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Trace Anomaly of the Stress-Energy Tensor for Massless Vector Particles Propagating in a General Background Metric*

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ABSTRACT

We reanalyze the problem of regularization of the stress-energy tensor for massless vector particles propagating in a general background metric, using covariant point separation techniques applied to the Hadamard elementary solution. We correct an error, pointed out by Wald, in the earlier formulation of Adler, Lieberman and Ng, and find a stress-energy tensor trace anomaly agreeing with that found by other regularization methods.

1. INTRODUCTION

The problem of regularization of the stress-energy tensor for particles propagating in a general background metric has been treated recently by many authors using differing calculational methods [1]. In an earlier paper by Adler, Lieberman and Ng [2], the regularization problem was approached by applying covariant point separation techniques to the Hadamard elementary solutions for the vector and scalar wave equations. The results of Ref.[2] appeared to contradict those obtained by other methods [1], in that no stress-energy tensor trace anomaly was found. Recently however, Wald [3] has found a mistake in the formal arguments of Sec. 5 and Appendix B of Ref.[2] which accounts for the discrepancy. The mistake consists of the assumption, made in Ref.[2], that the local and boundary-condition-dependent parts $G^{(1L)}$ and $G^{(1B)}$ of the Hadamard elementary solution $G^{(1)}$ are individually symmetric in their arguments x,x', as is their sum $G^{(1)}$. Wald [3] has shown that when this assumption is dropped, a reanalysis of the stress-energy tensor in the scalar particle case gives the standard result for the trace anomaly. The purpose of the present paper is to give the corrected stress-energy tensor calculation in the vector particle case; again, when the lack of symmetry of $G^{(1L)}_{\lambda\sigma'}$ and $G^{(1B)}_{\lambda\sigma'}$ is correctly taken into account, the standard result for the trace anomaly is obtained. In order to avoid needless repetition of formulas, this paper has been written in the form of a supplement to Ref.[2], with Eq.(N) of Ref.[2] indicated as Eq.(2.N) below.

2. THE CALCULATION

According to Eq.(2.4), the vector particle stress-energy tensor $T_{\alpha\beta}$ is a sum of Maxwell, gauge-breaking and ghost contributions,

$$T_{\alpha\beta} = T^{M}_{\alpha\beta} + T^{BR}_{\alpha\beta} + T^{GH}_{\alpha\beta} \quad . \tag{1}$$

Since the argument of Eqs.(2.43) - (2.45) showing that

$$\langle T^{BR}_{\alpha\beta} \rangle + \langle T^{GH}_{\alpha\beta} \rangle = 0 \tag{2}$$

does not depend on splitting the Hadamard elementary solution $G^{(1)}$ into (L) and (B) parts, it is unaffected by Wald's observation, and so we still have, as in Eqs.(2.5) - (2.6)

$$\begin{aligned} \langle T_{\alpha\beta}(x) \rangle &= \langle T^{M}_{\alpha\beta}(x) \rangle \\ &= (g_{\alpha}{}^{\mu} g_{\beta}{}^{\lambda} g^{\nu\sigma} - \tfrac{1}{4} g_{\alpha\beta} g^{\mu\lambda} g^{\nu\sigma}) P_{\mu\nu\lambda\sigma} \quad , \\ P_{\mu\nu\lambda\sigma} &= \lim_{x \to x'} \tfrac{1}{2} \langle F_{\mu\nu}(x)\, F_{\lambda\sigma}(x') + F_{\lambda\sigma}(x') F_{\mu\nu}(x) \rangle \quad . \end{aligned} \tag{3}$$

Expressing the expectation in Eq.(3) in terms of the vector particle Hadamard elementary solution $G^{(1)}_{\nu\sigma'}$, and splitting $G^{(1)}_{\nu\sigma'}$ into (L) and (B) parts, we get [as in Eq.(2.48)]

$$\begin{aligned} \langle T_{\alpha\beta}(x) \rangle &= \langle T^{(L)}_{\alpha\beta}(x) \rangle + \langle T^{(B)}_{\alpha\beta}(x) \rangle \quad , \\ \langle T^{(L/B)}_{\alpha\beta}(x) \rangle &= (g_{\alpha}{}^{\mu} g_{\beta}{}^{\lambda} g^{\nu\sigma} - \tfrac{1}{4} g_{\alpha\beta} g^{\mu\lambda} g^{\nu\sigma}) P^{(L/B)}_{\mu\nu\lambda\sigma} \quad , \\ P^{(L/B)}_{\mu\nu\lambda\sigma} &= 2\, G^{(L/B)}_{([\mu\nu][\lambda\sigma])} \quad , \\ G^{(L/B)}_{\mu\nu\lambda\sigma} &\equiv [D_{\mu} D_{\lambda'} G^{(1L/B)}_{\nu\sigma'}(x, x')] \quad , \end{aligned} \tag{4}$$

with the notation [ ] in the final line denoting the $x' \to x$ coincidence limit. We assume that the highly singular coincidence limit is regularized in such a manner that $\langle T_{\alpha\beta}(x)\rangle$ is finite and covariantly conserved, which implies that

$$D^{\alpha} \langle T^{(L)}_{\alpha\beta}(x)\rangle = -D^{\alpha} \langle T^{(B)}_{\alpha\beta}(x)\rangle \ . \tag{5}$$

