficult to calculate because of the very

strong nature of the interactions. Here gauge-gravity duality can map the problem into a weakly coupled problem in gravity in the presence of a black hole. A black hole is a natural object to introduce here, because it has a temperature just like a fluid. Although the fluid that is considered here is not exactly the quark-gluon plasma of QCD, the results obtained are in consistent with QCD experiments, perhaps indicating a broader range of applicability of the techniques.

In Chapter 2, we discuss the key concepts of the AdS/CFT correspondence and some particular details that will be relevant to us in the rest of the thesis.

Nonintegrability in models of Confinement

The mapping between theories of gravity and gauge theories has led to new approaches to building models of particle physics from string theory. One of the important features to model in this context, is the phenomenon of *confinement* in strong interaction in particle physics. Unlike the force of electromagnetism which falls off with the separation between two electromagnetically charged particles, the force of interaction between strongly interacting particles does not. Thus it is not possible to separate and isolate strongly interacting particles – they remain clumped together forming composite entities called “hadrons”. This property is known as confinement.

This feature of confinement is not exhibited in the gauge theory arising in the simplest of the examples of the duality, namely the correspondence between string theory in pure Anti-de Sitter spacetime and $\mathcal{N} = 4$ supersymmetric Yang-Mills gauge theory. However this $\mathcal{N} = 4$ supersymmetric Yang-Mills theory enjoys the property of being *integrable* – the theory can be exactly solved in terms of *conserved charges*. It is expected that if a more realistic theory is integrable, solvability of the theory would imply simple analytical expression for the quantities like masses of the hadrons in the theory. We discuss the definition and implications of integrability in greater detail in Chapter 3.

We move on to study gravitational theories whose gauge theory duals are more realistic in the sense that they exhibit the phenomenon of confinement. It turns out that such geometries cannot be infinite, but have to cap off like a cigar. The finiteness of the geometry sets a length scale to it, precisely the distance at which the cigar caps off, and this length scale in turn sets an energy scale in the dual gauge theory. There is a minimum energy of excitations or a “mass gap” in the dual theory, again a desirable feature from the point of view of real QCD. Many such “confining geometries” exist and we summarize some of the the more popular ones in Appendix C.

A main emphasis of this thesis is to study integrability in some of these confining backgrounds. In Chapter 4, we begin with one such background, the AdS soliton, which we use as an example throughout this thesis. We study the dynamics of closed strings in this geometry. A numerical analysis reveals that the system is chaotic. It shows an exponential sensitivity to initial conditions that chaotic systems do, and shows a typical transition to chaos as a nonlinear parameter is varied. Chaotic dynamical systems are known to be nonintegrable, and our analysis thus gives a negative result for integrability in the AdS soliton. In Chapter 5, we obtain the same conclusion

using analytical, as opposed to numerical methods. The full analysis of integrability involves sophisticated and relatively recently developed techniques from differential Galois theory, however the demonstration of nonintegrability can be achieved with the implementation of a simple algorithm to the differential equations governing the dynamical system. In Chapter 6, we extend our result to a large class of confining geometries, many of them being the most commonly cited ones. Our analysis thus rules out integrability in a large number of backgrounds – thus simple analytic expressions for the hadron excitations in the dual theories cannot be obtained.

Quantum Chaos

A natural question to ask at this stage is “What is the full quantum spectrum of the systems that we are studying?” The distribution in spacing of energy levels in the spectra of certain chaotic quantum systems have been observed to be qualitatively different from the universal distribution the same spacing for integrable systems. Interestingly, the distribution of the spacing of hadron excitations measured in the laboratory is found to match with level-spacing distribution of typical quantum chaotic systems. In Chapter 7, we numerically obtain the spectrum for the *AdS* soliton background and find an agreement with the typical quantum chaotic level-spacing distribution. The quantum analysis thus ends with a positive note confirming models of confinement coming from string theory as viable models of physics of the strong interaction.

Cosmological singularities

We devote Appendix A to review a model that attempts to understand “singularities” in general relativity using the correspondence. In General Relativity, “singularities” are regions where the theory itself breaks down and a quantum theory of gravity is expected to take over. Since the universe is expanding, our present model of cosmology tells us that at some very early time, the entire universe was contained in a very small region in space. In such a situation General Relativity would break down and such a singularity is known as the Big Bang. One can have models in cosmology (although this is probably not true for our own universe) where the universe stops expanding and begins to contract and shrinks to a very small region leading to a Big Crunch. Big Bangs and Big Crunches are examples of cosmological singularities. These singularities are very disturbing to physicists because they represent a “beginning of time” or “end of time”. There is no way for an observer to bypass them.

A quantum theory of gravitation is expected to smooth out or resolve these singularities. It might also provide some interpretation of cosmological data from the very early universe. Before the advent of String Theory there was no consistent quantum theory of gravity and understanding of cosmological singularities had been a major theoretical problem. Recently there has been some progress in understanding the nature of singularities using String Theory and in particular gauge-gravity duality.

As described above, the duality is between a gauge field theory and a string theory. When the gauge theory coupling is large, the string theory can be approximated by General Relativity in a conventional spacetime. When the gauge theory coupling is weak, General Relativity breaks down and there is no conventional notion of space and time. The gauge theory however is defined for all values of coupling even when General Relativity has ceased to be valid in the dual theory.

Several groups have tried to use this and obtain toy models of cosmological singularities whose gauge theory duals have a time-dependent parameter like the coupling. One would like to see if we can use the gauge theory to calculate the time evolution of the system in a regime where the gravitational description in terms of General Relativity is no longer valid. We have constructed such a toy model where the gauge theory coupling is slowly varying with time. It starts off with a large value in early times and goes on to a large value at late times but is small at intermediate times. In the dual theory there is a good spacetime description in early times which breaks down at intermediate times. The gauge theory however remains to be valid at all times and can be used to study the behavior of the dual gravity. The question that we ask is what happens at late times where a General Relativity description might again be applicable. There are two possibilities. One can be left with smooth spacetime like the one we started out with, the energy that is pumped in to the system during the contracting phase is extracted out during the expanding phase. Alternatively, some of the energy pumped into the system might get distributed between the various modes of the system and get thermalized. In that case one is left with a black hole in the gravity picture. Earlier attempts to obtain such “bouncing cosmologies” have often led to black holes that fill the universe.

We have developed a technique called “adiabatic approximation” in the field theory that is valid when the variation of the coupling is slow enough. In the regime of parameters where our approximation is valid, we have shown that a big black hole is never formed. If a black hole is formed, its size is small compared to the overall size of the universe. We are still trying to understand whether the energy is thermalized or we have a perfectly smooth spacetime at late times. Our approach is one of the very few methods where the question of bounce can be addressed in a controlled and self-consistent fashion.

Chapter 2

The AdS/CFT Correspondence

In Section 2.1, we discuss the AdS/CFT correspondence, an equivalence between gauge theories and gravity, first introduced by Maldacena (1998) [1] and more explicitly worked out by Gubser, Klebanov, Polyakov and Witten [2, 3] (for a standard review, see [4]). In Section 2.2, we discuss the dual description of confinement in field theories, that is going to be relevant to us.

2.1 The AdS/CFT Correspondence

The AdS/CFT correspondence relates a theory of gravity to a gauge theory with no dynamical gravity. In the simplest example of the correspondence the theory of gravity is a Type IIB String Theory on the $AdS_5 \times S^5$ background and the dual gauge theory is the $\mathcal{N} = 4$ supersymmetric $SU(N)$ Yang-Mills Theory. The AdS_5 background is usually written out in either one of the two popular coordinate choices:

- Global AdS , that covers the entire AdS spacetime, and
- Poincaré patch, does not cover the entire spacetime and has a coordinate horizon at $r = 0$.

The metric for $AdS_5 \times S^5$ in the two coordinate systems is given below:

$$\text{Global } AdS: \quad ds^2 = -\left(1 + \frac{r^2}{R_{AdS}^2}\right)dt^2 + \frac{dr^2}{1 + \frac{r^2}{R_{AdS}^2}} + r^2 d\Omega_3^2 + R_{AdS}^2 d\Omega_5^2. \quad (2.1)$$

$$\text{Poincaré patch:} \quad ds^2 = \frac{r^2}{R_{AdS}^2}(-dt^2 + d\mathbf{x}_3^2) + \frac{R_{AdS}^2}{r^2}dr^2 + R_{AdS}^2 d\Omega_5^2. \quad (2.2)$$

Here R_{AdS} is the length scale associated with the geometry, which we will often set to unity. Ω_5 are the coordinates of 5-sphere S^5 . For most of this thesis, this part of the geometry is also not going to be very important to us.¹ We will call r the radial direction. The transverse directions are then Ω_3 and \mathbf{x}_3 in the two cases, the coordinates on a 3-sphere S^3 and a 3-plane \mathbb{R}^3 respectively. The spacetime described above is “Anti de Sitter” – it has a constant negative curvature (Ricci scalar) of $-20/R_{AdS}^2$. The AdS spacetime has a boundary at $r \rightarrow \infty$. Although massive particles take infinite time to reach the boundary, massless fields reach in a finite time and are reflected back; so one needs to impose appropriate boundary conditions, usually reflecting, for fields in the AdS spacetime.

The $\mathcal{N} = 4$ supersymmetric $SU(N)$ Yang-Mills theory will often be referred to as the “gauge theory” or simply the “field theory”. Here \mathcal{N} is the number of possible

¹We will not be dealing with fields dependent in the S^5 directions. In other words, we will ignore the Kaluza-Klein modes coming from the dependence of fields in these internal directions.

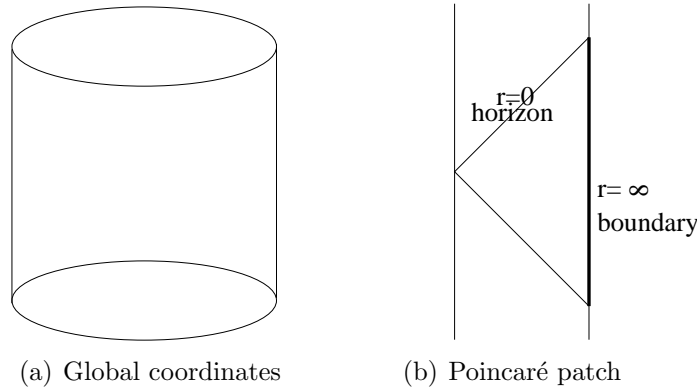


Figure 2.1: AdS spacetime in global and Poincaré coordinates.

supersymmetry transformations and N of $SU(N)$ is the number of colors in the gauge theory. An important thing to note here is that this is a theory with no dynamical gravity – there are no Einstein equations for the metric of the field theory. The field theory has a vanishing β function for the Yang-Mills coupling g_{YM} . The theory is thus independent of the scale at which it is being studied, and is a conformal field theory (CFT).

Holography

The gauge theory turns out to be defined on the boundary of the AdS spacetime, that is, S^3 or \mathbb{R}^3 depending on whether we are working with global or Poincaré AdS . The correspondence is thus a bulk to boundary correspondence, where the bulk gravity is mapped to a boundary gauge theory in a *lower* number of dimensions.

One might be a bit surprised at this equivalence and ask “How can the degrees of freedom of a higher and a lower dimensional theory be the same?” The answer lies in the fact that simple states in the theory of gravity in higher dimensions map to composite operators in the gauge theory in lower dimensions. Thus there is no contradiction.

Degrees of freedom and symmetries

The degrees of freedom on the gravity side are the metric $g_{\mu\nu}$ (or equivalently the graviton $h_{\mu\nu}$), a scalar field Φ called the dilaton, a self-dual 5-form field $F_{\mu\nu}^a$, fermionic superpartners of all the fields and the massive higher-string modes. Since supersymmetry is unbroken, one can consistently work in a basis where the fermionic fields are set to zero. The stringy excitations are characterized by an energy scale $1/\sqrt{\alpha'}$. If we can restrict ourselves to energy scales that are much smaller than the string energy scale, we will not excite any of the higher-string modes, and we will be able to restrict ourselves to the massless modes that are present in supergravity. This can be done if the length scale of our geometry R_{AdS} is much larger than the string length $l_s \sim \sqrt{\alpha'}$. In that limit, the Type IIB string theory reduces to the Type IIB supergravity, with

the action:

$$\mathcal{S}_{5D}^{\text{SUGRA}} = \frac{1}{2\kappa_{10}^2} \int d^5x \left(\mathcal{R}_{5D} - 2\Lambda - \frac{1}{2} \partial^\mu \Phi \partial_\mu \Phi - \frac{R_{AdS}^2}{8} F_{\mu\nu}^a F^{a\mu\nu} + \dots \right) \quad (2.3)$$

The degrees of freedom on the gauge theory side are the gauge field, four Weyl fermions and six real scalars, all in the adjoint representation of $SU(N)$. The internal symmetries of the gauge theory match the isometries of the dual spacetime. SYM in $3+1$ dimensions has a conformal group of $SO(4,2)$ and the R -symmetry group for a $\mathcal{N} = 4$ theory is $SU(4)$ which is same as $SO(6)$. The isometry of $AdS_5 \times S^5$ is $SO(4,2) \times SO(6)$.

Relationship between parameters

An important insight about AdS/CFT comes from the relationship between the parameters on either side of the correspondence. The parameters on the gravity side are the length scale associated with the geometry R_{AdS} , the string length l_s and the string coupling g_s .² The parameters on the gauge theory side are the Yang-Mills coupling g_{YM} and the N of the gauge group. It has been shown by 't Hooft that in the large N limit of gauge theories ($N \rightarrow \infty$, $g_{YM} \rightarrow 0$, $g_{YM}^2 N = \text{finite}$), the appropriate coupling is the combination $\lambda \equiv g_{YM}^2 N$, now known as the 't Hooft coupling. These parameters are related by

$$g_{YM}^2 = 4\pi g_s, \quad \lambda \equiv g_{YM}^2 N = \frac{R_{AdS}^4}{l_s^4}. \quad (2.4)$$

Thus the 't Hooft large N limit of the gauge theory corresponds to a *classical* limit string theory with the quantum corrections suppressed by $1/N$. Moreover, the above relations tell us that the large 't Hooft coupling limit ($\lambda \rightarrow \infty$), corresponds to the low-energy supergravity truncation of the string theory where the stringy modes are suppressed by α' . In this limit, the AdS/CFT correspondence can be viewed as an equivalence between Type IIB supergravity in AdS to a large 't Hooft coupling gauge theory.

Correspondence between fields and operators

The AdS/CFT correspondence, can be formulated more mathematically in the supergravity approximation. The mathematical statement of the correspondence is that the partition function of the field theory in the presence of a source J coupled to the operator $\hat{\mathcal{O}}$ is same as the exponential of the classical action with the solution of the classical field equations plugged in, where the boundary value of field Φ dual to the operator $\hat{\mathcal{O}}$ being the source J up to a scaling.

$$\mathcal{Z}_{4D}[J] \equiv \int \mathcal{D}\Phi \exp \left(i\mathcal{S} + i \int d^4x J \hat{\mathcal{O}} \right) = \exp(i\mathcal{S}[\Phi_{Cl}]) \quad (2.5)$$

²In String Theory, the coupling g_s is not an independent parameter, but is dynamically determined by the value of the dilaton $g_s = \Phi$.

$$\text{with} \quad \lim_{r \rightarrow \infty} r^\Delta \Phi_{\text{Cl}}(r, x) = J(x).$$

Here Δ is just the scaling dimension of the operator $\hat{\mathcal{O}}$. For a scalar field in AdS_{d+1} for example, $\Delta = \Delta_-$ where,

$$\Delta_{\pm} = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2 R_{AdS}^2}. \quad (2.6)$$

With this prescription, the quantum correlation functions in the field theory can be obtained by taking repeated derivatives of the classical action in supergravity with respect to the boundary value of the field. For example, for the simple case where $\Delta = 0$, the two-point function reads:

$$iG(x-y) \equiv \langle T \hat{\mathcal{O}}(x) \hat{\mathcal{O}}(y) \rangle = \frac{1}{\mathcal{Z}} \frac{\delta^2 \mathcal{Z}_{4D}}{\delta J(x) \delta J(y)} = \frac{-i \delta^2 \mathcal{S}[\Phi_{\text{Cl}}]}{\delta \Phi_{\text{Cl}}(x) \delta \Phi_{\text{Cl}}(y)}. \quad (2.7)$$

AdS/CFT at finite temperature

One can extend the *AdS/CFT* correspondence to a finite temperature. The dual of the $\mathcal{N} = 4$ SYM at a finite temperature is a black three brane metric given (in $R_{AdS} = 1$ units) by,

$$\text{Global } AdS: \quad ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_3^2, \quad f(r) = 1 + r^2 - \frac{r_h^2(1+r_h^2)}{r^2}. \quad (2.8)$$

$$\text{Poincaré patch:} \quad ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\mathbf{x}_3^2, \quad f(r) = r^2 \left(1 - \frac{r_h^4}{r^4}\right). \quad (2.9)$$

The Hawking temperature T_H in the two cases is given by

$$T_H = \frac{2r_h^2 + 1}{2\pi} \quad \text{and} \quad \frac{r_h}{\pi} \quad \text{respectively.} \quad (2.10)$$

The temperature T of the field theory is same as the temperature T_H of the geometry: $T = T_H$. The entropy of the field theory can be obtained from the entropy of the horizon of the dual geometry, which using the area-entropy relation is simply proportional to the area of horizon. The entropy density s of $\mathcal{N} = 4$ SYM at strong coupling is given by,

$$s = \frac{\pi^2}{2} N^2 T^3. \quad (2.11)$$

2.2 Confinement in *AdS/CFT*

In a confining theory, the vacuum expectation value of the Wilson loop has an area law behavior [5]

$$\langle W(C) \rangle \equiv \text{Tr} \left[P \exp \left(i \oint_C A \right) \right] \simeq \exp(-\sigma A(C)) \quad (2.12)$$

Type II B String Theory in $AdS_5 \times S^5$	\Longleftrightarrow	Large N limit of $\mathcal{N} = 4$, $SU(N)$ Super Yang Mills
Flux N	\leftrightarrow	Rank of $SU(N)$
String coupling $g_s = e^\Phi$	$=$	g_{YM}^2 Yang Mills coupling
String length $l_s \sim \sqrt{\alpha'}$	$=$	$g_{YM}^2 N \equiv \lambda$ 't Hooft coupling
Type II B classical SUGRA	\Longleftarrow	Large 't Hooft coupling
Stringy corrections α'		Planar corrections
Quantum corrections $\frac{l_{Pl}}{R_{AdS}}$		$\frac{1}{N}$ corrections
$e^{(iS[\Phi_{Cl}])}$	$=$	$\mathcal{Z}_{4D}[J]$
Pure $AdS_5 \times S^5$	\leftrightarrow	Vacuum of SYM
Normalizable modes	\leftrightarrow	Excited states of SYM
Non-normalizable modes	\leftrightarrow	Deformation of SYM action
Black-brane geometry	\leftrightarrow	SYM at finite temperature

Table 2.1: Summary of the AdS/CFT Correspondence

where $A(C)$ is the area of the loop enclosed by the loop C . The constant σ is called the string tension. The area law 2.12 is equivalent to the linear confining quark-antiquark potential $V(L) \sim \sigma L$. This can be seen simply by considering a rectangular loop C with sides of length T and L in Euclidean space as in Figure 2.2(a). For large values of T , we have, when $V(L) \sim \sigma L$ and interpreting T as the time direction,

$$\langle W(C) \rangle \sim \exp(-TV(L)) \quad (2.13)$$

The dual description of a quark-antiquark pair on the boundary field theory is a string stretching into the bulk of the geometry. The dual of the expectation value of the Wilson loop operator is the area of the string that minimizes the action. In the case of pure AdS , the string can stretch freely into the bulk and finally gives a result for the minimum action worldsheet that is consistent with the conformal invariance of $\mathcal{N} = 4$ SYM. However if the geometry is capped off, like that in Figure 4.1, then once the string hits the cap, with further separation between q and \bar{q} , the major contribution to the world sheet area comes from the “wall” sitting at the tip of the cap [Fig.2.2(b)]. Then with separation between q and \bar{q} , the area of the worldsheet increases linearly. This gives a linear dependence of the expectation of the Wilson loop with $L_{q\bar{q}}$ and thus we get a confining potential $V(L_{q\bar{q}}) \sim L_{q\bar{q}}$.

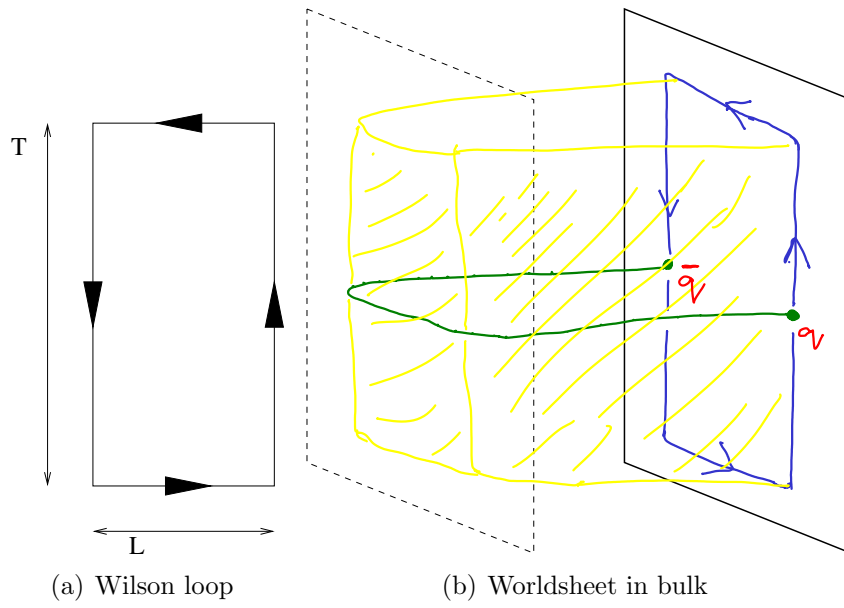


Figure 2.2: Confinement in a field theory is indicated by an area law for the Wilson loop. In the gravity perspective this maps to the effective area of a string world sheet stretching into the bulk. For a geometry with a confining wall, the area of the world sheet is essentially proportional to the separation between q and \bar{q} .

Chapter 3

Integrability and Chaos

In this chapter we go over the basics of integrability and chaos. In Section 3.1 we introduce the definition of an integrable systems and in Section 6.4 we introduce the properties of a chaotic system. In Section 3.3 we discuss the importance of integrability in the context string theory and the *AdS/CFT* correspondence. In section 3.4 we review the main results of [6], that 2-D sigma models and coset models are integrable.

3.1 What is Integrability?

In classical mechanics, a system is integrable when there are the same number of conserved quantities as the pairs of canonical coordinates. One can then make a canonical transformation to a coordinate system where these conserved quantities are the conserved momenta. In these “action-angle coordinates”, the conserved momenta are the action variables I_i and the corresponding coordinates are the angle variables θ_i .

$$\theta_i = I_i t, \quad I_i = \text{constant}. \quad (3.1)$$

The theory is thus exactly solvable in terms of these constants of motion. For a nonintegrable theory one can not write down a closed form analytic solution in general.

For a classical system with n degrees of freedom to be integrable, there are n conserved momenta p_i , which have zero Poisson bracket with each other and with the Hamiltonian H .

$$\{H, p_i\} = 0, \quad \{p_i, p_j\} = 0 \quad (3.2)$$

In quantum mechanics the Poisson brackets are replaced by the commutators. We have a net of n momenta that commute with themselves and the Hamiltonian.

$$[H, p_i] = 0, \quad [p_i, p_j] = 0 \quad (3.3)$$

3.2 What is Chaos?

Quoting from [7], “Chaos is the term used to describe the *apparently* complex behavior of what we consider to be simple, well-behaved systems. Chaotic behavior when looked at casually, looks erratic and almost random ...”. The behavior is seen in systems that are complete deterministic and the apparent randomness is just a manifestation of a strong sensitivity to initial conditions the system has – with a very slight change of initial conditions, a chaotic system can land up in a very different state. This sensitivity to initial conditions, whose meaning will be made more precise in §3.2.1, can be taken to be a working definition of chaos. Chaos is typically seen in *nonlinear* systems – dynamical systems governed by equations of motion that are nonlinear. One of the many universal properties of chaos is the universality of the

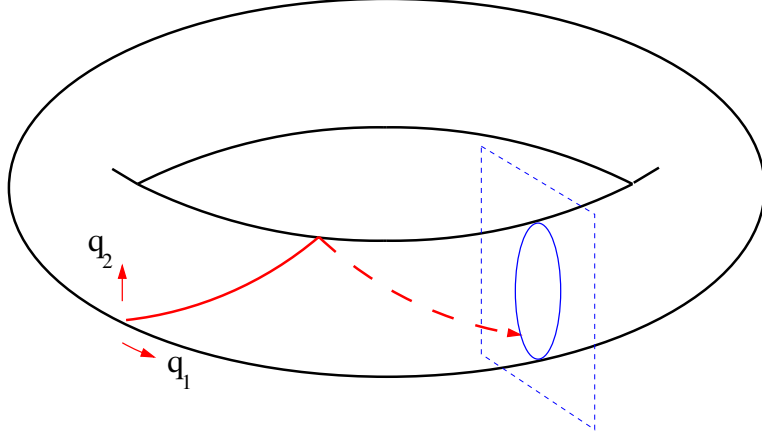


Figure 3.1: Poincaré section of phase-space for a 2-D dynamical system. As the particle moves, it goes around the torus in the q_1 direction and also winds around the torus in the q_2 direction. For each torus, the energies corresponding to each of the two degrees of freedom is conserved separately. For the tori of rational winding, the orbit goes around the torus q times and winds around it p times and comes back to the same point.

route to chaos a system takes as a nonlinear parameter is dialed. We look at the route of transition to chaos for Hamiltonian systems in §3.2.2.

3.2.1 Lyapunov exponent

One of the trademark signatures of chaos is the sensitive dependence on initial conditions, which means that for any point X in the phase space, there is (at least) one point arbitrarily close to X that diverges from X . The separation between the two is also a function of the initial location and has the form $\Delta X(X_0, \tau)$. The Lyapunov exponent is a quantity that characterizes the rate of separation of such infinitesimally close trajectories. Formally it is defined as,

$$\lambda = \lim_{\tau \rightarrow \infty} \lim_{\Delta X_0 \rightarrow 0} \frac{1}{\tau} \ln \frac{\Delta X(X_0, \tau)}{\Delta X(X_0, 0)} \quad (3.4)$$

3.2.2 Transition to Chaos in Hamiltonian systems

An integrable system has the same number of conserved quantities as degrees of freedom. A convenient way to understand these conserved charges is by looking at the phase space. Let us assume that we have a system with N position variables q_i with conjugate momenta p_i . The phase space is $2N$ -dimensional. Integrability means that there are N conserved charges $Q_i = f_i(p, q)$ which are constants of motion. One of them is the energy. These charges define a N -dimensional surface in the phase space which is a topological torus. The $2N$ -dimensional phase space is nicely foliated by these N -dimensional tori. In terms of action-angle variables (I_i, θ_i) these tori just become surfaces of constant action. With each torus there are N associated

frequencies $\omega_i(I_i)$, which are the frequencies of motion in each of the action-angle directions. An illustration for a 2 dimensional system is given in Figure 3.1. The tori which have rational ratios of frequencies, i.e. $m_i\omega_i = 0$ with $m \in \mathcal{Q}$ are called *resonant tori*. The total number of tori of rational winding is a set of Lebesgue measure zero. However infinitesimally close to a torus of rational winding, there exist tori of irrational winding. They are called the KAM (Kolmogorov-Arnold-Moser) tori.

It is interesting to study what happens to these tori when an integrable Hamiltonian is perturbed by a small nonintegrable piece. The KAM theorem states that most tori survive, but suffer a small deformation [8, 7]. However the *resonant* tori which have rational ratios of frequencies, i.e. $m_i\omega_i = 0$ with $m \in \mathcal{Q}$, get destroyed and motion on them become chaotic. For small values of the nonintegrable perturbations, these chaotic regions span a very small portion of the phase space and are not readily noticeable in a numerical study. As the strength of the nonintegrable interaction increases, more tori gradually get destroyed. A nicely foliated picture of the phase space is no longer applicable and the trajectories freely explore the entire phase space with energy as the only constraint. In such cases the motion is completely chaotic.

Hénon-Heiles equations

The Hénon-Heiles equation is a nonlinear *nonintegrable* Hamiltonian system. It is described by the potential

$$V(x, y) = \frac{1}{2} \left(x^2 + y^2 + 2x^2y - \frac{2}{3}y^3 \right). \quad (3.5)$$

Its equations of motion show typical characteristics of transition to chaos of a Hamiltonian system. In Figure 3.3 we look at the Poincaré section of phase space as the conserved energy E of the system is increased from 1/100 to 1/6. For small values of E , the most tori are intact. with increasing E , the KAM tori near the resonant tori start breaking into smaller elliptic tori in accordance with KAM theorem. For large values of E all the tori break and fill the entire phase space (Arnold diffusion). Here each color represents a different initial condition and hence a different torus. The mixing of colors in the final picture indicates the filling up of phase space by the tori.

3.3 Importance of Integrability in String Theory

The study of integrability has acquired a new importance in light of the *AdS/CFT* correspondence. For an excellent and relatively contemporary review refer to [9]. As we have already noted in Chapter 2, the *AdS/CFT* correspondence, in its simplest form maps Type II B String Theory in $AdS_5 \times S^5$ to a maximally supersymmetric Yang-Mills ($\mathcal{N} = 4$ SYM) theory. In the small coupling limit ($\lambda \rightarrow 0$) $\mathcal{N} = 4$ SYM has been shown to be integrable. The local operators of the theory have a one-to-one mapping with states of a certain quantum spin chain. The planar model is a generalization of the Heisenberg spin chain and the Hamiltonian is of the integrable kind. This implies that the spectrum can be solved efficiently by the corresponding

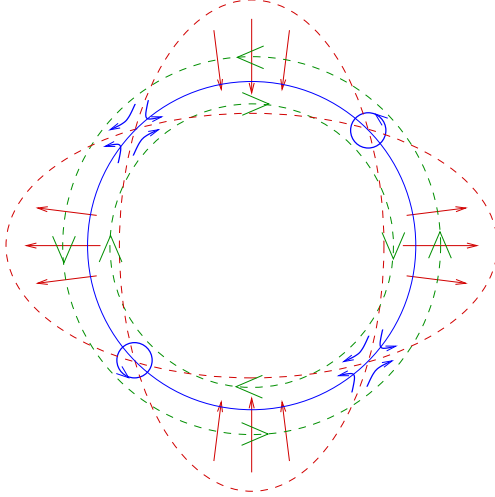


Figure 3.2: KAM tori breaking into elliptic and hyperbolic points. The blue circle is a section of a KAM torus of the unperturbed Hamiltonian close to a resonant torus of winding p/q . Here $q = 2$ and p is some winding number not divisible by 2. Each point on this blue circle is mapped to itself after q revolutions and p windings around the torus, $r_{n+q} = r_n$ and $\phi_{n+q} = \phi_n$. After the inclusion of a small nonintegrable perturbation, points on the blue circle are no longer mapped to themselves. However there are nearby orbits for which the r does not change after q revolutions. These are the green circles. Points on the outer green circle are mapped to the same r but a larger ϕ while points on the inner green circle are mapped to the same r but a smaller ϕ . Points on the outer (inner) circle therefore move counterclockwise (clockwise) after successive maps. Similarly there are nearby orbits for which the ϕ does not change. These are the red ellipses. Points move out along the horizontal axis and points move in along the vertical axis after successive maps. The original blue circle thus has alternate points around which the flow is elliptic and hyperbolic respectively. The original blue circle breaks up into q smaller circles. The picture is *self-similar* – for the small blue circle (near the elliptic fixed point), the same behavior is repeated again. Orbits thus break up forming cascades of smaller and smaller orbits.

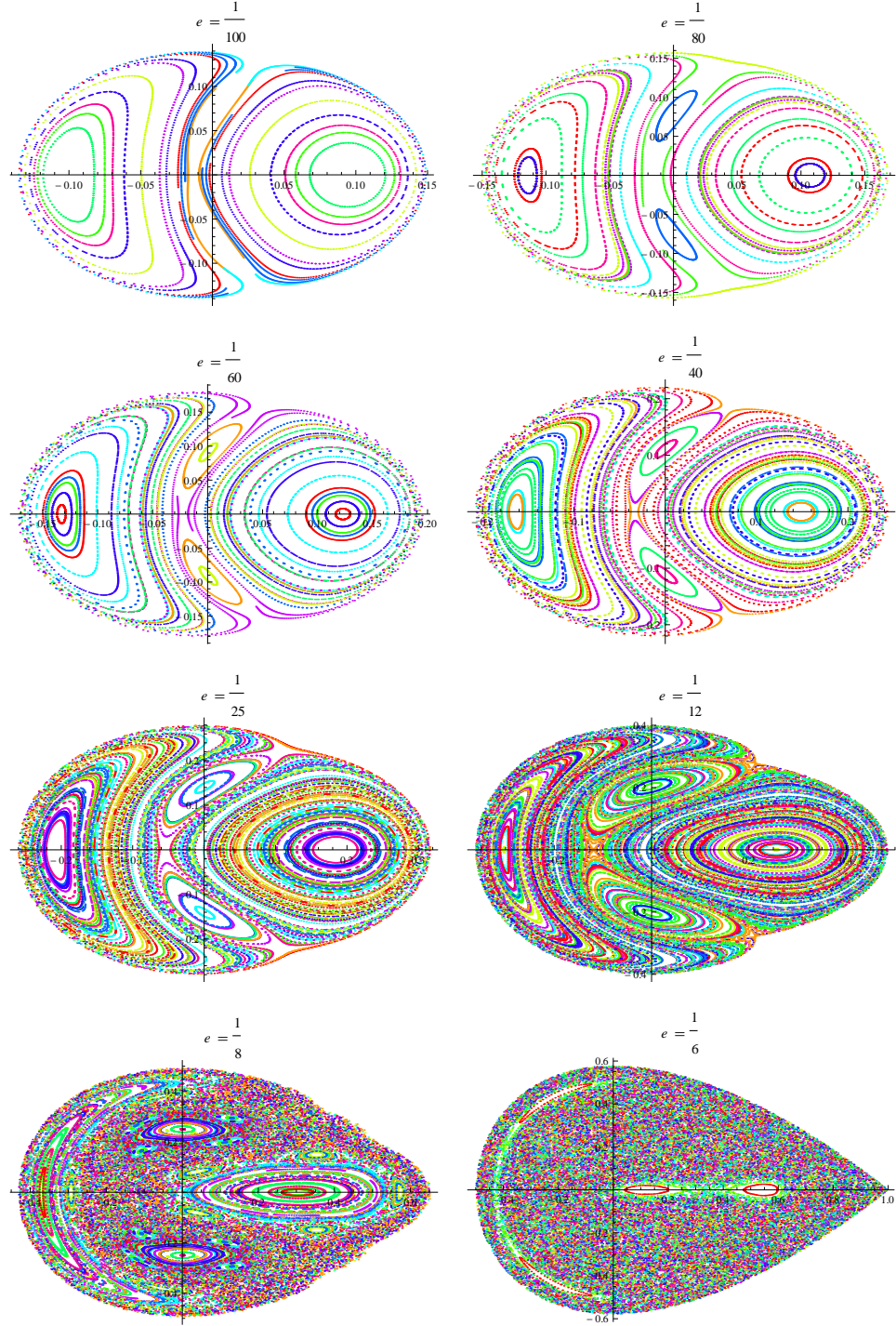


Figure 3.3: Poincaré sections for Hénon-Heiles Equations. This demonstrate breaking of the KAM tori en route to chaos. Each color represents a different initial condition. For smaller values of E the sections of the KAM tori are intact curves, except for the resonant ones. The tori near the resonant ones start breaking as E is increased. For very large values of E all the colors get mixed – this indicates that all the tori get broken and they fill the entire phase space.

Bethe Ansatz. The anomalous dimensions of the operators in the gauge theory can thus be calculated. Results have been obtained for one loop and higher loops and integrability is believed to be true to all loops at small coupling.

In the strong coupling limit ($\lambda \rightarrow \infty$), the dual string theory in $AdS_5 \times S^5$ has been shown to be integrable. The world sheet string theory maps to a sigma model in $\frac{PSU(2,2|4)}{SO(4,1) \times SO(5)}$ and integrability has been established in the bosonic sector of the sigma model in [10] and completed with the inclusion of fermions in [6]. One has to show that string motion in $AdS_5 \times S^5$ has an infinite number of conserved charges¹. Following [6], we go through a sketch of the proof of integrability in maximally symmetric coset spaces in Section 3.4.

The question now is what this integrability can be useful for. For one thing, integrability has allowed us to obtain many classical solutions of the theory that would otherwise have been impossible to find [11, 12]. Known to be solvable in the weak and strong coupling limits, one can now look into a finite coupling window in an integrable theory like $\mathcal{N} = 4$ SYM. For certain states of the theory, one can find a complete and exact description of the spectrum far beyond what is possible using just ordinary QFT.

What exactly do we mean by solving a theory? In an example like the harmonic oscillator or the Hydrogen atom, one actually obtains the spectrum as a simple algebraic expression. It would be too much to hope to expect such a simplistic behavior in our models. The best we can expect to find is a system of algebraic equations whose solution determines the spectrum. What integrability achieves here, is that it vastly reduces the complexity of the spectral problem by bypassing almost all the steps of standard QFT methods, Feynman graphs, loop integrals, regularization, etc. The integrability approach should directly predict the algebraic equations determining the scaling dimension D : $f(\lambda, D) = 0$ – an equation that includes the coupling constant λ in a functional form. This is what we call a *solution* to a spectral problem.

An important open question is whether integrability can be extended to more QCD-like theories.² The original form of AdS/CFT duality was for conformally invariant $\mathcal{N} = 4$ SYM theory but this can be deformed in various ways to produce string duals to confining gauge theories with less or no supersymmetry. The construction of [10, 6] does not readily generalize to these less symmetric backgrounds. One prime example of a confining background is the AdS soliton [13, 14]. Similar geometries have been used extensively to model various aspects of QCD in the context of holography [15]. Here we will look at the question of integrability of bosonic strings on an AdS soliton background. Although it is much more interesting to explore full quantum integrability, to begin with we may ask whether we can find enough conserved charges even at a purely classical level. The answer turns out to be negative. By choosing a class of simple classical string configurations, we show that the Lagrangian reduces to a set of coupled harmonic and anharmonic oscillators that correspond to the size fluctuation and the center of mass of motion of the string. The oscillators decouple

¹Although established classically, the quantum commutator algebra of such charges is not fully understood. However there are strong evidences to suspect that the full quantum theory is integrable

²This is one of the motivations discussed in the introduction of [6].

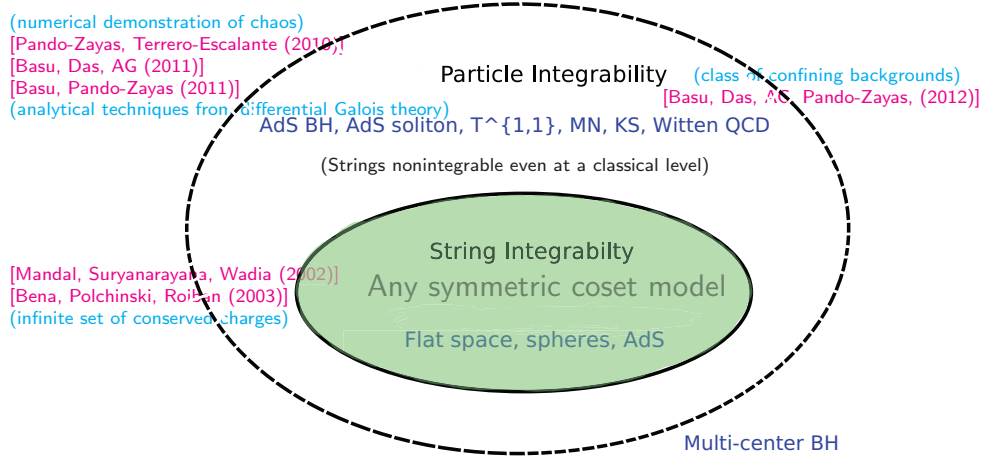


Table 3.1: Summary of Integrability in String Theory

in the low energy limit. With increasing energy the oscillators become nonlinearly coupled. Many such systems are well known to be chaotic and nonintegrable [8, 7]. It is no surprise that our system also shows a similar behaviour. Possibly chaotic behaviour of a test string has been argued previously in black hole backgrounds [16, 17]. However our problem is somewhat different as we are looking at a zero temperature geometry without a horizon. In [18] non-integrability of string theory in $AdS_5 \times T^{1,1}$ is discussed.