An explicit calculation of the right-hand side of Eq.(5), to be given below, shows that

$$D^{\alpha} \langle T^{(B)}_{\alpha\beta}(x)\rangle = D^{\alpha}\ t^{(L)}_{\alpha\beta}(x) \ , \tag{6}$$

with $t^{(L)}_{\alpha\beta}(x)$ a tensor local in the Riemann curvature and its second covariant derivatives. Thus, from Eq.(5), we have

$$D^{\alpha}\left[\langle T^{(L)}_{\alpha\beta}(x)\rangle + t^{(L)}_{\alpha\beta}(x)\right] = D^{\alpha}\left[-\langle T^{(B)}_{\alpha\beta}(x)\rangle + t^{(L)}_{\alpha\beta}(x)\right] = 0, \tag{7}$$

which implies that $\langle T^{(L)}_{\alpha\beta}(x)\rangle + t^{(L)}_{\alpha\beta}(x)$ is a tensor, local in the Riemann tensor and its covariant derivatives, which furthermore is covariantly conserved. Since no parameters with dimension of mass appear in the problem, on dimensional grounds this tensor must have the structure

$$\begin{aligned}
\langle T^{(L)}_{\alpha\beta}(x)\rangle + t^{(L)}_{\alpha\beta}(x) &= c_1\, I_{\alpha\beta}(x) + c_2\, J_{\alpha\beta}(x) \ , \\
I_{\alpha\beta} = \frac{1}{(-g)^{\frac{1}{2}}}\,\frac{\delta}{\delta g^{\alpha\beta}} \int d^4x(-g)^{\frac{1}{2}}\ R^2 &= -2\, g_{\alpha\beta} R_{,\theta}{}^{\theta} + 2R_{,\alpha\beta} - 2R\, R_{\alpha\beta} + \frac{1}{2}\, g_{\alpha\beta} R^2 , \\
J_{\alpha\beta} = \frac{1}{(-g)^{\frac{1}{2}}}\,\frac{\delta}{\delta g^{\alpha\beta}} \int d^4x(-g)^{\frac{1}{2}}\ R^{\rho\tau}R_{\rho\tau} &= -\frac{1}{2}\, g_{\alpha\beta} R_{,\theta}{}^{\theta} + \frac{1}{2}\, g_{\alpha\beta} R_{\rho\theta}R^{\rho\theta} \\
&\quad + R_{,\alpha\beta} - R_{\alpha\beta,\theta}{}^{\theta} - 2\, R^{\rho\theta}R_{\rho\alpha\theta\beta} \ ,
\end{aligned} \tag{8}$$

with c_1 and c_2 arbitrary coefficients. The regularized stress-energy tensor thus becomes

$$\langle T_{\alpha\beta}(x)\rangle = c_1 I_{\alpha\beta}(x) + c_2 J_{\alpha\beta}(x) - t^{(L)}_{\alpha\beta}(x) + \langle T^{(B)}_{\alpha\beta}(x)\rangle \quad , \tag{9}$$

which by Eqs.(7) and (8) is automatically covariantly conserved. To evaluate the trace of $\langle T_{\alpha\beta}(x)\rangle$, we note that from Eq.(4) we have $g^{\alpha\beta}\langle T^{(B)}_{\alpha\beta}(x)\rangle = 0$. Hence we get

$$g^{\alpha\beta}\langle T_{\alpha\beta}(x)\rangle = -2(3c_1 + c_2)R_{,\theta}{}^{\theta} - g^{\alpha\beta}t^{(L)}_{\alpha\beta}(x) \,, \tag{10}$$

which is the trace anomaly [and, as is evident from Eqs.(43) and (44) below, is nonvanishing irrespective of the value of $3c_1 + c_2$].

In order to evaluate $t^{(L)}_{\alpha\beta}(x)$ we must carefully calculate $D^{\alpha}\langle T^{(B)}_{\alpha\beta}(x)\rangle$, keeping in mind the fact that while $G^{(1)}_{\nu\sigma'}(x,x')$ is symmetric under the interchange $\nu,x \leftrightarrow \sigma',x'$, $G^{(1L)}_{\nu\sigma'}$ is not symmetric and hence neither is $G^{(1B)}_{\nu\sigma'}$. Following the notation of Appendix B of Ref.[2], we write

$$W_{\nu\sigma'}(x,x') = \frac{1}{2} G^{(B)}_{\nu\sigma'}(x,x') \quad , \tag{11}$$

which on substitution into Eq.(4) gives

$$D^{\mu}\langle T^{(B)}_{\mu\lambda}\rangle = g_{\lambda}{}^{\gamma} g^{\beta\delta} D^{\alpha} P^{(B)}_{\alpha\beta\gamma\delta} - \frac{1}{4} g^{\alpha\gamma} g^{\beta\delta} D_{\lambda} P^{(B)}_{\alpha\beta\gamma\delta} \quad , \tag{12}$$

with

$$P^{(B)}_{\alpha\beta\gamma\delta} = \left[W_{\beta\delta',\alpha\gamma'} + W_{\alpha\gamma',\beta\delta'} - W_{\alpha\delta',\beta\gamma'} - W_{\beta\gamma',\alpha\delta'} \right] . \tag{13}$$