3.4 Integrability in Sigma Models and Coset Spaces

We consider a 2-D nonlinear sigma model where the field $g(x) \in$ gauge group G , with the Lagrangian $L = Tr(dg^{-1} \wedge *dg)$. The following current, which corresponds to the global symmetry of left multiplication, can be regarded as a flat gauge connection in the Lie algebra \mathcal{G} .

$$j = -(dg)g^{-1}, \quad d * j = 0, \quad dj + j \wedge j = 0. \quad (3.6)$$

By taking linear combinations of j and $*j$, one can show that there are an infinite number of flat connections.

$$a = \alpha j + \beta * j, \quad da + a \wedge a = (\alpha^2 - \alpha - \beta^2)j \wedge j = 0, \\ \text{if } \alpha = \frac{1}{2}(1 \pm \cosh \lambda), \quad \beta = \frac{1}{2} \sinh \lambda. \quad (3.7)$$

Given any flat connection, the equation $dU = -aU$ is integrable: action on both sides with d gives zero. On a simply connected space, given an initial value $U(x_0, x_0) = 1$, this defines a group element $U(x, x_0)$. This is just the Wilson line, defining the parallel transport with the connection a ,

$$U(x, x_0) = P \exp \left(- \int_C a \right), \quad (3.8)$$

where C is any contour running from x_0 to x , and P denotes path ordering. The flatness of the connection implies that the Wilson line is invariant under continuous deformations of C . Following [19, 20], one can immediately construct an infinite number of conserved charges by taking the outbound spatial Wilson line at fixed time,

$$Q^a(t) = U^a(\infty, t; -\infty, t). \quad (3.9)$$

An analogous construction of conserved charges for coset models G/H is carried out in [6]. The construction is also extended to the Green-Schwarz superstring on $AdS_5 \times S^5$.

Chapter 4

Chaos in the AdS Soliton Background

In this chapter, we study the dynamics of closed strings in the AdS soliton background. In Section 4.1, introduce the background. In Section 4.2, we set up the dynamics of the string in the background and obtain the equations of motion. In Section 4.3, we study the dynamics of the string numerically. In a certain regime of parameter space the system shows a zigzag aperiodic motion characteristic of a chaotic system. Then look at the phase space – integrability implies the existence of a regular foliation of the phase space by invariant manifolds, known as KAM (Kolmogorov-Arnold-Moser) tori, such that the Hamiltonian vector fields associated with the invariants of the foliation span the tangent distribution. Our numerics shows how this nice foliation structure is gradually lost as we increase the energy of the system. To complete the discussion we also calculate Lyapunov indices for various parameter ranges and find large positive values in chaotic regimes indicating an exponential sensitivity to initial conditions. The bulk of this chapter is based on the work done in [21]¹.

4.1 The AdS soliton background

The AdS soliton metric for an asymptotically AdS_{d+1} background is given by [13],

$$ds^2 = L^2 \alpha' \left\{ e^{2u} (-dt^2 + T_{2\pi}(u) d\theta^2 + dw_i^2) + \frac{1}{T_{2\pi}(u)} du^2 \right\},$$

where $T_{2\pi}(u) = 1 - \left(\frac{d}{2} e^u \right)^{-d}.$ (4.1)

At large u , $T_{2\pi}(u) \approx 1$ and (4.1) reduces to AdS_{d+1} in Poincaré coordinates. However one of the spatial boundary coordinates θ is compactified on a circle. The remaining boundary coordinates w_i and t remain non-compact. The dual boundary theory may be thought of as a Scherk-Schwarz compactification on the θ cycle. The θ cycle shrinks to zero at a finite value of u , smoothly cutting off the IR region of AdS . This cutoff dynamically generates a mass scale in the theory, very much like in real QCD. The resulting theory is confining and has a mass gap.

Here we will work with² $d = 4$ and make a coordinate transformation $u = u_0 + ax^2$ with $u_0 = \log(2/d)$ and $a = d/4$, such that $T_{2\pi}(u_0) = 0$ and for small x the x - θ part of the metric looks flat, $ds^2 \approx dx^2 + x^2 d\theta^2$. In these coordinates the metric is

¹The convention employed in choosing the overall prefactors of the Hamiltonian is slightly different in this thesis than in [21]. Also, we work with AdS_{d+1} with $d = 4$ here as opposed to $d = 5$ in [21]. Therefore the numbers we see here are slightly different from that in our earlier work.

²The analysis for any other $d \geq 4$ proceeds along the same lines and almost identical results can be obtained.

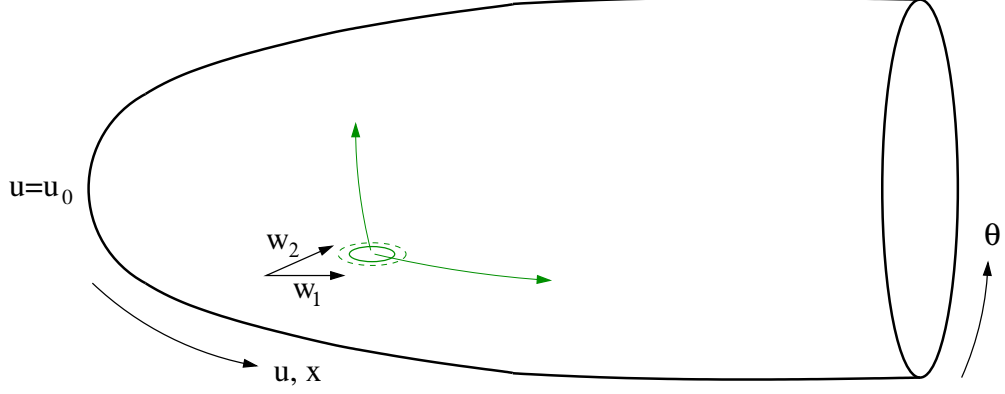


Figure 4.1: A classical string in the AdS soliton background.

$$ds^2 = L^2 \alpha' \left\{ e^{2u_0 + 2ax^2} (-dt^2 + T(x) d\theta^2 + dw_i^2) + \frac{4a^2 x^2}{T(x)} dx^2 \right\},$$

where $T(x) = 1 - e^{-dax^2}$.

(4.2)

4.2 Classical string in AdS -soliton

The motion of strings on the world-sheet is described by the Polyakov action [22]:

$$S_P = -\frac{1}{2\pi\alpha'} \int d\tau d\sigma \sqrt{-\gamma} \gamma^{ab} G_{\mu\nu} \partial_a X^\mu \partial_b X^\nu$$
(4.3)

where X^μ are the coordinates of the string, $G_{\mu\nu}$ is the spacetime metric of the fixed background, γ_{ab} is the worldsheet metric, the indices a, b represent the coordinates on the worldsheet of the string which we denote as (τ, σ) . We work in the conformal gauge $\gamma_{ab} = \eta_{ab}$ and use the following embedding for a closed string (partially motivated by [16]):

$$t = t(\tau), \quad \theta = \theta(\tau), \quad x = x(\tau),$$

$$w_1 = R(\tau) \cos(\phi(\sigma)), \quad w_2 = R(\tau) \sin(\phi(\sigma)) \quad \text{with } \phi(\sigma) = \alpha\sigma.$$
(4.4)

The string is located at a certain value of u and is wrapped around a pair of w -directions as a circle of radius R . It is allowed to move along the potential in u direction and change its radius R . Here $\alpha \in \mathbb{Z}$ is the winding number of the string. The test string Lagrangian takes the form:

$$\begin{aligned} \mathcal{L} &\propto \frac{1}{2} e^{2ax^2} \left\{ -\dot{t}^2 + T(x) \dot{\theta}^2 + \dot{w}_i^2 - w_i'^2 \right\} + \frac{d^2 a^2 x^2}{2T(x)} \dot{x}^2 \\ &= \frac{1}{2} e^{2ax^2} \left\{ -\dot{t}^2 + T(x) \dot{\theta}^2 + \dot{R}^2 - \alpha^2 R^2 \right\} + \frac{d^2 a^2 x^2}{2T(x)} \dot{x}^2, \end{aligned}$$
(4.5)

where dot and prime denote derivatives w.r.t τ and σ respectively. The coordinates t and θ are *ignorable* and the corresponding momenta are constants of motion. The test string Lagrangian differs from a test particle Lagrangian because of the potential term in $R(\tau)$. The coordinate R would be ignorable without a potential term. In general it can be easily argued that for a generic motion of a test particle in an AdS soliton background, all the coordinates other than x are ignorable and the equations of motion can be reduced to a Lagrangian dynamics in one variable x . This implies integrability.

Here the conserved momenta conjugate to t and θ are,

$$p_t = -e^{2ax^2}\dot{t} \equiv -E \quad , p_\theta = e^{2ax^2}T(x)\dot{\theta} \equiv k. \quad (4.6)$$

The conjugate momenta corresponding to the other coordinates are:

$$p_R = e^{2ax^2}\dot{R}, \quad p_x = \frac{d^2a^2x^2}{T(x)}\dot{x}. \quad (4.7)$$

With these we can construct the Hamiltonian density:

$$\mathcal{H} = \frac{1}{2} \left\{ \left(-E^2 + \frac{k^2}{T(x)} + p_R^2 \right) e^{-2ax^2} + \frac{T(x)p_x^2}{d^2a^2x^2} + \alpha^2 R^2 e^{2ax^2} \right\} \quad (4.8)$$

Hamilton's equations of motion give:

$$\begin{aligned} \dot{R} &= p_R e^{-2ax^2}, & \dot{p}_R &= R\alpha^2 e^{2ax^2}, & \dot{x} &= \frac{T(x)p_x}{d^2a^2x^2}, \\ \dot{p}_x &= -\frac{1}{2} \left\{ 4ax \left[\left(E^2 - \frac{k^2}{T(x)} - p_R^2 \right) e^{-2ax^2} + \alpha^2 R^2 e^{2ax^2} \right] \right. \\ &\quad \left. - \frac{2T(x)p_x^2}{d^2a^2x^3} + \left[\frac{p_x^2}{d^2a^2x^2} - \frac{k^2 e^{-2ax^2}}{T(x)^2} \right] \partial_x T(x) \right\} \end{aligned} \quad (4.9)$$

We also have the constraint equations:

$$\begin{aligned} G_{\mu\nu} (\partial_\tau X^\mu \partial_\tau X^\nu + \partial_\sigma X^\mu \partial_\sigma X^\nu) &= 0, \\ G_{\mu\nu} \partial_\tau X^\mu \partial_\sigma X^\nu &= 0. \end{aligned} \quad (4.10)$$

The first equation takes the form $\mathcal{H} = 0$ ³ and the second equation is automatically satisfied for our embedding.

4.3 Dynamics of the system

At $k = 0$, an exact solution to the EOM's is a fluctuating string at the tip of the geometry, given by

$$x(\tau) = 0 \quad (4.11)$$

³The Hamiltonian constraint could be tuned to a nonzero value by adding a momentum in a decoupled compact direction. For example if the space is $\mathcal{M} \times S^5$ then giving a non-zero angular momentum in an S^5 direction would do the job. However we choose to confine the motion within \mathcal{M} here.

$$R(\tau) = A \sin(\tau + \phi). \quad (4.12)$$

where A, ϕ are integration constants. No such solution with constant $x(\tau)$ exists for $k \neq 0$. However one may construct approximate quasi-periodic solutions for small $R(\tau), p_R(\tau)$. It should be noted that with $R, p_R = 0$ the zero energy condition Eqn.(4.10) becomes similar to the condition for a massless particle and the string escapes from AdS following a null geodesic. For small nonzero values of R_0, p_R , the motion in the x -direction will have a long time period. However the fluctuations in the radius will have a frequency proportional to the winding number which is of $\mathcal{O}(1)$. This is a perfect setup to do a two scale analysis. In the equation for \dot{p}_R we may replace $R(\tau)^2$ by a time average value. With this approximation, motion in the x -direction becomes an anharmonic problem in one variable which is solvable in principle. The motion is also periodic [Fig.4.2(a)]. On the other hand to solve for $R(\tau)$ we treat $x(\tau)$ as a slowly varying field. In this approximation the solution for $R(\tau)$ is given by

$$R(\tau) \approx \exp(-a x(\tau)^2) A \sin(\tau + \phi). \quad (4.13)$$

Hence $R(\tau)$ is quasi-periodic [Fig.4.2(a)]. We have verified that in the small R regime, the semi-analytic solution matches quite well with our numerics.

Once we start moving away from the small R limit the above two scale analysis breaks down and the nonlinear coupling between two oscillators gradually becomes important. In short the coupling between oscillators tends to increase as we increase the energy of the string. Due to the nonlinearity, the fluctuations in the x - and R -coordinates influence each other and the motions in both coordinates become aperiodic. Eventually the system becomes completely chaotic [Fig.4.2(c)]. The power spectrum changes from peaked to noisy as chaos sets in [Fig.4.2]. As we discuss in the next subsection, the pattern follows general expectations from the KAM theorem.

4.3.1 Poincaré sections and the KAM theorem

We look at the Poincaré section of the phase space of the system to investigate the transition to chaos for Hamiltonian systems described in Section 3.2.2. For our system, the phase space has four variables x, R, p_x, p_R . If we fix the energy we are in a three dimensional subspace. Now if we start with some initial condition and time-evolve, the motion is confined to a two dimensional torus for the integrable case. This 2d torus intersects the $R = 0$ hyperplane at a circle. Taking repeated snapshots of the system as it crosses $R = 0$ and plotting the value of (x, p_x) , we can reconstruct this circle. Furthermore varying the initial conditions (in particular we set $R(0) = 0$, $p_x(0) = 0$, vary $x(0)$ and determine $p_R(0)$ from the energy constraint), we can expect to get the foliation structure typical of an integrable system.

Indeed we see that for smaller value of energies, a distinct foliation structure exists in the phase space [Fig.4.3(a)]. However as we increase the energy some tori get gradually dissolved [Figs.4.3(b)-4.3(f)]. The tori which are destroyed sometimes get broken down into smaller tori [Figs.4.3(c)-4.3(d)]. Eventually the tori disappear and become a collection of scattered points known as cantori. However the breadths of

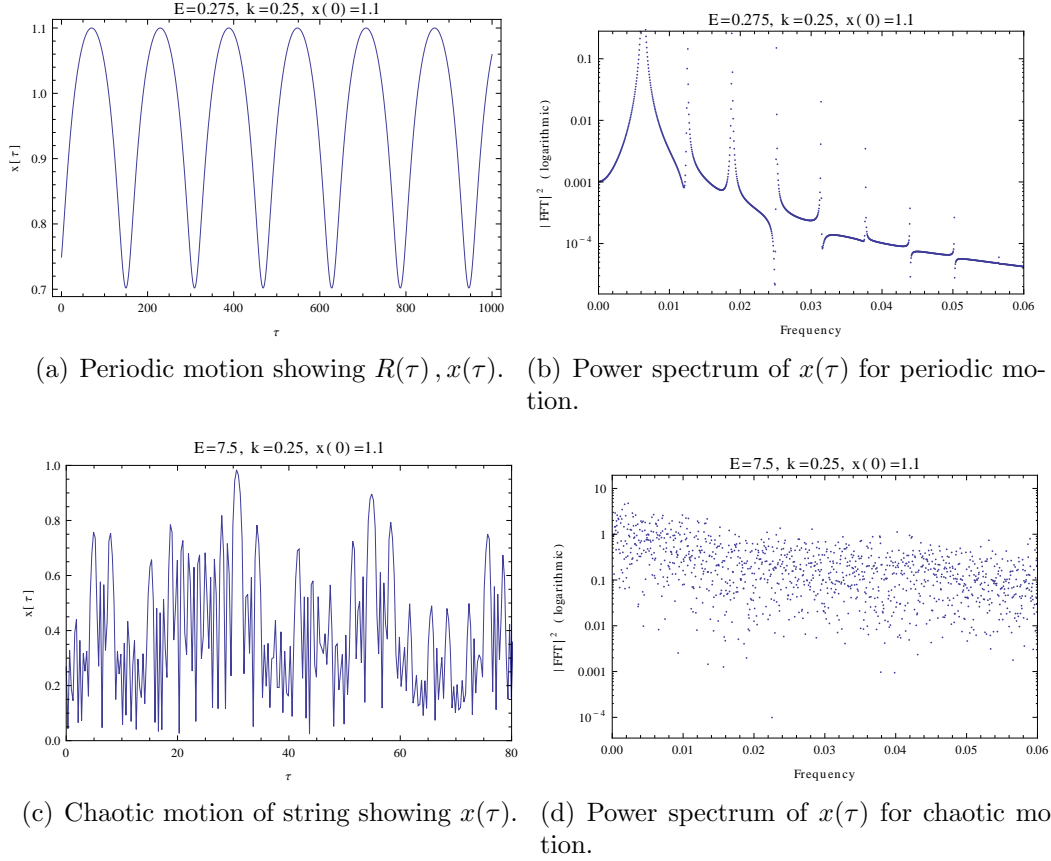


Figure 4.2: Numerical simulation of the motion of the string and the corresponding power spectra for small and large values of E . The initial momenta $p_x(0), p_R(0)$ have been set to zero. For a small value of $E = 0.22$, we see a (quasi-)periodicity in the oscillations. The power spectrum shows peaks at discrete harmonic frequencies. However for a larger value of $E = 3.0$, the motion is no longer periodic. We only show $x(\tau)$ but $R(\tau)$ is similar. The power spectrum is white.

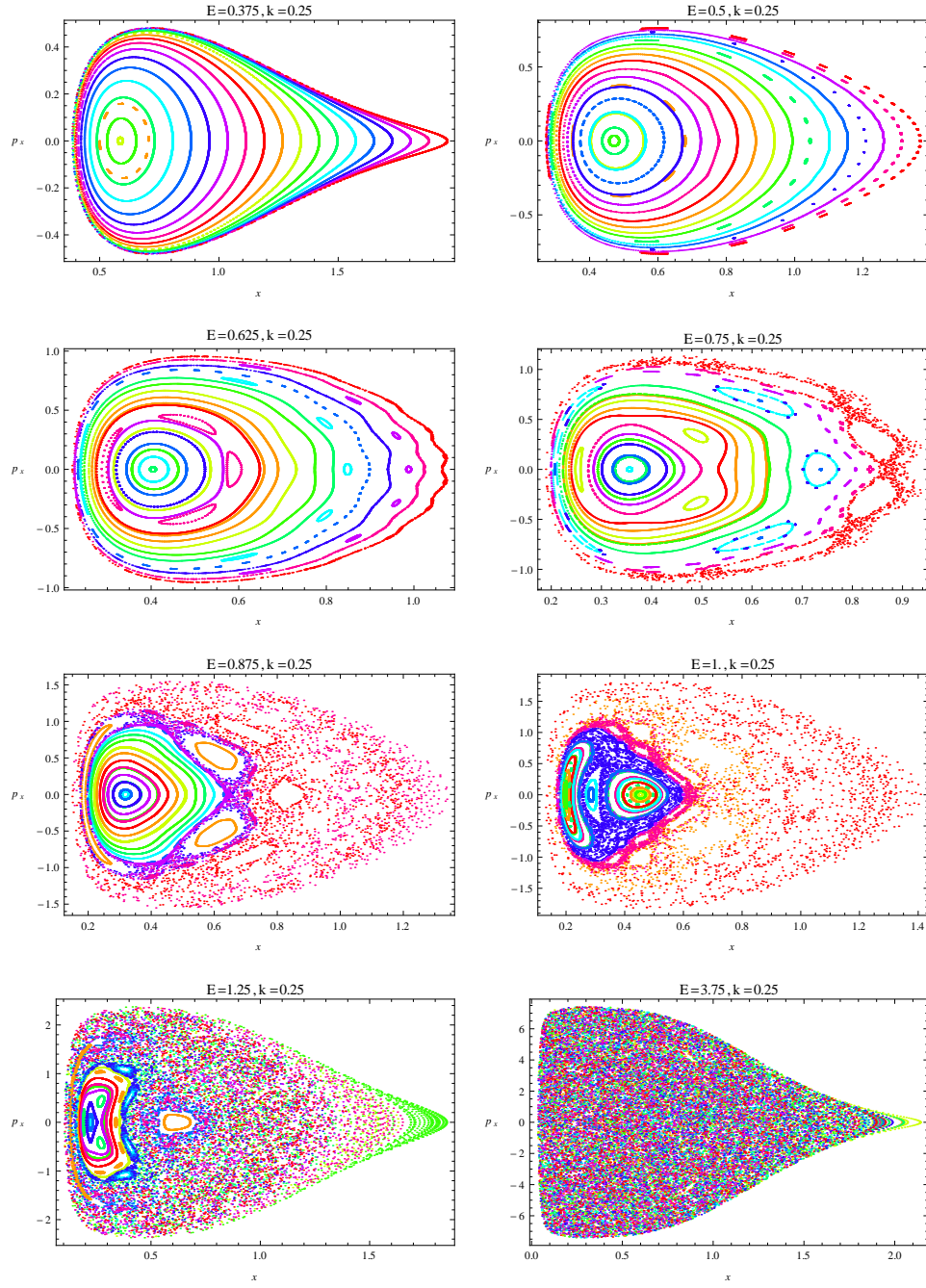
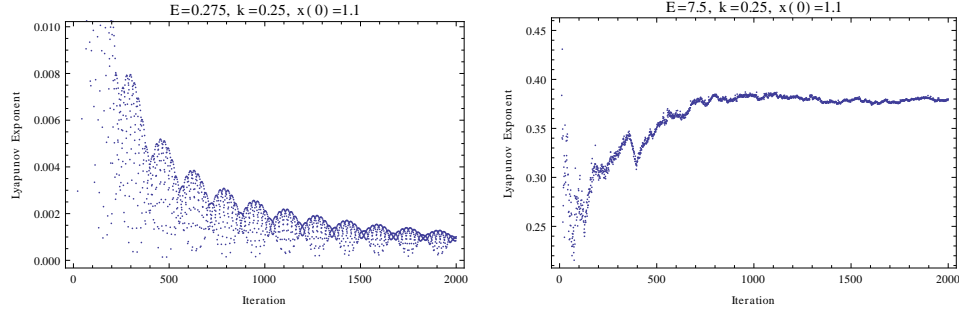


Figure 4.3: Poincaré sections demonstrate breaking of the KAM tori en route to chaos.