In evaluating Eq.(12) we can use the equation of motion at x,

$$W_{\nu\sigma',\mu}{}^{\mu} - R_{\nu}{}^{\theta} W_{\theta\sigma'} = 0 \quad , \tag{14}$$

but we cannot (as was done in Appendix B of Ref.[2]) assume symmetry of $W_{\nu\sigma'}$, or the Lorentz gauge condition of Eq.(2.B3). Beginning with the second term on the right-hand side of Eq.(12), and using Synge's theorem, we get

$$D_\lambda P^{(B)}_{\alpha\beta\gamma\delta} = [W_{\alpha\gamma',\beta\delta'\lambda} - W_{\alpha\delta',\beta\gamma'\lambda} + W_{\alpha\gamma',\beta\delta'\lambda'} - W_{\alpha\delta',\beta\gamma'\lambda'} - (\alpha \leftrightarrow \beta)]$$

$$= \left[W_{\alpha\gamma',\beta\delta'\lambda} - W_{\alpha\delta',\beta\gamma'\lambda} - W_{\gamma\alpha',\delta\beta'\lambda'} + W_{\delta\alpha',\gamma\beta'\lambda'} - \begin{pmatrix} \alpha \leftrightarrow \beta \\ \text{or} \\ \alpha' \leftrightarrow \beta' \end{pmatrix}\right] \qquad \text{(a)}$$

$$+ \left[W_{\gamma\alpha',\delta\beta'\lambda'} - W_{\delta\alpha',\gamma\beta'\lambda'} + W_{\alpha\gamma',\beta\delta'\lambda'} - W_{\alpha\delta',\beta\gamma'\lambda'} - \begin{pmatrix} \alpha \leftrightarrow \beta \\ \text{or} \\ \alpha' \leftrightarrow \beta' \end{pmatrix}\right] . \qquad \text{(b)} \quad (15)$$

Substituting Eq.(15) into the second term in Eq.(12), relabeling dummy indices and combining like terms, we get

$$-\frac{1}{4} g^{\alpha\gamma} g^{\beta\delta} D_\lambda P^{(B)}_{\alpha\beta\gamma\delta}$$

$$= -\frac{1}{2} \left[(D_\lambda - D_{\lambda'}) (W^\gamma{}_{\gamma',}{}^{\delta}{}_{\delta'} - W^\gamma{}_{\delta',}{}^{\delta}{}_{\gamma'}) \right] \qquad \text{(a)}$$

$$-\frac{1}{2} \left[(W^\gamma{}_{\gamma',}{}^{\delta}{}_{\delta'} - W^\gamma{}_{\delta',}{}^{\delta}{}_{\gamma'} - W^\delta{}_{\gamma',}{}^{\gamma}{}_{\delta'} + W^\delta{}_{\delta',}{}^{\gamma}{}_{\gamma'})_{,\lambda'} \right] . \qquad \text{(b)} \quad (16)$$

We next consider the first term on the right-hand side of Eq.(12). Again using Synge's theorem, we have

$$D^\alpha P^{(B)}_{\alpha\beta\gamma\delta} = \left[W_{\beta\delta',\alpha\gamma'}{}^{\alpha} + W_{\alpha\gamma',\beta\delta'}{}^{\alpha} - W_{\alpha\delta',\beta\gamma'}{}^{\alpha} - W_{\beta\gamma',\alpha\delta'}{}^{\alpha}\right. \qquad \text{(c)}$$

$$\left. + W_{\beta\delta',\alpha\gamma'}{}^{\alpha'} + W_{\alpha\gamma',\beta\delta'}{}^{\alpha'} - W_{\alpha\delta',\beta\gamma'}{}^{\alpha'} - W_{\beta\gamma',\alpha\delta'}{}^{\alpha'}\right] . \qquad \text{(d)} \quad (17)$$

Using the Ricci identity and Eq.(14) to simplify the first line of Eq.(17), and using the cyclic identity, just as is done in Appendix B of Ref.[2], to simplify the second line of Eq.(17), we get on substitution into the first term of Eq.(12)

$$g_\lambda{}^\gamma g^{\beta\delta} D^\alpha P^{(B)}_{\alpha\beta\gamma\delta} = \left[W_{\alpha\lambda',}{}^{\alpha\delta}{}_{\delta'} - W_{\alpha\delta',}{}^{\alpha\delta}{}_{\lambda'}\right] \qquad \text{(c)}$$

$$+ \frac{1}{2} \left[(W^\gamma{}_{\gamma',}{}^{\delta}{}_{\delta'} - W^\gamma{}_{\delta',}{}^{\delta}{}_{\gamma'} - W^\delta{}_{\gamma',}{}^{\gamma}{}_{\delta'} + W^\delta{}_{\delta',}{}^{\gamma}{}_{\gamma'})_{,\lambda'} \right] . \qquad \text{(d)} \quad (18)$$

Term (d) of Eq.(18) cancels term (b) of Eq.(16), giving the result

$$D^{\mu}\langle T^{(B)}_{\mu\lambda}\rangle = -\frac{1}{2}\left[(D_\lambda - D_{\lambda'})(W^{\gamma}{}_{\gamma',}{}^{\delta}{}_{\delta'} - W^{\gamma}{}_{\delta',}{}^{\delta}{}_{\gamma'})\right] \tag{19a}$$

$$+\left[W_{\alpha\lambda',}{}^{\alpha\delta}{}_{\delta'} - W_{\alpha\delta',}{}^{\alpha\delta}{}_{\lambda'}\right] \quad . \tag{19b}$$

_If_ $W_{\nu\sigma'}(x,x')$ were symmetric under the interchange $\nu x \leftrightarrow \sigma' x'$, the expression in Eq.(19a) would vanish, and _if_ the gauge condition of Eq.(2.B3) $[W_{\nu\sigma',}{}^{\nu} = -W_{,\sigma'}$ with W a scalar$]$ were valid, the expression in Eq.(19b) would vanish, giving the result $D^{\mu}\langle T^{(B)}_{\mu\lambda}\rangle = 0$ found in Appendix B of Ref.[2]. In fact, as we shall now show, both the expressions in Eq.(19a) and (19b) are nonvanishing, providing the origin of the tensor $t^{(L)}_{\alpha\beta}(x)$ appearing in Eq. (6).

We begin with the evaluation of Eq.(19a). Substituting

$$W_{\nu\sigma'} = \frac{1}{(4\pi)^2}\,\Delta^{\frac{1}{2}}\,w^{(B)}_{\nu\sigma'} = \frac{1}{(4\pi)^2}\,\Delta^{\frac{1}{2}}\left(w_{\nu\sigma'} - w^{(L)}_{\nu\sigma'}\right) \tag{20}$$

and noting that (i) $w_{\nu\sigma'}(x,x')$ _is_ symmetric, and so makes no contribution to Eq.(19a), (ii) in the series $w^{(L)}_{\nu\sigma'} = w^{(L)}_{1\nu\sigma'}\sigma(x,x') + w^{(L)}_{2\nu\sigma'}\sigma(x,x')^2 + \ldots$, only the first term contributes to the coincidence limit in Eq.(19a), and (iii) terms with Δ differentiated make no contribution to the coincidence limit in Eq.(19a), we get

$$\begin{aligned}
&-\frac{1}{2}\left[(D_\lambda - D_{\lambda'})(W^{\gamma}{}_{\gamma',}{}^{\delta}{}_{\delta'} - W^{\gamma}{}_{\delta',}{}^{\delta}{}_{\gamma'})\right] \\
&= \frac{1}{2}\frac{1}{(4\pi)^2}\left[(D_\lambda - D_{\lambda'})\left((\sigma\, w_1^{(L)\gamma}{}_{\gamma'})_{,}{}^{\delta}{}_{\delta'} - (\sigma\, w_1^{(L)\gamma}{}_{\delta'})_{,}{}^{\delta}{}_{\gamma'}\right)\right] \\
&= \frac{1}{2}\frac{1}{(4\pi)^2}\left[\,5\,(D_{\lambda'} - D_\lambda)\, w_1^{(L)\gamma}{}_{\gamma'} - 2\, D_{\gamma'}\, w_1^{(L)\gamma}{}_{\lambda'} + 2\, D^{\delta}\, w_1^{(L)}{}_{\lambda\delta'}\right] \quad .
\end{aligned} \tag{21}$$

The next step is to evaluate the coincidence limits appearing in

Eq.(21), following the procedure used by Wald [3] in the scalar case. Writing the recursion relations (2.20) and (2.38) for $v_{1\alpha\sigma'}$ and $w^{(L)}_{1\alpha\sigma'}$ in the form

$$v_1{}^{\alpha}{}_{\sigma'} + \frac{1}{2} s \frac{Dv_1{}^{\alpha}{}_{\sigma'}}{ds} = -\frac{1}{4}\left[\Delta^{-\frac{1}{2}} D_\mu D^\mu (\Delta^{\frac{1}{2}} v_0{}^{\alpha}{}_{\sigma'}) - R^{\alpha\gamma} v_{0\gamma\sigma'}\right] ,$$

$$w_1^{(L)\alpha}{}_{\sigma'} + \frac{1}{2} s \frac{Dw_1^{(L)\alpha}{}_{\sigma'}}{ds} = -\frac{1}{2} v_1{}^{\alpha}{}_{\sigma'} + \frac{1}{4}\left[\Delta^{-\frac{1}{2}} D_\mu D^\mu (\Delta^{\frac{1}{2}} v_0{}^{\alpha}{}_{\sigma'}) - R^{\alpha\gamma} v_{0\gamma\sigma'}\right] , \tag{22}$$

with s the arc length along the geodesic joining x' to x, one finds the unique solution regular at $s = 0$

$$w_1^{(L)\alpha}{}_{\sigma'} = -v_1{}^{\alpha}{}_{\sigma'} - \frac{1}{s^2} \int_0^s g^{\alpha}{}_{\bar{\alpha}}(s,\bar{s})\, v_1{}^{\bar{\alpha}}{}_{\sigma'}(\bar{x},x')\, \bar{s}\, d\bar{s} . \tag{23}$$

Taking the coincidence limit of Eq.(23) and its first covariant derivative gives

$$[w_1^{(L)\alpha}{}_{\sigma'}] = \left(-1 - \frac{1}{s^2}\int_0^s \bar{s}\, d\bar{s}\right)[v_1{}^{\alpha}{}_{\sigma'}] = -\frac{3}{2}[v_1{}^{\alpha}{}_{\sigma'}] ,$$

$$[D_\lambda w_1^{(L)\alpha}{}_{\sigma'}] = \left(-1 - \frac{d}{ds}\frac{1}{s^2}\int_0^s \bar{s}^2\, d\bar{s}\right)[D_\lambda v_1{}^{\alpha}{}_{\sigma'}] = -\frac{4}{3}[D_\lambda v_1{}^{\alpha}{}_{\sigma'}] . \tag{24}$$