(a) Lyapunov index for periodic motion. (b) Lyapunov index for chaotic motion.

Figure 4.4: Lyapunov indices for the same values of parameters as in Fig.(4.2). For $E = 0.22$, the Lyapunov exponent falls off to zero. For $E = 3.0$, the Lyapunov exponent converges to a positive value of about 0.38.

these cantori are restricted by the undissolved tori and other dynamical elements. Usually they do not span the whole phase space [Figs.4.3(c)-4.3(f)]. For sufficiently large values of energy there are no well defined tori. In this case phase space trajectories are all jumbled up and trajectories with very different initial conditions come arbitrary close to each other [Fig.4.3(h)]. The mechanism is very similar to what happens in well known nonintegrable systems like Hénon-Heiles models [8, 7].

4.3.2 Lyapunov exponent

We next calculate the Lyapunov exponent defined in Section 3.2.1. We use an algorithm by Sprott [23], which calculates λ over short intervals and then takes a time average. We should expect to observe that, as time τ is increased, λ settles down to oscillate around a given value. For trajectories belonging to the KAM tori, λ is zero, whereas it is expected to be non-zero for a chaotic orbit. We verify such expectations for our case. We calculate λ with various initial conditions and parameters. For apparently chaotic orbits we observe a nicely convergent positive λ [Fig.4.4].

Chapter 5

Analytic Nonintegrability

In this chapter we review the features of analytic integrability that are required to understand Hamiltonian dynamical systems. In Section 5.1 we summarize the main results of [24, 25, 26]. The gist of the discussion is a theorem and an algorithm:

- The *Ramis-Ruiz Theorem*, that guarantees integrability of the *Normal Variational Equation* (NVE) if the full set of equations is integrable, and
- The *Kovačič Algorithm*, that provides a systematic procedure of calculation of the first integral of the NVE, if it exists.

In Section 5.2 we obtain the NVE for the *AdS* soliton background introduced in Chapter 4 and employ the Kovačič Algorithm to solve it. The algorithm fails, implying nonintegrability of the system of equations¹.

The algorithm was first used in the context of string theory in [27] where nonintegrability of strings in $T^{p,q}$ and $Y^{p,q}$ backgrounds was discussed. It was also shown how the algorithm goes through for the integrable case of S^5 . These results are included in Appendix (B).

5.1 Analytic Integrability

Consider a general system of differential equations $\dot{\vec{x}} = \vec{f}(\vec{x})$. The general basis for proving nonintegrability of such a system is the analysis of the variational equation around a particular solution $\vec{x} = \vec{x}(t)$ which is called the straight line solution. The variational equation around $\vec{x}(t)$ is a linear system obtained by linearizing the vector field around $\vec{x}(t)$. If the nonlinear system admits some first integrals so does the variational equation. Thus, proving that the variational equation does not admit any first integral within a given class of functions implies that the original nonlinear system is nonintegrable. In particular when one works in the analytic setting where inverting the straight line solution $\vec{x}(t)$, one obtains a (noncompact) Riemann surface Γ given by integrating $dt = dw/\dot{\vec{x}}(w)$ with the appropriate limits. Linearizing the system of differential equations around the straight line solution yields the *Normal Variational Equation* (NVE), which is the component of the linearized system which describes the variational normal to the surface Γ .

The methods described here are useful for Hamiltonian systems, luckily for us, the Virasoro constraints in string theory provide a Hamiltonian for the systems we consider. This is particularly interesting as the origin of this constraint is strictly

¹Section 5.2 is part of an unpublished work done with Diptarka Das. An explicit application of the Kovačič algorithm in context of systems arising from string theory is not contained in any of the other papers. I would like to thank Diptarka Das for closely working out the steps of the algorithm with me and for initially writing it up.

stringy but allows a very intuitive interpretation from the dynamical system perspective. One important result at the heart of a analytic nonintegrability are Ziglin's theorems. Given a Hamiltonian system, the main statement of Ziglin's theorems is to relate the existence of a first integral of motion with the monodromy matrices around the straight line solution [28, 29]. The simplest way to compute such monodromies is by changing coordinates to bring the normal variational equation into a known form (hypergeometric, Lamé, Bessel, Heun, etc). Basically one needs to compute the monodromies around the regular singular points. For example, in the case where the NVE is a Gauss hypergeometric equation $z(1-z)\xi'' + (3/4)(1+z)\xi' + (a/8)\xi = 0$, the monodromy matrices can be expressed in terms of the product of monodromy matrices obtained by taking closed paths around $z = 0$ and $z = 1$. In general the answer depends on the parameters of the equation, that is, on a above. Thus, integrability is reduced to understanding the possible ranges of the parameter a .

Morales-Ruiz and Ramis proposed a major improvement on Ziglin's theory by introducing techniques of differential Galois theory [30, 31, 32]. The key observation is to change the formulation of integrability from a question of monodromy to a question of the nature of the Galois group of the NVE. In more classical terms, going back to Kovalevskaya's formulation, we are interested in understanding whether the KAM tori are resonant or not. In simpler terms, if their characteristic frequencies are rational or irrational (see the pedagogical introductions provided in [25, 33]). This statement turns out to be dealt with most efficiently in terms of the Galois group of the NVE. The key result is now stated as: If the differential Galois group of the NVE is non-virtually Abelian, that is, the identity connected component is a non-Abelian group, then the Hamiltonian system is nonintegrable. The calculation of the Galois group is rather intricate, as was the calculation of the monodromies, but the key simplification comes through the application of Kovačič algorithm [34]. Kovačič algorithm is an algorithmic implementation of Picard-Vessiot theory (Galois theory applied to linear differential equations) for second order homogeneous linear differential equations with polynomial coefficients and gives a constructive answer to the existence of integrability by quadratures.² So, once we write down our NVE in a suitable linear form it becomes a simple task to check their solvability in quadratures. An important property of the Kovačič algorithm is that it works if and only if the system is integrable, thus a failure of completing the algorithm equates to a proof of nonintegrability. This route of declaring systems nonintegrable has been successfully applied to various situations, some interesting examples include: [35, 36, 37, 38]. See also [39] for nonintegrability of generalizations of the Hénon-Heiles system [33]. A nice compilation of examples can be found in [25].

²Kovačič algorithm is implemented in most computer algebra software including Maple and Mathematica. It will be evident from the treatment in 5.2.1 that it is a little tedious but straightforward to go through the steps of the algorithm manually.

5.2 Nonintegrability in the AdS soliton background

We begin with the test string Lagrangian in the AdS soliton background (4.5), reproduced here for the convenience of the reader.

$$\mathcal{L} = \frac{1}{2}e^{2ax^2} \left\{ -\dot{t}^2 + T(x)\dot{\theta}^2 + \dot{R}^2 - \alpha^2 R^2 \right\} + \frac{d^2 a^2 x^2}{2T(x)} \dot{x}^2, \quad (5.1)$$

In the following analysis, We set $\dot{\theta} = 0$. Expanding our Lagrangian to quadratic order for small x and R , we obtain,

$$\mathcal{L} \approx c_0 \dot{x}^2 + c_1 x^2 + c_2 (\dot{R}^2 - \alpha^2 R^2) + c_3 x^2 (\dot{R}^2 - \alpha^2 R^2) \\ \text{where } c_0 = \frac{da}{2}, \quad c_1 = aE^2, \quad c_2 = \frac{1}{2}, \quad c_3 = a. \quad (5.2)$$

The Lagrangian is that of a pair of coupled harmonic oscillators, except for the opposite sign in the potential term for x . The equations of motion are a set of second order differential equations in (x, p_x, R, p_R) . If we can consistently set $(p_x = 0, x = \text{constant})$, we can analytically obtain the *straight line solution* for R which is just the harmonic oscillator solution,

$$R = A \sin \alpha t. \quad (5.3)$$

We can obtain the *normal variational equation* (NVE) for x by plugging in the above solution into the e.o.m. for x :

$$c_0 \ddot{x} = x[c_1 + c_3(\dot{R}^2 - \alpha^2 R^2)] \\ \implies \frac{d^2 x}{dR^2} - \frac{R}{A^2 - R^2} \frac{dx}{dR} - \frac{c_1 + c_3 \alpha^2 (A^2 - 2R^2)}{c_0 \alpha^2 (A^2 - R^2)} x = 0 \quad (5.4)$$

If there are two conserved quantities H and Q , then the NVE has a first integral. If the NVE does not have a first integral, the Ramis-Ruis theorem implies nonintegrability of the system of equations. The *Kovačič algorithm* provides a systematic procedure for calculating the first integral of the NVE, if it exists. If a first integral does not exist, the Kovačič algorithm fails.

5.2.1 Kovačič algorithm

In order to employ the Kovačič algorithm, we need to cast the above equation to its reduced invariant form, where prime now denotes derivative w.r.t. R ,

$$\xi'' + g\xi = 0 \quad (5.5)$$

And then g can be expressed as,

$$g = g(R) = -\frac{s(R)}{t(R)} \quad (5.6)$$

For our case,

$$\begin{aligned} s(R) &= 4c_3\alpha^2(A^4 - 3A^2R^2 + R^4) + 4c_1(A^2 - R^2) + \alpha^2R^2 \\ t(R) &= 4c_0(A^2 - R^2)^2\alpha^2, \end{aligned} \quad (5.7)$$

In the following discussion, we closely follow the steps of the algorithm as described in Section (2.7) of [25]. The algorithm is devoted to the computation of the minimal polynomial $Q(v)$. The degree n of $Q(v)$ belong to the set

$$L_{max} = \{1, 2, 4, 6, 12\}$$

The following function h is defined on the set L_{max}

$$h(1) = 1, \quad h(2) = 4, \quad h(4) = h(6) = h(12) = 12.$$

The **First Step** of the algorithm determines the subset L of L_{max} of the possible values of n . The **Second Step** and **Third Step** are devoted to the computation of $Q(v)$, if it exists. If the polynomial does not exist, the algorithm fails indicating that the equation (5.5) is not integrable and that its Galois group is $SL(2, \mathbf{C})$.

First Step

We factorize $t(R)$ in relatively monic polynomials:

$$t(R) = (R - A)^2(R + A)^2.$$

1.1 Let Γ' be the set of roots of $t(R)$: $\Gamma' = \{A, -A\}$. Let $\Gamma = \Gamma' \cup \infty$ be the set of singular points: $\Gamma = \{A, -A, \infty\}$. The order of a root at a singular point $c \in \Gamma'$ is, as usual, $o(c) = i$ if c is a root of multiplicity i of $t(x)$. The order at infinity is defined by $o(\infty) = \max(0, 4 + \deg(s) - \deg(t))$.

$$o(A) = o(-A) = 2, \quad o(\infty) = 4.$$

Now, m^+ is defined to be the maximum value of the order that appears at the singular points in Γ and Γ_i the set of singular points of order $i \leq m^+$:

$$m^+ = 4, \quad \Gamma_2 = \{A, -A\}, \quad \Gamma_4 = \{\infty\}.$$

1.2 If $m^+ > 2$, we write $\gamma_2 = \text{card}(\Gamma_2)$, else $\gamma_2 = 0$. Then we compute $\gamma = \gamma_2 + \text{card}(\bigcup_{3 \leq k \leq m^+} \Gamma_k)$. For our case,

$$\gamma = \gamma_2 = 2.$$

1.3 For the singular points of order one or two, $c \in \Gamma_2 \cup \Gamma_1$, we compute the principal parts of g :

$$\begin{aligned} g &= \alpha_c(R - c)^{-2} + \beta_c(R - c) + O(1) && \text{if } c \in \Gamma', \\ \text{and } g &= \alpha_\infty R^{-2} + \beta_\infty R^{-3} + O(R^{-4}) && \text{for } c = \infty. \end{aligned}$$

We obtain the following coefficients,

$$\alpha_A = \alpha_{-A} = \frac{3}{16}, \quad \beta_A = -\beta_{-A} = -\frac{\alpha^2 c_0 - 8c_1 + 8A^2 \alpha^2 c_3}{16A\alpha^2 c_0}.$$

1.4 We define the subset L' (of possible values for the degree of the minimal polynomial $Q(v)$) as $\{1\} \in L'$ if $\gamma = \gamma_2$, $\{2\} \in L'$ if $\gamma' \geq 2$ and $\{4, 6, 12\} \in L'$ if $m^+ \leq 2$. For us, the first two conditions are satisfied, but not the last one. So,

$$L' = \{1, 2\} \tag{5.8}$$

1.5 Since $m^+ > 2$, $L = L' = \{1, 2\}$.

1.6 Since $L \neq \emptyset$, the algorithm does not fail at this stage and we can move on to the **Second** and the **Third** steps.

Second Step

We consider the cases $n \in L$, that is $n = 1$ and $n = 2$.

2.1 For us ∞ does not have order 0. So we ignore this step.

2.2 For us, there is no c with order 1. So we ignore this step.

2.3 For $n = 1$, for each c of order 2, that is $c \in \{A, -A\}$, we define

$$E_c = \left\{ \frac{1}{2}(1 + \sqrt{1 + 4\alpha_c}), \frac{1}{2}(1 - \sqrt{1 + 4\alpha_c}) \right\}.$$

Thus for $n = 1$ we get,

$$E_A = E_{-A} = \left\{ \frac{2 + \sqrt{7}}{4}, \frac{2 - \sqrt{7}}{4} \right\}.$$

2.4 For $n \geq 2$, for each c of order 2 we define,

$$E_c = \mathbf{Z} \cap \left\{ \frac{h(n)}{2}(1 - \sqrt{1 + 4\alpha_c}) + \frac{h(n)}{n}k\sqrt{1 + 4\alpha_c} : k = 0, 1, \dots, n \right\}.$$

Thus for $n = 2$ we obtain;

$$E_A = E_{-A} = \{2\}$$

2.5 For $n = 1$, for each singular point of order 2ν , with $\nu > 1$, we compute the numbers α_c and β_c . For $c = \infty$, the prescription is to expand g as

$$g = \left\{ \alpha_\infty x^{\nu-2} + \sum_{i=0}^{\nu-3} \mu_{i,\infty} x^i \right\}^2 - \beta_\infty x^{\nu-3} + O(x^{\nu-4}).$$

Then, upto a sign ϵ , that is irrelevant for our case,

$$E_c = \left\{ \frac{1}{2}(\nu + \epsilon \frac{\beta_c}{\alpha_c}) \right\}$$

Our expansion reads,

$$\alpha_\infty^2 = -\frac{2c_3}{c_0} \quad \beta_\infty = 0.$$

Thus for $n = 1$ we obtain,

$$E_\infty = \{1\}$$

2.6 For $n = 2$, for each c of order ν , with $\nu > 3$, we write $E_c = \{\nu\}$. Thus for $n = 2$,

$$E_\infty = \{4\}.$$

Summarizing the results of the **Second** step:

$$\begin{aligned} n = 1 : \quad E_A = E_{-A} &= \left\{ \frac{2 + \sqrt{7}}{4}, \frac{2 - \sqrt{7}}{4} \right\}, \quad E_\infty = \{1\}. \\ n = 2 : \quad E_A = E_{-A} &= 2, \quad E_\infty = \{4\}. \end{aligned} \tag{5.9}$$

Third Step

3.1 We start with the smallest n in L . We try to obtain elements $\mathbf{e} = (e_c)_{c \in \Gamma}$ in the Cartesian product $\prod_{c \in \Gamma} E_c$ such that:

- (i) $d(\mathbf{e}) := n - \frac{n}{h(n)} \sum_{c \in \Gamma} e_c$ is a non-negative integer
- (ii) If $n = 2$, then there is at least one odd number in \mathbf{e}

If no element \mathbf{e} is obtained, we select the next value of n in L .

$n = 1$:

The possible choices for $\mathbf{e} \in E_A \otimes E_{-A} \otimes E_\infty$ are $(\frac{2 \pm \sqrt{7}}{4}, \frac{2 \pm \sqrt{7}}{4}, 1)$. The possible values of $d(\mathbf{e})$ are $\{-1, -1 \pm \frac{\sqrt{7}}{2}\}$. Clearly $d(\mathbf{e})$ cannot be a non-negative integer. This fails criterion (i).

$n = 2$:

There is only one choice for $\mathbf{e} \in E_A \otimes E_{-A} \otimes E_\infty$, that is $(2, 2, 4)$. $d(\mathbf{e}) = -2$ and clearly we cannot construct any \mathbf{e} with an odd component. This fails both criteria (i) and (ii).

Unable to compute the polynomial $Q(v)$, the **Kovačič algorithm fails** at this stage. This indicates that Eqn.(5.4) is nonintegrable and its Galois group is $SL(2, \mathbf{C})$.

Chapter 6

Nonintegrability in Generic Confining Backgrounds

6.1 Closed spinning strings in generic supergravity backgrounds

The Polyakov action and the Virasoro constraints characterizing the classical motion of the fundamental string are:

$$\mathcal{L} = -\frac{1}{2\pi\alpha'} \sqrt{-g} g^{ab} G_{MN} \partial_a X^M \partial_b X^N, \quad (6.1)$$

where G_{MN} is the spacetime metric of the fixed background, X^μ are the coordinates of the string, g_{ab} is the worldsheet metric, the indices a, b represent the coordinates on the worldsheet of the string which we denote as (τ, σ) . We will use to work in the conformal gauge in which case the Virasoro constraints are

$$\begin{aligned} 0 &= G_{MN} \dot{X}^M X'^N, \\ 0 &= G_{MN} \left(\dot{X}^M \dot{X}^N + X'^M X'^N \right), \end{aligned} \quad (6.2)$$

where dot and prime denote derivatives with respect to τ and σ respectively.

We are interested in the classical motion of the strings in background metrics G_{MN} that preserve Poincaré invariance in the coordinates (X^0, X^i) where the dual field theory lives:

$$ds^2 = a^2(r) dx_\mu dx^\mu + b^2(r) dr^2 + c^2(r) d\Omega_d^2. \quad (6.3)$$

Here $x^\mu = (t, x_1, x_2, x_3)$ and $d\Omega_d^2$ represents the metric on a d -dimensional sub-space that, can also have r -dependent coefficients. In the case of supergravity backgrounds in IIB, we have $d = 5$ but we leave it arbitrary to also accommodate backgrounds in 11-d supergravity in which case $d = 6$.

The relevant classical equations of motion for the string sigma model in this background are

$$\begin{aligned} \partial_a (a^2(r) \eta^{ab} \partial_b x^\mu) &= 0, \\ \partial_a (b^2(r) \eta^{ab} \partial_b r) &= \frac{1}{2} \partial_r (a^2(r)) \eta^{ab} \partial_a x_\mu \partial_b x^\mu + \frac{1}{2} \partial_r (b^2(r)) \eta^{ab} \partial_a r \partial_b r. \end{aligned} \quad (6.4)$$

They are supplemented by the Virasoro constraints. We will construct spinning strings by starting with the following Ansatz (**Ansatz I**). The world-sheet coordinate σ wraps around the r direction. We will also call this the σ -embedding.

$$\begin{aligned} x^0 &= e \tau, \\ x^1 &= f_1(\tau) g_1(\sigma), \quad x^2 = f_2(\tau) g_2(\sigma), \end{aligned}$$

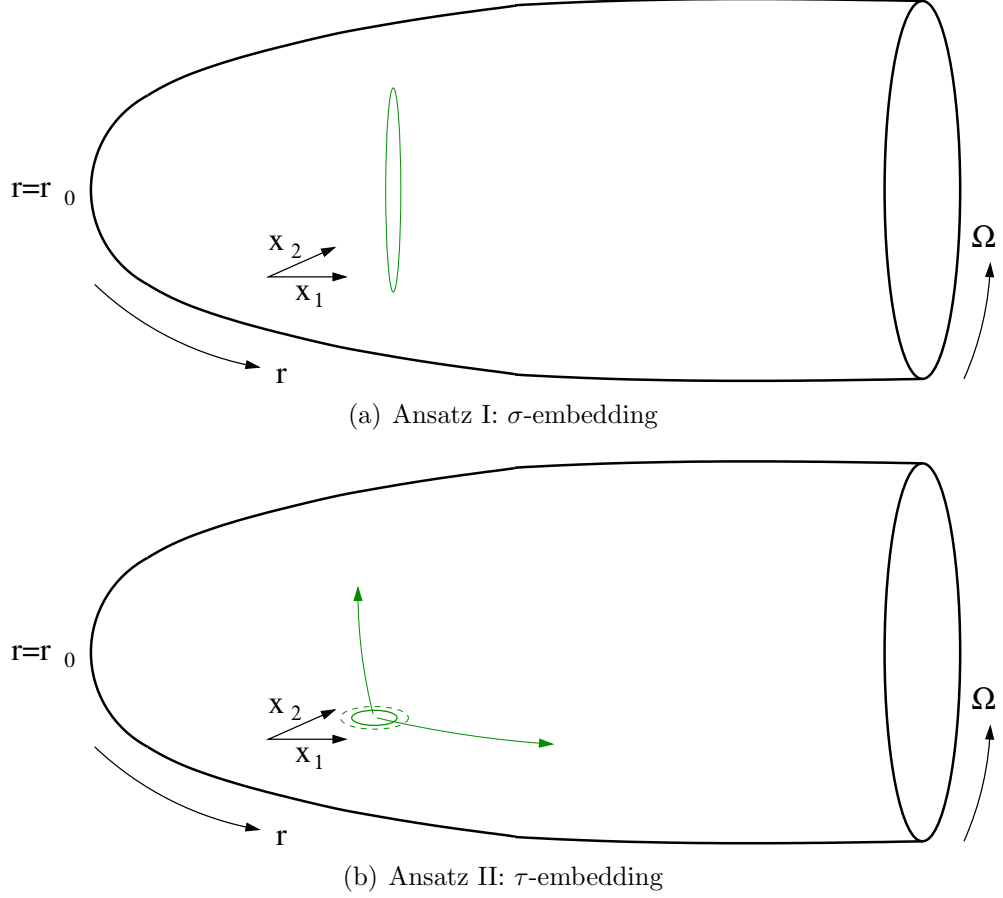


Figure 6.1: σ and τ embeddings of a string in a generic confining background given by Ansätze (6.5) and (6.6) respectively.

$$x^3 = \text{constant}, \quad r = r(\sigma). \quad (6.5)$$

We will also consider a slight modification of the above Ansatz as follows (**Ansatz II**). Here the string does not wrap around the r direction, but the r coordinate is a function of the world-sheet time τ . We will also call this the τ embedding.

$$\begin{aligned} x^0 &= e\tau, \\ x^1 &= f_1(\tau)g_1(\sigma), & x^2 &= f_2(\tau)g_2(\sigma), \\ x^3 &= \text{constant}, & r &= r(\tau). \end{aligned} \quad (6.6)$$

An illustration of the two embeddings for a confining geometry is given in Fig.(6.1).

6.1.1 Motion of the string with Ansatz I

With Ansatz I (6.5) the equation of motion for x^0 is trivially satisfied. Let us first show that the form of the functions f_i is fairly universal for this Ansatz. The equation of motion for x^i is

$$-a^2 g_i \ddot{f}_i + f_i \partial_\sigma (a^2 g'_i) = 0, \quad (6.7)$$

where a dot denotes a derivative with respect to τ and a prime denotes a derivative with respect to σ . Enforcing a natural separation of variables we see that

$$\ddot{f}_i + (e\omega)^2 f_i = 0, \quad \partial_\sigma(a^2 g'_i) + (e\omega)^2 a^2 g_i = 0. \quad (6.8)$$

The radial equation of motion is

$$(b^2 r')' = \frac{1}{2} \partial_r(a^2) [e^2 - g_i^2 \dot{f}_i^2 + f_i^2 g_i'^2] + \frac{1}{2} \partial_r(b^2) r'^2. \quad (6.9)$$

Finally the nontrivial Virasoro constraint becomes

$$b^2 r'^2 + a^2 [-e^2 + g_i^2 \dot{f}_i^2 + f_i^2 g_i'^2] = 0. \quad (6.10)$$

We are particularly interested in the integrals of motion describing the energy and the angular momentum

$$E = \frac{e}{2\pi\alpha'} \int a^2 d\sigma, \quad (6.11)$$

$$J = \frac{1}{2\pi\alpha'} \int a^2 [x_1 \partial_\tau x_2 - x_2 \partial_\tau x_1] d\sigma = \frac{1}{2\pi\alpha'} \int a^2 g_1 g_2 [f_1 \partial_\tau f_2 - f_2 \partial_\tau f_1] d\sigma \quad (6.12)$$

The above system can be greatly simplified by further taking the following particular solution:

$$f_1 = \cos e\omega \tau, \quad f_2 = \sin e\omega \tau, \quad \text{and} \quad g_1 = g_2 = g. \quad (6.13)$$

Under these assumptions the equation of motion for r and the Virasoro constraint become

$$(b^2 r')' - \frac{1}{2} \partial_r(a^2) [e^2 - (e\omega)^2 g^2 + g'^2] - \frac{1}{2} \partial_r(b^2) r'^2 = 0, \quad (6.14)$$

$$b^2 r'^2 + a^2 [-e^2 + (e\omega)^2 g^2 + g'^2] = 0. \quad (6.15)$$

The angular momentum is then

$$J = \frac{e\omega}{2\pi\alpha'} \int a^2 g^2 d\sigma. \quad (6.16)$$

6.1.2 Motion of the string with Ansatz II

In this subsection we consider the Ansatz given in equation (6.6). Note that the analysis given in the previous sections can be applied *mutatis mutandis* to this Ansatz. In particular, the separation of variables described in equation (6.8) can be performed in a symmetric way and one obtains:

$$g_i'' + \alpha^2 g_i = 0, \quad \partial_\tau(a^2 \partial_\tau f_i) + \alpha^2 a^2 f_i = 0. \quad (6.17)$$

The Ansatz given in (6.6, 6.13) becomes

$$\begin{aligned} t &= t(\tau), & r &= r(\tau), \\ x_1 &= R(\tau) \sin \alpha\sigma, & x_2 &= R(\tau) \cos \alpha\sigma. \end{aligned} \quad (6.18)$$

The Polyakov action is:

$$\mathcal{L} \propto a^2(r) [-\dot{t}^2 + \dot{R}^2 - \alpha^2 R^2] + b^2(r) \dot{r}^2. \quad (6.19)$$

The above Ansatz satisfies the first constraint automatically and the second constraint leads to a Hamiltonian constraint:

$$a^2(r) [\dot{t}^2 + \dot{R}^2 + \alpha^2 R^2] + b^2(r) \dot{r}^2 = 0. \quad (6.20)$$

We also have that

$$\dot{t} = E/a^2(r), \quad (6.21)$$

where E is an integration constant. This gives

$$\mathcal{L} \propto -\frac{E^2}{a^2(r)} + a^2(r) [\dot{R}^2 - \alpha^2 R^2] + b^2(r) \dot{r}^2. \quad (6.22)$$

From the above Lagrangian density the equations of motion for $r(\tau)$ and $R(\tau)$ are

$$\begin{aligned} \frac{d}{d\tau} \left(b^2(r) \frac{d}{d\tau} r(\tau) \right) &= \frac{E^2}{a^3(r)} \frac{d}{dr} a(r) + a(r) \frac{d}{dr} a(r) [\dot{R}^2 - \alpha^2 R^2] + b(r) \frac{d}{dr} b(r) \left(\frac{d}{d\tau} r \right)^2, \\ \frac{d}{d\tau} \left(a^2(r) \frac{d}{d\tau} R(\tau) \right) &= -\alpha^2 a^2(r) R(\tau). \end{aligned} \quad (6.23)$$

6.2 Confining backgrounds

Let us show that there exists a simple solution of the equations of motion (6.14) for any gravity background dual to a confining gauge theory. The conditions for a SUGRA background to be dual to a confining theory have been exhaustively explored [40, 41] using the fact that the corresponding Wilson loop in field theory should exhibit area law behavior. The main idea is to translate the condition for the vev of the rectangular Wilson loop to display an area law into properties that the metric of the supergravity background must satisfy through the identification of the vacuum expectation value of the Wilson loop with the value of the action of the corresponding classical string. It has been established that one set of necessary conditions is for g_{00} to have a nonzero minimum at some point r_0 usually known as the end of the space wall [40, 41]. Note that precisely these two conditions ensure the existence of a solution of (6.14). Namely, since $g_{00} = a^2$ we see that for a point $r = r_0 = \text{constant}$ is a solution if

$$\partial_r(g_{00})|_{r=r_0} = 0, \quad g_{00}|_{r=r_0} \neq 0. \quad (6.24)$$

The first condition solves the first equation in (6.14) and the second condition makes the second equation nontrivial. Interestingly, the second condition can be interpreted as enforcing that the quark-antiquark string tension be nonvanishing as it determines the value of the string action. It is worth mentioning that due to the UV/IR correspondence in the gauge/gravity duality the radial direction is identified with the energy scale. In particular, $r \approx r_0$ is the gravity dual of the IR in the gauge theory. Thus, the string we are considering spins in the region dual to the IR of the gauge theory. Therefore we can conclude that it is dual to states in the field theory that are characteristic of the IR.

6.2.1 Regge trajectories from closed spinning strings in confining backgrounds

Since we are working in Poincaré coordinates the quantity canonically conjugate to time is the energy of the corresponding state in the four dimensional theory. The angular momentum of the string describes the spin of the corresponding state. Thus a spinning string in the Poincaré coordinates is dual to a state of energy E and spin J . In order for our semiclassical approximation to be valid we need the value of the action to be large, this imply that we are considering gauge theory states in the IR region of the gauge theory with large spin and large energy. In the cases we study, expressions (6.11) and (6.16) yield a dispersion relation that can be identified with Regge trajectories.

Let us now explicitly display the Regge trajectories. The classical solution is given by (6.5) with $g(\sigma)$ solving the second equation from (6.14), that is, $g(\sigma) = (1/\omega) \sin(e\omega\sigma)$. Imposing the periodicity $\sigma \rightarrow \sigma + 2\pi$ implies that $e\omega = 1$ and hence

$$x^0 = e\tau, \quad x^1 = e \cos \tau \sin \sigma, \quad x^2 = e \sin \tau \sin \sigma. \quad (6.25)$$

The expressions for the energy and angular momentum of the string states are:

$$E = 4 \frac{e g_{00}(r_0)}{2\pi\alpha'} \int d\sigma = 2\pi g_{00}(r_0) T_s e, \quad J = 4 \frac{g_{00}(r_0) e^2}{2\pi\alpha'} \int \sin^2 \sigma d\sigma = \pi g_{00}(r_0) T_s e^2. \quad (6.26)$$

Defining the effective string tension as $T_{s, eff} = g_{00}(r_0)/(2\pi\alpha')$ and $\alpha'_{eff} = \alpha'/g_{00}$ we find that the Regge trajectories take the form

$$J = \frac{1}{4\pi T_{s, eff}} E^2 \equiv \frac{1}{2} \alpha'_{eff} t. \quad (6.27)$$

Notice that the main difference with respect to the result in flat space dating back to the hadronic models of the sixties is that the slope is modified to $\alpha'_{eff} = \alpha'/g_{00}$. It is expected that a confining background will have states that align themselves in Regge trajectories.

6.3 Analytic Nonintegrability in Confining Backgrounds

For confining backgrounds we have that the conditions on g_{00} described in (6.24) imply that:

$$a(r) \approx a_0 - a_2(r - r_0)^2, \quad (6.28)$$

where a_0 is the nonzero minimal value of $g_{00}(r_0)$ and the absence of a linear terms indicates that the first derivative at r_0 vanishes.

6.3.1 With Ansatz I

In this region is easy to show that both equations above can be satisfied. The equation for $r(\sigma)$ is satisfied by $r = r_0$ and $dr/d\sigma = 0$. The equation for $R(\sigma)$ is simply

$$\frac{d^2}{d\sigma^2} R(\sigma) + \omega^2 R(\sigma) = 0, \longrightarrow R(\sigma) = A \sin(\omega\sigma + \phi_0). \quad (6.29)$$

We can now write down the NVE equation by considering an expansion around the *straight line* solution, that is,

$$r = r_0 + \eta(\sigma). \quad (6.30)$$

We obtain

$$\eta'' - \frac{a_2 E^2}{2b_0^2 a_0^3} \left[1 - \frac{2\omega^2 A^2 a_0^4}{E^2} \cos 2\omega\sigma \right] \eta = 0. \quad (6.31)$$

The question of integrability of the system (6.23) has now turned into whether or not the NVE above can be solved in quadratures. The above equation can be easily recognized as the Mathieu equation. The analysis above has naturally appeared in the context of quantization of Regge trajectories and other classical string configurations. For example, [42, 43] derived precisely such equation in the study of quantum corrections to the Regge trajectories, those work went on to compute one-loop corrections in both, fermionic and bosonic sectors. Our goal here is different, for us the significance of (6.31) is as the Normal Variational Equation around the dynamical system (6.23) whose study will inform us about the integrability of the system. The general solution to the above equation is

$$\eta(\sigma) = c_1 C\left(-\frac{\alpha}{\omega^2}, -\frac{\alpha\beta}{2\omega^2}, \omega\sigma\right) + c_2 S\left(-\frac{\alpha}{\omega^2}, -\frac{\alpha\beta}{2\omega^2}, \omega\sigma\right), \quad (6.32)$$

where c_1 and c_2 are constants and

$$\alpha = \frac{a_2 E^2}{2b_0^2 a_0^3}, \quad \beta = \frac{2\omega^2 A^2 a_0^4}{E^2}. \quad (6.33)$$

Notice, crucially, that although the system obtain here is similar to the one discussed in the main text there is a key difference. Namely, that the effective “time” variable σ is now periodic. This periodicity precludes us from talking about asymptotic properties which lies at the heart of chaotic behavior. Most indicators of chaos, the largest Lyapunov exponent prominently, are based on the late time asymptotics of the system.

6.3.2 With Ansatz II

In this region is easy to show that both equations in (6.23) can be satisfied. The equation for $r(\tau)$ is satisfied by $r = r_0$ and $dr/d\tau = 0$. The straight line equation for $R(\tau)$ is simply

$$\frac{d^2}{d\tau^2} R(\tau) + \alpha^2 R(\tau) = 0, \longrightarrow R(\tau) = A \sin(\alpha\tau + \phi_0). \quad (6.34)$$

We can now write down the NVE equation by considering an expansion around the *straight line* solution, that is,

$$r = r_0 + \eta(\tau). \quad (6.35)$$

We obtain

$$\ddot{\eta} + \frac{a_2 E^2}{2b_0^2 a_0^3} \left[1 + \frac{2\alpha^2 A^2 a_0^4}{E^2} \cos 2\alpha\tau \right] \eta = 0. \quad (6.36)$$

The question of integrability of the system (6.23) has now turned into whether or not the NVE above can be solved in quadratures. The above equation can be easily recognized as the Mathieu equation. The analysis above has naturally appeared in the context of quantization of Regge trajectories and other classical string configurations. For example, [42, 43] derived precisely such equation in the study of quantum corrections to the Regge trajectories, those work went on to compute one-loop corrections in both, fermionic and bosonic sectors. Our goal here is different, for us the significance of (6.36) is as the Normal Variational Equation around the dynamical system (6.23) whose study will inform us about the integrability of the system.

The solution to the above equation (6.36) in terms of Mathieu functions is

$$\eta(\tau) = c_1 C\left(\frac{\theta}{\alpha^2}, \frac{\theta\beta}{2\alpha^2}, \alpha\tau\right) + c_2 S\left(\frac{\theta}{\alpha^2}, \frac{\theta\beta}{2\alpha^2}, \alpha\tau\right), \quad (6.37)$$

where c_1 and c_2 are constants and

$$\theta = \frac{a_2 E^2}{2b_0^2 a_0^3}, \quad \beta = \frac{2\alpha^2 A^2 a_0^4}{E^2}. \quad (6.38)$$

A beautiful description of a similar situation is presented in [44] where non-integrability of some Hamiltonians with rational potentials is discussed. In particular, the extended Mathieu equation is considered as an NVE equation

$$\ddot{y} = (a + b \sin t + c \cos t)y. \quad (6.39)$$

Our equation 6.36 is of this form with $2\alpha\tau \rightarrow t$ and $b = 0$. To aid the mathematically minded reader, and to make connection with our introduction to non-integrability in the beginning of section 5.1, we show that the extended Mathieu equation can be brought to an algebraic form using $x = e^{it}$ which leads to:

$$y'' + \frac{1}{x}y' + \frac{(b+c)x^2 + 2ax + c - b}{2x^3}y = 0. \quad (6.40)$$

The above equation is perfectly amenable to the application of Kovacic's algorithm. It was shown explicitly in [44] that our case ($b \neq -c$ above) corresponds to a non-integrable equation. More precisely, the Galois group is the connected component of $SL(2, \mathbb{C})$ and the identity component of the Galois group for (6.39) is exactly $SL(2, \mathbb{C})$, which is a non-Abelian group.

6.4 Explicit Chaotic Behavior in the MN Background

To logically close the circle we should also show explicit chaotic behavior of string motion in confining backgrounds. In this section, we choose the Maldacena-Núñez background which belongs to the class of confining backgrounds we are interested in. String motion in the AdS soliton background, which also belongs to this class, has already been demonstrated to be chaotic in Chapter 4.

The expression for the functions a and b in the main dynamical system (6.23) can be read directly from the MN background (C.10) (see Appendix C for details of the background).