Thus, making use of Synge's theorem and the fact that $v_1{}^{\gamma}{}_{\gamma'}$ is a symmetric biscalar function of x and x', which implies $[D_\lambda v_1{}^{\gamma}{}_{\gamma'}] = \frac{1}{2} D_\lambda [v_1{}^{\gamma}{}_{\gamma'}]$, we get the evaluations

$$[D_{\lambda'} w_1^{(L)\gamma}{}_{\gamma'}] = -\frac{5}{6} D_\lambda [v_1{}^{\gamma}{}_{\gamma'}]$$

$$[D_\lambda w_1^{(L)\gamma}{}_{\gamma'}] = -\frac{2}{3} D_\lambda [v_1{}^{\gamma}{}_{\gamma'}]$$

$$[D_{\gamma'} w_1^{(L)\gamma}{}_{\lambda'}] = \frac{4}{3}[D_\alpha (v_1{}^{\alpha}{}_{\lambda'})] - \frac{3}{2} D_\alpha [v_1{}^{\alpha}{}_{\lambda'}]$$

$$[D^{\delta} w^{(L)}_{1\,\lambda\delta'}] = \frac{4}{3}[D_\alpha (v_1{}^{\alpha}{}_{\lambda'})] - \frac{4}{3} D_\alpha [v_1{}^{\alpha}{}_{\lambda'}] . \tag{25}$$

Substituting these into Eq.(21), we get

$$- \frac{1}{2} \left[(D_\lambda - D_{\lambda'}) \left(W^{\gamma}{}_{\gamma',}{}^{\delta}{}_{\delta'} - W^{\gamma}{}_{\delta',}{}^{\delta}{}_{\gamma'} \right) \right]$$

$$= \frac{1}{2} \frac{1}{(4\pi)^2} \left\{ - \frac{5}{6} D_\lambda [v_1{}^{\gamma}{}_{\gamma'}] + \frac{1}{3} D_\alpha [v_1{}^{\alpha}{}_{\lambda'}] \right\} . \tag{26}$$

We turn next to the evaluation of Eq.(19b). We will need, as auxilliary formulas, some consequences of the gauge condition of Eq. (2.26),

$$G^{(1)}_{\nu\sigma',}{}^{\nu} + G^{(1)}_{0,\sigma'} = 0 \quad . \tag{27}$$

Substituting the Hadamard formulas

$$G^{(1)}_{\nu\sigma'} = \frac{2\Delta^{\frac{1}{2}}}{(4\pi)^2} \left(\frac{2\, g_{\nu\sigma'}}{\sigma} + v_{\nu\sigma'} \ln \sigma + w_{\nu\sigma'} \right)$$

$$G^{(1)}_{0} = \frac{2\Delta^{\frac{1}{2}}}{(4\pi)^2} \left(\frac{2}{\sigma} + v \ln \sigma + w \right) \tag{28}$$

into Eq.(27) and equating to zero the coefficient of $\ln \sigma$ gives

$$(\Delta^{\frac{1}{2}} v_{\nu\sigma'})_{,}{}^{\nu} + (\Delta^{\frac{1}{2}} v)_{,\sigma'} = 0 \ , \tag{29}$$

while the remainder gives

$$2\Delta^{\frac{1}{2}}{}_{,\sigma'} + 2(\Delta^{\frac{1}{2}} g_{\nu\sigma'})_{,}{}^{\nu} + \Delta^{\frac{1}{2}} v\sigma_{,\sigma'} + \Delta^{\frac{1}{2}} v_{\nu\sigma'}\sigma_{,}{}^{\nu} + \sigma(\Delta^{\frac{1}{2}} w)_{,\sigma'} + \sigma(\Delta^{\frac{1}{2}} w_{\nu\sigma'})_{,}{}^{\nu} = 0 \, . \tag{30}$$

Substituting the series expansions

$$v_{\nu\sigma'} = \sum_{n=0}^{\infty} v_{n\nu\sigma'}\, \sigma^n \qquad w_{\nu\sigma'} = \sum_{n=0}^{\infty} w_{n\nu\sigma'}\, \sigma^n$$

$$v = \sum_{n=0}^{\infty} v_n \sigma^n \qquad w = \sum_{n=0}^{\infty} w_n \sigma^n \tag{31}$$

into Eqs.(29) and (30) and equating coefficients order by order in σ, we get the recursion relations

$$2\Delta^{\frac{1}{2}}{}_{,\sigma'} + 2(\Delta^{\frac{1}{2}} g_{\nu\sigma'})_{,}{}^{\nu} + \Delta^{\frac{1}{2}} v_0{}^{\sigma}{}_{,\sigma'} + \Delta^{\frac{1}{2}} v_{0\nu\sigma'}{}^{\sigma}{}_{,}{}^{\nu} = 0 \tag{32a}$$

$$\Delta^{\frac{1}{2}} v_n{}^{\sigma}{}_{,\sigma'} + \Delta^{\frac{1}{2}} v_{n\nu\sigma'}{}^{\sigma}{}_{,}{}^{\nu} = -\frac{1}{n}\left\{(\Delta^{\frac{1}{2}} v_{n-1})_{,\sigma'} + (\Delta^{\frac{1}{2}} v_{n-1\nu\sigma'})_{,}{}^{\nu}\right\},\ n \geq 1 \tag{32b}$$

$$\begin{aligned}\Delta^{\frac{1}{2}} w_n{}^{\sigma}{}_{,\sigma'} + \Delta^{\frac{1}{2}} w_{n\nu\sigma'}{}^{\sigma}{}_{,}{}^{\nu} = {} & \frac{1}{n^2}\left\{(\Delta^{\frac{1}{2}} v_{n-1})_{,\sigma'} + (\Delta^{\frac{1}{2}} v_{n-1\nu\sigma'})_{,}{}^{\nu}\right\} \\ & - \frac{1}{n}\left\{(\Delta^{\frac{1}{2}} w_{n-1})_{,\sigma'} + (\Delta^{\frac{1}{2}} w_{n-1\nu\sigma'})_{,}{}^{\nu}\right\},\ n \geq 1\ .\end{aligned} \tag{32c}$$

In particular, we will need the coincidence limit of the $n = 2$ case of Eqs.(32b,c), which give

$$[v_{1\nu\sigma'}{}_{,}{}^{\nu} + v_{1,\sigma'}] = 0$$

$$[w_{1\nu\sigma'}{}_{,}{}^{\nu} + w_{1,\sigma'}] = 0 \quad , \tag{33}$$

and the $n = 1$ case of Eqs.(32b,c), which combined give

$$\begin{aligned}(\Delta^{\frac{1}{2}} w_0)_{,\sigma'} + (\Delta^{\frac{1}{2}} w_{0\nu\sigma'})_{,}{}^{\nu} = {} & -\Delta^{\frac{1}{2}} w_1{}^{\sigma}{}_{,\sigma'} - \Delta^{\frac{1}{2}} w_{1\nu\sigma'}{}^{\sigma}{}_{,}{}^{\nu} \\ & -\Delta^{\frac{1}{2}} v_1{}^{\sigma}{}_{,\sigma'} - \Delta^{\frac{1}{2}} v_{1\nu\sigma'}{}^{\sigma}{}_{,}{}^{\nu}\ .\end{aligned} \tag{34}$$

We now proceed as follows. Substituting Eq.(20) into Eq.(19b), and keeping only those terms in the series expansion of Eq.(31) which make a nonvanishing contribution in the coincidence limit, we get

$$[W_{\alpha\lambda'}{}_{,}{}^{\alpha\delta}{}_{\delta'} - W_{\alpha\delta'}{}_{,}{}^{\alpha\delta}{}_{\lambda'}]$$

$$= \frac{1}{(4\pi)^2}\left[(\Delta^{\frac{1}{2}} w^{(B)}_{0\alpha\lambda'} + \Delta^{\frac{1}{2}} w^{(B)}_{1\alpha\lambda'}\sigma)_{,}{}^{\alpha\delta}{}_{\delta'} - (\Delta^{\frac{1}{2}} w^{(B)}_{0\alpha\delta'} + \Delta^{\frac{1}{2}} w^{(B)}_{1\alpha\delta'}\sigma)_{,}{}^{\alpha\delta}{}_{\lambda'}\right]$$

$$= \frac{1}{(4\pi)^2}\left[(\Delta^{\frac{1}{2}} w_{0\alpha\lambda'})_{,}{}^{\alpha\delta}{}_{\delta'} - (\Delta^{\frac{1}{2}} w_{0\alpha\delta'})_{,}{}^{\alpha\delta}{}_{\lambda'}\right] \tag{a}$$

$$+ \frac{1}{(4\pi)^2}\left[(\Delta^{\frac{1}{2}} w^{(B)}_{1\alpha\lambda'}\sigma)_{,}{}^{\alpha\delta}{}_{\delta'} - (\Delta^{\frac{1}{2}} w^{(B)}_{1\alpha\delta'}\sigma)_{,}{}^{\alpha\delta}{}_{\lambda'}\right]\ , \tag{b) (35}$$

where in the next to last line we have replaced $w^{(B)}_{0\alpha\lambda'}$ by $w_{0\alpha\lambda'}$, which is justified since $w^{(L)}_{0\alpha\lambda'} = 0$. To evaluate term (a) in Eq.(35), we substitute Eq.(34), giving

$$\begin{aligned}
(a) = \frac{1}{(4\pi)^2} \Big[& \left(-(\Delta^{\frac{1}{2}} w_0)_{,\lambda'} - \Delta^{\frac{1}{2}} w_1 \sigma_{,\lambda'} - \Delta^{\frac{1}{2}} w_{1\alpha\lambda'} \sigma_{,}{}^{\alpha} - \Delta^{\frac{1}{2}} v_1 \sigma_{,\lambda'} - \Delta^{\frac{1}{2}} v_{1\alpha\lambda'} \sigma_{,}{}^{\alpha} \right)_{,}{}^{\delta}{}_{\delta'} \\
& - \left(-(\Delta^{\frac{1}{2}} w_0)_{,\delta'} - \Delta^{\frac{1}{2}} w_1 \sigma_{,\delta'} - \Delta^{\frac{1}{2}} w_{1\alpha\delta'} \sigma_{,}{}^{\alpha} - \Delta^{\frac{1}{2}} v_1 \sigma_{,\delta'} - \Delta^{\frac{1}{2}} v_{1\alpha\delta'} \sigma_{,}{}^{\alpha} \right)_{,}{}^{\delta}{}_{\lambda'} \Big] \\
= \frac{1}{(4\pi)^2} \Big[& -(\Delta^{\frac{1}{2}} w_1 \sigma_{,\lambda'} + \Delta^{\frac{1}{2}} w_{1\alpha\lambda'} \sigma_{,}{}^{\alpha})_{,}{}^{\delta}{}_{\delta'} + (\Delta^{\frac{1}{2}} w_1 \sigma_{,\delta'} + \Delta^{\frac{1}{2}} w_{1\alpha\delta'} \sigma_{,}{}^{\alpha})_{,}{}^{\delta}{}_{\lambda'} \Big] \\
+ \frac{1}{(4\pi)^2} \Big[& -(\Delta^{\frac{1}{2}} v_1 \sigma_{,\lambda'} + \Delta^{\frac{1}{2}} v_{1\alpha\lambda'} \sigma_{,}{}^{\alpha})_{,}{}^{\delta}{}_{\delta'} + (\Delta^{\frac{1}{2}} v_1 \sigma_{,\delta'} + \Delta^{\frac{1}{2}} v_{1\alpha\delta'} \sigma_{,}{}^{\alpha})_{,}{}^{\delta}{}_{\lambda'} \Big] \\
= -\frac{1}{(4\pi)^2} \Big[& (\Delta^{\frac{1}{2}} w_{1\alpha\lambda'})_{,}{}^{\alpha'} - (\Delta^{\frac{1}{2}} w_{1\alpha\lambda'})_{,}{}^{\alpha} - (\Delta^{\frac{1}{2}} w_{1\alpha}{}^{\alpha'})_{,\lambda'} + (\Delta^{\frac{1}{2}} w_{1\lambda\beta'})_{,}{}^{\beta} + 3(\Delta^{\frac{1}{2}} w_1)_{,\lambda'} \Big] \\
- \frac{1}{(4\pi)^2} \Big[& (\Delta^{\frac{1}{2}} v_{1\alpha\lambda'})_{,}{}^{\alpha'} - (\Delta^{\frac{1}{2}} v_{1\alpha\lambda'})_{,}{}^{\alpha} - (\Delta^{\frac{1}{2}} v_{1\alpha}{}^{\alpha'})_{,\lambda'} + (\Delta^{\frac{1}{2}} v_{1\lambda\beta'})_{,}{}^{\beta} + 3(\Delta^{\frac{1}{2}} v_1)_{,\lambda'} \Big] .
\end{aligned} \tag{36}$$