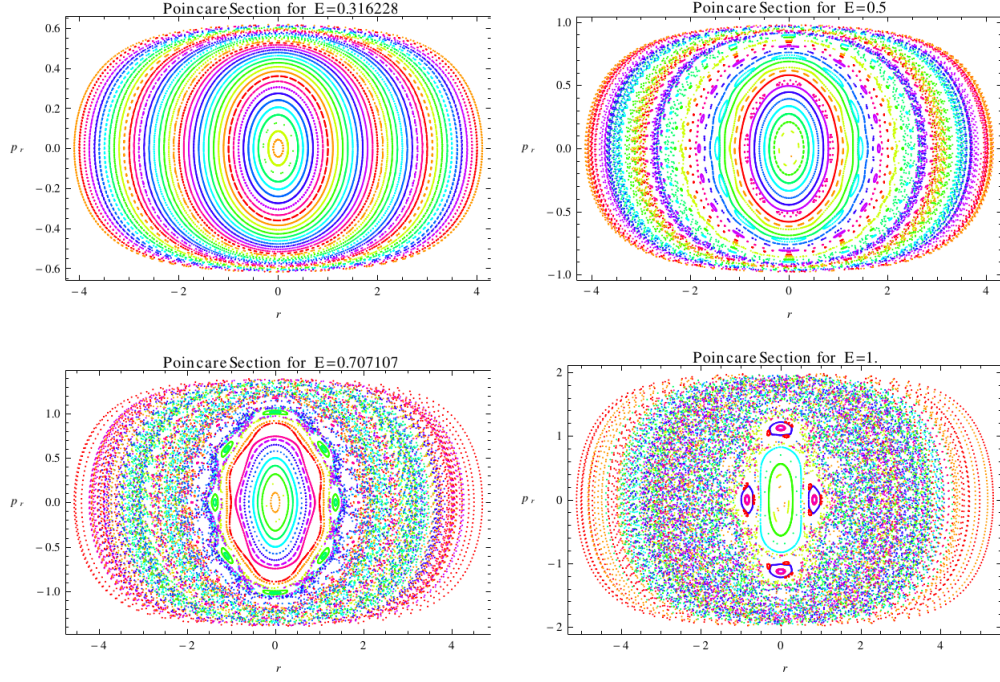


Figure 6.2: Poincaré sections for the Maldacena-Núñez background demonstrating breaking of the KAM tori en route to chaos.

$$a(r)^2 = e^{-\phi_0} \frac{\sqrt{\sinh(2r)/2}}{(r \coth 2r - \frac{r^2}{\sinh^2(2r)} - \frac{1}{4})^{1/4}}, \quad b(r)^2 = \alpha' g_s N a(r)^2. \quad (6.41)$$

It is crucial that

$$\lim_{r \rightarrow 0} a(r)^2 \rightarrow e^{-\phi_0}, \quad (6.42)$$

which is a nonzero constant that determines the tension of the confining string. The equations of motion can be obtained by plugging the above expressions in Eqn.(6.23).

Following an analysis very similar to Section 4.3, we obtain the Poincaré sections [Fig. 6.2], which show a KAM behavior of transition to chaos. A Lyapunov calculated in the chaotic regime [Fig. 6.3] is also found to converge to a positive value.

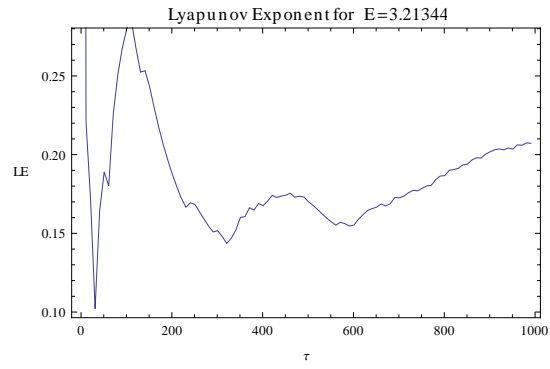


Figure 6.3: The Lyapunov Exponent has a positive value of about 0.2.

Chapter 7

Quantum Chaos in String Theory

In this chapter we look at the possible signatures of quantum chaos in confining backgrounds coming from string theory. In Section 7.1, we give an introduction to quantum chaos, with a special emphasis on observed signatures on quantum chaos in hadron spectra (§7.1.1). In Section 7.2 we look at the quantum level spacing distribution in the AdS soliton background and find signatures of quantum chaos in the distribution.

7.1 What is Quantum Chaos?

Quantum chaos is the study of quantum properties of systems whose classical limit is chaotic. For good books on the subject refer to [45, 46]. An interesting property to study turns out to be the distribution of the spacing of adjacent energy levels of the system. For a quantum system, whose Hamiltonian is classically integrable, it was shown by Berry and Tabor (1977) [47] that this level-spacing distribution is same as that for a sequence of uncorrelated levels, i.e. quite universally a Poisson distribution:

$$P(s) \simeq \exp(-s). \quad (7.1)$$

The next question to ask is whether there is any such universal distribution for systems which are classically chaotic. The models most frequently studied in this context are Billiard systems of Sinai, which also come up frequently while talking about classically chaotic systems. It was shown by Berry (1981) [48] that in the quantum spectrum of these systems small differences of eigenvalues are avoided (*level repulsion*, cf. Fig.7.2). It was further demonstrated by Bohigas, Giannoni and Schmit (1984) [49] that the level spacing distribution of eigenvalues calculated numerically perfectly agree with that of a Gaussian orthogonal ensemble (GOE) of random matrices, first discussed by Wigner (1958) [50].

$$P(s) \simeq \frac{\pi s}{2} \exp\left(-\frac{\pi s^2}{4}\right). \quad (7.2)$$

As seen from Figure (7.1), the principal qualitative difference between the level spacing distributions for the integrable and chaotic systems is that $P(s)$ goes to zero as $s \rightarrow 0$ in the Wigner GOE distribution for the chaotic cases, but it has a maximum at $s = 0$ in the Poisson distribution for the integrable case. This is an indication of the phenomenon of *level repulsion* for nonintegrable systems [Figure (7.2)]. Since degeneracies are avoided, a system parameter is dialed, the levels tend to repel. Thus there are only a small number of eigenvalues with small s : $P(0) \approx 0$ for the nonintegrable case. For the integrable case, the eigenvalues coming from the different separable sectors are independent of each other. Hence there is no such repulsion.

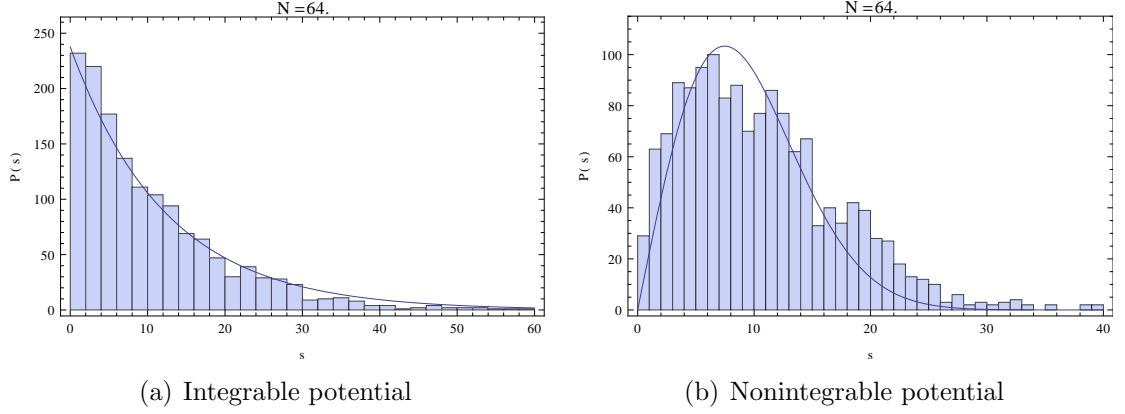


Figure 7.1: Distribution of level spacing for an integrable and a nonintegrable potential demonstrating Poisson and Wigner GOE distribution respectively. A box of dimensions $\sqrt[4]{e} \times 1$ and in the nonintegrable case, deformed by a potential $V(x, y) = \exp(\alpha(x - y - 1))$ with $\alpha = 138.2$. Eigenvalues have been calculated using pseudospectral method on a 64×64 Tchebychef grid.

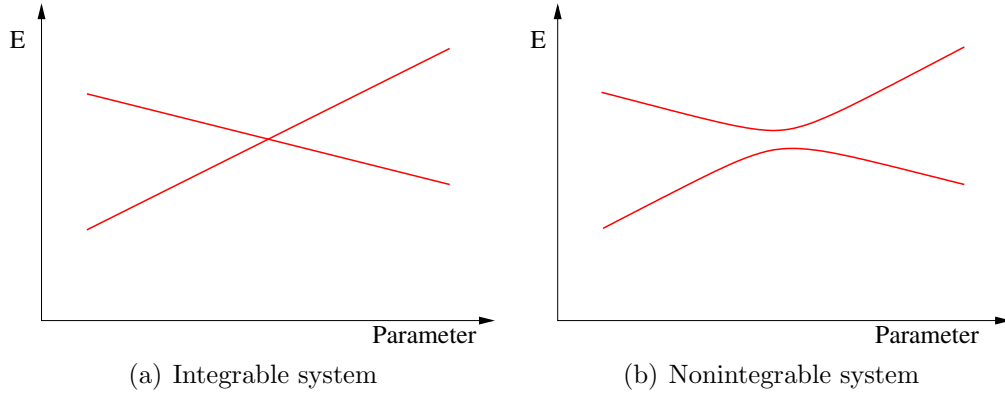


Figure 7.2: Level repulsion in a nonintegrable quantum system. In an integrable system the eigenvalues can cross and have degeneracies. In a nonintegrable system the eigenvalues repel. Hence small differences between eigenvalues are avoided.

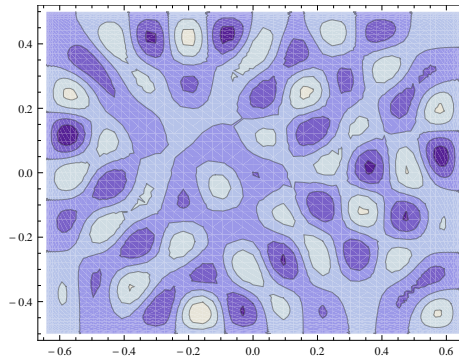


Figure 7.3: Contour plot of the wavefunction of a quantum chaotic system.

7.1.1 Quantum chaos in the Hadronic Spectrum

Signatures of chaos in the real hadron spectrum was discovered following Wigner's surmise. Experimental groups at Columbia and TUNL gathered data that permitted Haq, Pandey and Bohigan (1982) [51] to establish a statistically significant agreement between the measured fluctuations and Wigner's random matrix model. The data consisted of 1407 resonance energies corresponding to 30 sequences of 27 different nuclei. The Hydrogen atom in a magnetic field was also established to be quantum chaotic [52, 53]. The subject is thoroughly reviewed in [54, 55, 56, 57, 58]. Supersymmetric QCD at finite density [59] shows an even more exotic feature. While at zero chemical potential μ , the Wigner GOE distribution is obtained, for finite μ the level-spacing distribution agrees with the spacing of eigenvalues of non-Hermitian ensembles like the Ginibre ensemble [60].

7.2 Quantum Chaos in the AdS soliton background

Recall:

$$\mathcal{H} = \frac{1}{2} \left\{ \left(-E^2 + \frac{k^2}{T(x)} + p_R^2 \right) e^{-2ax^2} + \frac{T(x)}{d^2 a^2 x^2} p_x^2 + \alpha^2 R^2 e^{2ax^2} \right\} \quad (7.3)$$

We need to find the eigenvalues of \mathcal{E} using,

$$\mathcal{E}^2 \psi(x, R) = \frac{k^2}{T(x)} \psi(x, R) - \partial_R^2 \psi(x, R) - f(x) \partial_x^2 \psi(x, R) + V(x, R) \psi(x, R), \quad (7.4)$$

where,

$$f(x) = \frac{T(x)}{d^2 a^2 x^2}, \quad V(x, R) = \alpha^2 R^2 e^{2ax^2}. \quad (7.5)$$

With a transformation $dy = dx/\sqrt{f(x)}$, Eqn.(7.4) can be mapped to a canonical form,

$$\mathcal{E}^2 \tilde{\psi}(y, R) = \frac{k^2}{\tilde{T}(y)} - \partial_R^2 \tilde{\psi}(y, R) - \partial_y^2 \tilde{\psi}(y, R) + \tilde{V}(y, R) \tilde{\psi}(y, R). \quad (7.6)$$

Here,

$$\begin{aligned} \tilde{T}(y) &= T(x(y)), \quad \tilde{V}(y, R) = V(x(y), R) + \frac{1}{2} f''(x(y)) \\ \text{and} \quad \tilde{\psi}(y, R) &= f(x(y)) \psi(x(y), R). \end{aligned} \quad (7.7)$$

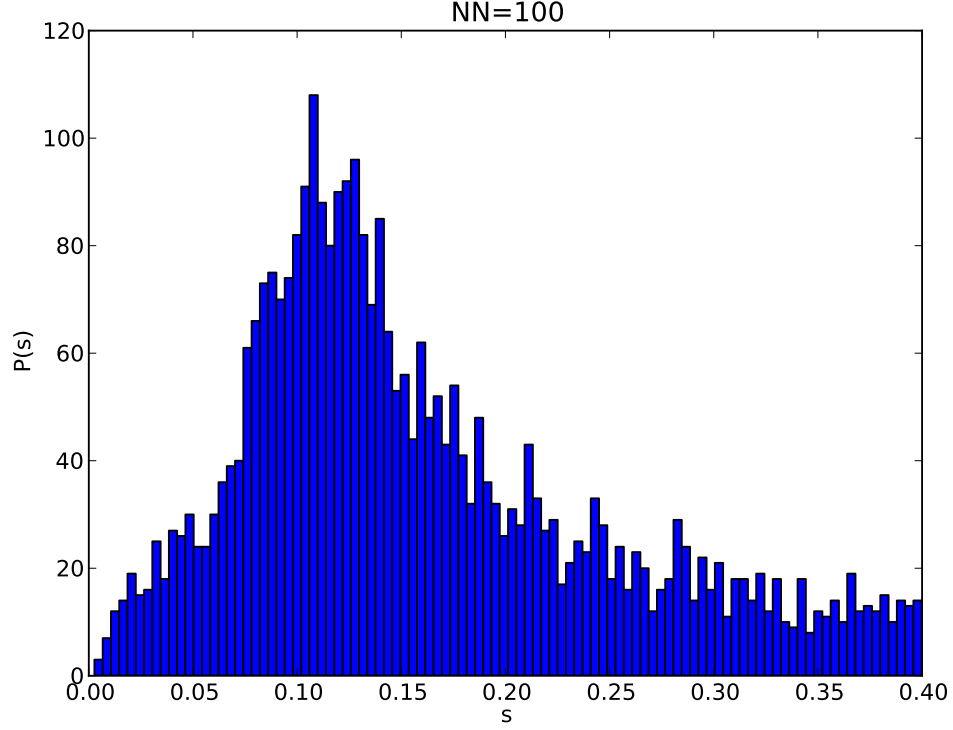


Figure 7.4: The distribution of level-spacings in the AdS soliton shows a strong level-repulsion and qualitative features of the Wigner distribution, indicating quantum chaos in this background. The eigenvalues of the Hamiltonian (7.4) with $d = 4$, $a = 1$, $\alpha = 1$ have been numerically obtained using Fourier spectral method on a 80×80 grid between $-12 < R < 12$ and $0.047 < y < 1.180$.

We calculate the eigenvalues of the Hamiltonian (7.4) and plot the histogram of the difference of nearest eigenvalues [Fig.(7.4)]. The distribution shows a strong level repulsion and is quite similar to the Wigner distribution, indicating of quantum chaos in the AdS soliton background.

Appendix A

Singularity Resolution in Cosmologies Using AdS/CFT

As an application of the AdS/CFT correspondence, in this appendix we review an example in which a dual gauge theory helps better understand the gravitational dynamics of a cosmological model.

A.1 Motivation and Overview

The theory of general relativity has a unique property of predicting its own failure – it breaks down at places called *singularities*. Our usual notions of spacetime break down near these singularities. At spacelike or null singularities, like those in the interiors of neutral black holes, “time” begins or ends, the meaning of which is not clear. Particularly disturbing are cosmological singularities like the big bang or the big crunch, which no observer can bypass.

Near these singularities, where the curvatures have become large compared to some scale, a quantum theory of gravity is expected to take over.

Holographic correspondences like AdS/CFT arising from string theory provide a nonperturbative formulation of quantum gravity. Here gravity in the bulk is an effective description of a non-gravitational field theory in a lower number of dimensions. We ask the question whether the dual description following these correspondences can give us some insight as to what happens near or past these singularities. In these models the spectrum at late times can be interpreted as the primordial fluctuations that lead to structure formation in the early universe. An incomplete but long list of attempts made in this direction is given in [61, 62, 63, 64, 65, 66, 67, 68, 69, 70]. For reviews refer to [71, 72]. The bulk of this article is based on the work done in [73].

A.2 The AdS/CFT Correspondence

The simplest example of the correspondence relevant to our discussion. We work in global coordinates in which the AdS_5 background has the metric [Eqn.(2.1)]:

$$ds^2 = -(1 + r^2)dt^2 + \frac{dr^2}{1 + r^2} + r^2 d\Omega_3^2 + d\Omega_5^2 \quad (A.1)$$

Here we are working with units where $R_{AdS} = 1$. Recall the AdS/CFT relations [Eqn.(2.4)]

$$g_{YM}^2 = 4\pi g_s, \quad \lambda \equiv g_{YM}^2 N = \frac{R_{AdS}^4}{l_s^4}. \quad (A.2)$$

In the large N limit ($N \rightarrow \infty$, $g_{YM} \rightarrow 0$, $\lambda \equiv g_{YM}^2 N = \text{finite}$), the dual description is a classical string theory with the quantum corrections suppressed by $1/N$. Moreover, the above relations tell us that the large 't Hooft coupling limit ($\lambda \rightarrow \infty$), corresponds

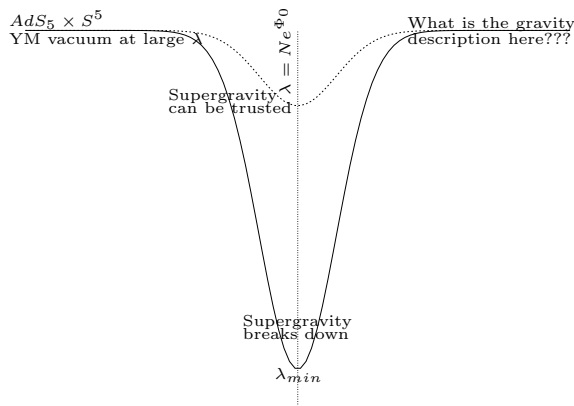


Figure A.1: A slowly-varying dilaton profile.

to the low-energy supergravity truncation of the string theory where the stringy modes are suppressed by α' . This is because, the massive string modes, which have masses of $O(1/\sqrt{\alpha'})$, are heavy compared to the energy scale of the theory $1/R_{AdS}$, and are not excited. So we are left only with the massless supergravity modes.

A.3 Setup in Supergravity

We consider a Yang-Mills theory with a time-dependent 't Hooft coupling $\lambda(t)$. In the dual bulk (via AdS/CFT), this corresponds to a non-normalizable dilaton mode whose boundary value $\Phi_0(t)$ is being dialled. The dilaton evolves following the supergravity equations of motion and produces a non-trivial metric by backreaction. The equations to solve are thus the Einstein-dilaton equations:

$$R_{ab} = \Lambda g_{ab} + \frac{1}{2} \partial_a \Phi \partial_b \Phi, \quad (\text{with } \Lambda = -4), \quad \nabla^2 \Phi = 0. \quad (\text{A.3})$$

We choose a profile such that the 't Hooft coupling λ is large at early and late times. The dual theory thus has a good gravity description at these times. We choose the initial state to be the vacuum of the gauge theory, which in the bulk corresponds to pure $AdS_5 \times S^5$. At intermediate times, λ is allowed to become $O(1)$ – curvatures in the bulk reach the string scale at this stage and supergravity description breaks down. We fix N to remain large – the curvatures do not reach the Planck scale and the system remains classical. Thus the singularities that we describe are rather restrictive.

Furthermore, we vary the 't Hooft coupling, or equivalently the boundary value of the dilaton *slowly*. The curvature scale of the AdS or equivalently the radius of S^3 on which the field theory is defined sets a scale in the theory. We can quantify this as

$$\Phi_0 = f(\epsilon t), \quad \epsilon \ll 1 \quad \dot{\Phi}_0 = \epsilon f'(\epsilon t), \quad (\text{A.4})$$

where each time derivative comes with an additional power of ϵ .

Although the change is slow, the total change in the value of the dilaton accrues over time and we can eventually have a large change in the value of the dilaton. Our solution therefore includes the case where the value of the dilaton becomes small, the bulk curvatures become large compared to the string scale and supergravity approximation in the bulk breaks down.

A.3.1 Supergravity Solution

We expand the dilaton and the bulk metric order by order in ϵ

$$\begin{aligned}\Phi(r, t) &= \Phi_0(t) + \epsilon \Phi_1(r, t) + \epsilon^2 \Phi_2(r, t) + \dots, \\ g_{ab} &= g_{ab}^{(0)} + \epsilon g_{ab}^{(1)} + \epsilon^2 g_{ab}^{(2)} + \dots,\end{aligned}\tag{A.5}$$

and plug them in the Einstein equations. Solving to the lowest non-trivial order in ϵ , we get

$$\begin{aligned}\Phi_2(r, t) &= \Phi_0(t) + \frac{1}{4} \ddot{\Phi}_0(t) \left[\frac{1}{r^2} \log(1 + r^2) - \frac{1}{2} (\log(1 + r^2))^2 - \text{dilog}(1 + r^2) - \frac{\pi^2}{6} \right], \\ g_{tt} &= 1 + r^2 - \frac{1}{12} \dot{\Phi}_0^2 \left[3 - \frac{1}{r^2} \ln(1 + r^2) \right], \quad g^{rr} = 1 + r^2 - \frac{1}{12} \dot{\Phi}_0^2 \left[1 - \frac{1}{r^2} \ln(1 + r^2) \right].\end{aligned}\tag{A.6}$$

The solution is smooth everywhere – there are no horizons and no black holes are formed. This is a consequence of the slow variation. For a variation that is fast enough we can form black holes even within the supergravity approximation [74].

We will now consider the situation where the value of the dilaton has become small at intermediate times and supergravity approximation has broken down. We turn to the gauge theory to see if it can give us any clue as to what happens in the stringy regime and what happens afterwards when supergravity is restored.

A.4 Field Theory Analysis

The boundary gauge theory is a quantum mechanical system with a time dependent parameter – the coupling – that is *slowly varying*. For the field theory on S^3 , the spectrum has a gap of order $1/R_{AdS}$ for all values of 't Hooft coupling.¹ So we can use the *adiabatic approximation* in quantum mechanics to study the system. The standard adiabatic approximation of quantum mechanics will turn out to be valid only if $N\epsilon \ll 1$, which is rather stringent. We will formulate an alternate *adiabatic approximation in terms of coherent states*.

A.4.1 Standard Adiabatic Approximation

Consider a quantum mechanical system with Hamiltonian $H(\zeta(t))$ that depends on a slowly-varying parameter $\zeta(t)$. In our case $\zeta(t)$ is the time dependent 't Hooft

¹This follows from the fact that the states provide a unitary representation of the conformal group for any value of the coupling.

coupling $\lambda(t)$ or equivalently the boundary value of the dilation field $\Phi_0(t)$. The instantaneous eigenstates of the Hamiltonian are

$$H(\zeta)|\phi_m(\zeta)\rangle = E_m(\zeta)|\phi_m(\zeta)\rangle. \quad (\text{A.7})$$

The evolution of the ground state $|\phi_0\rangle$ of $H(\zeta_0)$ in the adiabatic approximation is simply given by

$$|\psi_0(t)\rangle \simeq |\phi_0(\zeta)\rangle e^{-i \int_{-\infty}^t E_0(\zeta) dt}, \quad (\text{A.8})$$

with first corrections of the form

$$\begin{aligned} |\psi^1(t)\rangle &= \sum_{n \neq 0} a_n(t) |\phi_n(\zeta)\rangle e^{-i \int_{-\infty}^t E_n dt}, \\ a_n(t) &= - \int_{-\infty}^t dt' \frac{\langle \phi_n(\zeta) | \frac{\partial H}{\partial \zeta} | \phi_0(\zeta) \rangle}{E_0 - E_n} \dot{\zeta} e^{-i \int_{-\infty}^{t'} (E_0 - E_n) dt'} \end{aligned} \quad (\text{A.9})$$

The approximation is good if the first corrections are small:

$$|\langle \phi_n | \frac{\partial H}{\partial \zeta} | \phi_0 \rangle \dot{\zeta}| \ll (\Delta E)^2 \quad (\text{A.10})$$

where ΔE is the energy difference between the ground and the first excited states. In our case, $\dot{\zeta}(t) = \dot{\Phi}_0(t) \sim O(\epsilon)$ and using *AdS/CFT*²,

$$\langle \phi_n | \frac{\partial H}{\partial \Phi_0} | \phi_0 \rangle = - \langle \phi_n | \hat{\mathcal{O}}_{l=0} | \phi_0 \rangle \sim N. \quad (\text{A.11})$$

Thus, the adiabatic approximation is valid if $N\epsilon \ll 1$. This condition is however more stringent than the condition $\epsilon \ll 1$ that we had in the supergravity analysis.

A.4.2 Adiabatic Approximation with Coherent States

A general coherent state has the form

$$|\Psi(t)\rangle = \exp \left[\sum_n \lambda_n(t) a_n^\dagger \right] |0\rangle, \quad (\text{A.12})$$

where a_n^\dagger are the creation operators and $\lambda_n(t)$ are the coherent state parameters. For large values of λ_n , the system goes over to a classical configuration. In the large 't Hooft coupling regime they have a good description in terms of local fields. In our problem the creation and annihilation operators need to be replaced by the creation and annihilation parts $\hat{\mathcal{O}}_{(\pm)}$ of $\hat{\mathcal{O}}$. The evolution of $\lambda_n(t)$ is determined by the algebra of the operators $\hat{\mathcal{O}}_{(\pm)}$ and the Schrödinger equation. In general the operators $\hat{\mathcal{O}}_{(\pm)}$ have a non-trivial operator algebra which mixes all of them and makes the evolution of $\lambda_n(t)$ and quite complicated. However, the situation drastically simplifies for a slow

²Turning on a field Φ with a boundary value Φ_0 corresponds the deformation of the gauge theory action by a gauge invariant operator $\hat{\mathcal{O}} = \frac{\delta \mathcal{S}_{\text{Cl}}}{\delta \Phi_0}$ where \mathcal{S}_{Cl} is the classical supergravity action.

variation. The expectation value of the three-point function taken in a coherent state can be shown to be $\langle \psi | \hat{\mathcal{O}}_1 \hat{\mathcal{O}}_2 \hat{\mathcal{O}}_3 | \psi \rangle \sim \epsilon$. For $\epsilon \ll 1$, the operators $\hat{\mathcal{O}}_{(\pm)}$ *decouple*. The three and higher point functions are thus parametrically suppressed and all we are left with are the two point functions. Their algebra essentially reduces to that of a free harmonic oscillator.

$$\hat{\mathcal{O}}_{l=0} = N \sum_{n=1}^{\infty} F(2n) [A_{2n} e^{-i2nt} + A_{2n}^{\dagger} e^{i2nt}] \quad \text{with} \quad [A_{2m}, A_{2n}^{\dagger}] = \delta_{m,n'} \quad (\text{A.13})$$

We can now construct coherent states $|\psi\rangle = \hat{N}(t) e^{(\sum_n \lambda_n A_{2n}^{\dagger})} |\phi_0\rangle$ and find the Schrödinger evolution of the coherent state parameter.

$$\lambda_n(t) = \frac{e^{-i2nt} N F(2n)}{(2n)} \int_{-\infty}^t \dot{\Phi}_0(t') e^{i2nt'} dt' \quad (\text{A.14})$$

$$= -\frac{i N F(2n) \dot{\Phi}_0(t)}{(2n)^2} + \frac{i e^{-i2nt} N F(2n)}{(2n)^2} \int_{-\infty}^t dt' \ddot{\Phi}_0(t') e^{i2nt'} \quad (\text{A.15})$$

$$= -\frac{i N F(2n) \dot{\Phi}_0(t)}{(2n)^2} + \frac{N F(2n) \ddot{\Phi}_0(t)}{(2n)^3} + \dots \quad (\text{A.16})$$

Subsequent integration by parts leads to a perturbative expansion for λ_n . The above expansion is valid if the second and subsequent terms are small compared to the first term i.e. $|\frac{\ddot{\Phi}_0}{2n\dot{\Phi}_0}| \ll 1 \forall n$. This is identical to the condition $\epsilon \ll 1$. Moreover the state represents a “classical” deformation only when the coherent state parameter λ is large. That holds when $N\epsilon \gg 1$.

A.4.3 Results

For small 't Hooft coupling, the coupling between the oscillators are still individually suppressed by ϵ and most of the above analysis goes through. However now we have $O(N^2)$ *stringy modes* whose frequencies are comparable to those of the supergravity modes. *So there is a possibility of thermalization.* The energy that has been pumped in might get distributed between the various modes. Then at late times one could be left with $O(N^2 \epsilon^2)$ thermalized energy in the system. It is difficult to say whether thermalization would indeed occur since the time scale of variation is same as the thermalization time scale.

However we can assume that all the energy left behind in the system has been thermalized ($E_{\text{Thermal}} \sim N^2 \epsilon^2$) and from entropic considerations in supergravity figure out the worst case scenarios for various values of ϵ . For $\epsilon \ll \lambda^{5/4} N^{-1}$, a gas of *supergravity modes* is favoured. For $\epsilon \gtrsim \lambda^{5/4} N^{-1}$ we can have a gas of *massive string modes*. For $\epsilon \gtrsim \lambda^{-7/8}$ a *small black hole* can form. In order to form a *large black hole* one would require $\epsilon \gtrsim 1$. So a large black hole is never formed in our slowly varying solution.

A.5 Conclusions and Outlook

We have an example of a situation in which the gauge theory has a smooth time-evolution across a region whose dual bulk region has a singularity. An explicit calculation of the final state is possible in principle, but made difficult in practice because we do not have a complete *AdS/CFT* dictionary in the stringy regime. Of the possible outcomes, a thermal distribution at late times would imply a gaussian spectrum while formation of a black hole would source a nongaussianity.

Appendix B

$T^{p,q}$ and $Y^{p,q}$ Backgrounds

B.1 Wrapped strings in general $AdS_5 \times X^5$

The methods of analytic non-integrability can be applied to a large class of spaces in string theory. Let us start by considering a five-dimensional Einstein space X_5 , with $R_{ij} = \lambda g_{ij}$, where λ is a constant. Any such Einstein space furnishes a solution to the type IIB supergravity equations known as a Freund-Rubin compactification [75]. Namely, the solution takes the form

$$ds^2 = ds^2(AdS_5) + ds^2(X_5), \quad F_5 = (1 + \star)\text{vol}(AdS_5), \quad (\text{B.1})$$

where vol is the volume five-form and \star is the Hodge dual operator. Basically F_5 is a the sum of the volume forms on AdS_5 and the Einstein space X_5 . Of particular interest in string theory is the case when X_5 is Sasaki-Einstein, that is, on top of being Einstein it admits a spinor satisfying $\nabla_\mu \epsilon \sim \Gamma_\mu \epsilon$. The case of Sasaki-Einstein structure is particularly interesting from the string theory point of view as it preserves supersymmetry which is a mechanism that provides extra computational power.

We consider spaces X_5 that are a $U(1)$ fiber over a four-dimensional manifold, again, this is largely inspired by the Sasaki-Einstein class but clearly goes beyond that. In the case of topologically trivial fibration we are precluded from applying our argument, those manifolds can be considered separately and probably on a case by case basis.

The configuration that we are interested in exploring is a string sitting at the center of AdS_5 and winding in the *circles* provided by the base space. More explicitly, consider the AdS_5 metric in global coordinates

$$ds^2 = -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\mathcal{O}_3^2. \quad (\text{B.2})$$

Then, our solutions is localized at $\rho = 0$. As noted before, in the search for solutions of the form (B.1) a prominent place is taken by deformations of S^5 that preserve some amount of supersymmetry, they are given by Sasaki-Einstein spaces. The general structure of Sasaki-Einstein metrics is

$$ds_{X_{S^5-E}}^2 = (d\psi + \frac{i}{2}(K_{,i}dz^i - K_{,\bar{i}}d\bar{z}^{\bar{i}}))^2 + K_{,i\bar{j}}dz^i d\bar{z}^{\bar{j}}, \quad (\text{B.3})$$

where K is a Kähler potential on the complex base with coordinates z_i with $i = 1, 2$. This is the general structure that will serve as our guiding principle but we will not be limited to it. Roughly our Ansatz for the classical string configuration is

$$z_i = r_i(\tau)e^{i\alpha_i\sigma} \quad (\text{B.4})$$

where τ and σ are the worldsheet coordinates of the string. Note crucially we have introduced winding of the strings characterized by the constants α_i . The goal is to solved for the functions $r_i(\tau)$.

A summary from the previous section instructs us to:

- Select a particular solution, that is, define the *straight line solution*.
- Write the normal variational equation (NVE).
- Check if the identity component of the differential Galois group of the NVE is Abelian, that is, apply the Kovacic's algorithm to determine if the NVE is integrable by quadrature.

Given this Ansatz above we can now summarize the general results. Namely, in this section we prove that the corresponding effective Hamiltonian systems have two degrees of freedom and admit an invariant plane $\Gamma = \{r_2 = \dot{r}_2 = 0\}$ whose normal variational equation around integral curves in Γ we study explicitly.

B.1.1 $T^{p,q}$

These 5-manifolds are not necessarily Sasaki-Einstein, however, some of them are Einstein which allow for a consistent string backgrounds of the form described in equation (B.1). More importantly, some of these spaces provide exact conformal sigma models description [76] and are thus exact string backgrounds in all orders in α' . In this section we provide a unified treatment of this class for generic values of p and q . The metric has the form

$$ds^2 = a^2(d\psi + p \cos \theta_1 d\phi_1 + q \cos \theta_2 d\phi_2)^2 + b^2(d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2) + c^2(d\theta_2^2 + \sin^2 \theta_2 d\phi_2^2). \quad (\text{B.5})$$

The classical string configuration we are interested is

$$\begin{aligned} \theta_1 &= \theta_1(\tau), & \theta_2 &= \theta_2(\tau), & \psi &= \psi(\tau), & t &= t(\tau), \\ \phi_1 &= \alpha_1 \sigma, & \phi_2 &= \alpha_2 \sigma, \end{aligned} \quad (\text{B.6})$$

where α_i are constants quantifying how the string winds along the ϕ_i directions. Recall that t is from AdS_5 . The Polyakov Lagrangian is

$$\begin{aligned} \mathcal{L} &= -\frac{1}{2\pi\alpha'} \left[\dot{t}^2 - b^2 \dot{\theta}_1^2 - c^2 \dot{\theta}_2^2 - a^2 \dot{\psi}^2 \right. \\ &\quad \left. + \alpha_1^2 (b^2 - a^2 p^2) \sin^2 \theta_1 + \alpha_2^2 (c^2 - a^2 q^2) \sin^2 \theta_2 + 2\alpha_1 \alpha_2 p q a^2 \cos \theta_1 \cos \theta_2 \right] \quad (\text{B.7}) \end{aligned}$$

There are several conserved quantities, the corresponding nontrivial equations are

$$\begin{aligned} \ddot{\theta}_1 &+ \frac{\alpha_1}{b^2} \sin \theta_1 \left[\alpha_1 (b^2 - a^2 p^2) \cos \theta_1 - a^2 \alpha_2 p q \cos \theta_2 \right] = 0, \\ \ddot{\theta}_2 &+ \frac{\alpha_2}{c^2} \sin \theta_2 \left[\alpha_2 (c^2 - a^2 q^2) \cos \theta_2 - a^2 \alpha_1 p q \cos \theta_1 \right] = 0. \end{aligned} \quad (\text{B.8})$$

There is immediately some insight into the role of the fibration structure. Note that the topological winding in the space which is roughly described by p and q intertwines with the wrapping of the strings α_1 and α_2 . The effective number that appears in the interaction part of the equations is $\alpha_1 p$ and $\alpha_2 q$. For example, from the point of view of the interactions terms, taking $p = 0$ or $q = 0$ is equivalent to taking one of the $\alpha_i = 0$ which leads to an integrable system of two non-interacting gravitational pendula.

Following the structure of the discussion of section ??, we take the straight line solution to be

$$\theta_2 = \dot{\theta}_2 = 0. \quad (\text{B.9})$$

The equation for θ_1 becomes

$$\ddot{\theta}_1 + \frac{\alpha_1}{b^2} \left[\alpha_1 (b^2 - a^2 p^2) \cos \theta_1 - a^2 \alpha_2 p q \right] \sin \theta_1 = 0. \quad (\text{B.10})$$

Let us denote the solution to this equation $\bar{\theta}_1$, it can be given explicitly but we will not need the precise form. This solution also defines the Riemann surface Γ introduced in section ?. The NVE is obtained by considering small fluctuations in θ_2 around the above solutions and takes the form:

$$\ddot{\eta} + \frac{\alpha_2}{c^2} \left[\alpha_2 (c^2 - a^2 q^2) - \alpha_1 p q \cos \bar{\theta}_1 \right] \eta = 0. \quad (\text{B.11})$$

Our goal is to study the NVE. To make the equation amenable to the Kovacic's algorithm we introduce the following substitution

$$\cos(\bar{\theta}_1) = z. \quad (\text{B.12})$$

In this variable the NVE takes a form similar to Lamé equation (see section 2.8.4 of [25]),

$$f(z) \eta''(z) + \frac{1}{2} f'(z) \eta'(z) + \frac{\alpha_2}{c^2} \left[\alpha_2 (c^2 - a^2 q^2) - \alpha_1 p q z \right] \eta(z) = 0 \quad (\text{B.13})$$

where prime now denotes differentiation with respect to z .

$$f(z) = \dot{\theta}_1^2 \sin^2(\bar{\theta}_1) = \left(6E^2 - \frac{1}{3} (4\alpha_1 \alpha_2 z + \alpha_2^2 (1 - z^2)) \right) (1 - z^2) \quad (\text{B.14})$$

Equation (B.13) is a second order homogeneous linear differential equation with polynomial coefficients and it is, therefore, ready for the application of Kovacic's algorithm. For generic values of the parameters above the Kovacic's algorithm does not produce a solution meaning the system defined in equations (B.8) is not integrable.

B.1.2 NVE for $T^{1,1}$

It is worth taking a pause to discuss the case of $T^{1,1}$ explicitly. The NVE equation takes a simpler form:

$$\ddot{\eta} + \frac{1}{3} (\alpha_1^2 - 2\alpha_1 \alpha_2 \cos(\theta_1(t))) \eta = 0, \quad (\text{B.15})$$

where η , as above, is the variation in θ_2 . Substituting $\cos(y) = z$ this equation takes a form similar to Lamé equation

$$f(z)\eta''(z) + \frac{1}{2}f'(z)\eta'(z) + \frac{1}{3}(\alpha_1^2 - 2\alpha_1\alpha_2z)\eta(z) = 0 \quad (\text{B.16})$$

Similarly we can obtain an expression for the function $f(z)$ as

$$f(z) = \dot{y}^2 \sin^2(y) = \left(6E^2 - \frac{1}{3}(4\alpha_1\alpha_2z + \alpha_2^2(1 - z^2)) \right) (1 - z^2). \quad (\text{B.17})$$

Consequently, this system is also non-integrable.

The case of $T^{1,1}$ is particularly interesting because in the case the supergravity solution is supersymmetric and a lot of attention has been paid to extending configurations of $AdS_5 \times S^5$ to the case of $AdS_5 \times T^{1,1}$ [77, 78, 79, 80, 81, 82, 83].