Substituting Eq.(33), we can eliminate the terms $(\Delta^{\frac{1}{2}} w_1)_{,\lambda'}$ and $(\Delta^{\frac{1}{2}} v_1)_{,\lambda'}$, giving finally

$$\begin{aligned}
(a) = -\frac{1}{(4\pi)^2} \Big[& (\Delta^{\frac{1}{2}} w_{1\alpha\lambda'})_{,}{}^{\alpha'} - 4(\Delta^{\frac{1}{2}} w_{1\alpha\lambda'})_{,}{}^{\alpha} - (\Delta^{\frac{1}{2}} w_{1\alpha}{}^{\alpha'})_{,\lambda'} + (\Delta^{\frac{1}{2}} w_{1\lambda\beta'})_{,}{}^{\beta} \Big] \\
- \frac{1}{(4\pi)^2} \Big[& (\Delta^{\frac{1}{2}} v_{1\alpha\lambda'})_{,}{}^{\alpha'} - 4(\Delta^{\frac{1}{2}} v_{1\alpha\lambda'})_{,}{}^{\alpha} - (\Delta^{\frac{1}{2}} v_{1\alpha}{}^{\alpha'})_{,\lambda'} + (\Delta^{\frac{1}{2}} v_{1\lambda\beta'})_{,}{}^{\beta} \Big] .
\end{aligned} \tag{37}$$

Similarly carrying out the differentiations in term (b) of Eq.(35), we get

$$(b) = \frac{1}{(4\pi)^2} \Big[(\Delta^{\frac{1}{2}} w^{(B)}_{1\alpha\lambda'})_{,}{}^{\alpha'} - 4(\Delta^{\frac{1}{2}} w^{(B)}_{1\alpha\lambda'})_{,}{}^{\alpha} - (\Delta^{\frac{1}{2}} w^{(B)}_{1}{}_{\alpha}{}^{\alpha'})_{,\lambda'} + (\Delta^{\frac{1}{2}} w^{(B)}_{1\lambda\beta'})_{,}{}^{\beta} \Big] , \tag{38}$$

which when added to the expression in Eq.(37) gives

$$
\begin{aligned}
&[W_{\alpha\lambda',}{}^{\alpha\delta}{}_{\delta'} - W_{\alpha\delta',}{}^{\alpha\delta}{}_{\lambda'}] \\
&= -\frac{1}{(4\pi)^2}\left[D^{\alpha'} w^{(L)}_{1\alpha\lambda'} - 4D^{\alpha} w^{(L)}_{1\alpha\lambda'} - D_{\lambda'} w_{1}^{(L)}{}_{\alpha}{}^{\alpha'} + D^{\beta} w^{(L)}_{1\lambda\beta'}\right] \\
&\quad -\frac{1}{(4\pi)^2}\left[D^{\alpha'} v_{1\alpha\lambda'} - 4D^{\alpha} v_{1\alpha\lambda'} - D_{\lambda'} v_{1\alpha}{}^{\alpha'} + D^{\beta} v_{1\lambda\beta'}\right] \quad . \qquad (39)
\end{aligned}
$$