B.1.3 $Y^{p,q}$

These spaces have played a central role in developments of the AdS/CFT correspondence as they provided an infinite class of dualities. These spaces are Sasaki-Einstein but they are not coset spaces as was the case for the $Y^{p,q}$ discussed above. Following the general discussion of Sasaki-Einstein spaces above, we write the metric on these spaces as

$$\frac{1}{R^2}ds^2 = \frac{1}{9}(d\psi - (1 - cy) \cos \theta d\phi + y d\beta)^2 + \frac{1 - cy}{6}(d\theta^2 + \sin^2 \theta d\phi^2) + \frac{p(y)}{6}(d\beta + c \cos \theta d\phi)^2. \quad (\text{B.18})$$

$$p(y) = \frac{a - 3y^2 + 2cy^3}{3(1 - cy)}. \quad (\text{B.19})$$

The classical string configuration is described by the Ansatz

$$\begin{aligned} \theta &= \theta(\tau), & y &= y(\tau), \\ \phi &= \alpha_1 \sigma, & \beta &= \alpha_2 \sigma, \end{aligned} \quad (\text{B.20})$$

The Polyakov Lagrangian is simply:

$$\begin{aligned} \mathcal{L} &= -\frac{1}{2\pi\alpha'} \left[\dot{t}^2 - \frac{1 - cy}{6} \dot{\theta}^2 - \frac{1}{6p(y)} \dot{y}^2 - \frac{1}{9} \dot{\psi}^2 \right. \\ &\quad \left. + \frac{1 - cy}{6} \alpha_1^2 \sin^2 \theta + \frac{p(y)}{6} (\alpha_2 + c \alpha_1 \cos \theta)^2 + \frac{1}{9} (\alpha_2 y - \alpha_1 (1 - cy) \cos \theta)^2 \right] \end{aligned} \quad (\text{B.21})$$

As in previous cases the equations of motion for t and ψ are integrated immediately leaving only two nontrivial equations for θ and y

$$\begin{aligned}\ddot{\theta} &= \frac{c}{1-cy}\dot{y}\dot{\theta} + \alpha_1 \left(\alpha_1 \cos \theta - \frac{cp(y)}{1-cy}(\alpha_2 + c\alpha_1 \cos \theta) \right. \\ &\quad \left. - \frac{2}{3}(\alpha_2 y - \alpha_1(1-cy) \cos \theta) \right) \sin \theta = 0. \\ \ddot{y} &= \frac{p'}{p}\dot{y}^2 + \frac{pp'}{2}(\alpha_2 + c\alpha_1 \cos \theta)^2 - \frac{cp}{2}\alpha_1^2 \sin^2 \theta \\ &\quad + \frac{2}{3}p(\alpha_2 + c\alpha_1 \cos \theta)(\alpha_2 y - \alpha_1(1-cy) \cos \theta) = 0.\end{aligned}\tag{B.22}$$

B.1.3.1 θ straight line

The straight line solution can be taken to be $\theta = \dot{\theta} = 0$. Then the equation for y is simplified to

$$\ddot{y} - \frac{p'}{p}\dot{y}^2 + \frac{pp'}{2}(\alpha_2 + c\alpha_1)^2 + \frac{2}{3}p(\alpha_2 + c\alpha_1)(y(\alpha_2 + c\alpha_1) - \alpha_1) = 0.\tag{B.23}$$

The Normal Variational Equation takes the form

$$\ddot{\eta} - \frac{c\dot{y}_s}{1-cy_s}\dot{\eta} + \alpha_1 \left(\alpha_1 - \frac{cp(y_s)}{1-cy_s}(\alpha_2 + c\alpha_1) - \frac{2}{3}((\alpha_2 + c\alpha_1)y_s - \alpha_1) \right) \eta = 0.\tag{B.24}$$

To be able to write it in a form conducive to the application of Kovacic's algorithm we substitute $y_s(t) = y$ and the NVE takes the form

$$\begin{aligned}(\ddot{\eta}(t)(1-cy) - c\dot{\eta}^2(t)) \frac{d\eta}{dy} + (1-cy)\dot{\eta}^2(t) \frac{d^2\eta}{dy^2} + q(y)n(y) &= 0 \tag{B.25} \\ \dot{\eta}^2(t) = 6(E + p(y)V(y, 0)) = 6p(y) \left(\frac{p(y)}{6}(\alpha_2 + c\alpha_1)^2 + \frac{1}{9}(\alpha_2 y - \alpha_1(1-cy)) \right) \\ \ddot{\eta}(t) &= 3 \frac{d}{dy}(p(y)V(y, 0)) \\ q(y) &= \alpha_1(1-cy) \left(5/3 \alpha_1 - \frac{c(a-3y^2+2cy^3)(\alpha_2+c\alpha_1)}{(3-3cy)(1-cy)} - 2/3 (\alpha_2 + c\alpha_1) y \right)\end{aligned}$$

With this identifications we have rewritten the NVE as a homogeneous second order linear differential equation. The Kovacic's algorithm again fails to yield a solution pointing to the fact that the system is generically non-integrable.

B.1.4 The exceptional case: S^5

In this section we provide an integrable example where the Kovacic's algorithm should succeed. To expose the Sasaki-Einstein structure of S^5 , it is convenient to write the metric as a $U(1)$ fiber over \mathbb{P}^2 . The round metrics on S^5 may be elegantly expressed in terms of the left-invariant one-forms of $SU(2)$. For $SU(2)$, the left-invariant one-forms can be written as,

$$\begin{aligned}
\sigma_1 &= \frac{1}{2}(\cos(d\psi)d\theta + \sin(\psi)\sin(\theta)d\phi), \\
\sigma_2 &= \frac{1}{2}(\sin(\psi)d\theta - \cos(\psi)\sin(\theta)d\phi), \\
\sigma_3 &= \frac{1}{2}(d\psi + \cos(\theta)d\phi).
\end{aligned} \tag{B.26}$$

In terms of these 1-forms, the metrics on \mathbb{P}^2 and S^5 may be written,

$$\begin{aligned}
ds_{\mathbb{P}^2}^2 &= d\mu^2 + \sin^2(\mu) (\sigma_1^2 + \sigma_2^2 + \cos^2(\mu)\sigma_3^2), \\
ds_{S^5}^2 &= ds_{\mathbb{P}^2}^2 + (d\chi + \sin^2(\mu)\sigma_3)^2
\end{aligned} \tag{B.27}$$

where χ is the local coordinate on the Hopf fibre and $A = \sin^2(\mu)\sigma_3 = \sin^2(\mu)(d\psi + \cos(\theta)d\phi)/2$ is the 1-form potential for the Kähler form on \mathbb{P}^2 [84].

The classical string configuration is

$$\begin{aligned}
\theta &= \theta(\tau), & \mu &= \mu(\tau), \chi = \chi(\tau), \\
\phi &= \alpha_1\sigma, & \psi &= \alpha_2\sigma,
\end{aligned} \tag{B.28}$$

The Lagrangian is

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{2\pi\alpha'} \left[\dot{t}^2 - \dot{\mu}^2 - \frac{1}{4} \sin^2 \mu \dot{\theta}^2 - \dot{\chi}^2 \right. \\
&\quad \left. + \frac{1}{4} \sin^2 \mu (\alpha_1^2 \sin^2 \theta + (\alpha_2 + \alpha_1 \cos \theta)^2) \right].
\end{aligned} \tag{B.29}$$

The nontrivial equations of motion are

$$\begin{aligned}
\ddot{\mu} + \frac{1}{8} \sin(2\mu) \left[\dot{\theta}^2 - 2\alpha_1\alpha_2 \cos \theta - \alpha_1^2 - \alpha_2^2 \right] &= 0, \\
\ddot{\theta} + 2\dot{\mu}\dot{\theta} \cot(\mu) + \alpha_1\alpha_2 \sin \theta &= 0.
\end{aligned} \tag{B.30}$$

Inspection of the above system shows that we have various natural choices. We discussed the two natural choices of straight line solutions in what follows.

B.1.4.1 θ straight line

Let us assume $\theta = \dot{\theta} = 0$, then the equation for μ becomes

$$\ddot{\mu} - \frac{1}{8}(\alpha_1 + \alpha_2)^2 \sin(2\mu) = 0. \tag{B.31}$$

We call the solution of this equation μ_s . The NVE is

$$\ddot{\eta} + 2 \cot(\mu_s) \dot{\mu}_s \dot{\eta} + \alpha_1 \alpha_2 \eta = 0. \tag{B.32}$$

With $\sin(\mu) = z$ the NVE may be written as,

$$p(z) \frac{d^2}{dz^2} \eta(z) + q(z) \frac{d}{dz} \eta(z) + \alpha_1 \alpha_2 z^2 \eta(z) = 0 \quad (\text{B.33})$$

$$p(z) = z^2 (2E + 1/8 (\alpha_1 + \alpha_2)^2 (1 - 2z^2)) (1 - z^2) \quad (\text{B.34})$$

$$q(z) = -1/8z (-32E + 48Ez^2 - 2\alpha_1^2 + 9\alpha_1^2 z^2 - 8\alpha_1^2 z^4 - 4\alpha_1 \alpha_2 + 18\alpha_1 \alpha_2 z^2 - 16\alpha_1 \alpha_2 z^4 - 2\alpha_2^2 + 9\alpha_2^2 z^2 - 8\alpha_2^2 z^4)$$

This equation is now on the form conducive to Kovacic's algorithm which succeeds and gives a solution. Since the above approach obscures the nature of integrability of $AdS_5 \times S^5$ we consider another example which leaves no doubt about the integrability.

B.1.4.2 μ straight line

Let us assume the straight line is now given by $\mu = \pi/2, \dot{\mu} = 0$. The the equation for θ becomes

$$\ddot{\theta} + \alpha_1 \alpha_2 \sin \theta = 0. \quad (\text{B.35})$$

Let us call the solution to this equation θ_s . Then the NVE is

$$\ddot{\eta} + \frac{1}{4} \left(\dot{\theta}_s^2 - 2\alpha_1 \alpha_2 \cos(\theta_s) - \alpha_1^2 - \alpha_2^2 \right) \eta = 0. \quad (\text{B.36})$$

Note that the equation of motion for θ_s implies

$$\begin{aligned} & \ddot{\theta} + \alpha_1 \alpha_2 \sin \theta = 0 \\ \rightarrow & \frac{d}{d\tau} \left(\dot{\theta}_s^2 - 2\alpha_1 \alpha_2 \cos \theta_s \right) = 0, \\ \rightarrow & \dot{\theta}_s^2 - 2\alpha_1 \alpha_2 \cos \theta_s = C_0 \end{aligned} \quad (\text{B.37})$$

Thus the NVE equation can be written as a simple harmonic equation

$$\ddot{\eta} + \frac{1}{4} (C_0 - \alpha_1^2 - \alpha_2^2) \eta = 0. \quad (\text{B.38})$$

We do not require Kovacic's algorithm to tell us that there is an analytic solution for this equation. The power of differential Galois theory also guarantees that the result is really independent of the straight line solution (Riemann surface) that one chooses.

We conclude this subsection with the jovial comment that we now know a very precise sense in which *String theory in $AdS_5 \times S^5$ is like a harmonic oscillator*.

Appendix C

Straight-line Solution and NVE in Confining Backgrounds

In this appendix we show explicitly that the prototypical supergravity backgrounds in the gauge/gravity correspondence conform to the analysis presented in the main text. We consider the KS and MN backgrounds explicitly.

C.1 The Klebanov-Strassler background

We begin by reviewing the KS background, which is obtained by considering a collection of N regular and M fractional D3-branes in the geometry of the deformed conifold [85]. The 10-d metric is of the form:

$$ds_{10}^2 = h^{-1/2}(\tau) dX_\mu dX^\mu + h^{1/2}(\tau) ds_6^2, \quad (\text{C.1})$$

where ds_6^2 is the metric of the deformed conifold:

$$ds_6^2 = \frac{1}{2} \varepsilon^{4/3} K(\tau) \left[\frac{1}{3K^3(\tau)} (d\tau^2 + (g^5)^2) + \cosh^2\left(\frac{\tau}{2}\right) [(g^3)^2 + (g^4)^2] + \sinh^2\left(\frac{\tau}{2}\right) [(g^1)^2 + (g^2)^2] \right]. \quad (\text{C.2})$$

where

$$K(\tau) = \frac{(\sinh(2\tau) - 2\tau)^{1/3}}{2^{1/3} \sinh \tau}, \quad (\text{C.3})$$

and

$$\begin{aligned} g^1 &= \frac{1}{\sqrt{2}} [-\sin \theta_1 d\phi_1 - \cos \psi \sin \theta_2 d\phi_2 + \sin \psi d\theta_2], \\ g^2 &= \frac{1}{\sqrt{2}} [d\theta_1 - \sin \psi \sin \theta_2 d\phi_2 - \cos \psi d\theta_2], \\ g^3 &= \frac{1}{\sqrt{2}} [-\sin \theta_1 d\phi_1 + \cos \psi \sin \theta_2 d\phi_2 - \sin \psi d\theta_2], \\ g^4 &= \frac{1}{\sqrt{2}} [d\theta_1 + \sin \psi \sin \theta_2 d\phi_2 + \cos \psi d\theta_2], \\ g^5 &= d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2. \end{aligned} \quad (\text{C.4})$$

The warp factor is given by an integral expression for h is

$$h(\tau) = \alpha \frac{2^{2/3}}{4} I(\tau) = (g_s M \alpha')^2 2^{2/3} \varepsilon^{-8/3} I(\tau), \quad (\text{C.5})$$

where

$$I(\tau) \equiv \int_\tau^\infty dx \frac{x \coth x - 1}{\sinh^2 x} (\sinh(2x) - 2x)^{1/3}. \quad (\text{C.6})$$

The above integral has the following expansion in the IR:

$$I(\tau \rightarrow 0) \rightarrow a_0 - a_2 \tau^2 + \mathcal{O}(\tau^4) , \quad (\text{C.7})$$

where $a_0 \approx 0.71805$ and $a_2 = 2^{2/3} 3^{2/3}/18$. The absence of a linear term in τ reassures us that we are really expanding around the end of space, where the Wilson loop will find it more favorable to arrange itself.

C.1.1 The straight line solution in KS

We consider the quadratic fluctuations and their influence on the Regge trajectories (6.27). In the notation used in the bulk of the paper we have:

$$\begin{aligned} a^2(r) &= h^{-1/2}(r), \\ b^2(r) &= \frac{\varepsilon^{4/3}}{6K^2(r)} h^{1/2}(r). \end{aligned} \quad (\text{C.8})$$

Let us first consider the metric. The part of the metric perpendicular to the world volume, which is the deformed conifold metric, does not enter in the classical solution which involves only world volume fields. Noting that the value r_0 of section 6.2 is $\tau = 0$, we expand the deformed conifold up to quadratic terms in the coordinates:

$$ds_6^2 = \frac{\varepsilon^{4/3}}{2^{2/3} 3^{1/3}} \left[\frac{1}{2} g_5^2 + g_3^2 + g_4^2 + \frac{1}{2} d\tau^2 + \frac{\tau^2}{4} (g_1^2 + g_2^2) \right]. \quad (\text{C.9})$$

Let us further discuss the structure of this metric. It is known on very general grounds that the deformed conifold is a cone over a space that is topologically $S^3 \times S^2$ [86]. We can see that the S^3 roughly spanned by (g_3, g_4, g_5) has finite size, while the S^2 spanned by (g_1, g_2) shrinks to zero size at the apex of the deformed conifold. More importantly for us is the fact that if we do not allow non-trivial behavior in the directions (g_1, g_2) they cannot contribute to the NVE around the straight line solution characterized by $\tau = 0$. Therefore, we have that the NVE equation for the spinning string in the KS background is precisely of the form (6.36).

C.2 The Maldacena-Núñez background

The MN background [87] whose IR regime is associated with $\mathcal{N} = 1$ SYM theory is that of a large number of D5 branes wrapping an S^2 . To be more precise: (i) the dual field theory to this SUGRA background is the $\mathcal{N} = 1$ SYM contaminated with KK modes which cannot be de-coupled from the IR dynamics, (ii) the IR regime is described by the SUGRA in the vicinity of the origin where the S^2 shrinks to zero size. The full MN SUGRA background includes the metric, the dilaton and the RR three-form. It can also be interpreted as uplifting to ten dimensions a solution of seven dimensional gauged supergravity [88]. The metric and dilaton of the background are

$$ds^2 = e^\phi \left[dX^a dX_a + \alpha' g_s N (d\tau^2 + e^{2g(\tau)} (e_1^2 + e_2^2) + \frac{1}{4} (e_3^2 + e_4^2 + e_5^2)) \right],$$

$$\begin{aligned}
e^{2\phi} &= e^{-2\phi_0} \frac{\sinh 2\tau}{2e^{g(\tau)}}, \\
e^{2g(\tau)} &= \tau \coth 2\tau - \frac{\tau^2}{\sinh^2 2\tau} - \frac{1}{4},
\end{aligned} \tag{C.10}$$

where,

$$\begin{aligned}
e_1 &= d\theta_1, & e_2 &= \sin \theta_1 d\phi_1, \\
e_3 &= \cos \psi d\theta_2 + \sin \psi \sin \theta_2 d\phi_2 - a(\tau) d\theta_1, \\
e_4 &= -\sin \psi d\theta_2 + \cos \psi \sin \theta_2 d\phi_2 - a(\tau) \sin \theta_1 d\phi_1, \\
e_5 &= d\psi + \cos \theta_2 d\phi_2 - \cos \theta_1 d\phi_1, & a(\tau) &= \frac{\tau^2}{\sinh^2 \tau}.
\end{aligned} \tag{C.11}$$

where $\mu = 0, 1, 2, 3$, we set the integration constant $e^{\phi_{D_0}} = \sqrt{g_s N}$.

Note that we use notation where x^0, x^i have dimension of length whereas ρ and the angles $\theta_1, \phi_1, \theta_2, \phi_2, \psi$ are dimensionless and hence the appearance of the α' in front of the transverse part of the metric.

C.2.1 The straight line solution in MN

The position referred to as r_0 in section (6.1) is $\tau = 0$. Therefore, we will expand the metric around that value. Let us first identify some structures in the metric that are similar to the deformed conifold considered in the previous subsection. Notice that $e_1^2 + e_2^2$ is precisely an S^2 . Moreover, near $\tau = 0$ we have that $e^{2g} \approx \tau^2 + \mathcal{O}(\tau^4)$. Thus (τ, e_1, e_2) span an \mathbb{R}^3 in the limit

$$d\tau^2 + e^{2g(\tau)}(e_1^2 + e_2^2). \tag{C.12}$$

This means that without exciting the KK modes corresponding to (e_1, e_2) in our Ansatz II, the NVE equation is precisely of the form (6.36). Certainly $e_3^2 + e_4^2 + e_5^2$ parametrizes a space that is topologically a three sphere fibered over the S^2 spanned by (e_1, e_2) . However, near $\tau = 0$ we have a situation very similar to the structure of the metric in the deformed conifold. Namely, at $\tau = 0$ there we have that: $e_5 \rightarrow g_5$, $e_3 \rightarrow \sqrt{2}g_4$, $e_4 \rightarrow \sqrt{2}g_3$ (up to a trivial identification $\theta_1 \rightarrow -\theta_1$, $\phi_1 \rightarrow -\phi_1$). This allows us to identify this combination as a round S^3 of radius 2 and therefore can not alter the form of the NVE (6.36).

C.3 The Witten QCD background

The ten-dimensional string frame metric and dilaton of the Witten QCD model are given by

$$\begin{aligned}
ds^2 &= \left(\frac{u}{R}\right)^{3/2} (\eta_{\mu\nu} dx^\mu dx^\nu + \frac{4R^3}{9u_0} f(u) d\theta^2) + \left(\frac{R}{u}\right)^{3/2} \frac{du^2}{f(u)} + R^{3/2} u^{1/2} d\Omega_4^2, \\
f(u) &= 1 - \frac{u_0^3}{u^3}, & R &= (\pi N g_s)^{\frac{1}{3}} \alpha'^{\frac{1}{2}},
\end{aligned}$$

$$e^\Phi = g_s \frac{u^{3/4}}{R^{3/4}} . \quad (\text{C.13})$$

The geometry consists of a warped, flat 4-d part, a radial direction u , a circle parameterized by θ with radius vanishing at the horizon $u = u_0$, and a four-sphere whose volume is instead everywhere non-zero. It is non-singular at $u = u_0$. Notice that in the $u \rightarrow \infty$ limit the dilaton diverges: this implies that in this limit the completion of the present IIA model has to be found in M-theory. The background is completed by a constant four-form field strength

$$F_4 = 3R^3 \omega_4 , \quad (\text{C.14})$$

where ω_4 is the volume form of the transverse S^4 .

We will be mainly interested in classical string configurations localized at the horizon $u = u_0$, since this region is dual to the IR regime of the dual field theory. In this case the coordinate u is not suitable because the metric written in this coordinate looks singular at $u = u_0$. Then, as a first step, let us introduce the radial coordinate

$$r^2 = \frac{u - u_0}{u_0} , \quad (\text{C.15})$$

so that the metric expanded to quadratic order around $r = 0$ becomes

$$ds^2 \approx \left(\frac{u_0}{R}\right)^{3/2} \left[1 + \frac{3r^2}{2}\right] (\eta_{\mu\nu} dx^\mu dx^\nu) + \frac{4}{3} R^{3/2} \sqrt{u_0} (dr^2 + r^2 d\theta^2) + R^{3/2} u_0^{1/2} \left[1 + \frac{r^2}{2}\right] d\Omega_4^2 . \quad (\text{C.16})$$

C.3.1 The straight line solution in WQCD

In this section we consider the closed string configuration corresponding to the glueball Regge trajectories. The relevant closed folded spinning string configuration dual to the Regge trajectories and constituting the straight line solution in our analysis is

$$X^0 = k\tau , \quad X^1 = k \cos \tau \sin \sigma , \quad X^2 = k \sin \tau \sin \sigma , \quad (\text{C.17})$$

and all the other coordinates fixed.

To understand the NVE around the straight line solution given above, we need only look at (C.16) and realize that the only possible contribution to the NVE given in (6.36) can come only from KK modes in the S^4 of equation (C.16). We conclude that, in this case, as well the NVE is precisely of the form given in (6.36).

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