In writing Eq.(39) we have used the fact, noted above, that derivatives of Δ do not contribute in the coincidence limit. Eq.(39) can be reduced to final form by using the following relations [some of which were already given in Eq.(25) above]

$$
\begin{aligned}
[D^{\alpha'} v_{1\alpha\lambda'}] &= \tfrac{1}{2} D_{\lambda}[v_1] + D_{\alpha}[v_1{}^{\alpha}{}_{\lambda'}] \\
[D^{\alpha} v_{1\alpha\lambda'}] &= -\tfrac{1}{2} D_{\lambda}[v_1] \\
[D_{\lambda'} v_{1\alpha}{}^{\alpha'}] &= \tfrac{1}{2} D_{\lambda}[v_{1\alpha}{}^{\alpha'}] \\
[D^{\beta} v_{1\lambda\beta'}] &= \tfrac{1}{2} D_{\lambda}[v_1] + D_{\alpha}[v_1{}^{\alpha}{}_{\lambda'}] \\
[D^{\alpha'} w^{(L)}_{1\alpha\lambda'}] &= -\tfrac{2}{3} D_{\lambda}[v_1] - \tfrac{3}{2} D_{\alpha}[v_1{}^{\alpha}{}_{\lambda'}] \\
[D^{\alpha} w^{(L)}_{1\alpha\lambda'}] &= \tfrac{2}{3} D_{\lambda}[v_1] \\
[D_{\lambda'} w_{1}^{(L)}{}_{\alpha}{}^{\alpha'}] &= -\tfrac{5}{6} D_{\lambda}[v_{1\alpha}{}^{\alpha'}] \\
[D^{\beta} w^{(L)}_{1\lambda\beta'}] &= -\tfrac{2}{3} D_{\lambda}[v_1] - \tfrac{4}{3} D_{\alpha}[v_1{}^{\alpha}{}_{\lambda'}] \; , \qquad (40)
\end{aligned}
$$

which when substituted into Eq.(39) yield

$$
\begin{aligned}
&[W_{\alpha\lambda',}{}^{\alpha\delta}{}_{\delta'} - W_{\alpha\delta',}{}^{\alpha\delta}{}_{\lambda'}] \\
&= \frac{1}{(4\pi)^2}\left\{ -\tfrac{1}{3} D_{\lambda}[v_{1\gamma}{}^{\gamma'}] + \tfrac{5}{6} D_{\alpha}[v_1{}^{\alpha}{}_{\lambda'}] + D_{\lambda}[v_1] \right\} \quad . \qquad (41)
\end{aligned}
$$

Combining Eqs.(6), (19), (26) and (41), we determine the tensor $t_{\alpha\beta}^{(L)}$ to be

$$t_{\alpha\beta}^{(L)}(x) = \frac{1}{(4\pi)^2}\left\{ -\frac{3}{4}\, g_{\alpha\beta}[v_1{}^{\gamma}{}_{\gamma'}] + [v_{1\alpha\beta'}] + g_{\alpha\beta}[v_1]\right\} \quad . \tag{42}$$

Using Eqs.(2.33) - (2.34) and the formulas of Appendix D of Ref.[2] to evaluate the coincidence limits appearing in Eq.(42),

$$[v_{1\alpha\beta'}] = \frac{1}{2}\, a_{2\alpha\beta} \quad , \qquad [v_1] = \frac{1}{2}\, a_2^{V=0} \quad ,$$

$$a_{2\alpha\beta} = \frac{1}{2}\left(\frac{1}{6}\, g_{\alpha}{}^{\theta}R - R_{\alpha}{}^{\theta}\right)\left(\frac{1}{6}\, g_{\theta\beta}R - R_{\theta\beta}\right) + \frac{1}{30}\, g_{\alpha\beta}R_{,\theta}{}^{\theta} - \frac{1}{180}\, g_{\alpha\beta}R^{\rho\theta}R_{\rho\theta}$$

$$+ \frac{1}{180}\, g_{\alpha\beta}R^{\rho\tau\kappa\theta}R_{\rho\tau\kappa\theta} - \frac{1}{6}\, R_{\alpha\beta,\theta}{}^{\theta} - \frac{1}{12}\, R^{\rho\tau\theta}{}_{\alpha}R_{\rho\tau\theta\beta} \; ,$$

$$a_2^{V=0} = \frac{1}{72}\, R^2 + \frac{1}{30}\, R_{,\theta}{}^{\theta} + \frac{1}{180}\, R^{\rho\tau\kappa\theta}R_{\rho\tau\kappa\theta} - \frac{1}{180}\, R^{\rho\theta}R_{\rho\theta} \; , \tag{43}$$

we get as our final results for the regularized stress-energy tensor and its trace anomaly,

$$\langle T_{\alpha\beta}(x)\rangle = c_1\, I_{\alpha\beta}(x) + c_2\, J_{\alpha\beta}(x) + \langle T_{\alpha\beta}^{(B)}(x)\rangle$$

$$- \frac{1}{(4\pi)^2}\,\frac{1}{2}\left\{-\frac{3}{4}\, g_{\alpha\beta}a_{2\gamma}{}^{\gamma} + a_{2\alpha\beta} + g_{\alpha\beta}a_2^{V=0}\right\} \quad ,$$

$$g_{\alpha\beta}\langle T_{\alpha\beta}(x)\rangle = -2(3c_1 + c_2)R_{,\theta}{}^{\theta} + \frac{1}{(4\pi)^2}\left(a_{2\gamma}{}^{\gamma} - 2a_2^{V=0}\right) \quad . \tag{44}$$

Apart from the undetermined multiple of $R_{,\theta}{}^{\theta}$ arising from the undetermined multiples of $I_{\alpha\beta}$ and $J_{\alpha\beta}$ in $\langle T_{\alpha\beta}(x)\rangle$, the trace anomaly given in Eq.(44) agrees with that found by other calculational methods. We note, in conclusion, that in the present formulation of the regularization calculation the "curvature-dependent modified averaging" prescription of Ref.[2] plays no role, the regularized local part $T^{(L)}_{\alpha\beta}(x)$ having been identified, by general arguments, to have the value given in Eq.(8).

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